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Contraction in Rational Forms in the Framework of Super Metric Spaces

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Abstract: In this paper, we investigate contractions in a rational form in the context of the supermetric space, which is a very interesting generalization of the metric space. We consider an illustrative example to support this new result on supermetric space.

Keywords: fixed point; metric space; supermetric space; rational form

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1. Introduction

The metric fixed point theory is one of the most useful and attractive topics of nonlinear functional analysis. Considering Banach's pioneering fixed point theorem, in the last hundred years, a large number of results have been observed and published on this subject [1–7]. Basically, there are two mainstream concepts on the advances of the metric fixed point: The first is changing (weakening) the conditions of the contraction mapping, and the second is changing the abstract structure. So far, several generalizations and extensions of metric spaces have been introduced. Among these are the quasi-metric space, b-metric space, symmetric space, fuzzy metric space, dislocated metric space, partial metric space, 2-metric spaces, modular metric spaces, cone metric spaces, ultra metric spaces, and a lot more of their combinations.

It is worth noting that the fixed point theory is very functional and useful in solving many problems in various fields. For this reason, a lot of research has been performed on this subject, and the results of these research works have been published in the form of articles and books. On the other hand, in the last decades, observational articles have indicated that the results of a significant number of publications either coincide, overlap, or are equivalent to other existing results in the literature. These observations underline the fact that the there is congestion and squeezing with regard to the fixed point theory. For example, most of the fixed results for cone metric spaces are equivalent to the corresponding results in the setting of standard metric space. The same conclusion can be reached for the G-metric space.

Consequently, the most important reason for us to write this article is to put forward a proposal to remove this congestion. Therefore in this paper, we propose a new result in the context of a new structure, namely the supermetric space [8]. We were able to obtain certain fixed point theorems in this structure, and we think this approach may help to overcome the aforementioned congestion and squeezing.

Before stating the definition of supermetric, we recall some basic definitions, notations, and results. We first consider two interesting generalizations of metric spaces: Let $\mathfrak{X}$ be a non-empty set and $\mathsf{b}, \mathcal{D} : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be two given mappings. We can then say the following:

(A) b is a *b-metric* ([9]) on $\mathfrak{X}$ if it satisfies the following conditions:

(A_1)For every $(\chi, y) \in \mathfrak{X} \times \mathfrak{X}$, we have $\mathsf{b}(\chi, y) = 0 \Leftrightarrow \chi = y$;
(A_2)For every $(\chi, y) \in \mathfrak{X} \times \mathfrak{X}$, we have $\mathsf{b}(\chi, y) = \mathsf{b}(y, \chi)$;
(A_3)There exists $\mathsf{s} \geq 1$ such that for every $(\chi, y, v) \in \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}$, we have

$$\mathsf{b}(\chi, y) \leq \mathsf{s}[\mathsf{b}(\chi, v) + \mathsf{b}(v, y)].$$

The tripled $(\mathfrak{X}, \mathsf{b}, \mathsf{s})$ is called a *b-metric space*.

(B) $\mathcal{D}$ is a *generalized metric* ([10]) on $\mathfrak{X}$ if it satisfies the following conditions:

(B_1) For every $(\chi, y) \in \mathfrak{X} \times \mathfrak{X}$, we have $\mathcal{D}(\chi, y) = 0 \to \chi = y$;
(B_2) For every $(\chi, y) \in \mathfrak{X} \times \mathfrak{X}$, we have $\mathcal{D}(\chi, y) = \mathcal{D}(y, \chi)$;
(B_3) There exist $C > 0$ such that if $(\chi, y) \in \mathfrak{X} \times \mathfrak{X}$, $\{\chi_n\} \in \mathcal{C}(\mathcal{D}, \mathfrak{X}, \chi)$, then

$$\mathcal{D}(\chi, y) \leq C \limsup_{n \to \infty} \mathcal{D}(\chi_n, y),$$

where $\mathcal{C}(\mathcal{D}, \mathfrak{X}, \chi) = \Big\{ \{\chi_n\} \in \mathfrak{X} : \lim_{n \to \infty} \mathcal{D}(\chi_n, \chi) = 0 \Big\}$.

The tripled $(\mathfrak{X}, \mathcal{D}, C)$ is called a *generalized metric space*.

Let $\mathsf{T} : \mathfrak{X} \to \mathfrak{X}$ be a mapping and $\{\mathsf{T}^n \chi\}_{n \geq 0}$ be the Picard iteration for the initial point $\chi \in \mathfrak{X}$, where T^n denotes the n-th iterates of T. Following [11], we then say that the Picard sequence is

- *infinite* if
$$\chi_n \neq \chi_p \text{ for all } n, p \in \mathbb{N}, n \neq p; \tag{1}$$
- *almost periodic* if there exists $k_0, N \in \mathbb{N}$, such that
$$\chi_{k_0+k+Nm} = \chi_{k_0+k} \text{ for all } m \in \mathbb{N} \text{ and all } k \in \{0, 1, 2, ..., N-1\}. \tag{2}$$

Therefore,

$$\{\chi_k : k \geq k_0\} = \{\chi_{k_0}, \chi_{k_0+1}, \chi_{k_0+2}, ..., \chi_{k_0+N-1}\} = \{\chi_k : k \geq m_0\}, \tag{3}$$

for all $m_0 \geq k_0$ (see [11]).

The mapping T is *asymptotically regular* if $\lim_{k \to \infty} m(\mathsf{T}^k \chi, \mathsf{T}^{k+1} \chi) = 0$ for every $\chi \in \mathfrak{X}$.

A fixed point of a mapping $\mathsf{T} : \mathfrak{X} \to \mathfrak{X}$ is an element $\omega \in \mathfrak{X}$, such that $\mathsf{T}\omega = \omega$.

2. Main Results

We begin this section with the definition of the supermetric.

