mathematics

Article

Existence of Positive Solutions for a Fully Fourth-Order Boundary Value Problem

Yongxiang Li * and Weifeng Ma

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
* Correspondence: liyxnwnu@163.com; Tel.: +86-0931-7971111

Abstract: This paper deals with the existence of positive solutions of the fully fourth-order boundary value pqroblem $u^{(4)} = f(t, u, u', u'', u''')$ on $[0, 1]$ with the boundary condition $u(0) = u(1) = u''(0) = u''(1) = 0$, which models a statically bending elastic beam whose two ends are simply supported, where $f : [0, 1] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}^+$ is continuous. Some precise inequality conditions on f guaranteeing the existence of positive solutions are presented. The inequality conditions allow that $f(t, u, v, w, z)$ may be asymptotically linear, superlinear, or sublinear on u, v, w, and z as $|(u, v, w, z)| \to 0$ and $|(u, v, w, z)| \to \infty$. Our discussion is based on the fixed point index theory in cones.

Keywords: fully fourth-order boundary value problem; simply supported beam equation; positive solution; cone; fixed point index

MSC: 34B18; 47H11

check for updates

Citation: Li, Y.; Ma, W. Existence of Positive Solutions for a Fully Fourth-Order Boundary Value Problem. *Mathematics* **2022**, *10*, 3063. https://doi.org/10.3390/math10173063

Academic Editors: Janusz Brzdęk and Luigi Rodino

Received: 30 June 2022
Accepted: 19 August 2022
Published: 25 August 2022

1. Introduction and Main Results

The deformations of an elastic beam in an equilibrium state, whose two ends are simply supported, can be described by the fourth-order ordinary differential equation boundary value problem (BVP)

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & t \in [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1}$$

where $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ is continuous. In mechanics, the problem is called the beam equation with simple support, and in the equation, the physical meaning of the derivatives of the deformation function $u(t)$ is as follows: $u^{(4)}$ is the load density stiffness, u''' is the shear force stiffness, u'' is the bending moment stiffness, and u' is the slope [1–4]. Owing to its importance in physics, some special cases of this problem have been studied by many authors, see [5–26] and references therein. However, just a few writers address the fully nonlinear BVP (1). In some practice, only its positive solution is significant. The positive solution u of BVP (1) means that $u \in C^4[0, 1]$ is a solution of BVP (1) and it satisfies $u(t) > 0$ for every $t \in (0, 1)$. In this paper, we discuss the existence of positive solutions for BVP (1).

For the special case of BVP (1) that f does not contain any derivative term, namely the simple beam equation

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), & t \in [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{2}$$

here $f : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, the existence of positive solutions has been studied by some authors, see [5,7,9,11,23]. In [23], Ma and Wang showed the existence of positive solutions of BVP (2) under that $f(t, u)$ is either superlinear or sublinear on u by employing the fixed point theorem of cone extension or compression in $C[0, 1]$. In [5] (Theorem 3.5), Bai and Wang improved this result and proved that if $f(t, u)$ satisfies one of the following conditions

(A1) $f^0 < \pi^4, \quad f_\infty > \pi^4$;
(A2) $f_0 > \pi^4, \quad f^\infty < \pi^4$,

where

$$\begin{aligned} f_0 &= \liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u}, \qquad & f^0 &= \limsup_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,x)}{x}, \\ f_\infty &= \liminf_{u\to +\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u}, \qquad & f^\infty &= \limsup_{u\to +\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u}, \end{aligned} \tag{3}$$

then BVP (2) has at least one positive solution. Clearly, Condition (A1) covers the situation in which $f(t, u)$ is the superlinear growth on u, while Condition (A2) includes the case in which $f(t, u)$ is the sublinear growth on u. Since Condition (A1) and (A2) allow that $f(t, u)/u$ is near to the first eigenvalue $\lambda_1 = \pi^4$, by the Fredholm alternative, (A1) and (A2) are optimal upper and lower limits Conditions to the existence of positive solutions. By definition (3), we can verify that (A1) holds if and only if f satisfies conditions (B1) and (B2), and and (A2) holds if and only if f satisfies (B3) and (B4):

(B1) there exists constants $a \in (0, \pi^4)$ and $\delta > 0$ such that

$$f(t, u) \le a\,u, \qquad \forall\, t \in [0, 1], \quad u \in [0, \delta];$$

(B2) there exists constants $b > \pi^4$ and $H > 0$ such that

$$f(t, u) \ge b\,u, \qquad \forall\, t \in [0, 1], \quad u \ge H,$$

(B3) there exists constants $b > \pi^4$ and $\delta > 0$ such that

$$f(t, u) \ge b\,u, \quad \forall\, t \in [0, 1], \quad u \in [0, \delta];$$

(B4) there exists constants $a \in (0, \pi^4)$ and $H > 0$ such that

$$f(t, u) \le a\,u, \qquad \forall\, t \in [0, 1], \quad u \ge H.$$

For the special case of BVP (1) that f only contains second-order derivative term u'', namely the elastic beam equation with bending moment term

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)), \qquad t \in [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{4}$$

where $f : [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}^+$ is continuous, the existence of positive solutions has also been discussed by several authors; see [15,16,20–22,24]. In [21], Ma obtained the existence of positive solutions of BVP (4) under that $f(t, u, w)$ is superlinear or sublinear on w. In [15], Li extended this result and showed that BVP (4) has a positive solution when $f(t, u, w)$ satisfies the following superlinear or sublinear growth condition on u and v:

(C1) $f^{20} < 36/7, \quad f_{2\infty} > 128$;
(C2) $f_{20} > 128, \quad f^{2\infty} < 36/7$,

where

$$\begin{aligned} f_{20} &= \liminf_{|u|+|w|\to 0} \min_{t\in[0,1]} \frac{f(t,u,w)}{|u|+|w|}, \quad f^{20} = \limsup_{|u|+|w|\to 0} \max_{t\in[0,1]} \frac{f(t,u,w)}{|u|+|w|}, \\ f_{2\infty} &= \liminf_{|u|+|w|\to \infty} \min_{t\in[0,1]} \frac{f(t,u,w)}{|u|+|w|}, \quad f^{2\infty} = \limsup_{|u|+|w|\to \infty} \max_{t\in[0,1]} \frac{f(t,u,w)}{|u|+|w|}. \end{aligned} \tag{5}$$

In [16], Li further improved Condition (C1) to the following inequality conditions (D1) and (D2), and Condition (C2) to the following inequality conditions (D3) and (D4):

(D1) There exist $a,\ c \ge 0,\ \frac{a}{\pi^4} + \frac{c}{\pi^2} < 1$, and $\delta > 0$, such that

$$f(t,u,w) \le a\,u - c\,w\,, \qquad \forall\, t \in [0,1], \quad u \in [0,\delta], \quad w \in [-\delta, 0]\,;$$

(D2) There exist $a_1,\ c_1 \ge 0,\ \frac{a_1}{\pi^4} + \frac{c_1}{\pi^2} > 1$, and $H > 0$, such that

$$f(t,u,w) \ge a_1\,u - c_1\,w\,, \qquad \forall\, t \in [0,1], \quad |u| + |w| \ge H\,,$$

(D3) There exist $a,\ c \ge 0,\ \frac{a}{\pi^4} + \frac{c}{\pi^2} > 1$, and $\delta > 0$, such that

$$f(t,u,w) \ge a\,u - c\,w\,, \qquad \forall\, t \in [0,1], \quad u \in [0,\delta], \quad w \in [-\delta, 0]\,;$$

(D4) There exist $a_1,\ c_1 \ge 0,\ \frac{a_1}{\pi^4} + \frac{c_1}{\pi^2} < 1$, and $H > 0$, such that

$$f(t,u,w) \le a_1\,u - c_1\,w\,, \qquad \forall\, t \in [0,1], \quad |u| + |w| \ge H\,.$$

From definition (5), we easily see that

$$\text{(C1)} \implies \text{(D1) and (D2) hold;}$$

$$\text{(C2)} \implies \text{(D3) and (D4) hold.}$$

Since the straight line

$$\ell_1 = \left\{ (a,c) \;\middle|\; \frac{a}{\pi^4} + \frac{c}{\pi^2} = 1 \right\} \tag{6}$$

is the first eigenline of the two-parameter linear eigenvalue problem (LEVP)

$$\begin{cases} u^{(4)}(t) \ + cu''(t) \ - au(t) = 0\,, \\ u(0) = u(1) = u''(0) = u''(1) = 0\,, \end{cases} \tag{7}$$

we conclude that Conditions (D1), (D2), (D3), and (D4) are precise. If f does not contain v, then Conditions (D1)–(D4) are simplified to (B1)–(B4), respectively, by letting $c = c_1 = 0$. Hence, the results in [16] unify and extend the ones in [5,15,20,21,23].

