mathematics

Article

Tilting and Cotilting in Functor Categories

Junfu Wang [1] and Tiwei Zhao [2,*]

[1] Changzhou College of Information Technology, Changzhou 213164, China
[2] School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China
* Correspondence: tiweizhao@qfnu.edu.cn

Abstract: In this paper, we introduce the notion of n-tilting (resp. n-cotilting) objects in functor categories and give some characterizations of n-tilting objects and n-tilting classes (resp. n-cotilting objects and n-cotilting classes).

Keywords: functor categories; tilting; cotilting; cotorsion pairs

MSC: 18A25; 18G10

1. Introduction

Tilting theory traces its history back to the fundamental work in [1], and later, was generalized by Brenner and Butler in [2]. The notion of tilting modules over finite dimensional algebras and the beginning of the extensive study of tilting theory and tilted algebras are principally due to Happel and Ringel [3], Bongartz [4], and others. After that, some results of tilting theory in module categories were obtained by many authors, see [5–18].

As a higher dimensional generalization of tilting modules of a projective dimension over arbitrary rings, Bazzoni gave in [8] a characterization of n-tilting (resp. n-cotilting) modules in module categories over arbitrary rings, which provided an equivalent condition for a module to be tilting. Then, Wei in [17] characterized n-tilting modules in arbitrary module categories. Angeleri Hügel and Coelho characterized the classes $\mathcal{X}$ induced by generalized tilting modules in terms of the existence of $\mathcal{X}$-preenvelopes in [5].

Let $\mathscr{C}$ be a skeletally small preadditive category. By $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ (resp. $(\mathscr{C}, \mathrm{Ab})$) we denote the functor category whose objects are additive contravariant (resp. covariant) functors from $\mathscr{C}$ to the category Ab of abelian groups and morphisms as the natural transformations between two such functors. If $T, U \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$, we write $\mathrm{Nat}[T, U]$ (or $[T, U]$ for short) for the class of natural transformations from T to U. The induced cohomological group will be denoted by $\mathrm{Ext}^i[T, U]$. Functor categories are of interest in category theory, especially in representation theory of algebra and homological algebra (e.g., [19–26]). The reasons are as follows: on the one hand, many common categories are in fact functor categories, most results coming from functor categories are widely applicable; on the other hand, by applying the well-known Yoneda Lemma, every category can be embeded in a functor category, so that we often obtain our desired properties in the original category by studying the associated functor categories.

Based on the references above, some natural questions arise:

Question A. How can we define the tilting and cotilting objects in the functor categories felicitously?

Question B. Are the characterizations in the functor categories as good as those of the tilting objects in classical tilting theory?

The aim of this paper is to solve these questions for which we introduce the notions of n-tilting (resp. n-cotilting) objects and n-tilting (resp. n-cotilting) classes in the functor category $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ and then provide some of their characterizations.

check for updates

Citation: Wang, J.; Zhao, T. Tilting and Cotilting in Functor Categories. *Mathematics* **2022**, *10*, 3163. https://doi.org/10.3390/math10173163

Academic Editor: Askar Tuganbaev

Received: 6 August 2022
Accepted: 31 August 2022
Published: 2 September 2022

The paper is organized as follows. In Section 2, we give provide some preliminaries and terminology. Based on the result of Mitchell [27], we introduce the notions of n-tilting (resp. n-cotilting) objects and n-tilting (resp. n-cotilting) classes in the functor category $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ and then study some of their basic properties. In Section 3, we give our main results, namely some characterizations of tilting objects and tilting classes in the functor category $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$. The following are Theorems 1 and 2, respectively.

Theorem 1. *Let* $T, U \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$. *Then,*
(1) T *is n-tilting if and only if* $T^{\perp} = \mathrm{Gen}_n\, T$*;*
(2) U *is n-cotilting if and only if* ${}^{\perp}U = \mathrm{Cogen}_n\, T$.

Theorem 2. *Let* $\mathcal{M} \subseteq (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ *be a class of objects. Then, the following assertions are equivalent.*
(1) $\mathcal{M}$ *is n-tilting.*
(2) $\mathcal{M}$ *is coresolving, special preenveloping, and closed under direct sums and direct summands and* ${}^{\perp}\mathcal{M} \subseteq \mathcal{P}_n$.

2. Preliminaries

In this section, $\mathscr{A}$ is an abelian category. For a subcategory of $\mathscr{A}$, we always mean a full and additive subcategory closed under isomorphisms and direct summands.

Definition 1 ([11], Definition 2.2.8, see also [28], Definition 16)**.** *Let* $\mathscr{A}$ *be an abelian category with enough projective and injective objects. A subcategory* $\mathscr{T}$ *of* $\mathscr{A}$ *is resolving if it is closed under extensions, kernels of epimorphisms and contains the projective objects in* $\mathscr{A}$*. Dually,* $\mathscr{T}$ *is coresolving if it is closed under extensions and cokernels of monomorphisms and contains the injective objects in* $\mathscr{A}$*.*

Assume that $\mathscr{A}$ has enough projective and injective objects. For every subcategory $\mathscr{T}$ of $\mathscr{A}$, we set

$$\mathscr{T}^{\perp} := \{X \in \mathscr{A} \mid \mathrm{Ext}^i_{\mathscr{A}}(C, X) = 0 \text{ for all } C \in \mathscr{T}, i \geq 1\},$$

$${}^{\perp}\mathscr{T} := \{X \in \mathscr{A} \mid \mathrm{Ext}^i_{\mathscr{A}}(X, C) = 0 \text{ for all } C \in \mathscr{T}, i \geq 1\},$$

and

$$\mathscr{T}^{\perp_1} := \{X \in \mathscr{A} \mid \mathrm{Ext}^1_{\mathscr{A}}(C, X) = 0 \text{ for all } C \in \mathscr{T}\},$$

$${}^{\perp_1}\mathscr{T} := \{X \in \mathscr{A} \mid \mathrm{Ext}^1_{\mathscr{A}}(X, C) = 0 \text{ for all } C \in \mathscr{T}\}.$$

A pair $(\mathbb{A}, \mathbb{B})$ of subcategories in $\mathscr{A}$ is called a *cotorsion pair* if $\mathbb{A} = {}^{\perp_1}\mathbb{B}$ and $\mathbb{B} = \mathbb{A}^{\perp_1}$ ([11], Definition 2.2.1). For every subcategory $\mathscr{T}$, ${}^{\perp}\mathscr{T}$ is resolving and $\mathscr{T}^{\perp}$ is coresolving.

