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General Chen Inequalities for Statistical Submanifolds in Hessian Manifolds of Constant Hessian Curvature

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Abstract: Chen's first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature was obtained by B.-Y. Chen et al. Other particular cases of Chen inequalities in a statistical setting were given by different authors. The objective of the present article is to establish the general Chen inequalities for statistical submanifolds in Hessian manifolds of constant Hessian curvature.

Keywords: Hessian manifolds; constant Hessian curvature; statistical submanifolds; Chen inequalities

MSC: 53C05; 53C40

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1. Motivation

In [1], the motivation of the definition of a statistical structure on a Riemannian manifold was given, starting from the notion of probability distribution, as follows:

Let $X \subset \mathbb{R}^m$ be a discrete (countable) set or $X = \mathbb{R}^m$. A map $p : X \to \mathbb{R}$ is called a *probability distribution* if:

(1) $p(x) \geq 0, \forall x \in X$.

(2) $\sum_{x \in X} p(x) = 1$, if X is discrete, or $\int_X p(x)dx = 1$, if $X = \mathbb{R}^m$.

The sum $\sum_{x \in X}$ is also denoted by $\int_X$.

The *expectation* of a function f on X with respect to a probability distribution p is defined by

$$E[f] = \int_X f(x)p(x)dx.$$

Let $\mathcal{P} = \{p(x, \lambda) | \lambda \in \Lambda\}$ be a family of probability distributions on X parametrized by $\lambda = [\lambda^1, \ldots, \lambda^n] \in \Lambda$ satisfying the following:

(P_1) $\Lambda \subset \mathbb{R}^n$ is a domain.

(P_2) $p(x; \lambda)$ is smooth with respect to λ.

(P_3) The operations of integration with respect to x and differentiation with respect to λ^i are commutative.

One denotes by $l_\lambda = l(x; \lambda) = \log p(x; \lambda)$ and by E_λ the expectation with respect to $p_\lambda = p(x; \lambda)$.

Define

$$g_{ij}(\lambda) = E_\lambda\left[\frac{\partial l_\lambda}{\partial \lambda^i}\frac{\partial l_\lambda}{\partial \lambda^j}\right] = \int_x \frac{\partial l(x;\lambda)}{\partial \lambda^i}\frac{\partial l(x;\lambda)}{\partial \lambda^j} p(x;\lambda)dx.$$

The matrix $g = [g_{ij}(\lambda)]$ is said to be the *Fisher information matrix*. We assume

(P_4) The Fisher information matrix $g = [g_{ij}(\lambda)]$ for a family of probability distributions $\mathcal{P} = \{p(x, \lambda) | \lambda \in \Lambda\}$ is positive definite.

Then, g may be regarded as a Riemannian metric on Λ.

Let Γ^i_{jk} be the Christoffel symbols of the Levi-Civita connection.

Denote

$$T_{ijk} = \frac{1}{2}E_\lambda\left[\frac{\partial l_\lambda}{\partial \lambda^i}\frac{\partial l_\lambda}{\partial \lambda^j}\frac{\partial l_\lambda}{\partial \lambda^k}\right],$$

$$\Gamma(t)_{kij} = \Gamma_{kij} - tT_{kij},$$

$$\Gamma(t)^{i}_{jk} = g^{is}\Gamma(t)_{sjk}.$$

It is easily seen that the $T_{ijk}(t)$ are symmetric. Then, they define a torsion-free connection $\nabla(t)$.

We point-out the following property:

$$Xg(Y,Z) = g(\nabla(t)_X Y, Z) + g(Y, \nabla(-t)_X Z),$$

i.e., $\nabla(t)$ and $\nabla(-t)$ are dual connections with respect to the Fisher information metric g.

2. Hessian Manifolds and Their Submanifolds

S. Amari [2] started the use of differential geometric methods in statistics and defined statistical structures on Riemannian manifolds. Because the geometry of such manifolds is based on dual connections, it is obviously closely related to affine differential geometry; the dual connections are also named conjugate connections (see [3]). Moreover, a statistical structure is a generalization of a Hessian one.