Definition 1. *Let $m : \mathfrak{X} \times \mathfrak{X} \to [0, +\infty)$, where $\mathfrak{X}$ is a nonempty set. We say that m is a* supermetric *if it satisfies the following axioms:*

(m_1) *For all $\chi, y \in \mathfrak{X}$, if $m(\chi, y) = 0$, then $\chi = y$;*
(m_2) *$m(\chi, y) = m(y, \chi)$ for all $\chi, y \in \mathfrak{X}$;*
(m_3) *There exists $\mathsf{s} \geq 1$ such that for every $y \in \mathfrak{X}$, there exist distinct sequences $(\chi_n), (y_n) \subset \mathfrak{X}$, with $m(\chi_n, y_n) \to 0$ when $n \to \infty$, such that*

$$\limsup_{n \to \infty} m(y_n, y) \leq \mathsf{s} \limsup_{n \to \infty} m(\chi_n, y). \tag{4}$$

The tripled $(\mathfrak{X}, m, \mathsf{s})$ is called a supermetric space.

The notions of convergence and the Cauchy sequence with respect to completeness of a supermetric space are defined as follows:

Definition 2. *On a supermetric space $(\mathfrak{X}, m, \mathsf{s})$, a sequence $\{x_n\}$:*

(*c*) *converges to x in $\mathfrak{X}$ if and only if $\lim\limits_{n\to\infty} m(x_n, x) = 0$;*

(*C*) *is a Cauchy sequence in $\mathfrak{X}$ if and only if $\lim\limits_{n\to\infty} \sup\{m(x_n, x_p) : p > n\} = 0$.*

Proposition 1. *On a supermetric space, the limit of a convergent sequence is unique.*

Proof. Let $x \in \mathfrak{X}$, and (x_n) be a sequence in $\mathfrak{X}$ such that $m(x_n, x) \to 0$ as $n \to \infty$. Thus, letting $y_n = x$ in (m_3), we get

$$m(x, y) \leq \mathsf{s} \limsup_{n\to\infty} m(x_n, y),$$

for any $y \in \mathfrak{X}$. Supposing that (x_n) converges to y, the above inequality leads to $m(x, y) = 0$. Consequently, taking (m_1) into account, it follows that $x = y$. □

Definition 3. *We say that a supermetric space $(\mathfrak{X}, m, \mathsf{s})$ is* complete *if and only if every Cauchy sequence is convergent in $\mathfrak{X}$.*

Example 1. *Let the set $\mathfrak{X} = \mathbb{R}$, $\mathsf{s} = 2$, and $m : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be an application defined as follows:*

$$\begin{aligned} m(x, y) &= (x - y)^2, \text{ for } x, y \in \mathbb{R} \setminus \{1\} \\ m(1, y) &= m(y, 1) = (1 - y^3)^2, \text{ for } y \in \mathbb{R} \end{aligned}$$

Of course, we can easily observe that the conditions (m_1) and (m_2) are satisfied. Let $y \in \mathbb{R} \setminus \{1\}$ and two sequences $\{x_n\}, \{y_n\}$ in $\mathbb{R} \setminus \{1\}$, such that $m(x_n, y_n) \to 0$ as $n \to \infty$. Thus, we get that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = v$ and

$$\limsup_{n\to\infty} m(y_n, y) = \limsup_{n\to\infty} (x_n - y)^2 = (v - y)^2 \leq \mathsf{s}(v - y)^2 = \limsup_{n\to\infty} m(x_n, y).$$

If $y = 1$, by choosing the same sequences $\{x_n\}, \{y_n\} \in \mathfrak{X}$, it follows that (m_1) indeed holds. Consequently, the tripled $(\mathfrak{X}, m, \mathsf{s})$ forms a supermetric space.

On the other hand, let $\mathcal{C}(m, \mathfrak{X}, 1) = \Big\{\{x_n\} \in \mathfrak{X} : \lim\limits_{n\to\infty} m(x_n, 1) = 0\Big\}$. If we can find $\mathsf{s} \geq 1$, such that

$$(1 - y^3)^2 = m(1, y) \leq \mathsf{s} \limsup_{n\to\infty} m(x_n, y) = \mathsf{s} \limsup_{n\to\infty} (x_n - y)^2 = \mathsf{s}(1 - y)^2,$$

for any $y \in \mathbb{R} \setminus \{1\}$, we get $(1 + y + y^2)^2 \leq \mathsf{s}$. Subsequently, we cannot find a bound for s by which

$$m(1, y) \leq \mathsf{s} \limsup_{n\to\infty} m(x_n, y).$$

This shows that $(\mathfrak{X}, m, \mathsf{s})$ is not a generalized metric space.

Example 2. *Let the set $\mathfrak{X} = [0, +\infty]$ and $m : \mathfrak{X} \times \mathfrak{X} \to [0, +\infty)$ be an application, defined as follows:*

$$\begin{aligned} m(x, y) &= \tfrac{|xy - 1|}{x + y + 1}, \text{ for } x, y \in [0, 1) \cup (1, +\infty], x \neq y \ ' \\ m(x, y) &= 0, \text{ for } x, y \in [0, +\infty) \ x = y, \\ m(x, 1) &= m(1, x) = |x - 1|, \text{ for } x \in [0, +\infty] \end{aligned}$$

We can easily see that m forms a supermetric on $\mathfrak{X}$. Indeed, for any $y \in \mathfrak{X}$, choosing the sequences (x_n), (y_n) in $(\mathfrak{X})$, such that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 1$, we have $m(x_n, y_n) \to 0$ as $n \to \infty$. Thus, the following can be stated:

1. *For* $y \neq 1$,

$$\limsup_{n\to\infty} m(y_n, y) = \limsup_{n\to\infty} \frac{|y_n y - 1|}{y_n + y + 1} = \frac{|y-1|}{y+2} \leq \mathsf{s}\frac{|y-1|}{y+2} = \mathsf{s}\limsup_{n\to\infty} m(\varkappa_n, y),$$

for any $\mathsf{s} \geq 1$*;*

2. *For* $y = 1$,

$$\limsup_{n\to\infty} m(y_n, 1) = \limsup_{n\to\infty} |y_n - 1| = 0 \leq \mathsf{s}\limsup_{n\to\infty} m(\varkappa_n, y),$$

for any $s \geq 1$.