Note that if $\mathscr{T}$ is resolving, then $\mathscr{T}^{\perp} = \mathscr{T}^{\perp_1}$; if $\mathscr{T}$ is coresolving, then ${}^{\perp}\mathscr{T} = {}^{\perp_1}\mathscr{T}$. A pair $(\mathbb{A}, \mathbb{B})$ is called a *hereditary cotorsion pair* if $\mathbb{A} = {}^{\perp}\mathbb{B}$ and $\mathbb{B} = \mathbb{A}^{\perp}$. A cotorsion pair $(\mathbb{A}, \mathbb{B})$ is hereditary if and only if $\mathbb{A}$ is resolving if and only if $\mathbb{B}$ is coresolving ([11], Lemma 2.2.10).

A concept very useful when dealing with cotorsion pairs is the notion of approximations via precovers and preenvelopes defined by Enochs in [29] as a generalization of the notion of right and left approximations introduced by Auslander and Smalø [30] in representation theory of finite dimensional algebras. We recall now these definitions.

Let $\mathscr{T}$ be a class of objects in $\mathscr{A}$. Following [29,30], we say that a morphism $\phi : C \to A$ in $\mathscr{A}$ is a $\mathscr{T}$*-precover* of A if $C \in \mathscr{T}$, and, for any morphism $f : C' \to A$ with $C' \in \mathscr{T}$, there is a morphism $g : C' \to C$ such that $\phi g = f$. A $\mathscr{T}$-precover $\phi : C \to A$ is said to be a $\mathscr{T}$*-cover* of A if every endomorphism $g : C \to C$ such that $\phi g = \phi$ is an isomorphism. A $\mathscr{T}$-precover $\phi : C \to A$ is said to be *special* if it is an epimorphism and $\mathrm{Ker}\, \phi \in \mathscr{T}^{\perp_1}$. Dually, we have the definitions of a $\mathscr{T}$*-preenvelope*, a $\mathscr{T}$*-envelope*, and a *special* $\mathscr{T}$*-preenvelope*. $\mathscr{T}$-covers ($\mathscr{T}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphisms.

A class $\mathscr{T}$ is said to be *precovering, covering, special precovering* (*preenveloping, enveloping, special preenveloping*), respectively, if every object in $\mathscr{A}$ admits a $\mathscr{T}$-precover, a $\mathscr{T}$-cover, a special $\mathscr{T}$-precover (a $\mathscr{T}$-preenvelope, a $\mathscr{T}$-envelope, a special $\mathscr{T}$-preenvelope) respectively.

A cotorsion pair $(\mathbb{A}, \mathbb{B})$ is said to be *complete* if every object in $\mathscr{A}$ admits a special $\mathbb{A}$-precover and a special $\mathbb{B}$-preenvelope. In fact, by ([11], Proposition 1.1), a cotorsion pair $(\mathbb{A}, \mathbb{B})$ in $\mathscr{A}$ is complete if and only if $\mathbb{A}$ is special precovering and if and only if $\mathbb{B}$ is special preenveloping.

In this sequel, we mainly work on the functor category $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$, where $\mathscr{C}$ is a skeletally small preadditive category. Note that the category $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ admits arbitrary coproducts; products and the direct products are exact, and it satisfies Grothendieck's AB5 condition, that is, it has exact filtered limits.

Let $\mathcal{M} \subseteq (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ be a class of additive contravariant functors from $\mathscr{C}^{\mathrm{op}}$ to Ab. We denote by $\mathrm{Add}\,\mathcal{M}$ (resp. $\mathrm{Prod}\,\mathcal{M}$) the subcategory consisting of all additive contravariant functors isomorphic to direct summands of direct sums (resp. direct products) of elements of $\mathcal{M}$. If $\mathcal{M} = \{M\}$ with $M \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$, then we shall denote these subcategories by $\mathrm{Add}\, M$ and $\mathrm{Prod}\, M$, respectively.

Given an object $M \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$, we write $\mathrm{Gen}\, M$ for the subcategory of all M-generated objects in $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$, that is, those objects X admitting an epimorphism $M_1 \to X$ with $M_1 \in \mathrm{Add}\, M$. The subcategory of M-cogenerated objects, that is, those objects X admitting a monomorphism $X \to M_1$ with $M_1 \in \mathrm{Prod}\, M$, is denoted by $\mathrm{Cogen}\, M$.

The following lemma is useful in this paper, it is cited from ([5], Proposition 1.1), see also [31]. Here, we talk about a similar version in functor categories, and give the proof for the reader's convenience.

Lemma 1. *Let $M \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$. Then,* $\mathrm{Add}\, M$ *is precovering, and* $\mathrm{Prod}\, M$ *is preenveloping.*

Proof. For any $T \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$, let $I = [M, T]$; then, the codiagonal map $M^{(I)} \to T$ induced by all homomorphisms is an $\mathrm{Add}\, M$-precover. Dually, for $J = [T, M]$ the diagonal map $T \to M^J$ is a $\mathrm{Prod}\, M$-preenvelope. □

Following Mitchell [27], one has that $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ is an abelian category with a projective generator and an injective cogenerator. Using it, we give the following definitions.