A *statistical manifold* is a Riemannian manifold $(\tilde{M}^m, \tilde{g})$ of dimension m, endowed with a pair of torsion-free affine connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ satisfying

$$Z\tilde{g}(X,Y) = \tilde{g}\big(\tilde{\nabla}_Z X, Y\big) + \tilde{g}\big(X, \tilde{\nabla}^*_Z Y\big),$$

for any X, Y and $Z \in \Gamma\big(T\tilde{M}^m\big)$.

It is always possible to find the dual connection $\tilde{\nabla}^*$ of any torsion-free affine connection $\tilde{\nabla}$; they are related by

$$\tilde{\nabla} + \tilde{\nabla}^* = 2\tilde{\nabla}^0,$$

where $\tilde{\nabla}^0$ is the Levi-Civita connection on $\tilde{M}^m$.

We denote by $\tilde{R}$ and $\tilde{R}^*$ the curvature tensor fields; they satisfy

$$\tilde{g}(\tilde{R}^*(X,Y)Z, W) = -\tilde{g}(Z, \tilde{R}(X,Y)W).$$

We say that a statistical manifold is of constant curvature $\varepsilon \in \mathbb{R}$ if

$$\tilde{R}(X,Y)Z = \varepsilon[\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y],$$

for any $X, Y, Z \in \Gamma(T\tilde{M}^m)$. In this case, the curvature tensor field $\tilde{R}^*$ has the same expression.

A *Hessian manifold* is a statistical manifold of constant curvature zero. On a Hessian manifold $(\tilde{M}^m, \tilde{\nabla})$, let $\gamma = \tilde{\nabla} - \tilde{\nabla}^0$. One defines the tensor field $\tilde{Q}$ of type (1,3) by $\tilde{Q}(X,Y) = [\gamma_X, \gamma_Y]$, $X, Y \in \Gamma(T\tilde{M}^m)$ and it is called the *Hessian curvature tensor* for $\tilde{\nabla}$. We refer to H. Shima [1] and B. Opozda [4].

The following relation holds:

$$\tilde{R}(X,Y) + \tilde{R}^*(X,Y) = 2\tilde{R}^0(X,Y) + 2\tilde{Q}(X,Y).$$

A Hessian sectional curvature can be defined on a Hessian manifold by using the Hessian curvature tensor $\tilde{Q}$ as follows.

Let $p \in \tilde{M}^m$ and π a plane in $T_p\tilde{M}^m$. Take an orthonormal basis $\{X, Y\}$ of π and set

$$\tilde{K}(\pi) = \tilde{g}(\tilde{Q}(X,Y)Y, X).$$

The number $\tilde{K}(\pi)$ is called the *Hessian sectional curvature* (it is independent of the choice of an orthonormal basis).

It is easily seen [1] that a Hessian manifold of constant Hessian sectional curvature c is a Riemannian space form of constant sectional curvature $-c$.

Let M^n be a submanifold of $\tilde{M}^m$ of dimension n. Then, the Gauss formulae are

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y),$$

for any $X, Y \in \Gamma(TM^n)$, where h and h^* are the imbedding curvature tensor of M^n in $\tilde{M}^m$ for $\tilde{\nabla}$ and the imbedding curvature tensor of M^n in $\tilde{M}^m$ for $\tilde{\nabla}^*$, respectively.

Because h and h^* are bilinear and symmetric, there exist linear transformations A_ξ and A^*_ξ given by

$$g(A_\xi X, Y) = \tilde{g}(h(X, Y), \xi),$$

$$g\left(A^*_\xi X, Y\right) = \tilde{g}(h^*(X, Y), \xi),$$

for any $\xi \in \Gamma(T^\perp M^n)$ and $X, Y \in \Gamma(TM^n)$.

The Weingarten formulae are

$$\tilde{\nabla}_X \xi = -A^*_\xi X + \nabla^\perp_X \xi,$$

$$\tilde{\nabla}^*_X \xi = -A_\xi X + \nabla^{*\perp}_X \xi,$$

for any $\xi \in \Gamma(T^\perp M^n)$ and $X \in \Gamma(TM^n)$. With respect to the induced metric on $\Gamma(T^\perp M^n)$, the normal connections $\nabla^\perp$ and $\nabla^{*\perp}$ are Riemannian dual connections.