Consequently, since (m_1), (m_2) *are obviously satisfied, it follows that* m *is a supermetric on* $\mathfrak{X}$*.*

However, for instance, by letting $\varkappa = n$, $y = 2n$, $n \in \mathbb{N}$, *and* $z = 0$, *if there exists* $s \geq 1$ *such that*

$$m(\varkappa, y) = \frac{|2n^2 - 1|}{3n+1} \leq \mathsf{s}\left[\frac{1}{n+1} + \frac{1}{2n+1}\right] = \mathsf{s}[m(\varkappa, 0) + m(0, y)],$$

we get that $s \geq \frac{|2n^2-1|(2n^2+3n+1)}{3n+2}$*, which is a contradiction because* $\mathfrak{X}$ *is unbounded. Consequently,* m *does not define a b-metric.*

At the same time, letting $y \in \mathfrak{X}$, $y \neq 1$ *and the sequence* $(\varkappa_n)$ *in* $\mathfrak{X}$*, such that* $\varkappa_n \to 1$ *as* $n \to \infty$,

$$m(1, y) = |y - 1| \leq \mathsf{s}\limsup_{n\to\infty} m(\varkappa_n, y) = \mathsf{s}\limsup_{n\to\infty} \frac{|\varkappa_n y - 1|}{\varkappa_n + y + 1} = \frac{|y-1|}{y+2},$$

which means that $\mathsf{s} \geq y + 2$*, which is a contradiction. Therefore,* m *it is not a generalized metric on* $\mathfrak{X}$*.*

Proposition 2. *Let* $\mathsf{T} : \mathfrak{X} \to \mathfrak{X}$ *be an asymptotically regular mapping on a complete supermetric space* $(\mathfrak{X}, m, \mathsf{s})$*. Then, the Picard iteration* $\{\mathsf{T}^n x\}$ *for the initial point* $\varkappa \in \mathfrak{X}$ *is a convergent sequence on* $\mathfrak{X}$*.*

Proof. For $\varkappa \in \mathfrak{X}$, setting $\varkappa_k = \mathsf{T}^k\varkappa$ for $n \in \mathbb{N} \cup \{0\}$, we have

$$\lim_{k\to\infty} m(\varkappa_k, \varkappa_{k+1}) = 0. \tag{5}$$

We can assume that the Picard sequence of T is infinite. If not, we can find a pair (k, p), $k, p \in \mathbb{N} \cup \{0\}$, $k < p$, such that $\varkappa_k = \varkappa_p$. Choosing (k_0, p_0) such that the difference of $N = p_0 - q_0$ is minimum, we can claim that

$$\varkappa_{k_0+Nq} = \varkappa_{k_0} \text{ for all } q \in \mathbb{N}. \tag{6}$$

To prove this, we use mathematical induction. Indeed, for $q = 0$, we have $\varkappa_{k_0} = \varkappa_{k_0}$, and for $q = 1$, we get $\varkappa_{k_0+N} = \varkappa_{p_0} = \varkappa_{k_0}$. Now, supposing that (6) holds for some $q \in \mathbb{N}$, we have

$$\varkappa_{k_0+N(q+1)} = \varkappa_{k_0+Nq+N} = \mathsf{T}^N(\varkappa_{k_0+Nq}) = \mathsf{T}^N\varkappa_{k_0} = \varkappa_{k_0+N} = \varkappa_{k_0},$$

which completes the proof of our claim. Moreover,

$$\varkappa_{k_0+k+Nq} = \mathsf{T}^k(\varkappa_{k_0+Nq}) = \mathsf{T}^k\varkappa_{k_0} = \varkappa_{k_0+k},$$

for all $q \in \mathbb{N}$ and all $k \in \{0, 1, 2, ..., N-1\}$, that is, the sequence $\{\varkappa_k\}$ is almost periodic. Now,

Case 1. If $N = 1$, we have $\varkappa_{k_0+q} = \varkappa_{k_0}$ for all $q \geq 0$, which means that for $k \geq k_0$, $\varkappa_k = \omega$, where $\omega \in \mathfrak{X}$. Therefore, $\mathsf{T}\omega = \mathsf{T}\varkappa_k = \varkappa_{k+1} = \omega$, so ω is a fixed point of the mapping T.

Case 2. If $N \geq 2$, then $\varkappa_{k_0+j} \neq \varkappa_{k_0+l}$ for all $0 \leq j < l \leq N-1$, because N was supposed to be the smallest integer such that (6) holds. Thus, for $l = j+1$, we have

$$2\varepsilon = \min_{0 \leq j \leq N-1} m(\varkappa_{k_0+j}, \varkappa_{k_0+j+1}) > 0.$$

On the other hand, by (5), there exists $r_0 \in \mathbb{N}$ with $r_0 \geq k_0$, such that

$$m(\varkappa_{r_0}, \varkappa_{r_0+1}) < \varepsilon. \tag{7}$$

If $(\widehat{r_0 - k_0})_{modN} = j_0$, with $j_0 \in \{0, 1, 2, ..., N-1\}$, there exists an unique integer $q \geq 0$, such that $r_0 - k_0 = Nq + j_0$ and

$$\varkappa_{r_0} = \varkappa_{k_0+j_0+Nq} = \varkappa_{k_0+j_0},$$

where $k_0 + j_0 \in \{k_0, k_0+1, ..., k_0+N-1\}$. Thus, we obtain

$$2\varepsilon = \min_{0 \leq j \leq N-1} m(\varkappa_{k_0+j}, \varkappa_{k_0+j+1}) \leq m(\varkappa_{k_0+j_0}, \varkappa_{k_0+j_0+1}) = m(\varkappa_{r_0}, \varkappa_{r_0+1}) < \varepsilon,$$

which is a contradiction. Consequently, we can assume that the Picard sequence $\{\varkappa_k\}$ of T is infinite. Thus, by using mathematical induction, we will show that

$$\lim_{p \to \infty} m(\varkappa_p, \varkappa_{p+n}) = 0, \quad \text{for all } n > 0. \tag{8}$$

