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What Knowledge Do Teachers Need to Predict the Mathematical Behavior of Students?

Eric Flores-Medrano [1,*], Danae Gómez-Arroyo [2], Álvaro Aguilar-González [3] and Laura Muñiz-Rodríguez [3]

[1] Department of Mathematics Didactics, Complutense University of Madrid, 28040 Madrid, Spain
[2] Physics and Mathematics School, Meritorious Autonomous University of Puebla, Puebla 72000, Mexico
[3] Department of Statistics and Operational Research and Mathematics Didactics, University of Oviedo, 33003 Oviedo, Spain
* Correspondence: erflores@ucm.es

Abstract: The aim of this study was to explore the specialized knowledge of five mathematics teachers who participated in a continuing training project. Teachers were asked to formulate conjectures about the type of mathematical work that students enrolled in a calculus course would develop when approaching the graphical representation of functions as an introductory activity to the calculation of the volume of solids of revolution. The data collected was analyzed using the categories of the MTSK (Mathematics Teacher's Specialized Knowledge) model. The results report how knowledge of topics and the knowledge of features of learning mathematics, particularly in relation to the knowledge of strengths and difficulties, served as fundamental pillars for the formulation of the conjectures about students' mathematical behavior.

Keywords: continuing training; learning conjectures; mathematical behaviour; MTSK; teacher's knowledge

MSC: 97B50

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1. Introduction

In recent years, there have been discussions in different forums and various studies have been carried out to understand the elements that enable the development of professional competences in mathematics teachers [1,2]. Among the range of professional competences that mathematics teachers should master [3], this study focuses on teachers' competence to predict how students would complete a mathematical task. This competence is crucial when it comes to making didactically powerful planning [4,5].

Some previous studies were interested in predicting students' academic achievement based on different variables such as their socio-economic, demographic, or cultural conditions, among others [6,7]. In a more epistemological-cognitive version but with the same interests in anticipating students' learning outcomes, constructs aiming to establish how students build their mathematical knowledge emerged. On the one hand, [8] presented a series of pedagogical implications due to the use of genetic decompositions, defined as a mapping done by the student when confronts a mathematical concept for the first time. Among the implications described in such study, it is important to remark the possibility of establishing links between the knowledge that students are expected to acquire and the knowledge that they genuinely have, highlighting that a person's mathematical view of a new concept is not always the same and, even so, it is possible to anticipate actions that lead learning achievement. These authors [8] concluded that such decompositions start from a complex system of preceding structures and require standardization in order to introduce the required concept.

On the other hand, Ref. [9] presented a model, called the Mathematics Teaching Cycle, which considers that instruction is adjusted to students' performance from a constructivist approach and, at the same time, it retains the traditional planning based on pre-established objectives. Within this cycle, the Hypothetical Learning Trajectories (HLT) emerge, constituted by three elements: the learning objectives for the students, the mathematical activities that will generate learning, and the conjectures or hypotheses about students' learning. Although the elements of the HLT are well defined, over the years some authors have interpreted and applied this idea differently, leading to the need to discuss, clarify points, and search for shared meanings [10]. Despite the potential of the HLT, Ref. [11] reported that these trajectories by themselves do not establish how to formulate conjectures. Thus, the formulation of conjectures is a deeper process of reflection, in which it is necessary to discuss the possible errors, the typical ways of solving an activity, the necessary prior knowledge, the difficulties related to the concepts involved in the resolution of the task, the strengths that the students may have, the mathematical characteristics of the concepts or processes, among others. Considering these ideas from the teachers' professional development perspective, this research addresses the following question: What knowledge do mathematics teachers need to predict the possible mathematical behavior of students when solving a mathematical task?

To answer this question, the Mathematics Teacher's Specialized Knowledge (MTSK) model, developed by [12], was used to analyze the knowledge evidenced by five mathematics teachers when discussing an activity to introduce the calculation of the volume of solids of revolution through the graphing of functions.