The purpose of this paper is to discuss the existence of a positive solution to the fully fourth-order boundary value problem (1). Li [27] discussed the existence of a positive solution to the following fully fourth-order nonlinear boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \qquad t \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0\,, \end{cases} \tag{8}$$

which models a statically elastic beam fixed at the left and freed one at the right, and is called a cantilever beam in mechanics. Some special cases of BVP (8) are studied in [26,28–33]. Owing to the boundary conditions of BVP (1) being different from ones of BVP (8), the discussed methods of [27] cannot be simply applied to BVP (1). For $h \in C^+(I)$, here $I = [0, 1]$, the solution of the linear boundary value problem corresponding to BVP (8)

$$\begin{cases} u^{(4)}(t) = h(t), \qquad t \in [0, 1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0\,, \end{cases} \tag{9}$$

has the sign-preserving property ([27], Lemma 2.2):

$$u \ge 0, \quad u' \ge 0, \quad u'' \ge 0, \quad u''' \le 0, \tag{10}$$

and hence BVP (8) can be converted to a fixed point problem of a cone mapping in $C^3(I)$. However, the solution of the linear boundary value problem corresponding to BVP (1)

$$\begin{cases} u^{(4)}(t) = h(t), \qquad t \in [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0\,, \end{cases} \tag{11}$$

has only the sign-preserving property ([16], Lemma 1):

$$u \ge 0, \qquad u'' \le 0, \tag{12}$$

u' and u''' are sign-changing, and BVP (1) cannot be treated in the same manner as BVP (8). We will use different methods to discuss BVP (1) and obtain optimal existence conditions of positive solutions. The existence conditions of positive solutions in [27] are not optimal. Let $I = [0, 1]$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0]$. Our main results are as follows:

Theorem 1. *Assume that* $f : I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}^+$ *is continuous and satisfies the following conditions*
(F0) Given any $M > 0$*, there is a positive continuous function* $g_M(\rho)$ *on* $\mathbb{R}^+$ *satisfying*

$$\int_0^{+\infty} \frac{\rho\, d\rho}{g_M(\rho) + 1} = +\infty, \tag{13}$$

such that

$$f(t, u, v, w, z) \le g_M(|z|), \qquad t \in [0, 1], \;\; |u|, |v|, |w| \le M, \;\; z \in \mathbb{R}. \tag{14}$$

(F1) There exist $a, b, c, d \ge 0$ *with* $\frac{a}{\pi^4} + \frac{b}{\pi^3} + \frac{c}{\pi^2} + \frac{d}{\pi} < 1$ *and* $\delta > 0$*, such that*

$$f(t, u, v, w, z) \le a\, u + b\,|v| + c\,|w| + d\,|z|,$$

for all $(t, u, v, w, z) \in I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R}$ *with* $|u|, |v|, |w|, |z| \le \delta$.
(F2) There exist $a_1, c_1 \ge 0$ *with* $\frac{a_1}{\pi^4} + \frac{c_1}{\pi^2} > 1$ *and* $H > 0$*, such that*

$$f(t, u, v, w, z) \ge a_1\, u + c_1\,|w|\,,$$

for all $(t, u, v, w, z) \in I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R}$ *with* $|u| + |v| + |w| + |z| \ge H$.
Then BVP (1) has at least one positive solution.

Theorem 2. *Assume that* $f : I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}^+$ *is continuous and satisfies the following conditions*
(F3) There exist $a, c \ge 0$ *with* $\frac{a}{\pi^4} + \frac{c}{\pi^2} > 1$ *and* $\delta > 0$*, such that*

$$f(t, u, v, w, z) \ge a\, u + c\,|w|\,,$$

for all $(t, u, v, w, z) \in I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R}$ *with* $|u|, |v|, |w|, |z| \le \delta$.

(F4) There exist $a_1, b_1, c_1, d_1 \geq 0$ *with* $\frac{a_1}{\pi^4} + \frac{b_1}{\pi^3} + \frac{c_1}{\pi^2} + \frac{d_1}{\pi} < 1$ *and* $H > 0$, *such that*

$$f(t, u, v, w, z) \leq a_1 u + b_1 |v| + c_1 |w| + d_1 |z|,$$

for all $(t, u, v, w, z) \in I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R}$ *with* $|u| + |v| + |w| + |z| \geq H$.
Then BVP (1) has at least one positive solution.

In Theorem 1, Condition (F0) is a Nagumo-type growth condition on z, in which for given $M > 0$, the control function $g_M(\rho)$ can be determined by

$$g_M(\rho) := \max\left\{ f(t, u, v, w, z) \;\middle|\; \begin{array}{l} (t, u, v, w, z) \in I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R}, \\ |u|,\ |v|,\ |w| \leq M,\ \ |z| \leq \rho \end{array} \right\}, \tag{15}$$

and it restricts f is at most quadratic growth with respect to z by (13) and (14). When f is independent of z, by (15), $g_M(\rho)$ is a positive constant and (F0) naturally holds. Conditions (F1) and (F2) allow that $f(t, u, v, w, z)$ is superlinear growth on u, v, w, z as $|(u, v, w, z)| \to 0$ and $|(u, v, w, z)| \to \infty$. For example, the power function

$$f(t, u, v, w, z) = |u|^{p_0} + |v|^{p_1} + |w|^{p_2} + |z|^{p_3} \tag{16}$$

satisfies Conditions (F1) and (F2) when $p_0, p_1, p_2, p_3 > 1$. However, only when $p_3 \leq 2$ does Assumption (F0) hold.

In Theorem 2, Conditions (F3) and (F4) allow that $f(t, u, v, w, z)$ is sublinear growth on u, v, w, z as $|(u, v, w, z)| \to 0$ and $|(u, v, w, z)| \to \infty$. For example, the power function defined by (16) satisfies (F3) and (F4) when $0 < p_0, p_1, p_2, p_3 < 1$.

In Theorems 1 and 2, if f is independent of v and z, we choose $c = d = c_1 = d_1 = 0$ in Conditions (F1) and (F4), Conditions (F1)–(F4) are just simplified to (D1)–(D4), respectively. Hence, Conditions (F1) and (F2) in Theorem 1 and Conditions (F3) and (F4) in Theorem 2 are optimal, and Theorems 1 and 2 extend the existing results mentioned above. Conditions (F1)–(F4) also allow that f may be asymptotically linear on u, v, w, z as $|(u, v, w, z)| \to 0$ and $|(u, v, w, z)| \to \infty$, see (H1)–(H4) in Section 4.

The proofs of Theorems 1 and 2 will be given in Section 3. Some preliminaries to discuss BVP (1) are presented in Section 2. Some applications of Theorems 1 and 2 are given in Section 4.

2. Preliminaries

Let $C(I)$ denote the Banach space of all continuous function u on I with norm $\|u\|_C = \max_{t\in I} |u(t)|$. Generally, for $n \in \mathbb{N}$, we use $C^n(I)$ to denote the Banach space of all nth-order continuous differentiable function on I with the norm

$$\|u\|_{C^n} = \max\{\, \|u\|_C, \|u'\|_C, \cdots, \|u^{(n)}\|_C \}.$$

Let $C^+(I)$ be the cone of nonnegative functions in $C(I)$. Let $L^2(I)$ be the usual Hilbert space with the interior product $(u, v) = \int_0^1 u(t)v(t)dt$ and the norm $\|u\|_2 = \left(\int_0^1 |u(t)|^2 dt \right)^{1/2}$. Let $H^n(I)$ be the usual Sobolev space. $u \in H^n(I)$ means that $u \in C^{n-1}(I)$, $u^{(n-1)}(t)$ is absolutely continuous on I and $u^{(n)} \in L^2(I)$. The norm of $H^n(I)$ is defined by $\|u\|_{H^n} = \max\{\, \|u\|_2, \|u'\|_2, \cdots, \|u^{(n)}\|_2 \}$.

To discuss BVP (1), we first consider the corresponding linear boundary value problem (LBVP)

$$\begin{cases} u^{(4)} = h(t)\,, \qquad t \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0\,, \end{cases} \tag{17}$$

where $h \in L^2(I)$.