Definition 2. *An object $T \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ is said to be n-tilting provided that:*

(T1) $\mathrm{pd}\, T \leq n$;

(T2) $\mathrm{Ext}^i[T, T^{(\lambda)}] = 0$ *for each $i > 0$ and for every cardinal λ;*

(T3) there exists a long exact sequence

$$0 \to P \to T_0 \to T_1 \to \cdots \to T_r \to 0,$$

where P is a projective generator in $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$, and $T_i \in$ $\mathrm{Add}\, T$ *for every $0 \leq i \leq r$.*

In this case, the associated class $T^{\perp} := \{M \mid \mathrm{Ext}^i[T, M] = 0$ for any $i > 0\}$ is called the n-tilting class induced by T. Clearly, $({}^{\perp_1}(T^{\perp}), T^{\perp})$ is a hereditary cotorsion pair in $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$, called the n-tilting cotorsion pair induced by T.

Dually, we have the following definition.

Definition 3. *An object $U \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ is said to be n-cotilting provided that:*

(C1) $\mathrm{id}\, U \leq n$;

(C2) $\mathrm{Ext}^i[U^{\lambda}, U] = 0$ *for each $i > 0$ and for every cardinal λ;*

(C3) there exists a long exact sequence

$$0 \to U_r \to \cdots \to U_1 \to U_0 \to Q \to 0,$$

where Q is an injective cogenerator in $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$, and $U_i \in$ $\mathrm{Prod}\, U$ *for every $0 \leq i \leq r$.*

In this case, the class ${}^{\perp}U$ is called the n-cotilting class induced by U. Clearly, $({}^{\perp}U, ({}^{\perp}U)^{\perp_1})$ is a hereditary cotorsion pair in $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$, called the n-cotilting cotorsion pair induced by U.

Definition 4. *(1) Let* $T \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$*. We write*

$$\begin{aligned}\mathrm{Gen}_\infty T =& \{H \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab}) \mid \text{ there exists an exact sequence} \\ & \cdots \to T^{(\lambda_n)} \to \cdots \to T^{(\lambda_2)} \to T^{(\lambda_1)} \to H \to 0 \text{ for some cardinals } \lambda_i\}; \\ \mathrm{Gen}_n T =& \{H \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab}) \mid \text{ there exists an exact sequence} \\ & T^{(\lambda_n)} \to \cdots \to T^{(\lambda_2)} \to T^{(\lambda_1)} \to H \to 0 \text{ for some cardinals } \lambda_i\}.\end{aligned}$$

In particular, $\mathrm{Gen}_1 T = \mathrm{Gen}\, T$*.*
(2) Let $U \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$*. We write*

$$\begin{aligned}\mathrm{Cogen}_\infty U =& \{G \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab}) \mid \text{ there exists an exact sequence} \\ & 0 \to G \to U^{\alpha_1} \to U^{\alpha_2} \to \cdots \to U^{\alpha_n} \to \cdots \text{ for some cardinals } \alpha_i\}; \\ \mathrm{Cogen}_n U =& \{G \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab}) \mid \text{ there exists an exact sequence} \\ & 0 \to G \to U^{\alpha_1} \to U^{\alpha_2} \to \cdots \to U^{\alpha_n} \text{ for some cardinals } \alpha_i\}.\end{aligned}$$

In particular, $\mathrm{Cogen}_1 U = \mathrm{Cogen}\, U$*.*

Lemma 2. *Let* $T, U \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$*.*
(1) If T *satisfies the conditions (T2) and (T3), then* $T^\perp \subseteq \mathrm{Gen}\, T$*.*
(2) If U *satisfies the conditions (C2) and (C3), then* ${}^\perp U \subseteq \mathrm{Cogen}\, U$*.*

Proof. (1) Consider the following sequence given by the condition (T3):

$$0 \to P \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \to \cdots \xrightarrow{f_n} T_n \to 0,$$

with P a projective generator and $T_i \in \mathrm{Add}\, T$ for every $0 \le i \le n$. Clearly, we have that $T_i \in T^\perp \cap {}^{\perp_1}(T^\perp)$ by (T2). Notice that ${}^{\perp_1}(T^\perp)$ is resolving, we infer that $K_i = \mathrm{Ker}\, f_i \in {}^{\perp_1}(T^\perp)$ for each $1 \le i \le n$. Let $G \in T^\perp$. There exists some cardinal λ, such that $g : P^{(\lambda)} \to G$ is epic. Consider the pushout diagram:

$$\begin{tikzcd} 0 \arrow[r] & P^{(\lambda)} \arrow[r, "f_0^{(\lambda)}"] \arrow[d, "g"] & T_0^{(\lambda)} \arrow[r] \arrow[d] & K_2^{(\lambda)} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & G \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, dashed] & K_2^{(\lambda)} \arrow[r] & 0. \\ & 0 & 0 & & \end{tikzcd}$$

Since $K_2^{(\lambda)} \in {}^{\perp_1}(T^\perp)$, the second row splits, so G is a direct summand of F. Since $F \in \mathrm{Gen}\, T_0 \subseteq \mathrm{Gen}\, T$, and $G \in \mathrm{Gen}\, T$. This implies that $T^\perp \subseteq \mathrm{Gen}\, T$.

The proof of (2) is the dual. □

Lemma 3. *Let* $T, U \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$*.*
(1) If T *satisfies the condition (T2) and* $T^\perp \subseteq \mathrm{Gen}\, T$*, then*
(i) for each $W \in T^\perp$*, there exists a short exact sequence*

$$0 \to F \to T_1 \to W \to 0,$$

with $T_1 \in \mathrm{Add}\, T$ *and* $F \in T^\perp$*;*
(ii) every map $V \to W$ *with* $V \in {}^\perp(T^\perp)$ *and* $W \in T^\perp$ *factors through* $\mathrm{Add}\, T$*. In particular, we have* $\mathrm{Add}\, T = T^\perp \cap {}^\perp(T^\perp)$*.*
(2) If U *satisfies the condition (C2), and* ${}^\perp U \subseteq \mathrm{Cogen}\, U$*, then*

(i) for each $W \in {}^{\perp}U$, there exists a short exact sequence

$$0 \to W \to U_1 \to G \to 0,$$

with $U_1 \in \operatorname{Prod} U$ and $G \in {}^{\perp}U$;

(ii) every map $W \to J$ with $J \in ({}^{\perp}U)^{\perp}$ and $W \in {}^{\perp}U$ factors through $\operatorname{Prod} U$. *In particular, we have* $\operatorname{Prod} U = {}^{\perp}U \cap ({}^{\perp}U)^{\perp}$.