2. Theoretical Background

This research builds on a conceptual framework covering three elements. On the one hand, we will describe the most relevant elements of the Mexican curriculum at the high school level to facilitate the understanding of the teachers' conjectures. Secondly, we briefly describe the domains, subdomains, and categories of the MTSK model used to carry out the analysis. On the other hand, we present the main difficulties that students have related to the concept of solid of revolution, as well as forms of students' interaction with this content, which allowed us to have theoretical sensitivity when interpreting the conjectures made by the teachers.

2.1. Mexican Curriculum Framework for the High School Level

In Mexico, high school corresponds to grades 10 to 12, i.e., students are fifteen or older, except in special cases. According to [13], the high schools can have general (without specialization area) or technological (with specialization areas) orientation, and are organized in five disciplinary fields (mathematics, communication, social sciences, humanities, and experimental sciences). The teachers and students involved in this study belonged to the technological high school system in the disciplinary field of mathematics, so, in this paper, we focus on the most relevant elements of this field.

The three years of the program are divided into six semesters, in which Algebra; Geometry and Trigonometry; Analytical Geometry; Differential Calculus; Integral Calculus; and Probability and Statistics are studied, respectively. Therefore, the Integral Calculus course is preceded by courses of Algebra, from which it takes up the algebraic treatments of simplification of expressions, as well as the graphing of functions from tabulation; Geometry and Trigonometry, from which it takes up definitions and synthetic properties of the circumference and its notable lines; Analytic Geometry, which includes the definition and graphing of functions, taking into account the components of these views as geometric places; and Differential Calculus, which includes the notion of limit and function from a formal point of view, as well as the concepts of differential and infinitesimal and various mathematical artifices to find the derivative of a variety of functions.

Within the Integral Calculus course, the study of solids of revolution is one of the suggested applications and is located at the end of the course. It is preceded by the definition of antiderivatives, their analytical treatment, the definite integral and the calculation of area under the curve and between two curves [13].

2.2. The MTSK Model

The MTSK model integrates, according to [14], the core professional knowledge of a mathematics teacher. It is composed of three domains: the Mathematical Knowledge (MK), the Pedagogical Content Knowledge (PCK), and teacher's beliefs on mathematics and mathematics teaching and learning. The two first knowledge domains have, in turn, three subdomains each, as shown in Figure 1. According to [12], the divisions between domains and subdomains represent an effort to facilitate analytical studies, being the limits with the central domain represented with dashed lines to recognize the influence that beliefs have on knowledge. Notwithstanding, the continuity reflected in the other segments does not intend to represent the lack of relationship between subdomains, as some analytical studies have shown [15].

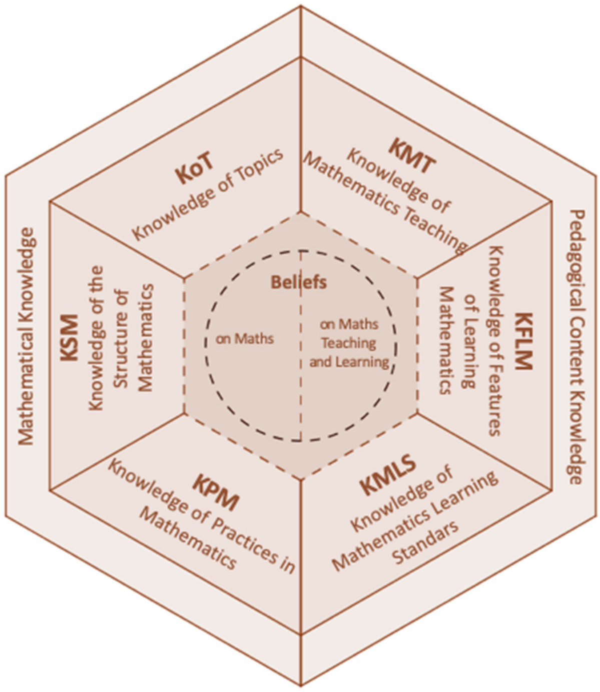

Figure 1. The MTSK model. Adapted with permission from Ref. [12] Copyright 2018.

The elements that constitute the MTSK model only apply to mathematics teachers [14]. In summary, the MK domain covers the knowledge of topics, how they are interrelated and the mathematical practices, referring to the ways in which a person proceeds and produces in mathematics. The PCK domain considers mathematical contents as objects of teaching, learning, as well as their curricular background.