The Gauss, Codazzi and Ricci equations are given by [5].

$$\begin{aligned}\tilde{g}(\tilde{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) + \tilde{g}(h(X, Z), h^*(Y, W)) \\ &\quad - \tilde{g}(h^*(X, W), h(Y, Z)), \\ (\tilde{R}(X, Y)Z)^\perp &= \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \\ &\quad - \left\{\nabla^\perp_Y h(X, Z) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z)\right\}, \\ \tilde{g}\left(R^\perp(X, Y)\xi, \eta\right) &= \tilde{g}(\tilde{R}(X, Y)\xi, \eta) + g\left(\left[A^*_\xi, A_\eta\right]X, Y\right),\end{aligned}$$

where R, R^* and $R^\perp$ are the curvature tensors of ∇, ∇^* and $\nabla^\perp$, respectively, $\xi, \eta \in \Gamma(T^\perp M^n)$ and $\left[A^*_\xi, A_\eta\right] = A^*_\xi A_\eta - A_\eta A^*_\xi$.

Let $p \in M^n$ and $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_m\}$ be orthonormal bases of $T_p M^n$ and $T^\perp_p M^n$, respectively. Then, the mean curvature vector fields are defined by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) = \frac{1}{n}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n} h^\alpha_{ii}\right)e_\alpha,\ h^\alpha_{ij} = \tilde{g}(h(e_i, e_j), e_\alpha),$$

and

$$H^* = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i, e_i) = \frac{1}{n}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n} h^{*\alpha}_{ii}\right)e_\alpha,\ h^{*\alpha}_{ij} = \tilde{g}(h^*(e_i, e_j), e_\alpha),$$

for $1 \le i, j \le n$ and $n+1 \le \alpha \le m$.

3. Chen's Invariants

The main Riemannian invariants are the curvature invariants. They play important roles in physics and biology; for example, by applying the laws of Newton, one shows that the magnitude of a necessary force to move an object with constant speed is a multiple (constant) of the curvature of the trajectory. Furthermore, the general theory of relativity of Einstein says that the motion of a body in a gravitational field is given by the curvature of spacetime. All kinds of shapes (red cells, soap bubbles, etc.) are precisely determined by certain curvatures.

The sectional curvature, the scalar curvature and the Ricci curvature are the most (natural) studied curvature invariants.

B.-Y. Chen [6,7] introduced new Riemannian invariants, which were different in nature from the classical ones. They are known as *Chen invariants* or δ-invariants.

Let M^n be a Riemannian manifold of dimension n. Denote by τ the scalar curvature of M^n, i.e., $\tau(p) = \sum_{1\leq i<j\leq n} K(e_i \wedge e_j)$, for any $p \in M^n$ and $\{e_1, \dots, e_n\}$ an orthonormal basis of T_pM^n, where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j. If $L \subset T_pM^n$ is an r-dimensional subspace, then its scalar curvature is given by $\tau(L) = \sum_{1\leq\alpha<\beta\leq r} K(e_\alpha \wedge e_\beta)$, where $\{e_1, \dots, e_r\} \subset L$ is an orthonormal basis.

Let $k \in \mathbb{N}^*$ and $n_1, \dots, n_k \geq 2$ be integers such that $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. For any $p \in M^n$, the Chen invariant $\delta(n_1, \dots, n_k)$ at p is defined by

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\},$$

where $L_1, \dots, L_k$ are mutually orthogonal subspaces of T_pM^n of dim $L_j = n_j$, $\forall j = 1, \dots, k$.

In particular, $\delta(2) = \tau - \inf K$ is the Chen first invariant.

B.-Y. Chen [7] established sharp estimates of the squared mean curvature $\|H\|^2$ in terms of Chen invariants for submanifolds M^n in Riemannian space forms $\tilde{M}^m(c)$.

$$\delta(n_1, \dots, n_k) \leq \frac{n^2(n + k - \sum_{j=1}^k n_j - 1)}{2(n + k - \sum_{j=1}^k n_j)}||H||^2 + \frac{1}{2}[n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)]c.$$

These inequalities are known as Chen inequalities (see also [8]).

After that, Chen inequalities for special classes of submanifolds in various space forms were obtained by several researchers.

Particular cases of Chen inequalities were also proven in statistical settings. The aim of this article is to prove the general Chen inequalities for statistical submanifolds in Hessian manifolds of constant Hessian curvature.