Let $G(t,\, s)$ be Green's function to the linear boundary value problem

$$-u'' = 0, \qquad u(0) = u(1) = 0,$$

which is expressed by

$$G(t,\, s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1\,, \\ s(1-t), & 0 \le s \le t \le 1\,. \end{cases} \tag{18}$$

It is easy to see that $G(t,s)$ has the following properties

(1) $G(t,\, s) > 0,\;\; t,\, s \in (0,1);$
(2) $G(t,\, s) \le G(s,\, s),\;\; t,\, s \in I;$
(3) $G(t,\, s) \ge G(t,\, t)\, G(s,\, s),\;\; t,\, s \in I.$

For any given $h \in L^2(I)$, it is easy to verify that the LBVP (17) has a unique solution $u \in H^4(I)$ given by

$$u(t) = \int_0^1 \int_0^1 G(t,\, \tau)\, G(\tau,\, s) h(s)\, ds d\tau\;, \tag{19}$$

and the second-order derivative u'' can be expressed by

$$u''(t) = -\int_0^1 G(t,\, s)\, h(s)\, ds\;. \tag{20}$$

Lemma 1. *For every $h \in L^2(I)$, LBVP (17) has a unique solution $u := Sh \in H^4(I)$, which satisfies*

$$\|u\|_2 \le \frac{1}{\pi}\,\|u'\|_2,\;\; \|u'\|_2 \le \frac{1}{\pi}\,\|u''\|_2,\;\; \|u''\|_2 \le \frac{1}{\pi}\,\|u'''\|_2,\;\; \|u'''\|_2 \le \frac{1}{\pi}\,\|u^{(4)}\|_2. \tag{21}$$

Moreover, the solution operator $S : L^2(I) \to H^4(I)$ is a linear bounded operator and its norm satisfies

$$\|S\|_{\mathcal{B}(L^2(I),\, H^4(I))} = 1. \tag{22}$$

When $h \in C(I)$, the solution $u = Sh \in C^4(I)$, and the solution operator $S : C(I) \to C^3(I)$ is completely continuous.

Proof. For any $h \in L^2(I)$, $u = Sh$, given by (19), belongs to $H^4(I)$ and is a unique solution of LBVP (17). Owing to the sine system, $\{\, \sin k\pi t \mid k \in \mathbb{N}\,\}$ is a complete orthogonal system of $L^2(I)$, every $h \in L^2(I)$ can be expressed by the Fourier series expansion

$$h(t) = \sum_{k=1}^{\infty} h_k\, \sin k\pi t\,, \tag{23}$$

where $h_k = 2\int_0^1 h(s) \sin k\pi s\, ds,\;\; k = 1,\, 2,\, \cdots$, and the Paserval equality

$$\|h\|_2{}^2 = \frac{1}{2} \sum_{k=1}^{\infty} |h_k|^2 \tag{24}$$

holds. Since $u = Sh \in H^4(I)$, u, u'', and $u^{(4)}$ belong to $L^2(I)$ and they can also be expressed by the Fourier series expansion of the sine system. Since $u^{(4)} = h$, by the integral formula of Fourier coefficient, we have

$$u(t) = \sum_{k=1}^{\infty} \frac{h_k}{k^4\pi^4}\ \sin k\pi t\,, \tag{25}$$

$$u''(t) = -\sum_{k=1}^{\infty} \frac{h_k}{k^2\pi^2}\ \sin k\pi t\,. \tag{26}$$

On the other hand, since the cosine system $\{\cos k\pi t \mid k = 0, 1, 2, \cdots\}$ is another complete orthogonal system of $L^2(I)$, every $v \in L^2(I)$ can be expressed by the cosine series expansion

$$v(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\pi t\,,$$

where $a_k = 2\int_0^1 h(s)\cos k\pi s\, ds,\ k = 0, 1, 2, \cdots$. For the above $u = Sh$, by the integral formula of the coefficient of the cosine series expansion, we obtain the cosine series expansions of u' and u''':

$$u'(t) = \sum_{k=1}^{\infty} \frac{h_k}{k^3\pi^3}\ \cos k\pi t\,, \tag{27}$$

$$u'''(t) = -\sum_{k=1}^{\infty} \frac{h_k}{k\pi}\ \cos k\pi t\,. \tag{28}$$

By (23), (25)–(28), and the Paserval equality, we obtain that

$$\|u\|_2{}^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^4\pi^4}\right|^2 \le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^3\pi^3}\right|^2 = \frac{1}{\pi^2}\,\|u'\|_2{}^2\,,$$

$$\|u'\|_2{}^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^3\pi^3}\right|^2 \le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^2\pi^2}\right|^2 = \frac{1}{\pi^2}\,\|u''\|_2{}^2\,,$$

$$\|u''\|_2{}^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^2\pi^2}\right|^2 \le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k\pi}\right|^2 = \frac{1}{\pi^2}\,\|u'''\|_2{}^2\,,$$

$$\|u'''\|_2{}^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k\pi}\right|^2 \le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}|h_k|^2 = \frac{1}{\pi^2}\,\|h\|_2{}^2 = \frac{1}{\pi^2}\,\|u^{(4)}\|_2{}^2\,.$$

Hence, (21) holds.

By expression (19) of the solution $u = Sh$, $S : L^2(I) \to H^4(I)$ is a linear bounded operator. By (21) we have

$$\begin{aligned}\|Sh\|_{H^4} = \|u\|_{H^4} &= \max\{\ \|u\|_2,\ \|u'\|_2,\ \|u''\|_2,\ \|u'''\|_2,\ \|u^{(4)}\|_2\ \}\\ &= \|u^{(4)}\|_2 = \|h\|_2.\end{aligned}$$

Hence, $\|S\|_{\mathcal{B}(L^2(I),\,H^4(I))} = 1$.

When $h \in C(I)$, by (19) and (20), $u \in C^4(I)$ and the solution operator $S : C(I) \to C^4(I)$ are bounded. By the compactness of the embedding $C^4(I) \hookrightarrow C^3(I)$, $S : C(I) \to C^3(I)$ is completely continuous. $\square$

Lemma 2. *Let $h \in C^+(I)$. Then the solution u of LBVP (17) has the following properties:*

(a) $u(t) \ge t(1-t)\,\|u\|_C\,,\ \forall\ t \in I;\quad \|u\|_C \le \frac{\pi^3}{4}\int_0^1 u(t)\ \sin \pi t\, dt;$
(b) $u''(t) \le -t(1-t)\,\|u''\|_C\,,\ \forall\ t \in I;\quad \|u''\|_C \le \frac{\pi^5}{4}\int_0^1 u(t)\ \sin \pi t\, dt;$
(c) $\|u\|_C \le \|u'\|_C \le \|u''\|_C \le \|u'''\|_C;$

(d) there exists $\xi \in (0, 1)$ *such that* $u'''(\xi) = 0$, $u'''(t) \leq 0$ *for* $t \in [0, \xi]$ *and* $u'''(t) \geq 0$ *for* $t \in [\xi, 1]$. *Moreover,* $\|u'''\|_C = \max\{-u'''(0), u'''(1)\}$.

Proof. Set $v = -u''$. Then from (20), we obtain that

$$v(t) = \int_0^1 G(t, s)\, h(s)\, ds\,, \tag{29}$$

and therefore $v \in C^+(I)$. Combining (19) and (29), we have

$$u(t) = \int_0^1 G(t, s)\, v(s)\, ds\,.$$

From this and property (2) of $G(t, s)$ we get that $\|u\|_C \leq \int_0^1 G(s, s)\, v(s)\, ds$. From this and property (3) of $G(t, s)$, we have

$$\begin{aligned} u(t) = \int_0^1 G(t, s)\, v(s)\, ds \;&\geq\; G(t, t) \int_0^1 G(s, s)\, v(s)\, ds \\ &\geq\; G(t, t)\, \|u\|_C = \; t(1-t)\, \|u\|_C. \end{aligned}$$

Multiplying this inequality by $\sin \pi t$ and integrating on I, we have

$$\int_0^1 u(t)\, \sin \pi t\, dt \geq \|u\|_C \int_0^1 t(1-t)\, \sin \pi t\, dt = \frac{4}{\pi^3}\, \|u\|_C\,.$$

Thus, conclusion (a) holds.

For $v = -u''$, by (29) with a similar argument to u, we have

$$v(t) \geq t(1-t)\, \|v\|_C, \quad \forall\, t \in I; \qquad \|v\|_C \leq \frac{\pi^3}{4} \int_0^1 v(t)\, \sin \pi t\, dt.$$

This implies that $u''(t) \leq -t(1-t)\, \|u''\|$ for every $t \in I$ and

$$\|u''\| \leq -\frac{\pi^3}{4} \int_0^1 u''(t)\, \sin \pi t\, dt = \frac{\pi^5}{4} \int_0^1 u(t)\, \sin \pi t\, dt.$$

Namely, conclusion (b) holds.