To facilitate the analysis, the subdomains of the MTSK model were split into categories, which were described by [12] and are listed in Table 1 (MK subdomains) and Table 2 (PCK subdomains).

Table 1. Categories of the MK subdomains.

Subdomain	Code	Category [1]
Knowledge of Topics (KoT)	KoT1	Procedures
	KoT2	Definitions, properties and foundations
	KoT3	Registers of representation
	KoT4	Phenomenology and applications
Knowledge of the Structure of Mathematics (KSM)	KSM1	Connections based on increased complexity
	KSM2	Connections based on simplification
	KSM3	Transverse connections
	KSM4	Auxiliary connections
Knowledge of Practices in Mathematics (KPM)	KPM1	Demonstrations
	KPM2	Definitions
	KPM3	Exemplification
	KPM4	Use of heuristics

[1] Note: All the categories were established in [12], except those relating to Knowledge of Practices in Mathematics, which correspond to the proposal made by [16], in view of improving the characterization of this subdomain.

Table 2. Categories of the PCK subdomains.

Subdomain	Code	Category [1]
Knowledge of Mathematics Teaching (KMT)	KMT1	Theories of mathematics teaching
	KMT2	Teaching resources (physical and digital)
	KMT3	Strategies, techniques, tasks, and examples
Knowledge of Features of Learning Mathematics (KFLM)	KFLM1	Theories of mathematical learning
	KFLM2	Strengths and weaknesses in learning mathematics
	KFLM3	Ways students interact with mathematical content
	KFLM4	Emotional aspects of learning mathematics
Knowledge of Mathematics Learning Standards (KMLS)	KMLS1	Demonstrations
	KMLS2	Definitions
	KMLS3	Exemplification
	KMLS4	Use of heuristics

[1] Note: All the categories were established in [12].

In this paper we focus on the MK and PCK domains, leaving for an upcoming study the implications between teachers' predictions and the domain of beliefs on mathematics and mathematics teaching and learning.

2.3. Students' Difficulties with Solids of Revolution

Although solids of revolution are present in different educational levels, in this paper we focus on the calculation of their volume using integral calculus methods. Ref. [17] mentioned that one difficulty in working with solids of revolution resides in the curricular sequencing of this topic. By studying first the calculation of areas between curves as an application of integrals and, subsequently, calculating the volumes as another application, students may think that it is the area what is rotated to obtain the solid. This error is also reproduced in some textbooks [17].

Another difficulty, found by [18], refers to the meaning that students attribute to $f(x)$, as it is not unambiguous. Instead, students must distinguish between three simultaneous meanings for $f(x)$: as the function to be integrated, as the image of the function and as the radius of the circles that form cylinders of height dx. These authors [18] explained that switching between registers of representation, especially between algebraic and graphical registers, represents another difficulty for students.

Other authors [19] mentioned that the main difficulties linked to the learning of the volume of solids of revolution are related to the visualization of rotations, the interpretation of the definite integral, and the integration of the concepts involved (drawing graphs, indicating and rotating the representative function, and indicating the correct formula for calculating the volume of the solid generated). In particular, these authors reported that, in tasks related to the calculation of the volume of solids of revolution, students have difficulties in drawing graphs, moving between 2D and 3D representations, understanding how a 2D object can rotate and become a 3D object, and using techniques to calculate integrals.

3. Methodology

This research used a qualitative approach and represents what [20] calls an instrumental case study. Five mathematics teachers who regularly teach Integral Calculus in high schools in Mexico agreed to participate in the research study. Their teaching experience ranged from 15 to 32 years. Data collection was carried out in a 120-min session through video recordings and two researchers acted as observers [21]. During this session, participants were asked to predict students' mathematical behavior when graphing the functions shown in Figure 2, which was part of a didactic sequence designed by the two researchers to deal with the topic of the volume of solids of revolution.

ACTIVITY 6: THEY ARE PROVIDED WITH A FUNCTION

In the following activity, the teacher should provide students with the following functions: $y=3$ on the interval [1,9], $y= 2x$ on the interval [0,3], $y = \sqrt{36 - x^2}$. on the interval [0,6] and $y = x^2 + 1$ on the interval [−2,5].