Without loss of generality, we can suppose that $\varkappa_p \neq \varkappa_{p+n}$. Indeed, for $n = 1$, by (5), $\lim_{p \to \infty} m(\varkappa_p, \varkappa_{p+1}) = 0$. Letting $n = 2$, by (m_3), we have

$$\limsup_{p \to \infty} m(\varkappa_p, \varkappa_{p+2}) \leq s \limsup_{p \to \infty} m(\varkappa_{p+1}, \varkappa_{p+2}) = 0;$$

it follows that $\limsup_{p \to \infty} m(\varkappa_p, \varkappa_{p+2}) = 0$. Now, supposing that $\limsup_{p \to \infty} m(\varkappa_p, \varkappa_{p+n}) = 0$, where $n > 0$, we have

$$\limsup_{p \to \infty} m(\varkappa_p, \varkappa_{p+n+1}) \leq s \limsup_{p \to \infty} m(\varkappa_p, \varkappa_{p+n}) = 0.$$

Consequently,

$$\lim_{n \to \infty} \sup\{m(\varkappa_p, \varkappa_n) : n > p\} = 0,$$

that is, the sequence $\{\varkappa_k\}$ is Cauchy. Since the space $(\mathfrak{X}, m, s)$ was supposed to be complete, we know that there exists $\omega \in \mathfrak{X}$, such that $\lim_{k \to \infty} m(\varkappa_k, \omega) = 0$. $\square$

Rational Contractions in Super Metric Space

Theorem 1. *Let $(\mathfrak{X}, m, s)$ be a complete supermetric space and $\mathsf{T} : \mathfrak{X} \to \mathfrak{X}$ be a mapping, such that there exists $\kappa \in [0, 1)$ and that*

$$m(\mathsf{T}\varkappa, \mathsf{T}y) \leq \kappa \max\left\{ m(\varkappa, y), \frac{m(\varkappa, \mathsf{T}\varkappa) m(y, \mathsf{T}y)}{m(\varkappa, y) + 1} \right\}. \tag{9}$$

Then, T has a unique fixed point.

Proof. Let $\varkappa \in \mathfrak{X}$ and $\{\varkappa_k\}$ be the Picard iteration of the mapping T. If there exists $k_0 \in \mathbb{N}$, such that $\varkappa_{k_0} = \varkappa_{k_0+1}$, from the way in which this sequence is defined, it follows that $\mathsf{T}\varkappa_{k_0} = \varkappa_{k_0+1} = \varkappa_{k_0}$, which means that $\varkappa_{k_0}$ is a fixed point of the mapping T. Therefore, we can assume that $\varkappa_k \neq \varkappa_{k+1}$ for all $k \in \mathbb{N}$. Hence, $m(\varkappa_{k_0}, \varkappa_{k_0+1}) > 0$, and taking (9) into account,

$$\begin{aligned} m(x_k, x_{k+1}) &= m(\mathsf{T}x_{k-1}, \mathsf{T}x_k) \le \kappa \max\Big\{ m(x_{k-1}, x_k), \tfrac{m(x_{k-1}, \mathsf{T}x_{k-1}) m(x_k, \mathsf{T}x_k)}{m(x_{k-1}, x_k)+1} \Big\} \\ &= \kappa \max\Big\{ m(x_{k-1}, x_k), \tfrac{m(x_{k-1}, x_k) m(x_k, x_{k+1})}{m(x_{k-1}, x_k)+1} \Big\} \\ &\le \kappa \max\{ m(x_{k-1}, x_k), m(x_k, x_{k+1}) \}. \end{aligned}$$

Since in the case of $\max\{m(x_{k-1}, x_k, m(x_k, x_{k+1})\} = m(x_k, x_{k+1})$ we get a contradiction ($m(x_k, x_{k+1}) \le \kappa m(x_k, x_{k+1}) < m(x_k, x_{k+1})$), it follows that $\max\{m(x_{k-1}, x_k), m(x_k, x_{k+1})\} = m(x_{k-1}, x_k)$. Thus, we have

$$0 < m(x_k, x_{k+1}) \le \kappa m(x_{k-1}, x_k) \le \kappa^2 m(x_{k-2}, x_{k-1}) \le \ldots \le \kappa^k m(x_0, x_1),$$

and in taking the limit from the above inequality, we get

$$\lim_{k\to\infty} m(x_k, x_{k+1}) = \lim_{k\to\infty} m(\mathsf{T}^{k-1}x, \mathsf{T}^k x) = 0.$$

Therefore, T is asymptotically regular, and from Proposition (2), the Picard iteration $\{\mathsf{T}^n x\}$ is a convergent sequence. Thus, there exists $\omega \in \mathfrak{X}$, such that $\lim\limits_{n\to\infty} m(x_n, \omega) = 0$.

We claim that ω is a fixed point of the mapping T. If not, $\omega \neq \mathsf{T}\omega$, and then $m(\omega, \mathsf{T}\omega) > 0$. On the other hand, since the sequence $\{x_k\}$ is supposed to be infinite, we can find a sub-sequence $\{x_{k_n}\}$ of the sequence $\{x_k\}$, such that $x_{k_n} \neq \omega$ for all $k_n \in \mathbb{N}$. Thus, by (m_3),

$$\begin{aligned} 0 &< m(x_{k_n+1}, \mathsf{T}\omega) = m(\mathsf{T}x_{k_n}, \mathsf{T}\omega) \\ &\le \kappa \max\Big\{ m(x_{k_n}, \omega), \tfrac{m(x_{k_n}, \mathsf{T}x_{k_n}) m(\omega, \mathsf{T}\omega)}{m(x_{k_n}, \omega)+1}, \Big\} \\ &= \kappa \max\Big\{ m(x_{k_n}, \omega), \tfrac{m(x_{k_n}, x_{k_n+1}) m(\omega, \mathsf{T}\omega)}{m(x_{k_n}, \omega)+1} \Big\} \\ &\le s \limsup_{n\to\infty} m(\omega, x_{k_n}) = 0. \end{aligned} \tag{10}$$

Consequently,

$$\lim_{n\to\infty} m(x_{k_n+1}, \mathsf{T}\omega) \le \kappa \lim_{n\to\infty} m(\omega, x_{k_n}) = 0,$$

and we obtain $\lim\limits_{n\to\infty} m(x_{k_n+1}, \mathsf{T}\omega) = 0$. That is, that $\mathsf{T}\omega$ is also a limit for the Picard iteration. However, from Proposition 1, it follows that $\mathsf{T}\omega = \omega$, so that ω is a fixed point of the mapping T.