Proof. We only prove (1), and (2) is dual.

(i) Let $W \in T^{\perp}$. By Lemma 1, there exists an $\operatorname{Add} T$-precover $g : T_1 \to W$ with $T_1 \in \operatorname{Add} T$. Clearly, g is an epimorphism, since $W \in T^{\perp} \subseteq \operatorname{Gen} T$. We claim that $F = \operatorname{Ker} g$ belongs to $T^{\perp}$. Indeed, we observe that $\operatorname{Ext}^1[T, F] = 0$ because $[T, g]$ is an epimorphism, and $\operatorname{Ext}^1[T, T_1] = 0$ (by (T2)). For $i \geq 1$, consider the sequence

$$\operatorname{Ext}^i[T, W] \to \operatorname{Ext}^{i+1}[T, F] \to \operatorname{Ext}^{i+1}[T, T_1] = 0.$$

Since $W \in T^{\perp}$, we obtain $\operatorname{Ext}^{i+1}[T, F] = 0$ for $i \geq 1$. So $F \in T^{\perp}$.

(ii) Let $f : V \to W$ be a map with $V \in {}^{\perp}(T^{\perp})$ and $W \in T^{\perp}$. By (i), there exists a short exact sequence

$$0 \to F \to T_1 \xrightarrow{g} W \to 0$$

with $T_1 \in \operatorname{Add} T$, we obtain f factors through g as required, since $\operatorname{Ext}^1[V, F] = 0$. For $\operatorname{Add} T = T^{\perp} \cap {}^{\perp}(T^{\perp})$, we observe that for $H \in T^{\perp} \cap {}^{\perp}(T^{\perp})$, its identity map id_H factors through $\operatorname{Add} T$, and so $H \in \operatorname{Add} T$. The other inclusion follows directly from the condition (T2). □

Proposition 1. *(1) Let $T \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$. If T is n-tilting, then $T^{\perp} = \operatorname{Gen}_n T$. In particular, $T^{\perp}$ is closed under direct sums. Moroever, $\operatorname{Gen}_n T = \operatorname{Gen}_{n+k} T = \operatorname{Gen}_{\infty} T$, for every $k \geq 0$.*

(2) Let $U \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$. If U is n-cotilting, then ${}^{\perp}U = \operatorname{Cogen}_n T$. In particular, ${}^{\perp}U$ is closed under direct products. Moroever, $\operatorname{Cogen}_n T = \operatorname{Cogen}_{n+k} T = \operatorname{Cogen}_{\infty} T$, for every $k \geq 0$.

Proof. We only prove (1), and (2) is dual.

Let T be an n-tilting object in $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$. We first claim that $T^{\perp} = \operatorname{Gen}_{\infty} T$. In fact, for any $W \in T^{\perp}$, by Lemma 3(1), there exists an exact infinite sequence of the form

$$\cdots \to T_n \to \cdots \to T_2 \to T_1 \to W \to 0$$

with $T_i \in \operatorname{Add} T$. So, by adding suitable direct sums of copies of T to T_i, we obtain the following sequence of the form

$$\cdots \to T^{(\alpha_n)} \to \cdots \to T^{(\alpha_2)} \to T^{(\alpha_1)} \to W \to 0$$

for some cardinals α_i, that is, $W \in \operatorname{Gen}_{\infty} T$. The other inclusion follows directly from dimension shifting. Clearly, $T^{\perp}$ is closed under direct sums by the claim. Next we prove the "MOROEVER", and then we complete the proof. Note that $\operatorname{Gen}_{\infty} T \subseteq \operatorname{Gen}_{n+k} T \subseteq \operatorname{Gen}_n T$ for every $k \geq 0$. Conversely, suppose $H \in \operatorname{Gen}_n T$; that is, there exists an exact sequence

$$T^{(\alpha_n)} \xrightarrow{f_n} \cdots \to T^{(\alpha_2)} \xrightarrow{f_2} T^{(\alpha_1)} \xrightarrow{f_1} H \to 0$$

for some cardinals α_i. Let $K_i = \operatorname{Ker} f_i$ for each $1 \leq i \leq n$. By dimension shifting, $\operatorname{Ext}^i[T, H] \cong \operatorname{Ext}^{i+n}[T, K_n]$, for each $i \geq 1$, and we obtain $H \in T^{\perp}$, since $\operatorname{pd} T \leq n$. Hence, $H \in \operatorname{Gen}_{\infty} T$ by the claim. □

The following lemma is important for the main results in Section 3, it is cited from ([11], Theorem 3.2.1). Here, we give a similar version in functor categories. We leave the details of the proof for the reader.

Lemma 4. *Let $\mathcal{S}$ be a set of objects in $(\mathscr{C}^{\rm op}, \mathrm{Ab})$. Then, $\mathcal{S}^{\perp_1}$ is special preenveloping.*

Recall from [7] that for a subcategory $\mathcal{X} \subseteq \mathscr{A}$, we denote by $\widehat{\mathcal{X}}$ the subcategory of $\mathscr{A}$ whose objects are the C for which there is some nonnegative integer n and an exact sequence

$$0 \to X_n \to \cdots \to X_0 \to C \to 0$$

with X_i in $\mathcal{X}$. Dually, we denote by $\widetilde{\mathcal{X}}$ the subcategory of $\mathscr{A}$ whose objects are the C for which there are some nonnegative integer n and an exact sequence

$$0 \to C \to X_0 \to \cdots \to X_n \to 0$$

with X_i in $\mathcal{X}$.