Since u is a solution of LBVP (17), by the boundary conditions of LBVP (17), there exists $\xi \in (0, 1)$ such that $u'(\xi) = 0$, and for every $t \in I$,

$$u(t) = \int_0^t u'(s)\, ds, \quad u'(t) = \int_\xi^t u''(s)\, ds, \quad u''(t) = \int_0^t u'''(s)\, ds.$$

Hence, we have

$$\begin{aligned} |u(t)| &= \Big| \int_0^t u'(s)\, ds \Big| \leq t\, \|u'\|_C \leq \|u'\|_C\,, \\ |u'(t)| &= \Big| \int_\xi^t u''(s)\, ds \Big| \leq |t - \xi|\, \|u''\|_C \leq \|u''\|_C\,, \\ |u''(t)| &= \Big| \int_0^t u'''(s)\, ds \Big| \leq t\, \|u'''\|_C \leq \|u'''\|_C\,. \end{aligned}$$

From these inequalities, we conclude that

$$\|u\|_C \leq \|u'\|_C, \quad \|u'\|_C \leq \|u''\|_C, \quad \|u''\|_C \leq \|u'''\|_C.$$

Hence, the conclusion (c) holds.

Since $u'' \leq 0$, from the boundary conditions of LBVP (17) we see that $u'''(0) \leq 0$ and $u'''(1) \geq 0$. Since $u^{(4)}(t) = h(t) \geq 0$ for every $t \in I$, it follows that $u'''(t)$ is a monotone non-decreasing function on I. From these we conclude that there exists $\xi \in (0, 1)$ such that $u'''(\xi) = 0$, $u'''(t) \leq 0$ for $t \in [0, \xi]$ and $u'''(t) \geq 0$ for $t \in [\xi, 1]$. Moreover $\|u'''\|_C = \max_{t\in I} |u'''(t)| = \max\{-u'''(0), u'''(1)\}$. Hence, the conclusion (d) holds. □

Consider BVP (1). Let $f : I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}^+$ be continuous. Define a closed convex cone K in $C^3(I)$ by

$$K = \{\, u \in C^3(I) \mid u(t) \geq 0, \;\; u''(t) \leq 0, \; \forall\, t \in I \,\}. \tag{30}$$

By Lemma 2(a) and (b), we have that $S(C^+(I)) \subset K$. For every $u \in K$, set

$$F(u)(t) := f(t, u(t), u'(t), u''(t), u'''(t)), \qquad t \in I. \tag{31}$$

Then $F : K \to C^+(I)$ is continuous and it maps every bounded in K into a bounded set in $C^+(I)$. Define a mapping $A : K \to K$ by

$$A = S \circ F. \tag{32}$$

By Lemma 1, $A : K \to K$ is a completely continuous mapping. By the definitions of S and K, the positive solution of BVP (1) is equivalent to the nonzero fixed point of A. We will find the nonzero fixed point of A by using the fixed point index theory in cones.

Let E be a Banach space and $K \subset E$ be a closed convex cone in E. Assume Ω is a bounded open subset of E with boundary $\partial\Omega$, and $K \cap \Omega \neq \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $Au \neq u$ for any $u \in K \cap \partial\Omega$, then the fixed point index $i\,(A, K \cap \Omega, K)$ is well defined. The following lemmas in [34,35] are needed in our discussion.

Lemma 3. *Let Ω be a bounded open subset of E with $\theta \in \Omega$, and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If $\mu\, Au \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \mu \leq 1$, then $i\,(A, K \cap \Omega, K) = 1$.*

Lemma 4. *Let Ω be a bounded open subset of E and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If there exists $v_0 \in K \setminus \{\theta\}$ such that $u - Au \neq \tau\, v_0$ for every $u \in K \cap \partial\Omega$ and $\tau \geq 0$, then $i\,(A, K \cap \Omega, K) = 0$.*

Lemma 5. *Let Ω be a bounded open subset of E, and A, $A_1 : K \cap \overline{\Omega} \to K$ be two completely continuous mappings. If $(1 - s)Au + sA_1u \neq u$ for every $u \in K \cap \partial\Omega$ and $0 \leq s \leq 1$, then $i\,(A, K \cap \Omega, K) = i\,(A_1, K \cap \Omega, K)$.*

3. Proof of the Main Results

Proof of Theorem 1. Let $E = C^3(I)$, $K \subset C^3(I)$ be the closed convex cone defined by (30) and $A : K \to K$ be the completely continuous mapping defined by (32). Then the positive solution of BVP (1) is equivalent to the nontrivial fixed point of A. Let $0 < r < R < +\infty$ and set

$$\Omega_1 = \{u \in C^3(I) \mid \|u\|_{C^3} < r\}, \qquad \Omega_2 = \{u \in C^3(I) \mid \|u\|_{C^3} < R\}. \tag{33}$$

We show that A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when r is small enough and R large enough.

Choose $r \in (0, \delta)$, where δ is the positive constant in Condition (F1). We prove that A satisfies the condition of Lemma 3 in $K \cap \partial\Omega_1$, namely

$$\mu\, Au \neq u, \qquad \forall\, u \in K \cap \partial\Omega_1, \;\; 0 < \mu \leq 1. \tag{34}$$

In fact, if (34) does not hold, there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \mu_0 \leq 1$ such that $\mu_0 \, Au_0 = u_0$. Since $u_0 = S(\mu_0 F(u_0))$, by the definition of S, $u_0 \in C^4(I)$ is the unique solution of LBVP (17) for $h = \mu_0 F(u_0) \in C^+(I)$. Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$0 \leq u_0(t),\ |u_0{}'(t)|,\ -u_0{}''(t),\ |u_0{}'''(t)| \leq \|u_0\|_{C^3} = r < \delta, \quad t \in I. \tag{35}$$

Hence, by Condition (F1) we have

$$\begin{aligned} 0 \leq F(u_0)(t) &= f(t,\, u_0(t),\, u_0{}'(t),\, u_0{}''(t),\, u_0{}'''(t)) \\ &\leq a\,|u_0(t)| + b\,|u_0{}'(t)| + c\,|u_0{}''(t)| + d\,|u_0{}'''(t)|, \quad t \in I. \end{aligned}$$

By this inequality and (21) we obtain that

$$\begin{aligned} \|F(u_0)\|_2 &\leq a\,\|u_0\|_2 + b\,\|u_0{}'\|_2 + c\,\|u_0{}''\|_2 + d\,\|u_0{}'''\|_2 \\ &\leq \Big(\frac{a}{\pi^4} + \frac{b}{\pi^3} + \frac{c}{\pi^2} + \frac{d}{\pi}\Big)\|u_0{}^{(4)}\|_2 \\ &\leq \Big(\frac{a}{\pi^4} + \frac{b}{\pi^3} + \frac{c}{\pi^2} + \frac{d}{\pi}\Big)\|u_0\|_{H^4}. \end{aligned}$$

Hence, by (22) we conclude that

$$\begin{aligned} \|u_0\|_{H^4} = \|S(\mu_0 F(u_0))\|_{H^4} &\leq \|S\|_{\mathcal{B}(L^2(I),\, H^4(I))} \cdot \|F(u_0)\|_2 \\ &\leq \Big(\frac{a}{\pi^4} + \frac{b}{\pi^3} + \frac{c}{\pi^2} + \frac{d}{\pi}\Big)\|u_0\|_{H^4}. \end{aligned} \tag{36}$$

Since $\|u_0\|_{H^4} > 0$, from this inequality it follows that $\frac{a}{\pi^4} + \frac{b}{\pi^3} + \frac{c}{\pi^2} + \frac{d}{\pi} > 1$, which contradicts the assumption in Condition (F1). Hence, (34) holds, namely A satisfies the condition of Lemma 3 in $K \cap \partial\Omega_1$. By Lemma 3, we have

$$i\,(A,\, K \cap \Omega_1,\, K) = 1. \tag{37}$$

Set $C_0 = \max\{\, |f(t,\, u,\, v,\, w,\, z) - a_1\, u + c_1\, w| \,:\, (t,\, u,\, v,\, w,\, z) \in I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R},\ |u| + |v| + |w| + |z| \leq H\} + 1$. Then, by Condition (F2) we have

$$f(t,\, u,\, v,\, w,\, z) \geq a_1\, u - c_1\, w - C_0, \quad \forall\, (t,\, u,\, v,\, w,\, z) \in I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R}. \tag{38}$$