Supposing that there exists another point, $\eta \in \mathfrak{X}$, such that $\mathsf{T}\eta = \eta \neq \omega = \mathsf{T}\omega$. Then, by (9), we have

$$\begin{aligned} m(\mathsf{T}\eta, \mathsf{T}\omega) &\le \kappa \max\Big\{ m(\eta, \omega), \tfrac{m(\eta, \mathsf{T}\eta) m(\omega, \mathsf{T}\omega)}{m(\eta, \omega)+1} \Big\} \\ &= \kappa m(\eta, \omega) < m(\eta, \omega), \end{aligned}$$

which is a contradiction. □

Example 3. *Let $\mathfrak{X} = [0,1]$, $\mathsf{s} = 1$, and the application $m : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be defined as follows:*

$$\begin{aligned} m(x, y) &= xy, \text{ for all } x \neq y,\ x, y \in (0,1); \\ m(x, y) &= 0, \text{ for all } x = y,\ x, y \in [0,1]; \\ m(0, y) &= m(y, 0) = y, \text{ for all } y \in (0,1]; \\ m(1, y) &= m(y, 1) = 1 - \tfrac{y}{2}, \text{ for all } y \in [0,1). \end{aligned}$$

We claim that m is a supermetric on $\mathfrak{X}$. Since the conditions (m_1), (m_2) are easy to verify, we will focus on (m_3). For any $y \in (0,1)$, we can choose the sequences $\{x_n\}$, $\{y_n\}$ in $\mathfrak{X}$, where

$$x_n = \frac{n^2+1}{n^2+2}, \text{ and } y_n = \frac{n+1}{n^2+1}, \text{ for any } n \in \mathbb{N}.$$

Since $\lim\limits_{n\to\infty} x_n = 1$ and $\lim\limits_{n\to\infty} y_n = 0$, we have $\lim\limits_{n\to\infty} m(x_n, y_n) = 0$. Thus,

$$\begin{aligned} \limsup_{n\to\infty} m(y_n, y) &= \limsup_{n\to\infty} y_n y = 0, \\ \limsup_{n\to\infty} m(x_n, y) &= \limsup_{n\to\infty} x_n y = y, \end{aligned}$$

and (m_1) holds.

If $y = 0$, using the same sequences, we get

$$\begin{aligned} \limsup_{n\to\infty} m(y_n, y) &= \limsup_{n\to\infty} y_n = 0, \\ \limsup_{n\to\infty} m(x_n, y) &= \limsup_{n\to\infty} x_n = 1, \end{aligned}$$

and again, (m_1) holds.

If $y = 1$, choosing $x_n = \frac{n+1}{n^2+2}$ and $y_n = \frac{n+2}{n+3}$, we have $m(x_n, y_n) = 0$ and

$$\begin{aligned} \limsup_{n\to\infty} m(y_n, y) &= \limsup_{n\to\infty} (1 - \frac{y_n}{2}) = \frac{1}{2}, \\ \limsup_{n\to\infty} m(x_n, y) &= \limsup_{n\to\infty} (1 - \frac{x_n}{2}) = 1. \end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} m(y_n, y) = \frac{1}{2} < 1 = \mathsf{s} \limsup_{n\to\infty} m(x_n, y).$$

Hence, our claim is proven. That is, m defines a supermetric on $\mathfrak{X}$.

On the other hand, let $\{z_n\}$ be a sequence in $\mathfrak{X}$, such that $\lim\limits_{n\to\infty} z_n = 1$. Since $m(1, y) = 1 - \frac{y}{2}$, for any $y \in (0,1)$, if there exists $C > 0$ such that

$$1 - \frac{y}{2} = m(1, y) \le C \limsup_{n\to\infty} m(z_n, y) = C \limsup_{n\to\infty} z_n y \le Cy, \tag{11}$$

we get $C > (1 - \frac{y}{2})/y$. Subsequently, we cannot find a bound for C, such that 11 *holds; that means m is not a generalized metric space.*

Now, let the mapping $\mathsf{T} : \mathfrak{X} \to \mathfrak{X}$, with $\mathsf{T}x = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,1) \\ \frac{1}{8}, & \text{if } x = 1 \end{cases}$. We then check if the mapping T satisfies (9), for $\kappa = \frac{1}{2}$. We consider the following cases:

1. *If $x, y \in (0,1)$, we have*

$$\begin{aligned} m(x, y) &= xy,\ m(\mathsf{T}x, \mathsf{T}y) = m(\tfrac{x}{2}, \tfrac{y}{2}) = \tfrac{xy}{4}, \\ m(\mathsf{T}x, \mathsf{T}y) &= \tfrac{xy}{4} \le \tfrac{xy}{2} = \kappa \cdot m(x, y) \le \kappa \cdot \max\Big\{ m(x, y), \tfrac{m(x, \mathsf{T}x) m(y, \mathsf{T}y)}{m(x,y)+1} \Big\}. \end{aligned}$$

2. *If $x = 0$, $y \in (0,1)$, we have*

$$\begin{aligned} m(0, y) &= y,\ m(\mathsf{T}0, \mathsf{T}y) = m(0, \tfrac{y}{2}) = \tfrac{y}{2}, \\ m(\mathsf{T}x, \mathsf{T}y) &= \tfrac{y}{2} \le \tfrac{y}{2} = \kappa \cdot m(0, y) \le \kappa \cdot \max\Big\{ m(0, y), \tfrac{m(0, \mathsf{T}0) m(y, \mathsf{T}y)}{m(0,y)+1} \Big\}. \end{aligned}$$

3. *If $\chi = 0$, $y = 0$ or $\chi = 1$, $y = 1$, we have $m(\chi, y) = 0 = m(\mathsf{T}\chi, \mathsf{T}y)$, and so (9) is obviously verified.*
4. *If $\chi = 0$, $y = 1$:*

$$\begin{aligned} m(0,1) &= 1,\ m(\mathsf{T}0,\mathsf{T}1) = m(0,\tfrac{1}{8}) = \tfrac{1}{8}, \\ m(\mathsf{T}0,\mathsf{T}1) &= \tfrac{1}{8} < \tfrac{1}{2} = \kappa \cdot m(0,1) \le \kappa \cdot \max\Big\{m(0,1), \tfrac{m(0,\mathsf{T}0)m(1,\mathsf{T}1)}{m(0,1)+1}\Big\}. \end{aligned}$$