4. An Algebraic Lemma

We prove the main result by using an algebraic lemma.

Lemma 1. *Let $n \geq 3, k \geq 1$ be two integers and $n_1, \dots, n_k \geq 2$ integers such that $n_1 < n$, $n_1 + \cdots + n_k \leq n$. Denote $N_0 = 0$, $N_j = n_1 + \cdots + n_j$ and $j = 1, \dots, k$. Then, for any real numbers $a_1, \dots, a_n$, we have*

$$\sum_{1\leq i<j\leq n} a_i a_j - \sum_{j=1}^{k} \sum_{N_{j-1}+1\leq\alpha_j<\beta_j\leq N_j} a_{\alpha_j} a_{\beta_j} \leq \frac{n + k - \sum_{j=1}^k n_j - 1}{2(n + k - \sum_{j=1}^k n_j)} \left(\sum_{i=1}^{n} a_i\right)^2.$$

Moreover, the equality holds if and only if

$$\sum_{\alpha_j=N_{j-1}+1}^{N_j} a_{\alpha_j} = a_{N_k+1} = \cdots = a_n, \ \forall j = 1, \dots, k.$$

Proof. We use the Cauchy–Schwarz inequality.

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \left(\sum_{j=1}^{k} \sum_{\alpha_j=N_{j-1}+1}^{N_j} a_{\alpha_j} + a_{N_k+1} + \cdots + a_n\right)^2$$

$$\leq (n + k - \sum_{j=1}^{k} n_j) \left[\sum_{j=1}^{k} \left(\sum_{\alpha_j=N_{j-1}+1}^{N_j} a_{\alpha_j}\right)^2 + a_{N_k+1}^2 + \cdots + a_n^2\right]$$

$$= (n+k-\sum_{j=1}^{k} n_j)\left[\sum_{j=1}^{k}\left(\sum_{\alpha_j=N_{j-1}+1}^{N_j} a_{\alpha_j}^2 + 2\sum_{N_{j-1}+1\leq\alpha_j<\beta_j\leq N_j} a_{\alpha_j}a_{\beta_j}\right) + a_{N_k+1}^2 + \cdots + a_n^2\right]$$

$$= (n+k-\sum_{j=1}^{k} n_j)\left[\left(\sum_{i=1}^{n} a_i\right)^2 - 2\sum_{1\leq i<j\leq n} a_i a_j + 2\sum_{j=1}^{k}\sum_{N_{j-1}+1\leq\alpha_j<\beta_j\leq N_j} a_{\alpha_j}a_{\beta_j}\right],$$

which implies the inequality to prove.

We have the equality if and only if the equality holds in the Cauchy–Schwarz inequality, i.e.,

$$\sum_{\alpha_j=N_{j-1}+1}^{N_j} a_{\alpha_j} = a_{N_k+1} = \cdots = a_n,\ \forall j = 1,\ldots,k.$$

□

5. General Chen Inequalities

In [9], the first author of the present paper et al. obtained geometric inequalities for statistical submanifolds in statistical manifolds with a constant curvature. The study of Chen invariants on statistical submanifolds was started by B.-Y. Chen et al. [10]. After that, particular cases of Chen inequalities in statistical settings were obtained (see [11–18]).

In [16], we recently proved a Chen inequality involving the Chen invariant $\delta(k)$ for submanifolds in Riemannian space forms, from where we derived the Chen first inequality and Chen–Ricci inequality. In addition, we established a corresponding inequality for statistical submanifolds. In that paper, we used a new algebraic lemma.

In the present paper, we establish the general Chen inequalities for statistical submanifolds in Hessian manifolds of constant Hessian curvature. In the proof of the main result, we use Lemma 1 from Section 4, which can be regarded as a generalization of the algebraic lemma from [16].

Theorem 1. *Let M^n be an n-dimensional statistical submanifold of a Hessian manifold $\tilde{M}^m(c)$ of constant Hessian curvature. Then, for any integers $k \in \mathbb{N}^*$ and $n_1,\ldots,n_k \geq 2$ such that $n_1 < n$, $n_1+\cdots+n_k \leq n$, we have:*

$$\tau - \sum_{j=1}^{k}\tau(L_j) \geq \tau_0 - \sum_{j=1}^{k}\tau_0(L_j) + [n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)]c$$

$$-\frac{n^2(n+k-\sum_{j=1}^{k} n_j - 1)}{4(n+k-\sum_{j=1}^{k} n_j)}[||H||^2 + ||H^*||^2],$$

where $L_1,\ldots,L_k$ are mutually orthogonal subspaces of T_pM^n with $\dim L_j = n_j, \forall j = 1,\ldots,k$.