Define a mapping $F_1 : K \to C^+(I)$ by

$$\begin{aligned} F_1(u)(t) &:= f(t,\, u(t),\, u'(t),\, u''(t),\, u'''(t)) + C_0 \\ &= F(u)(t) + C_0, \qquad t \in I, \end{aligned} \tag{39}$$

and set

$$A_1 = S \circ F_1. \tag{40}$$

Then $A_1 : K \to K$ is a completely continuous mapping. Let $R > \delta$, we show that A_1 satisfies that

$$i\,(A_1,\, K \cap \Omega_2,\, K) = 0. \tag{41}$$

Choose $v_0 = \sin \pi t$. Then $v_0 \in K \setminus \{\theta\}$ and $S(\pi^4 v_0) = v_0$. We show that A_1 satisfies the condition of Lemma 4 in $K \cap \partial\Omega_2$, namely

$$u - A_1 u \neq \tau\, v_0, \qquad \forall\, u \in K \cap \partial\Omega_2, \quad \tau \geq 0. \tag{42}$$

In fact, if (42) does not hold, there exist $u_1 \in K \cap \partial\Omega_2$ and $\tau_1 \geq 0$ such that $u_1 - A_1 u_1 = \tau_1 v_0$. Since $u_1 = A_1 u_1 + \tau_1 v_0 = S(F(u_1) + C_0 + \tau_1 \pi^4 v_0)$, by the definition of S, u_1 is the unique solution of LBVP (17) for $h = F(u_1) + C_0 + \tau_1 \pi^4 v_0 \in C^+(I)$. Hence, $u_1 \in C^4(I)$ satisfies the differential equation

$$\begin{cases} {u_1}^{(4)}(t) = f(t, u_1(t), {u_1}'(t), {u_1}''(t), {u_1}'''(t)) + C_0 + \tau_1 \pi^4 v_0(t), & t \in I, \\ u_1(0) = u_1(1) = {u_1}''(0) = {u_1}''(1) = 0\,. \end{cases} \tag{43}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, we have

$$u_1(t) \geq 0, \quad {u_1}''(t) \leq 0, \quad \forall\, t \in I.$$

Hence, by (38), we have

$$f(t, u_1(t), {u_1}'(t), {u_1}''(t), {u_1}'''(t)) \geq a_1\, u_1(t) - c_1\, {u_1}''(t) - C_0, \qquad t \in I.$$

From this and (43), we conclude that

$$\begin{aligned} {u_1}^{(4)}(t) &= f(t, u_1(t), {u_1}'(t), {u_1}''(t), {u_1}'''(t)) + C_0 + \tau_1 \pi^4 v_0(t) \\ &\geq a_1\, u_1(t) - c_1\, {u_1}''(t) + \tau_1 \pi^4 v_0(t) \\ &\geq a_1\, u_1(t) - c_1\, {u_1}''(t), \qquad t \in I. \end{aligned}$$

Multiplying this inequality by $\sin \pi t$ and integrating on I, then using integration by parts for the left side, we have

$$\pi^4 \int_0^1 u_1(t)\, \sin \pi t\, dt \;\geq\; (a_1 + c_1 \pi^2) \int_0^1 u_1(t)\, \sin \pi t\, dt. \tag{44}$$

By Lemma 2(a), $\int_0^1 u_1(t)\, \sin \pi t\, dt \geq \frac{4}{\pi^3} \|u_1\|_C > 0$. From (44) it follows that $\pi^4 \geq a_1 + c_1 \pi^2$, which contradicts the assumption $\frac{a_1}{\pi^4} + \frac{c_1}{\pi^2} > 1$ in (F2). Hence, (42) holds, namely A_1 satisfies the condition of Lemma 4 in $K \cap \partial\Omega_2$. By Lemma 4, (41) holds.

Next, we show that A and A_1 satisfy the condition of Lemma 5 in $K \cap \partial\Omega_2$ when R is large enough, namely

$$(1-s)Au + sA_1 u \neq u, \qquad \forall\, u \in K \cap \partial\Omega_2, \quad 0 \leq s \leq 1. \tag{45}$$

If (45) is not valid, there exist $u_2 \in K \cap \partial\Omega_2$ and $s_0 \in [0, 1]$, such that $(1 - s_0)Au_2 + s_0 A_1 u_2 = u_2$. Since $u_2 = S((1 - s_0)\, F(u_2) + s_0 F_1(u_2))$, by the definition of S, u_2 is the unique solution of LBVP (17) for $h = (1 - s_0)\, F(u_2) + s_0 F_1(u_2) \in C^+(I)$. Hence, $u_2 \in C^4(I)$ satisfies the differential equation

$$\begin{cases} {u_2}^{(4)}(t) = f(t, u_2(t), {u_2}'(t), {u_2}''(t), {u_2}'''(t)) + s_0\, C_0, & t \in I, \\ u_2(0) = u_2(1) = {u_2}''(0) = {u_2}''(1) = 0\,. \end{cases} \tag{46}$$

Since $u_2 \in K \cap \partial\Omega_2$, by the definition of K, we have

$$u_2(t) \geq 0, \quad {u_2}''(t) \leq 0, \qquad \forall\, t \in I.$$

Hence, by (38) we have

$$f(t, u_2(t), {u_2}'(t), {u_2}''(t), {u_2}'''(t)) \geq a_1\, u_2(t) - c_1\, {u_2}''(t) - C_0, \qquad t \in I.$$

From this and (46), we obtain that

$$\begin{aligned} {u_2}^{(4)}(t) &= f(t, u_2(t), {u_2}'(t), {u_2}''(t), {u_2}'''(t)) + s_0\, C_0 \\ &\geq a_1\, u_2(t) - c_1\, {u_2}''(t) - (1 - s_0)\, C_0, \\ &\geq a_1\, u_2(t) - c_1\, {u_2}''(t) - C_0, \qquad t \in I. \end{aligned} \tag{47}$$

Multiplying this inequality by $\sin \pi t$ and integrating on I, then using integration by parts, we have

$$\pi^4 \int_0^1 u_2(t)\, \sin \pi t\, dt \;\geq\; (a_1 + c_1 \pi^2) \int_0^1 u_2(t)\, \sin \pi t\, dt - \frac{2C_0}{\pi}.$$

From this inequality, it follows that

$$\int_0^1 u_2(t)\, \sin \pi t\, dt \leq \frac{2C_0}{\pi^5 (\frac{a_1}{\pi^4} + \frac{c_1}{\pi^2} - 1)}. \tag{48}$$

Hence, by Lemma 2(b),

$$\|{u_2}''\|_C \leq \frac{\pi^5}{4} \int_0^1 u_2(t)\, \sin \pi t\, dt \leq \frac{C_0}{2(\frac{a_1}{\pi^4} + \frac{c_1}{\pi^2} - 1)} := M\,. \tag{49}$$

From this and Lemma 2(c), we obtain that

$$\|u_2\|_C \leq \|{u_2}'\|_C \leq \|{u_2}''\|_C \leq M. \tag{50}$$

For this $M > 0$, by Assumption (F0), there is a positive continuous function $g_M(\rho)$ on $\mathbb{R}^+$ satisfying (13) such that (14) holds. By (50) and definition of K,

$$0 \leq u_2(t) \leq M, \quad |{u_2}'(t)| \leq M, \quad -M \leq {u_2}''(t) \leq 0, \qquad t \in I.$$

Hence, from (14) it follows that

$$f(t, u_2(t), {u_2}'(t), {u_2}''(t), {u_2}'''(t)) \leq g_M(|{u_2}'''(t)|), \qquad t \in I.$$

Combining this with (46), we have

$${u_2}^{(4)}(t) \leq g_M(|{u_2}'''(t)|) + C_0, \qquad t \in I. \tag{51}$$

From (13) we easily obtain that

$$\int_0^{+\infty} \frac{\rho\, d\rho}{g_M(\rho) + C_0} = +\infty.$$

Hence, there exists a positive constant $M_1 \geq M$ such that

$$\int_0^{M_1} \frac{\rho\, d\rho}{g_M(\rho) + C_0} > M. \tag{52}$$

By Lemma 2(d), there exists $\xi \in (0, 1)$ such that ${u_2}'''(\xi) = 0$, ${u_2}'''(t) \leq 0$ for $t \in [0, \xi]$, ${u_2}'''(t) \geq 0$ for $t \in [\xi, 1]$, and $\|{u_2}'''\|_C = \max\{-{u_2}'''(0), {u_2}'''(1)\}$. Hence, $\|{u_2}'''\|_C = -{u_2}'''(0)$ or $\|{u_2}'''\|_C = {u_2}'''(1)$. We only consider the case of that $\|{u_2}'''\|_C = -{u_2}'''(0)$, and the other case can be treated in the same way.