5. *If $\chi = 1$, $y \in (0,1)$:*

$$\begin{aligned} m(1,y) &= 1 - \tfrac{y}{2},\ m(\mathsf{T}1,\mathsf{T}y) = m(\tfrac{1}{8},\tfrac{y}{2}) = \tfrac{y}{16},\ m(1,\mathsf{T}1) = 1 - \tfrac{1}{16},\ m(y,\mathsf{T}y) = \tfrac{y^2}{2} \\ m(\mathsf{T}1,\mathsf{T}y) &= \tfrac{y}{16} \le \tfrac{1}{2}(1 - \tfrac{y}{2}) = \kappa \cdot m(1,y) \le \kappa \cdot \max\Big\{m(1,y), \tfrac{m(1,\mathsf{T}1)m(y,\mathsf{T}y)}{m(1,y)+1}\Big\}. \end{aligned}$$

Therefore, we conclude that the mapping T has a unique fixed point; that is, $\chi = 0$.

Theorem 2. *Let $(\mathfrak{X}, m, s)$ be a complete supermetric space and $\mathsf{T} : \mathfrak{X} \to \mathfrak{X}$ be an asymptotically regular mapping. If there exists $\kappa \in [0,1)$, such that*

$$m(\mathsf{T}\chi, \mathsf{T}y) \le \kappa \max \left\{ \begin{array}{c} m(\chi, y), \frac{m(\chi,\mathsf{T}y)+m(y,\mathsf{T}\chi)}{2s} \\ \frac{m(\chi,\mathsf{T}\chi)m(\chi,\mathsf{T}y)+m(y,\mathsf{T}y)m(y,\mathsf{T}\chi)}{m(\chi,\mathsf{T}y)+m(y,\mathsf{T}\chi)+1} \end{array} \right\}, \tag{12}$$

then T has a unique fixed point.

Proof. Let $\chi \in \mathfrak{X}$ be an arbitrary (but fixed) point in $\mathfrak{X}$ and $\{\chi_k\}$ the Picard sequence associated with the mapping T, which started in χ. Since T is an asymptotically regular mapping,

$$\lim_{k\to\infty} m(\chi_k, \chi_{k+1}) = 0, \tag{13}$$

and moreover, by Proposition 1, there exists $\omega \in \mathfrak{X}$, such that

$$\lim_{k\to\infty} m(\chi_k, \omega) = 0. \tag{14}$$

Supposing that $\chi_k \ne \chi_{k+1}$ for all $k \in \mathbb{N}$ (see the previous proof), replacing these in (12), we have

$$\begin{aligned} 0 &< m(\chi_{k+1}, \mathsf{T}\omega) = m(\mathsf{T}\chi_k, \mathsf{T}\omega) \\ &\le \kappa \max \left\{ \begin{array}{c} m(\chi_k, \omega), \frac{m(\chi_k,\mathsf{T}\omega)+m(\omega,\mathsf{T}\chi_k)}{2s}, \\ \frac{m(\chi_k,\mathsf{T}\chi_k)m(\chi_k,\mathsf{T}\omega)+m(\omega,\mathsf{T}\omega)m(\omega,\mathsf{T}\chi_k)}{m(\chi_k,\mathsf{T}\omega)+m(\omega,\mathsf{T}\chi_k)+1} \end{array} \right\} \\ &= \kappa \max \left\{ \begin{array}{c} m(\chi_k, \omega), \frac{m(\chi_k,\mathsf{T}\omega)+m(\omega,\chi_{k+1})}{2s}, \\ \frac{m(\chi_k,\mathsf{T}\omega)m(\chi_k,\chi_{k+1})+m(\omega,\mathsf{T}\omega)m(\omega,\chi_{k+1})}{m(\chi_k,\mathsf{T}\omega)+m(\omega,\chi_{k+1})+1} \end{array} \right\} \end{aligned}$$

Consequently, while keeping in mind (13), (14), and (m_3), we get

$$\begin{aligned}
\limsup_{k\to\infty} m(x_{k+1}, \mathsf{T}\omega) &\leq \kappa \limsup_{k\to\infty} \max\left\{\begin{array}{c} m(x_k,\omega), \frac{m(x_k,\mathsf{T}\omega)+m(\omega,x_{k+1})}{2\mathsf{s}}, \\ \frac{m(x_k,\mathsf{T}\omega)m(x_k,x_{k+1})+m(\omega,\mathsf{T}\omega)m(\omega,x_{k+1})}{m(x_k,\mathsf{T}\omega)+m(\omega,x_{k+1})+1} \end{array}\right\} \\
&\leq \kappa \limsup_{k\to\infty} \frac{m(x_k, \mathsf{T}\omega)}{2\mathsf{s}} \\
&\leq \kappa\mathsf{s} \limsup_{k\to\infty} \frac{m(x_{k+1}, \mathsf{T}\omega)}{2\mathsf{s}} \\
&= \kappa \limsup_{k\to\infty} m(x_{k+1}, \mathsf{T}\omega).
\end{aligned}$$

However, $\kappa \in [0,1)$, therefore

$$\limsup_{k\to\infty} m(x_{k+1}, \mathsf{T}\omega) = 0,$$

which means that $\mathsf{T}\omega$ is the limit of the Picard iteration, and Propsition 1 leads us to $\mathsf{T}\omega = \omega$.