Moreover, the equality holds at a point $p \in M^n$, if and only if there exist orthonormal bases $\{e_1,\ldots,e_n\}$ in T_pM^n and $\{e_{n+1},\ldots,e_m\}$ in $T_p^{\perp}M^n$ such that the shape operators take the following form:

$$A_{e_r} = \begin{pmatrix} A_1^r & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & A_k^r & \\ & 0 & & \mu_r I \end{pmatrix},\quad A_{e_r}^* = \begin{pmatrix} A_1^{*r} & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & A_k^{*r} & \\ & 0 & & \mu_r^* I \end{pmatrix};\quad r = n+1,\ldots,m,$$

where I is the identity matrix and A_j^r and A_j^{*r} are symmetric $n_j \times n_j$ submatrices with trace $A_j^r = \mu_r$, trace $A_j^{*r} = \mu_r^*$, *for all $j = 1,\ldots,k$.*

Proof. Let $p \in M^n$ and $\{e_1,\ldots,e_n\}$ and $\{e_{n+1},\ldots,e_m\}$ be orthonormal bases of T_pM^n and $T_p^{\perp}M^n$, respectively.

The Gauss equation implies

$$\begin{aligned}
\tau &= \frac{1}{2}\sum_{1\le i<j\le n}[g(R(e_i,e_j)e_j,e_i)+g(R^*(e_i,e_j)e_j,e_i)-2g(R^0(e_i,e_j)e_j,e_i)] \\
&= \frac{1}{2}\sum_{1\le i<j\le n}[g(h^*(e_i,e_i),h(e_j,e_j))+g(h(e_i,e_i),h^*(e_j,e_j)) \\
&\quad -2g(h(e_i,e_j),h^*(e_i,e_j))]-\tau_0 \\
&= \frac{1}{2}\sum_{r=n+1}^{m}\sum_{1\le i<j<n}(h_{ii}^{*r}h_{jj}^{r}+h_{ii}^{r}h_{jj}^{*r}-2h_{ij}^{r}h_{ij}^{*r})-\tau_0.
\end{aligned} \tag{1}$$

It is known that the components of the second fundamental form h^0 (with respect to the Levi-Civita connection $\tilde{\nabla}^0$) satisfy $2h_{ij}^{0r}=h_{ij}^{r}+h_{ij}^{*r}$, $i,j=1,\dots,n, r=n+1,\dots,m$. Then,

$$\begin{aligned}
\tau &= \frac{1}{2}\sum_{r=n+1}^{m}\sum_{1\le i<j\le n}[(h_{ii}^{r}+h_{ii}^{*r})(h_{jj}^{r}+h_{jj}^{*r})-h_{ii}^{r}h_{jj}^{r}-h_{ii}^{*r}h_{jj}^{*r} \\
&\quad -(h_{ij}^{r}+h_{ij}^{*r})^2+(h_{ij}^{r})^2+(h_{ij}^{*r})^2]-\tau_0 \\
&= \sum_{r=n+1}^{m}\sum_{1\le i<j\le n}\Big\{2[h_{ii}^{0r}h_{jj}^{0r}-(h_{ij}^{0r})^2]-\frac{1}{2}[h_{ii}^{r}h_{jj}^{r}-(h_{ij}^{r})^2] \\
&\quad -\frac{1}{2}[h_{ii}^{*r}h_{jj}^{*r}-(h_{ij}^{*r})^2]\Big\}-\tau_0.
\end{aligned} \tag{2}$$

Recall that $\tilde{M}^m(c)$ is a Riemannian space form of constant sectional curvature $-c$. Then, the Gauss equation with respect to the Levi-Civita connection gives

$$\tau_0=-n(n-1)\frac{c}{2}+\sum_{r=n+1}^{m}\sum_{1\le i<j\le n}[h_{ii}^{0r}h_{jj}^{0r}-(h_{ij}^{0r})^2]. \tag{3}$$