For a fixed nonnegative integer n, we use $\mathcal{P}_n$ (resp. $\mathcal{I}_n$) to denote the subcategory consisting of all objects in $(\mathscr{C}^{\rm op}, \mathrm{Ab})$ with projective (resp. injective) dimensions at most n.

Proposition 2. *Let $M \in (\mathscr{C}^{\rm op}, \mathrm{Ab})$ and n be a nonnegative integer.*
(1) If $\mathrm{pd}\, M \leq n$, then $\widetilde{M^{\perp}} = \mathrm{Mod}\,\mathscr{C}$, and ${}^{\perp}(M^{\perp}) \subseteq \mathcal{P}_n$.
(2) If $\mathrm{id}\, M \leq n$, then $\widehat{{}^{\perp}M} = \mathrm{Mod}\,\mathscr{C}$, and $({}^{\perp}M)^{\perp} \subseteq \mathcal{I}_n$.

Proof. We only prove (2), and (1) is the dual.

Let $X \in (\mathscr{C}^{\rm op}, \mathrm{Ab})$. Consider the long exact sequence

$$0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to X \to 0$$

with P_i projective. Since $\mathrm{id}\, M \leq n$, we have that $\mathrm{Ext}^i[K_n, M] \cong \mathrm{Ext}^{i+n}[X, M] = 0$ for all $i > 0$; that is, $K_n \in {}^{\perp}M$, and so $X \in \widehat{{}^{\perp}M}$. Let $Y \in ({}^{\perp}M)^{\perp}$. We obtain $\mathrm{Ext}^{i+n}[X, Y] \cong \mathrm{Ext}^i[K_n, Y] = 0$ for each $i > 0$, since $K_n \in {}^{\perp}M \subseteq {}^{\perp}Y$. By the former argument, X is arbitrary, and we infer that $\mathrm{id}\, Y \leq n$. □

Lemma 5 ([32], Theorem 1.1)**.** *Let $\mathscr{B} \subseteq \mathscr{A}$ be closed under extensions, and $\omega \subseteq \mathscr{B}$. Suppose there exists, for each $B \in \mathscr{B}$, a short exact sequence*

$$0 \to B \to W \to L \to 0$$

with $W \in \omega$ and $L \in \mathscr{B}$. Then, for each $C \in \widehat{\mathscr{B}}$, there exists short exact sequences

$$0 \to W_c \to B_c \to C \to 0, \text{ and}$$

$$0 \to C \to W^c \to B^c \to 0$$

with $B_c, B^c \in \mathscr{B}$ and $W_c, W^c \in \widehat{\omega}$.

Lemma 6. *Let $U \in (\mathscr{C}^{\rm op}, \mathrm{Ab})$ be an n-cotilting object. Then, ${}^{\perp}U$ is special precovering.*

Proof. Put $\mathscr{A} = (\mathscr{C}^{\rm op}, \mathrm{Ab}), \mathscr{B} = {}^{\perp}U$, and $\omega = \mathrm{Prod}\, U = {}^{\perp}U \cap ({}^{\perp}U)^{\perp}$ (by Lemma 3(2)). It follows from Lemma 3(2) that, for each $B \in \mathscr{B}$, there exists a short exact sequence

$$0 \to B \to W \to L \to 0$$

with $W \in \omega$ and $L \in \mathscr{B}$. By Lemma 5, for each $H \in \widehat{\mathscr{B}} = (\mathscr{C}^{\rm op}, \mathrm{Ab})$ (by Proposition 2(2)), we obtain a short exact sequence

$$0 \to F \to G \xrightarrow{f} H \to 0$$

with $G \in {}^{\perp}U$ and $F \in \widehat{\omega}$. Notice that $\omega \subseteq \mathscr{B}^{\perp}$; we infer that $F \in \widehat{\omega} \subseteq \mathscr{B}^{\perp} = ({}^{\perp}U)^{\perp}$. So, f is a special ${}^{\perp}U$-precover, as required. □

3. Main Results

In this section, we will give some characterizations of tilting objects and tilting classes in the functor category $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$. The dual versions for cotilting are also true. We first show that the converse of Proposition 1 holds. Here, we need the following lemma.

Lemma 7. *Let $T, U \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$.*
(1) Assume that $T^{\perp} = \mathrm{Gen}_n\, T$. Then, T satisfies (T1) and (T2).
(2) Assume that ${}^{\perp}U = \mathrm{Cogen}_n\, T$. Then, U satisfies (C1) and (C2).

Proof. We only prove (1), and (2) is dual.

Let $T^{\perp} = \mathrm{Gen}_n\, T$ ($\subseteq \mathrm{Gen}\, T$). Clearly, (T2) holds, since $T^{(\lambda)} \in \mathrm{Gen}_n\, T = T^{\perp}$ for every cardinal λ. We prove that $\mathrm{pd}\, T \leq n$. For any $H \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$, we consider an injective resolution of H:

$$0 \to H \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \to \cdots \to I_{j-1} \xrightarrow{f_j} I_j.$$

Let $K_m = \mathrm{Coker}\, f_{m-1}$ for $1 \leq m \leq j$. By Lemma 3(1), there exists a cardinal α_i and an exact sequence

$$0 \to W_i \to T^{(\alpha_i)} \to I_i \to 0$$

with $W_i \in T^{\perp}$ for every I_i. We claim that $K_m \in \mathrm{Gen}_m\, T$ for every $m \leq n$.