Since $u_2'''(t) \le 0$ for $t \in [0, \xi]$, multiplying both sides of the inequality (51) by $-u_2'''(t)$, we obtain that

$$\frac{-u_2^{(4)}(t)\, u_2'''(t)}{g_M(-u_2'''(t)) + C_0} \le -u_2'''(t), \qquad t \in [0, \xi].$$

Integrating both sides of this inequality on $[0, \xi]$ and making the variable transformation $\rho = -u_2'''(t)$ for the left side, we have

$$\int_0^{-u_2'''(0)} \frac{\rho\, d\rho}{g_M(\rho) + C_0} \le -u_2''(\xi) \le \|u_2''\|_C.$$

Since $\|u_2'''\|_C = -u_2'''(0)$, from this inequality and (50) it follows that

$$\int_0^{\|u_2'''\|_C} \frac{\rho\, d\rho}{g_M(\rho) + C_0} \le M. \tag{53}$$

Using this inequality and (52), we obtain that

$$\|u_2'''\|_C \le M_1. \tag{54}$$

Hence, from this and (50) it follows that

$$\|u_2\|_{C^3} \le M_1. \tag{55}$$

Let $R > \max\{M_1, \delta\}$. Since $u_2 \in K \cap \partial\Omega_2$, by the definition of Ω_2, $\|u_2\|_{C^3} = R > M_1$. This contradicts (55). Hence, (45) holds, namely A and A_1 satisfies the condition of Lemma 5 in $K \cap \partial\Omega_2$. By Lemma 5, we have

$$i\,(A, K \cap \Omega_2, K) = i\,(A_1, K \cap \Omega_2, K). \tag{56}$$

Hence, from (56) and (41) it follows that

$$i\,(A, K \cap \Omega_2, K) = 0. \tag{57}$$

Now using the additivity of the fixed point index, from (37) and (57), we conclude that

$$i\,(A, K \cap (\Omega_2 \setminus \overline{\Omega}_1), K) = i\,(A, K \cap \Omega_2, K) - i\,(A, K \cap \Omega_1, K) = -1.$$

Hence, A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive solution of BVP (1). The proof of Theorem 1 is completed. □

Proof of Theorem 2. Let $\Omega_1, \Omega_2 \subset C^3(I)$ be defined by (33). $K \subset C^3(I)$ is the close convex cone defined by (30). We prove that the completely continuous mapping $A : K \to K$ defined by (32) has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when r is small enough and R large enough.

Let $r \in (0, \delta)$, where δ is the positive constant in Condition (F3). Choose $v_0 = \sin \pi t$. Then $v_0 \in K \setminus \{\theta\}$ and $S(\pi^4 v_0) = v_0$. We show that A satisfies the condition of Lemma 4 in $K \cap \partial\Omega_1$, namely

$$u - Au \ne \tau\, v_0, \qquad \forall\, u \in K \cap \partial\Omega_1, \quad \tau \ge 0. \tag{58}$$

In fact, if (58) is not valid, there exist $u_0 \in K \cap \partial\Omega_1$ and $\tau_0 \ge 0$ such that $u_0 - Au_0 = \tau_0 v_0$. Since $u_0 = Au_0 + \tau_0 v_0 = S(F(u_0) + \tau_0 \pi^4 v_0)$, by the definition of S, u_0 is the unique solution of LBVP (17) for $h = F(u_0) + \tau_0 \pi^4 v_0 \in C^+(I)$. Hence, $u_0 \in C^4(I)$ satisfies the differential equation

$$\begin{cases} {u_0}^{(4)}(t) = f(t, u_0(t), {u_0}'(t), {u_0}''(t), {u_0}'''(t)) + \tau_0 \pi^4 v_0(t), \quad t \in I, \\ u_0(0) = u_0(1) = {u_0}''(0) = {u_0}''(1) = 0\,. \end{cases} \tag{59}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$0 \le u_0(t),\ |{u_0}'(t)|,\ -{u_0}''(t),\ |{u_0}'''(t)| \le \|u_0\|_{C^3} = r < \delta, \quad t \in I. \tag{60}$$

Hence, by Condition (F3), we have

$$f(t, u_0(t), {u_0}'(t), {u_0}''(t), {u_0}'''(t)) \ge a\, u_0(t) - c\, {u_0}''(t), \qquad t \in I.$$

From this inequality and Equation (59) it follows that

$${u_0}^{(4)}(t) \ge a\, u_0(t) - c\, {u_0}''(t), \qquad t \in I.$$

Multiplying this inequality by $\sin \pi t$ and integrating on I, then using integration by parts, we have

$$\pi^4 \int_0^1 u_0(t)\ \sin \pi t\, dt\ \ge\ (a + c\pi^2) \int_0^1 u_0(t)\ \sin \pi t\, dt. \tag{61}$$

By Lemma 2(a), $\int_0^1 u_0(t)\ \sin \pi t\, dt \ge \frac{4}{\pi^3} \|u_0\|_C > 0$. Hence, from (61) it follows that $\pi^4 \ge a + c\pi^2$, which contradicts to the assumption $\frac{a}{\pi^4} + \frac{c}{\pi^2} > 1$ in (F3). Hence, (58) holds. Hence, by Lemma 4, we have

$$i\,(A,\, K \cap \Omega_1,\, K) = 0. \tag{62}$$

Let $R > \delta$ be large enough. We show that A satisfies the condition of Lemma 3 in $K \cap \partial\Omega_2$, namely

$$\mu\, Au \neq u, \qquad \forall\, u \in K \cap \partial\Omega_2, \quad 0 < \mu \le 1. \tag{63}$$

In fact, if (63) is not valid, there exist $u_1 \in K \cap \partial\Omega_2$ and $0 < \mu_1 \le 1$ such that $\mu_1\, Au_1 = u_1$. Since $u_1 = S(\mu_0 F(u_1))$, by the definition of S, $u_1 \in C^4(I)$ is the unique solution of LBVP (17) for $h = \mu_1 F(u_1) \in C^+(I)$. Since $u_1 \in K \cap \partial\Omega_2$, by the definitions of K and Ω_2, we have

$$u_1(t) \ge 0, \quad {u_1}''(t) \le 0, \qquad t \in I. \tag{64}$$

Set $C_1 = \max\{\, |f(t, u, v, w, z) - (a_1\, u + b_1|v| - c_1\, w + d_1\, |z|)| : (t, u, v, w, z) \in I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R},\ |u| + |v| + |w| + |z| \le H\} + 1$. Then by Condition (F4), we have

$$\begin{aligned} f(t, u, v, w, z) \le{} & a_1\, u + b_1|v| + c_1\, |w| + d_1\, |z| + C_1, \\ & \text{for all } (t, u, v, w, z) \in I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R}. \end{aligned} \tag{65}$$

Combining this with (64), we obtain that

$$\begin{aligned} 0 \le F(u_1)(t) &= f(t, u_1(t), {u_1}'(t), {u_1}''(t), {u_1}'''(t)) \\ &\le a_1\, |u_1(t)| + b_1\, |{u_1}'(t)| + c_1\, |{u_1}''(t)| + d\, |{u_1}'''(t)| + C_1, \qquad t \in I. \end{aligned}$$

From this inequality and (21), we conclude that

$$\begin{aligned}\|F(u_1)\|_2 &\le a_1\,\|u_1\|_2 + b_1\,\|u_1{}'\|_2 + c_1\,\|u_1{}''\|_2 + d_1\,\|u_1{}'''\|_2 + C_1\\ &\le \Big(\frac{a_1}{\pi^4} + \frac{b_1}{\pi^3} + \frac{c_1}{\pi^2} + \frac{d_1}{\pi}\Big)\|u_1{}^{(4)}\|_2 + C_1\\ &\le \Big(\frac{a_1}{\pi^4} + \frac{b_1}{\pi^3} + \frac{c_1}{\pi^2} + \frac{d_1}{\pi}\Big)\|u_1\|_{H^4} + C_1.\end{aligned}$$

By this and (22) we have

$$\begin{aligned}\|u_1\|_{H^4} = \|S(\mu_1 F(u_1))\|_{H^4} &\le \|S\|_{\mathcal{B}(L^2(I),\,H^4(I))}\cdot\|F(u_1)\|_2\\ &\le \Big(\frac{a_1}{\pi^4} + \frac{b_1}{\pi^3} + \frac{c_1}{\pi^2} + \frac{d_1}{\pi}\Big)\|u_1\|_{H^4} + C_1,\end{aligned}$$

from which it follows that

$$\|u_1\|_{H^4} \le \frac{C_1}{1-\big(\frac{a_1}{\pi^4} + \frac{b_1}{\pi^3} + \frac{c_1}{\pi^2} + \frac{d_1}{\pi}\big)}.$$

Hence, by the boundedness of the Sobolev embedding $H^4(I) \hookrightarrow C^3(I)$, we have

$$\|u_1\|_{C^3} \le C\,\|u_1\|_{H^4} \le \frac{CC_1}{1-\big(\frac{a_1}{\pi^4} + \frac{b_1}{\pi^3} + \frac{c_1}{\pi^2} + \frac{d_1}{\pi}\big)} := M_2, \tag{66}$$

where C is the Sobolev embedding constant.