Substituting Equation (3) in (2), we get

$$\begin{aligned}
\tau =\; &\tau_0+n(n-1)c \\
&-\frac{1}{2}\sum_{r=n+1}^{m}\sum_{1\le i<j\le n}[h_{ii}^{r}h_{jj}^{r}-(h_{ij}^{r})^2]-\frac{1}{2}\sum_{r=n+1}^{m}\sum_{1\le i<j\le n}[h_{ii}^{*r}h_{jj}^{*r}-(h_{ij}^{*r})^2].
\end{aligned} \tag{4}$$

For any $j=1,\dots,k$, by using the Gauss equation, we have

$$\begin{aligned}
\tau(L_j) &= \frac{1}{2}\sum_{N_{j-1}+1\le\alpha_j<\beta_j\le N_j}[g(R(e_{\alpha_j},e_{\beta_j})e_{\beta_j},e_{\alpha_j})+g(R^*(e_{\alpha_j},e_{\beta_j})e_{\beta_j},e_{\alpha_j}) \\
&\quad -2g(R^0(e_{\alpha_j},e_{\beta_j})e_{\beta_j},e_{\alpha_j})] \\
&= \frac{1}{2}\sum_{N_{j-1}+1\le\alpha_j<\beta_j\le N_j}[g(h^*(e_{\alpha_j},e_{\alpha_j}),h(e_{\beta_j},e_{\beta_j}))+g(h(e_{\alpha_j},e_{\alpha_j}),h^*(e_{\beta_j},e_{\beta_j})) \\
&\quad -2g(h(e_{\alpha_j},e_{\beta_j}),h^*(e_{\alpha_j},e_{\beta_j}))]-\tau_0(L_j),
\end{aligned}$$

which implies

$$
\begin{aligned}
\tau(L_j) &= \frac{1}{2}\sum_{r=n+1}^{m}\sum_{N_{j-1}+1\le\alpha_j<\beta_j\le N_j}(h^{*r}_{\alpha_j\alpha_j}h^{r}_{\beta_j\beta_j}+h^{r}_{\alpha_j\alpha_j}h^{*r}_{\beta_j\beta_j}-2h^{r}_{\alpha_j\beta_j}h^{*r}_{\alpha_j\beta_j})-\tau_0(L_j)\\
&= \frac{1}{2}\sum_{r=n+1}^{m}\sum_{N_{j-1}+1\le\alpha_j<\beta_j\le N_j}[(h^{r}_{\alpha_j\alpha_j}+h^{*r}_{\alpha_j\alpha_j})(h^{r}_{\beta_j\beta_j}+h^{*r}_{\beta_j\beta_j})\\
&\quad -h^{r}_{\alpha_j\alpha_j}h^{r}_{\beta_j\beta_j}-h^{*r}_{\alpha_j\alpha_j}h^{*r}_{\beta_j\beta_j}-(h^{r}_{\alpha_j\beta_j}+h^{*r}_{\alpha_j\beta_j})^2+(h^{r}_{\alpha_j\beta_j})^2+(h^{*r}_{\alpha_j\beta_j})^2]-\tau_0(L_j)\\
&= \sum_{r=n+1}^{m}\sum_{N_{j-1}+1\le\alpha_j<\beta_j\le N_j}\Big\{2[h^{0r}_{\alpha_j\alpha_j}h^{0r}_{\beta_j\beta_j}-(h^{0r}_{\alpha_j\beta_j})^2]-\frac{1}{2}[h^{r}_{\alpha_j\alpha_j}h^{r}_{\beta_j\beta_j}-(h^{r}_{\alpha_j\beta_j})^2]\\
&\quad -\frac{1}{2}[h^{*r}_{\alpha_j\alpha_j}h^{*r}_{\beta_j\beta_j}-(h^{*r}_{\alpha_j\beta_j})^2]\Big\}-\tau_0(L_j).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tau(L_j) &= \tau_0(L_j)+n_j(n_j-1)c-\\
&\quad -\frac{1}{2}\sum_{r=n+1}^{m}\sum_{N_{j-1}+1\le\alpha_j<\beta_j\le N_j}\Big\{[h^{r}_{\alpha_j\alpha_j}h^{r}_{\beta_j\beta_j}-(h^{r}_{\alpha_j\beta_j})^2]+[h^{*r}_{\alpha_j\alpha_j}h^{*r}_{\beta_j\beta_j}-(h^{*r}_{\alpha_j\beta_j})^2]\Big\}.
\end{aligned}
\tag{5}
$$