If we can find another point, $\eta \in \mathfrak{X}$, such that $\eta = \mathsf{T}\eta$ and $\eta \neq \omega$, then

$$\begin{aligned}
0 &< m(\eta,\omega) = m(\mathsf{T}\eta, \mathsf{T}\omega) \leq \kappa \max\left\{\begin{array}{c} m(\eta,\omega), \frac{m(\eta,\mathsf{T}\omega)+m(\omega,\mathsf{T}\eta)}{2\mathsf{s}}, \\ \frac{m(\eta,\mathsf{T}\eta)m(\eta,\mathsf{T}\omega)+m(\omega,\mathsf{T}\omega)m(\omega,\mathsf{T}\eta)}{m(\eta,\mathsf{T}\omega)+m(\omega,\mathsf{T}\eta)+1} \end{array}\right\} \\
&= \kappa m(\eta,\omega) < m(\eta,\omega),
\end{aligned}$$

which is a contradiction. Therefore, the mapping T has a unique fixed point. □

Example 4. *Let the set* $\mathfrak{X} = \{1,2,3,4\}$ *and* $m : \mathfrak{X} \times \mathfrak{X} \to [0,+\infty)$ *be an application, such that*

$$\begin{aligned}
m(x,y) &= m(y,x) = |x-y|^2, \ \text{ for } x,y \in \{2,3,4\} \\
m(1,x) &= m(x,1) = (1-x^3)^2, \ \text{ for } x \in \mathfrak{X}
\end{aligned}$$

It is easy to check that m *forms a supermetric on* $\mathfrak{X}$, *with* $\mathsf{s} = 2$. *Now, let the mapping* $\mathsf{T} : \mathfrak{X} \to \mathfrak{X}$, *where*

$$\mathsf{T}1 = \mathsf{T}4 = 2, \ \mathsf{T}2 = \mathsf{T}3 = 3.$$

First of all, we observe that T *is an asymptotically regular mapping since* $\mathsf{T}^n x = 3$ *for any* $n \in \{2,3,\ldots\}$. *We must then consider the following cases:*

1. *For* $x = 1, y = 4$, *respectively* $x = 2, y = 3$, *we have* $m(\mathsf{T}x, \mathsf{T}y) = 0$, *and (12) holds for any* $\kappa \in (0,1)$.
2. *For* $x = 1, y = 2$, *we have* $m(1,2) = 49$, $m(\mathsf{T}1,\mathsf{T}2) = m(2,3) = 1$, *and (12) holds for any* $\kappa \in (0,1)$.
3. *For* $x = 1, y = 3$, *we have* $m(1,3) = 26^2$, $m(\mathsf{T}1,\mathsf{T}3) = m(2,3) = 1$, *and (12) holds for any* $\kappa \in (0,1)$.
4. *For* $x = 2, y = 4$, *we have* $m(2,4) = 16$, $m(\mathsf{T}2,\mathsf{T}4) = m(2,3) = 1$, *and (12) holds for any* $\kappa \in (0,1)$.
5. *For* $x = 3, y = 4$, *we have* $m(3,4) = 1$, $m(\mathsf{T}3,\mathsf{T}4) = m(2,3) = 1$, *and*

$$\begin{aligned}
m(3,\mathsf{T}3) &= m(3,3) = 0, \ m(4,\mathsf{T}4) = m(4,2) = 4, \\
m(3,\mathsf{T}4) &= m(3,2) = 1, \ m(4,\mathsf{T}3) = m(4,3) = 1.
\end{aligned}$$

Thus,

$$\max\left\{\begin{array}{c} m(3,4), \frac{m(3,\mathsf{T}3)+m(4,\mathsf{T}3)}{2s} \\ \frac{m(3,\mathsf{T}3)m(3,\mathsf{T}4)+m(4,\mathsf{T}4)m(4,\mathsf{T}3)}{m(3,\mathsf{T}4)+m(4,\mathsf{T}3)+1} \end{array}\right\} = \max\left\{1, \frac{2}{4}, \frac{4}{3}\right\} = \frac{4}{3}.$$

Therefore, in choosing $\kappa = \frac{7}{8}$, for example, we have

$$m(\mathsf{T}3,\mathsf{T}4) = 1 \le \frac{7}{6} = \kappa \max\left\{1, \frac{2}{4}, \frac{4}{3}\right\} = \kappa \max\left\{\begin{array}{c} m(3,4), \frac{m(3,\mathsf{T}4)+m(4,\mathsf{T}3)}{2s} \\ \frac{m(3,\mathsf{T}3)m(3,\mathsf{T}4)+m(4,\mathsf{T}4)m(4,\mathsf{T}3)}{m(3,\mathsf{T}4)+m(4,\mathsf{T}3)+1} \end{array}\right\},$$

and (12) holds. Hence, according to Theorem 2, we can conclude that the mapping T has a unique fixed point, this being $\chi = 3$.
In the end, we observe that Theorem 1 cannot be applied because by letting $\chi = 3$, $y = 4$ in (9), we have

$$m(\mathsf{T}3,\mathsf{T}4) = 1 \le \kappa = \kappa \max\left\{m(3,4), \frac{m(3,\mathsf{T}3)m(4,\mathsf{T}4)}{m(3,4)+1}\right\},$$

which is a contradiction.

Theorem 3. *Let $(\mathfrak{X}, m, s)$ be a complete supermetric space and $\mathsf{T} : \mathfrak{X} \to \mathfrak{X}$ be an asymptotically regular mapping. If there exist $\kappa \in [0,1)$ and $\mathcal{L} \ge 0$, such that*

$$m(\mathsf{T}\chi,\mathsf{T}y) \le \kappa \cdot m(\chi,y) + \mathcal{L} \cdot \min\left\{m(\chi,\mathsf{T}\chi), m(y,\mathsf{T}y), \frac{m(\chi,\mathsf{T}\chi)m(y,\mathsf{T}y)}{m(\chi,y)+1}, \frac{m(\chi,\mathsf{T}y)m(y,\mathsf{T}\chi)}{m(\chi,y)+1}\right\}, \tag{15}$$

then T has a fixed point.