Choose $R > \max\{M_2,\,\delta\}$. Since $u_1 \in K \cap \partial\Omega_2$, by the definition of Ω_2, we see that $\|u_1\|_{C^3} = R > M_2$, which contradicts (66). Hence, (63) holds. By Lemma 3, we have

$$i\,(A,\,K\cap\Omega_2,\,K) = 1. \tag{67}$$

Now, from (62) and (67) it follows that

$$i\,(A,\,K\cap(\Omega_2\setminus\overline{\Omega}_1),\,K) = i\,(A,\,K\cap\Omega_2,\,K) - i\,(A,\,K\cap\Omega_1,\,K) = 1.$$

Hence, A has a fixed-point in $K\cap(\Omega_2\setminus\overline{\Omega}_1)$, which is a positive solution of BVP (1). The proof of Theorem 2 is completed. □

4. Applications

In this section, we use Theorems 1 and 2 to present some existing results of positive solutions for BVP (1). Theorems 1 and 2 are also applicable to the case that $f(t,\,u,\,v,\,w,\,z)$ is asymptotically linear as $|(u,v,w,z)| \to 0$ and $|(u,v,w,z)| \to \infty$, here $|(u,v,w,z)| = |u| + |v| + |w| + |z|$. For this case, we have:

Theorem 3. *Let $f: I\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^-\times\mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following conditions*
(H1) There exist constants $a,\,b,\,c,\,d \ge 0$, $\frac{a}{\pi^4} + \frac{b}{\pi^3} + \frac{c}{\pi^2} + \frac{d}{\pi} < 1$, such that

$$f(t,\,u,\,v,\,w,\,z) = a\,u + b\,|v| - c\,w + d\,|z| + o(|(u,v,w,z)|), \quad |(u,v,w,z)| \to 0;$$

(H2) There exist constants $a_1,\,b_1,\,c_1,d_1 > 0$, $\frac{a_1}{\pi^4} + \frac{c_1}{\pi^2} > 1$, such that

$$f(t,\,u,\,v,\,w,\,z) = a_1 u + b_1|v| - c_1 w + d_1|z| + o(|(u,v,w,z)|), \quad |(u,v,w,z)| \to \infty.$$

Then BVP (1) has at least one positive solution.

Theorem 4. *Let $f: I\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^-\times\mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following conditions*

(H3) There exist constants $a,\ b,\ c,\ d > 0$, $\frac{a}{\pi^4} + \frac{c}{\pi^2} > 1$, *such that*

$$f(t,\ u,\ v,\ w,\ z) = a\,u + b\,|v| - c\,w + d\,|z| + o(|(u,v,w,z)|), \quad |(u,v,w,z)| \to 0;$$

(H4) There exist constants $a_1,\ b_1,\ c_1, d_1 \geq 0$, $\frac{a_1}{\pi^4} + \frac{b_1}{\pi^3} + \frac{c_1}{\pi^2} + \frac{d_1}{\pi} < 1$, *such that*

$$f(t,\ u,\ v,\ w,\ z) = a_1 u + b_1|v| - c_1 w + d_1|z| + o(|(u,v,w,z)|), \quad |(u,v,w,z)| \to \infty.$$

Then BVP (1) has at least one positive solution.

Proof. Clearly, we have

$$\begin{aligned} &(\mathrm{H1}) \implies (\mathrm{F1}) \text{ holds;} \\ &(\mathrm{H2}) \implies (\mathrm{F2}) \text{ and } (\mathrm{F0}) \text{ hold;} \\ &(\mathrm{H3}) \implies (\mathrm{F3}) \text{ holds;} \\ &(\mathrm{H4}) \implies (\mathrm{F4}) \text{ holds.} \end{aligned}$$

Hence, by Theorems 1 and 2, the conclusions of Theorems 3 and 4 hold. □

Example 1. *Consider the following nonlinear fourth-order boundary value problem with all derivative terms*

$$\begin{cases} u^{(4)}(t) = \frac{(a_1 u(t) + b_1|u'(t)| - c_1 u''(t) + d_1|u'''(t)|)^2}{a_1 u(t) + b_1|u'(t)| - c_1 u''(t) + d_1|u'''(t)| + 1}, \quad t \in [0,\ 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{68}$$

where $a_1,\ b_1,\ c_1,\ d_1$ *are positive constants. If* $\frac{a_1}{\pi^4} + \frac{c_1}{\pi^2} > 1$, *the corresponding nonlinearity*

$$f(t,\ u,\ v,\ w,\ z) = \frac{(a_1 u + b_1|v| - c_1 w + d_1|z|)^2}{a_1 u + b_1|v| - c_1 w + d_1|z| + 1} \tag{69}$$

satisfies Condition (H2). From definition (69), we easily see that f *also satisfies (H1) for* $a = b = c = d = 0$. *By Theorem 3, BVP (68) has at least one positive solution.*

Example 2. *Consider the following nonlinear fourth-order boundary value problem*

$$\begin{cases} u^{(4)}(t) = \frac{a\,u(t) + b\,|u'(t)| + c\,|u''(t)| + d\,|u'''(t)|}{1 + u^2(t) + u'^2(t) + u''^3(t) + u'''^2(t)}, \quad t \in [0,\ 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{70}$$

where $a,\ b,\ c,\ d$ *are positive constants with* $\frac{a}{\pi^4} + \frac{c}{\pi^2} > 1$. *We easily verify that the corresponding nonlinearity*

$$f(t,\ u,\ v,\ w,\ z) = \frac{a\,u + b|v| + c|w| + d|z|}{1 + u^2 + v^2 + w^2 + z^2} \tag{71}$$

satisfies Conditions (H3) and (H4). By Theorem 4, BVP (70) has at least one positive solution.

We introduce the following notations

$$
\begin{aligned}
f_{40} &= \liminf_{|(u,v,w,z)|\to 0}\ \min_{t\in I} \frac{f(t,\,u,\,w,\,v,\,z)}{|(u,v,w,z)|}\,,\\
f^{40} &= \limsup_{|(u,v,w,z)|\to 0}\ \max_{t\in I} \frac{f(t,\,u,\,w,\,v,\,z)}{|(u,v,w,z)|}\,,\\
f_{4\infty} &= \liminf_{|(u,v,w,z)|\to \infty}\ \min_{t\in I} \frac{f(t,\,u,\,w,\,v,\,z)}{|(u,v,w,z)|}\,,\\
f^{4\infty} &= \limsup_{|(u,v,w,z)|\to \infty}\ \max_{t\in I} \frac{f(t,\,u,\,w,\,v,\,z)}{|(u,v,w,z)|}\,.
\end{aligned} \tag{72}
$$

Usually, the growth of nonlinearity f as $|(u,v,w,z)| \to 0$ or $|(u,v,w,z)| \to \infty$ is described by these upper and lower limits. In Theorems 1 and 2, we use the inequality conditions to describe the growth of the nonlinearity f. Our inequality conditions are precise and include the upper and lower limit conditions. In fact, by definition (72), we can conclude that

$$
\begin{aligned}
f^{40} &< \alpha_\pi \quad \Longrightarrow \quad \text{(F1) holds;}\\
f_{4\infty} &> \beta_\pi \quad \Longrightarrow \quad \text{(F2) holds;}\\
f_{40} &> \beta_\pi \quad \Longrightarrow \quad \text{(F3) holds;}\\
f^{4\infty} &< \alpha_\pi \quad \Longrightarrow \quad \text{(F4) holds,}
\end{aligned}
$$

where

$$
\alpha_\pi = \frac{\pi^4}{1+\pi+\pi^2+\pi^3}, \qquad \beta_\pi = \frac{\pi^4}{1+\pi^2}. \tag{73}
$$

Hence, by Theorems 1 and 2, we obtain that

Theorem 5. *Let $f : I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}^+$ be continuous. If f satisfies Assumption* (F0) *and the following condition*
(H5) $f^{40} < \alpha_\pi$, $f_{4\infty} > \beta_\pi$,
then BVP (1) has at least one positive solution.