By summing after $j=1,\ldots,k$ the relations (5) and subtracting from (4), we obtain

$$
\begin{aligned}
&\left(\tau-\sum_{j=1}^{k}\tau(L_j)\right)-\left(\tau_0-\sum_{j=1}^{k}\tau_0(L_j)\right)\ge[n(n-1)-\sum_{j=1}^{k}n_j(n_j-1)]c\\
&-\frac{1}{2}\sum_{r=n+1}^{m}\left[\left(\sum_{1\le i<j\le n}h^{r}_{ii}h^{r}_{jj}-\sum_{j=1}^{k}\sum_{N_{j-1}+1\le\alpha_j<\beta_j\le N_j}h^{r}_{\alpha_j\alpha_j}h^{r}_{\beta_j\beta_j}\right)\right.\\
&\left.+\left(\sum_{1\le i<j\le n}h^{*r}_{ii}h^{*r}_{jj}-\sum_{j=1}^{k}\sum_{N_{j-1}+1\le\alpha_j<\beta_j\le N_j}h^{*r}_{\alpha_j\alpha_j}h^{*r}_{\beta_j\beta_j}\right)\right].
\end{aligned}
$$

By using Lemma 1, one has

$$
\sum_{1\le i<j\le n}h^{r}_{ii}h^{r}_{jj}-\sum_{j=1}^{k}\sum_{N_{j-1}+1\le\alpha_j<\beta_j\le N_j}h^{r}_{\alpha_j\alpha_j}h^{r}_{\beta_j\beta_j}\le\frac{n+k-\sum_{j=1}^{k}n_j-1}{2(n+k-\sum_{j=1}^{k}n_j)}\left(\sum_{i=1}^{n}h^{r}_{ii}\right)^2,
$$

$$
\sum_{1\le i<j\le n}h^{*r}_{ii}h^{*r}_{jj}-\sum_{j=1}^{k}\sum_{N_{j-1}+1\le\alpha_j<\beta_j\le N_j}h^{*r}_{\alpha_j\alpha_j}h^{*r}_{\beta_j\beta_j}\le\frac{n+k-\sum_{j=1}^{k}n_j-1}{2(n+k-\sum_{j=1}^{k}n_j)}\left(\sum_{i=1}^{n}h^{*r}_{ii}\right)^2.
$$

It follows that

$$
\tau-\sum_{j=1}^{k}\tau(L_j)\ge\tau_0-\sum_{j=1}^{k}\tau_0(L_j)+[n(n-1)-\sum_{j=1}^{k}n_j(n_j-1)]c
$$

$$
-\frac{n^2(n+k-\sum_{j=1}^{k}n_j-1)}{4(n+k-\sum_{j=1}^{k}n_j)}[||H||^2+||H^*||^2].
$$

The equality case follows from the equality case of Lemma 1. □

Corollary 1. *Let M^n be an n-dimensional statistical submanifold of a Hessian manifold $\tilde{M}^m(c)$ of constant Hessian curvature. If there exists a point $p \in M^n$ such that*

$$\tau - \sum_{j=1}^{k} \tau(L_j) < \tau_0 - \sum_{j=1}^{k} \tau_0(L_j) + [n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)]c,$$

then M^n is nonminimal in $\tilde{M}^m(c)$, i.e., either $H \neq 0$ or $H^* \neq 0$.

In particular, for $k = 1$ and $n_1 = 2$, one finds the main result from [10].

Corollary 2. *Let M^n be an n-dimensional statistical submanifold of a Hessian manifold $\tilde{M}^m(c)$ of constant Hessian curvature. Then, for any $p \in M^n$ and any plane section $\pi \subset T_pM$, we have:*

$$\tau - K(\pi) \geq \tau_0 - K_0(\pi) + (n-2)(n+1)c - \frac{n^2(n-2)}{4(n-1)}[||H||^2 + ||H^*||^2].$$

6. Conclusions

The above Lemma 1 allows to obtain Chen inequalities for different classes of submanifolds in various space forms, not only in statistical settings.

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