We proceed with the proof by induction on m. If $m = 1$, notice that $I_0 \in \mathrm{Gen}\, T$, and we have $K_1 \in \mathrm{Gen}\, T$. We assume that the claim is true for K_m ($m < n$). Then, we have two exact sequences

$$0 \to K_m \to I_m \to K_{m+1} \to 0, \text{ and}$$
$$0 \to W_m \to T^{(\alpha_m)} \to I_m \to 0.$$

Consider the following pullback diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & W_m \arrow[r, equal] \arrow[d] & W_m \arrow[d] & & \\
0 \arrow[r] & X \arrow[r, dashed] \arrow[d, dashed] & T^{(\alpha_m)} \arrow[r] \arrow[d] & K_{m+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_m \arrow[r] \arrow[d] & I_m \arrow[r] \arrow[d] & K_{m+1} \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

To prove $K_{m+1} \in \mathrm{Gen}_{m+1}\, T$, it suffices to check $X \in \mathrm{Gen}_m\, T$. Consider the second column

$$0 \to W_m \to X \to K_m \to 0$$

with $W_m \in T^{\perp} = \mathrm{Gen}_n\, T \subseteq \mathrm{Gen}_m\, T$ and $K_m \in \mathrm{Gen}_m\, T$. By Lemma 3(1) and the Horseshoe Lemma, it is not hard to prove that $X \in \mathrm{Gen}_m\, T$.

So, in particular, $K_n \in \mathrm{Gen}_n\, T = T^{\perp}$. By dimension shifting, we obtain $\mathrm{Ext}^{n+1}[T, H] \cong \mathrm{Ext}^1[T, K_n] = 0$; that is, $\mathrm{pd}\, T \leq n$, since H is arbitrary. □

Now we give the main results in this paper.

Theorem 1. *Let $T, U \in (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$. Then,*
(1) T is n-tilting if and only if $T^{\perp} = \mathrm{Gen}_n\, T$;
(2) U is n-cotilting if and only if ${}^{\perp}U = \mathrm{Cogen}_n\, T$.

Proof. We only prove (1), and (2) is dual.

The necessity is trivial by Proposition 1(1). For the sufficiency, let $T^{\perp} = \mathrm{Gen}_n T$ ($\subseteq \mathrm{Gen}\, T$). Then, (T1) and (T2) follow from Lemma 7(1).

Next, we show that T satisfies (T3). Since $\mathrm{pd}\, T \leq n$ (by (T1)), there exists a projective resolution of T

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to T \to 0$$

with the syzygies $S_0 = T, \cdots, S_n = P_n$. Take $S = \bigoplus_{i \leq n} S_i$; then, $T^{\perp} = S^{\perp_1}$. So, by Lemma 4, there exists a $T^{\perp}$-preenvelope $f : V \to W$ for every $V \in {}^{\perp}(T^{\perp})$. By Lemma 3(1), f factors through a map $g : V \to W'$ with $W' \in \mathrm{Add}\, T$; hence, g is an $\mathrm{Add}\, T$-preenvelope for every $V \in {}^{\perp}(T^{\perp})$ since $\mathrm{Add}\, T \subseteq T^{\perp}$ (by (T2)). From (ii) of Lemma 3(1), we infer that all homomorphisms $V \to H$ with $V \in {}^{\perp}(T^{\perp})$ and $H \in T^{\perp}$ factor through $\mathrm{Add}\, T$ and therefore factor through g. In particular, this applies to any monomorphism $V \to I$ with I injective, showing that g is a monomorphism. We claim that $K = \mathrm{Coker}\, g \in {}^{\perp}(T^{\perp})$. In fact, for any $X \in T^{\perp}$ we have that $[g, X]$ is an epimorphism, and $\mathrm{Ext}^1[W', X] = 0$, which implies $\mathrm{Ext}^1[K, X] = 0$; that is, $K \in {}^{\perp_1}(T^{\perp})$. Notice that $T^{\perp}$ is coresolving, we obtain $K \in {}^{\perp}(T^{\perp})$. Let us now take $V = P$, where P is a projective generator in $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$. Iterating the above construction, we obtain an exact sequence

$$0 \to P \to T_0 \to T_1 \to \cdots \to T_{n-1} \to K_n \to 0$$

with $T_i \in \mathrm{Add}\, T$, and all cokernels in ${}^{\perp}(T^{\perp})$. So, we infer that $\mathrm{Ext}^i[T, K_n] \cong \mathrm{Ext}^{i+n}[T, P] = 0$ for all $i > 0$; hence, $K_n \in T^{\perp} \cap {}^{\perp}(T^{\perp}) = \mathrm{Add}\, T$ by Lemma 3(1), and the above sequence gives the one required in condition (T3). □

Proposition 3. *Let T be an n-tilting $\mathscr{C}$-module and U an n-cotilting $\mathscr{C}$-module. Then,*

(1) the n-tilting cotorsion pair $({}^{\perp_1}(T^{\perp}), T^{\perp})$ is complete, and ${}^{\perp_1}(T^{\perp}) \subseteq \mathcal{P}_n$;

(2) the n-cotilting cotorsion pair $({}^{\perp}U, ({}^{\perp}U)^{\perp_1})$ is complete, and $({}^{\perp}U)^{\perp_1} \subseteq \mathcal{I}_n$.

Proof. (1) Let

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to T \to 0$$

be a projective resolution of T with the syzygies $K_0 = T, \cdots, K_n = P_n$. Take $S = \bigoplus_{i \leq n} K_i$; then, $T^{\perp} = S^{\perp_1}$. Clearly, the cotorsion pair $({}^{\perp_1}(T^{\perp}), T^{\perp})$ is complete by Lemma 4. Notice that ${}^{\perp_1}(T^{\perp}) = {}^{\perp}(T^{\perp})$, and the second part follows from Proposition 2(1).

(2) The first part follows from Lemma 6 and the second part follows from Proposition 2(2). □

The following result, due to Angeleri Hügel and Coelho [5], is proved for module categories over rings, see also Trlifaj [11]. Here, we give the counterpart in functor categories.

Theorem 2. *Let $\mathcal{M} \subseteq (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ be a class of objects. Then, the following assertions are equivalent.*

(1) $\mathcal{M}$ is n-tilting.