Proof. Let $\chi \in \mathfrak{X}$ be an arbitrary (but fixed) point in $\mathfrak{X}$ and $\{\chi_k\}$ the Picard sequence associated with the mapping T, which started in χ. Since T is an asymptotically regular mapping,

$$\lim_{k\to\infty} m(\chi_k, \chi_{k+1}) = 0, \tag{16}$$

and moreover, by Proposition 1, there exists $\omega \in \mathfrak{X}$, such that

$$\lim_{k\to\infty} m(\chi_k, \omega) = 0. \tag{17}$$

We can then claim that ω is a fixed point of the mapping T. Supposing that $\chi_k \ne \chi_{k+1}$ for all $k \in \mathbb{N}$ (see the previous proof), by replacing this in (15), we have

$$\begin{aligned} 0 &< m(\chi_{k+1},\mathsf{T}\omega) = m(\mathsf{T}\chi_k,\mathsf{T}\omega) \\ &\le \kappa \cdot m(\chi_k,\omega) + \mathcal{L} \cdot \min\left\{m(\chi_k,\mathsf{T}\chi_k), m(\omega,\mathsf{T}\omega), \tfrac{m(\chi_k,\mathsf{T}\chi_k)m(\omega,\mathsf{T}\omega)}{m(\chi_k,\omega)+1}, \tfrac{m(\chi_k,\mathsf{T}\omega)m(\omega,\mathsf{T}\chi_k)}{m(\chi_k,\omega)+1}\right\}, \\ &= \kappa \cdot m(\chi_k,\omega) + \mathcal{L} \cdot \min\left\{m(\chi_k,\chi_{k+1}), m(\omega,\mathsf{T}\omega), \tfrac{m(\chi_k,\chi_{k+1})m(\omega,\mathsf{T}\omega)}{m(\chi_k,\omega)+1}, \tfrac{m(\chi_k,\mathsf{T}\omega)m(\omega,\chi_{k+1})}{m(\chi_k,\omega)+1}\right\}. \end{aligned}$$

Letting $k \to \infty$ in the above inequality, we have

$$\lim_{k\to\infty} m(\chi_{k+1},\mathsf{T}\omega) \le \lim_{k\to\infty}\left[\kappa m(\chi_k,\omega) + +\mathcal{L} \cdot \min\left\{\begin{array}{c} m(\chi_k,\chi_{k+1}), m(\omega,\mathsf{T}\omega), \\ \frac{m(\chi_k,\chi_{k+1})m(\omega,\mathsf{T}\omega)}{m(\chi_k,\omega)+1}, \frac{m(\chi_k,\mathsf{T}\omega)m(\omega,\chi_{k+1})}{m(\chi_k,\omega)+1} \end{array}\right\}\right] = 0.$$

Therefore, $\lim_{k\to\infty} m(\chi_{k+1},\mathsf{T}\omega) = 0$, and then $\mathsf{T}\omega = \omega$. □

Example 5. *Let the set* $\mathfrak{X} = [0, +\infty)$ *and* $m : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ *be the supermetric* ($\mathsf{s} = 2$) *defined as follows:*

$$\begin{aligned} m(\chi, y) &= (\chi - y)^2, \ \text{for } \chi, y \in \mathbb{R} \setminus \{1\}, \\ m(1, y) &= m(y, 1) = (1 - y^3)^2, \ \text{for } y \in \mathbb{R}. \end{aligned}$$

Let $\mathsf{T} : \mathfrak{X} \to \mathfrak{X}$ *be a mapping defined by* $\mathsf{T}\chi = \begin{cases} 3, & \text{for } \chi \in [0, 6] \\ 4, & \text{for } \chi \in (6, +\infty) \end{cases}$. *Since* T *is an asymptotically regular mapping, we must make sure that (15) holds. Let* $\kappa = \frac{1}{2}$.

For any $\chi, y \in [0, 6]$ *and respectively* $\chi, y \in (6, +\infty)$, *we have* $m(\mathsf{T}\chi, \mathsf{T}y) = 0$, *and obviously, (15) holds.*

If $\chi = 3$, $y \in (6, +\infty)$,

$$m(3, y) = |3 - y|^2, \ m(\mathsf{T}3, \mathsf{T}y) = 1, m(3, \mathsf{T}3) = m(3, 3) = 0,$$

and

$$m(\mathsf{T}3, \mathsf{T}y) = 1 \le \frac{|3 - y|^2}{2} = \kappa \cdot m(3, y).$$

Therefore, (15) holds.

If $\chi = 4$, $y \in (6, +\infty)$,

$$m(4, y) = |4 - y|^2, \ m(\mathsf{T}4, \mathsf{T}y) = 1, m(4, \mathsf{T}y) = m(4, 4) = 0,$$

$$m(\mathsf{T}4, \mathsf{T}y) = 1 \le \frac{|4 - y|^2}{2} = \kappa \cdot m(4, y),$$

and hence, (15) holds.

Similarly, if $\chi = 1$, $y \in (6, +\infty)$, *we have*

$$m(1, y) = (1 - y^3)^2, m(\mathsf{T}1, \mathsf{T}y) = 1,$$

and

$$m(\mathsf{T}1, \mathsf{T}y) = 1 \le \frac{(1 - y^3)^2}{2} = \kappa \cdot m(1, y).$$

If $\chi \in [0, 1) \cup (1, 3) \cup (3, 4) \cup (4, 6]$, $y \in (6, +\infty)$,

$$m(\chi, y) = |\chi - y|^2, \ m(\mathsf{T}\chi, \mathsf{T}y) = 1,$$

and $\min\left\{ m(\chi, \mathsf{T}\chi), m(y, \mathsf{T}y), \frac{m(\chi,\mathsf{T}\chi)m(y,\mathsf{T}y)}{m(\chi,y)+1}, \frac{m(\chi,\mathsf{T}y)m(y,\mathsf{T}\chi)}{m(\chi,y)+1} \right\} \neq 0$. *Consequently, we can find an* $\mathcal{L} > 0$, *such that (15) is satisfied.*

3. Conclusions

In the last decades, in relation to the metric fixed point theory, a vast number of the fixed point results have been re-discovered or have overlapped the existing ones; additionally, equivalent versions have been published due to some false assumptions. The main reason for these situations is that the theory is squeezed. In this paper, we propose a new structure in which the existence and uniqueness of the fixed point of certain operators can be discussed. The notion of the supermetric is possibly a very good candidate for expanding the metric fixed point theory. In this paper, we gave some fixed point theorems for this new structure. We believe that a good examination of this structure will give priority to overcoming the congestion of the metric fixed point theory.

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