Theorem 6. *Let $f : I \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following condition*
(H6) $f_{40} > \beta_\pi$, $f^{4\infty} < \alpha_\pi$.
Then BVP (1) has at least one positive solution.

Example 3. *Consider the superlinear fourth-order boundary value problem*

$$
\begin{cases}
u^{(4)}(t) = u^3(t) + u'^4(t) - u''^5(t) + u'''^2(t), & t \in [0,\,1],\\
u(0) = u(1) = u''(0) = u''(1) = 0\,.
\end{cases} \tag{74}
$$

We easily verify that the corresponding nonlinearity

$$
f(t,\,u,\,v,\,w,\,z) = u^3 + v^4 - w^5 + z^2
$$

satisfies Conditions (F0) and (H5). By Theorem 3, BVP (74) has at least one positive solution.

Example 4. *Consider the sublinear fourth-order boundary value problem*

$$
\begin{cases}
u^{(4)}(t) = \sqrt[3]{|u(t)|^2} + \sqrt[3]{|u'(t)|} + \sqrt[4]{|u''(t)|^3} + \sqrt{|u'''(t)|}, & t \in [0,\,1],\\
u(0) = u(1) = u''(0) = u''(1) = 0\,.
\end{cases} \tag{75}
$$

It is easy to see that the corresponding nonlinearity

$$f(t, u, v, w, z) = \sqrt[3]{|u|^2} + \sqrt[3]{|v|} + \sqrt[4]{|w|^3} + \sqrt{|z|}$$

satisfies Condition (H6). By Theorem 4, BVP (75) has at least one positive solution.

Author Contributions: Methodology, Y.L.; Y.L. and W.M. carried out the first draft of this manuscript, Y.L. prepared the final version of the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research is supported by NNSFs of China (12061062, 11661071).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The manuscript has no associate data.

Conflicts of Interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Aftabizadeh, A.R. Existence and uniqueness theorems for fourth-order boundary value problems. *J. Math. Anal. Appl.* **1986**, *116*, 415–426. [CrossRef]
2. Agarwal, R.P. *Boundary Value Problems for Higher Order Differential Equations*; World Scientific: Singapore, 1986.
3. Gupta, C.P. Existence and uniqueness theorems for a bending of an elastic beam equation. *Appl. Anal.* **1988**, *26*, 289–304. [CrossRef]
4. Gupta, C.P. Existence and uniqueness results for the bending of an elastic beam equation at resonance. *J. Math. Anal. Appl.* **1988**, *135*, 208–225. [CrossRef]
5. Bai, Z.; Wang, H. On positive solutions of some nonlinear fourth-order bean equations. *J. Math. Anal. Appl.* **2002**, *270*, 357–368. [CrossRef]
6. Cabada, A.; Cid, J.A.; Sanchez, L. Positivity and lower and upper solutions for fourth order boundary value problems. *Nonlinaer Anal.* **2007**, *67*, 1599–1612. [CrossRef]
7. Dalmasso, R. Uniqueness of positive solutions for some nonlinear fourth-order equations. *J. Math. Anal. Appl.* **1996**, *201*, 152–168. [CrossRef]
8. de Coster, C.; Fabry, C.; Munyamarere, F. Nonresonance conditions for fourth-order nonlinear boundary value problems. *Inter. J. Math. Math. Sci.* **1994**, *17*, 725–740. [CrossRef]
9. de Coster, C.; Sanchez, L. Upper and lower solutions, Ambrosetti-Prodi problem and positive solutions for fourth O.D.E. *Riv. Mat. Pura Appl.* **1994**, *14*, 57–82.
10. del Pino, M.A.; Manasevich, R.F. Existence for a fourth-order boundary value problem under a two-parameter nonresonance condition. *Proc. Am. Math. Soc.* **1991**, *112*, 81–86. [CrossRef]
11. Han, G.; Li, F. Multiple solutions of some fourth-order boundary value problems. *Nonlinaer Anal.* **2007**, *66*, 2591–2603. [CrossRef]
12. Korman, P. A maximum principle for fourth-order ordinary differential equations. *Appl. Anal.* **1989**, *33*, 267–373. [CrossRef]
13. Li, F.; Zhang, Q.; Liang, Z. Existence and multipicity of solutions of a kind of fourth-order boundary value problem. *Nonlinaer Anal.* **2005**, *62*, 803–816. [CrossRef]
14. Li, Y. Positive solutions of fourth-order boundary value problems with two parameters. *J. Math. Anal. Appl.* **2003**, *281*, 477–484. [CrossRef]
15. Li, Y. Existence and multiplicity of positive solutions for fourth-order boundary value problems. *Acta Math. Appl. Sin.* **2003**, *26*, 109–116. (In Chinese)
16. Li, Y. On the existence of positive solutions for the bending elastic beam equations. *Appl. Math. Comput.* **2007**, *189*, 821–827. [CrossRef]
17. Li, Y.; Gao, Y. Existence and uniqueness results for the bending elastic beam equations. *Appl. Math. Lett.* **2019**, *95*, 72–77. [CrossRef]
18. Li, Y.; Liang, Q. Existence results for a fully fourth-order boundary value problem. *J. Funct. Spaces Appl.* **2013**, *2013*, 641617. [CrossRef]
19. Lin, X.; Jiang, D.; Li, X. Existence and uniqueness of solutions for singular fourth-order boundary value problems. *J. Comput. Appl. Math.* **2006**, *196*, 155–161. [CrossRef]
20. Liu, B. Positive solutions of fourth-order two point boundary value problems. *Appl. Math. Comput.* **2004**, *148*, 407–420. [CrossRef]
21. Ma, R. Positive solutions of fourth-order two point boundary value problems. *Ann. Diff. Equ.* **1999**, *15*, 305–313.
22. Ma, R. Existence of positive solutions of a fourth-order boundary value problem. *Appl. Math. Comput.* **2005**, *168*, 1219–1231. [CrossRef]

23. Ma, R.; Wang, H. On the existence of positive solutions of fourth-order ordinary differential equations. *Appl. Anal.* **1995**, *59*, 225–231.
24. Ma, R.; Xu, L. Existence of positive solutions of a nonlinear fourth-order boundary value problem. *Appl. Math. Lett.* **2010**, *23*, 537–543. [CrossRef]
25. Ma, R.; Zhang, J.; Fu, S. The method of lower and upper solutions for fourth-order two-point boundary value problems. *J. Math. Anal. Appl.* **1997**, *215*, 415–422.
26. Yang, Y.; Zhang, J. Existence of solutions for some fourth-order boundary value problems with parameters. *Nonlinaer Anal.* **2008**, *69*, 1364–1375. [CrossRef]
27. Li, Y. Existence of positive solutions for the cantilever beam equations with fully nonlinear terms. *Nonlinear Anal. RWA* **2016**, *27*, 221–237. [CrossRef]
28. Alves, E.; Ma, T.F.; Pelicer, M.L. Monotone positive solutions for a fourth order equation with nonlinear boundary conditions. *Nonlinear Anal.* **2009**, *71*, 3834–3841. [CrossRef]
29. Wei, M.; Li, Y. Solvability for a fully elastic beam equation with left-end fixed and right-end simply supported. *Math. Probl. Eng.* **2021**, *2021*, 5528270. [CrossRef]
30. Cabada, A.; Tersian, S. Multiplicity of solutions of a two point boundary value problem for a fourth-order equation. *Appl. Math. Comput.* **2013**, *219*, 5261–5267. [CrossRef]
31. Yao, Q. Monotonically iterative method of nonlinear cantilever beam equations. *Appl. Math. Comput.* **2008**, *205*, 432–437. [CrossRef]
32. Yao, Q. Local existence of multiple positive solutions to a singular cantilever beam equation. *J. Math. Anal. Appl.* **2010**, *363*, 138–154. [CrossRef]
33. Yao, Q. Positive solutions of nonlinear beam equations with time and space singularities. *J. Math. Anal. Appl.* **2011**, *374*, 681–692. [CrossRef]
34. Deimling, K. *Nonlinear Functional Analysis*; Springer: New York, NY, USA, 1985.
35. Guo, D.; Lakshmikantham, V. *Nonlinear Problems in Abstract Cones*; Academic Press: New York, NY, USA, 1988.