(2) $\mathcal{M}$ is coresolving, special preenveloping, closed under direct sums and direct summands, and ${}^{\perp}\mathcal{M} \subseteq \mathcal{P}_n$.

Proof. (1)⇒(2) Let $\mathcal{M}$ be an n-tilting class, that is, there exists an n-tilting object T such that $\mathcal{M} = T^{\perp}$. This follows from Proposition 3(1) and Proposition 1(1).

(2)⇒(1) First, ${}^{\perp}\mathcal{M} = {}^{\perp_1}\mathcal{M}$, since $\mathcal{M}$ is coresolving.

Let P be a projective generator in $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$. Because $\mathcal{M}$ is special preenveloping, there exists a short exact sequence

$$0 \to P \to M_0 \to K_1 \to 0,$$

with $M_0 \in \mathcal{M}$ and $K_1 \in {}^{\perp}\mathcal{M} \subseteq \mathcal{P}_n$. We have $M_0 \in \mathcal{M} \cap {}^{\perp}\mathcal{M}$, since $P \in {}^{\perp}\mathcal{M}$. By induction, we obtain short exact sequences

$$0 \to K_i \to M_i \to K_{i+1} \to 0,$$

with $M_i \in \mathcal{M} \cap {}^{\perp}\mathcal{M}$ and $K_{i+1} \in {}^{\perp}\mathcal{M} \subseteq \mathcal{P}_n$ for any i. Since $K_{n+1} \in \mathcal{P}_n$, we have $\mathrm{Ext}^1[K_{n+1}, K_n] \cong \mathrm{Ext}^{n+1}[K_{n+1}, P] = 0$; then, the sequence

$$0 \to K_n \to M_n \to K_{n+1} \to 0$$

splits. So, we can assume that $K_{n+1} = 0$ and form the long exact sequence

$$0 \to P \to M_0 \to M_1 \to \cdots \to M_{n-1} \to M_n \to 0$$

with $M_i \in \mathcal{M} \cap {}^{\perp}\mathcal{M}$ for all $i \leq n$. Put $T = \bigoplus_{i \leq n} M_i$. We will prove that T is n-tilting. Clearly, (T1) holds since $T \in \mathcal{M} \cap {}^{\perp}\mathcal{M} \subseteq \mathcal{P}_n$, and the long exact sequence above gives (T3). Since $\mathcal{M}$ is closed under direct sums, $T^{(\lambda)} \in \mathcal{M}$ for each cardinal λ, and (T2) holds.

Next, we show that $\mathcal{M} = T^{\perp}$. First, we observe that $\mathcal{M} \subseteq T^{\perp}$ since $T \in {}^{\perp}\mathcal{M}$. Conversely, suppose $H \in T^{\perp}$. Since $\mathcal{M}$ is special preenveloping, repeatedly, it follows from the former argument that there exists an exact sequence of finite length

$$0 \to H \xrightarrow{f_0} V_0 \xrightarrow{f_1} V_1 \to \cdots \to V_{n-1} \xrightarrow{f_n} V_n \to 0$$

with $V_i \in \mathcal{M} \subseteq T^{\perp}$ for all $i < n$, and $V_n \in \mathcal{M} \cap {}^{\perp}\mathcal{M} \subseteq \mathcal{P}_n$. Since $H \in T^{\perp}$, and $T^{\perp}$ is coresolving, we infer that $L_i = \mathrm{Coker}\, f_{i-1} \in T^{\perp}$ for all $1 \leq i \leq n-1$. We claim that $\mathcal{M} \cap {}^{\perp}\mathcal{M} \subseteq {}^{\perp_1}(T^{\perp})$. So f_n splits, and by induction, f_0 splits; that is, $H \in \mathcal{M}$ since $\mathcal{M}$ is closed under direct summands.

Proof of the claim: suppose $W \in \mathcal{M} \cap {}^{\perp}\mathcal{M}$. We observe that $W \in T^{\perp} \cap \mathcal{P}_n$. Notice that T is n-tilting, by Lemma 3(1), it is easy to show that there exists a long exact sequence

$$0 \to T_n \to \cdots \to T_1 \to T_0 \xrightarrow{\varphi_0} W \to 0$$

with $T_i \in \mathrm{Add}\, T$ for all $i \leq n$. Since $\mathcal{M}$ is closed under direct sums and direct summands, $T_i \in \mathrm{Add}\, T \subseteq \mathcal{M}$, and we infer that $\mathrm{Ker}\, \varphi_0 \in \mathcal{M}$ since $\mathcal{M}$ is coresolving; then, the sequence

$$0 \to \mathrm{Ker}\, \varphi_0 \to T_0 \to W \to 0$$

splits. So $W \in \mathrm{Add}\, T \subseteq {}^{\perp_1}(T^{\perp})$. □

Corollary 1. *Let $\mathfrak{C} = (\mathbb{A}, \mathbb{B})$ be a cotorsion pair in $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$. Then, the following assertions are equivalent.*

(1) $\mathfrak{C}$ is an n-tilting cotorsion pair.

(2) $\mathfrak{C}$ is complete and hereditary, $\mathbb{A} \subseteq \mathcal{P}_n$ and $\mathbb{B}$ is closed under direct sums.

Proof. Easy. □

Using Lemma 3(2), Propositions 1(2) and 3(2), we can obtain the dual versions of Theorem 2 and Corollary 1. We leave the details of the proof for the reader.

Theorem 3. *Let $\mathcal{M} \subseteq (\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ be a class of objects. Then, the following assertions are equivalent.*

(1) $\mathcal{M}$ is n-cotilting.

(2) $\mathcal{M}$ is resolving, special precovering, closed under direct products and direct summands, and $\mathcal{M}^{\perp} \subseteq \mathcal{I}_n$.

Corollary 2. *Let $\mathfrak{C} = (\mathbb{A}, \mathbb{B})$ be a cotorsion pair in $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$. Then, the following assertions are equivalent.*

(1) $\mathfrak{C}$ is an n-cotilting cotorsion pair.

(2) $\mathfrak{C}$ is complete and hereditary, $\mathbb{B} \subseteq \mathcal{I}_n$ and $\mathbb{A}$ is closed under direct products.

Author Contributions: Writing—original draft preparation, J.W. and T.Z.; writing—review and editing, J.W. and T.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by NSFC (11901341, 11971225) and the project ZR2019QA015 supported by Shandong Provincial Natural Science Foundation.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the referees for the helpful suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bernstein, I.N.; Gelfand, I.M.; Ponomarev, V.A. Coxeter functors and Gabriel's theorem. *Usp. Mat. Nauk* **1973**, *28*, 19–33.
2. Brenner, S.; Butler, M. Generalizations of the Bernstein-Gelfand-Ponomarev refection functors. In Proceedings of the Second International Conference on Representations of Algebras, Ottawa, ON, Canada, 13–25 August 1979; LNM 832; Springer: Berlin, Germany, 1980; pp. 103–169.
3. Happel, D.; Ringel, C.M. Tilted algebras. *Trans. Am. Math. Soc.* **1982**, *274*, 399–443.
4. Bongartz, K. Tilted algebras. In Proceedings of the Third International Conference on Representations of Algebras, Puebla, Mexico, 4–8 August 1980; LNM 903; Springer: Berlin, Germany, 1981; pp. 26–38.
5. Angeleri Hügel, L.; Coelho, F.U. Infinitely generated tilting modules of finite projective dimension. *Forum Math.* **2001**, *13*, 239–250.
6. Angeleri Hügel, L.; Tonolo, A.; Trlifaj, J. Tilting preenvelopes and cotilting precovers. *Algebr. Represent. Theory* **2001**, *4*, 155–170. [CrossRef]
7. Auslander, M.; Reiten, I. Applications of Contravariantly Finite Subcategories. *Adv. Math.* **1991**, *86*, 111–152.
8. Bazzoni, S. A characterization of n-cotilting and n-tilting modules. *J. Algebra* **2004**, *273*, 359–372.
9. Colby, R.R.; Fuller, K.R. Tilting, cotilting and serially tilted rings. *Commun. Algebra* **1990**, *18*, 1585–1615.
10. Colby, R.R.; Trlifaj, J. Tilting modules and tilting torsion theories. *J. Algebra* **1995**, *178*, 614–634.
11. Göbel, R.; Trlifaj, J. *Approximations and Endomorphism Algebras of Modules*; De Gruyter: Berlin, Germany, 2006; Volume 41.
12. Huang, Z. Generalized tilting modules with finite injective dimension. *J. Algebra* **2007**, *311*, 619–634.
13. Miyashita, Y. Tilting modules of finite projective dimension. *Math. Z.* **1986**, *193*, 113–146.
14. Smalø, S. Torsion theories and tilting modules. *Bull. Lond. Math. Soc.* **1984**, *16*, 518–522.
15. Šťovíček, J. All n-cotilting modules are pure-injective. *Proc. Am. Math. Soc.* **2006**, *134*, 1891–1897.
16. Wei, J. Equivalences and the tilting theory. *J. Algebra* **2005**, *283*, 584–595.
17. Wei, J. n-star modules and n-tilting modules. *J. Algebra* **2005**, *283*, 711–722.
18. Wei, J.; Huang, Z.; Tong, W.; Huang, J. Tilting modules of finite projective dimension and a generalization of $*$-modules. *J. Algebra* **2003**, *268*, 404–418.
19. Asadollahi, J.; Hafezi, R.; Vahed, R. On the recollements of functor categories. *Appl. Categor. Struct.* **2016**, *24*, 331–371.
20. Asadollahi, J.; Hafezi, R.; Vahed, R. Derived equivalences of functor categories. *J. Pure Appl. Algebra* **2019**, *223*, 1073–1096.
21. Mao, L. On covers and envelopes in some functor categories. *Commun. Algebra* **2013**, *41*, 1655–1684.
22. Mao, L. On strongly flat and Ω-Mittag-Leffler objects in the category $((R-\mathrm{mod})^{\mathrm{op}}, Ab)$. *Mediterr. J. Math.* **2013**, *10*, 655–676.
23. Mao, L.; Ding, N. On covers and envelopes under Hom and tensor functors. *Commun. Algebra* **2015**, *43*, 4334–4349.
24. Martínez-Villa, R.; Ortiz-Morales, M. Tilting theory and functor categories I: Classical tilting. *Appl. Categor. Struct.* **2014**, *22*, 595–646.
25. Martínez-Villa, R.; Ortiz-Morales, M. Tilting theory and functor categories II: Generalized tilting. *Appl. Categor. Struct.* **2013**, *21*, 311–348.
26. Martínez-Villa, R.; Ortiz-Morales, M. Tilting theory and functor categories III: The maps category. *Int. J. Algebra* **2011**, *5*, 529–561.
27. Mitchell, B. Rings with several objects. *Adv. Math.* **1972**, *8*, 1–161.
28. Tan, L.; Liu, L. Resolution dimension relative to resolving subcategories in extriangulated categories. *Mathematics* **2021**, *9*, 980.
29. Enochs, E.E. Injective and flat covers, envelopes and resolvents. *Israel J. Math.* **1981**, *39*, 33–38.
30. Auslander, M.; Smalø, S. Preprojective modules over artin algebras. *J. Algebra* **1980**, *66*, 61–122.

31. Rada, J.; Saorín, M. Rings characterized by (pre)envelopes and (pre)covers of their modules. *Commun. Algebra* **1998**, *26*, 899–912.
32. Auslander, M.; Buchweitz, R. The homological theory of maximal Cohen-Macaulay approximations. *Mem. Soc. Math. Fr. Suppl. Nouv. Ser.* **1989**, *38*, 5–37. [CrossRef]