- The student must draw the given functions and the solids of revolution generated from them, with the convention that the axis of rotation will be the horizontal axis.
- Once the solid of revolution generated by each function has been drawn, the student should concentrate on some point on the axis of rotation and approximate the distance between that point and some other points on the outline of the solid on the circumference that would be formed by cutting perpendicular to the axis at the height of that point. Then, compare the resulting measurements and repeat the same procedure for some other points on the axis of rotation.
- The teacher will generate a discussion about what this instance looks like at different points on the axis of rotation, and what the variation is due to or depends on.

Figure 2. Graphing activity. Note: The activity was delivered to teachers in Spanish.

In order to formulate conjectures, teachers were first asked to solve the activity in pairs as if they were students. In order to be able to explore a variety of conjectures, each participant was randomly and anonymously assigned a student profile with specific characteristics of three different performance levels (low, medium or high). All participants solved and discussed the activity, first assuming the assigned role and then with their own arguments, i.e., without considering the role. This discussion generated a series of five conjectures about students' mathematical behavior regarding the aforementioned task. Next, in order to obtain more detailed explanations of their conjectures, participants watched in video format how the activity had been carried out with a group of high school students. This was followed by a new discussion oriented to check whether their conjectures matched students' mathematical behaviors.

Data analysis was performed under a mixed Bottom-Up and Top-Down approach [22]. On the one hand, the Bottom-Up analysis was used to group teachers' contributions to generate a set of five conjectures about students' mathematical behavior when facing the task. On the other hand, the Top-Down analysis was used to assign the categories of knowledge that supported the contributions that generated such conjectures.

4. Results

Five conjectures of students' mathematical behavior emerged during the discussion. In this section, evidence of mathematics teachers' knowledge that led to these predictions is analyzed. To organize the presentation of the results, teachers are named as T1, T2, T3, T4, and T5. The interventions of the researchers (named as R1 and R2) are also included, but their knowledge is not analyzed as this has never been the intention of this study. It is important to mention that some teachers' predictions were made under the specific assigned role, while others were made when participants were asked not to assume such role. To distinguish between both, an * is placed next to the participant's name when the teacher was under the role. Note that T1 and T2 were in the low performance role, T3 and T4 in the high performance role and T5 in the medium performance role.

Conjecture 1. *The majority of students will choose tabulation as their preferred tool for graphing.*

When discussing how to carry out the activity under the assigned role, T3 wanted to use analytical tools, while T2 pointed out that he did not remember the meaning of the concepts being mentioned.

T3 *	The second function is a straight line passing through the origin and whose slope is 2. Then it would look like this [he makes a gesture with his hand reflecting an increasing straight line].
T2 *	I remember having seen these topics in Analytic Geometry, but I have forgotten what the slope refers to. I do remember about the origin.
T3 *	All you have to do is to stand at the origin and move, for each unit on the x-axis [abscissa axis], two units on the y-axis [ordinate axis].
T2 *	But wouldn't it be easier to give values for x in the function and get values for y?
T3 *	In this case it is easier, but not in all cases.

In this extract, T3 reflects knowledge of the meaning of the elements of a straight line (KoT2) and of how such elements change due to transformations both in the analytical and graphical registers (KoT3).

After this conversation, both teachers made the remaining graphs using tabulation. Later, R2 questioned this decision.

R2	I heard from some of you that some parameters of the functions were being used to graph their behavior. However, I saw that many of you graphed using the tabulation. T3, you had proposed using the slope, right?
T3	What T2 and I discussed is that it is easier for students to tabulate. But I think that, if they have already learned other techniques, we should force them to use them.
T5	I don't think it should be forced, but that each student should use what he/she feels more comfortable with. They should be familiar with various graphing techniques, but then it's up to the individual to decide which one to use.
T3	But if you plot a parabola like this, you will not get a good drawing, you will get a peak or a horizontal part depending on the values you consider.
R1	And what do you think the students would do in this part of the activity?
T2	I believe they would tabulate.

All teachers agreed that tabulation is the form of graphing mainly chosen by students, although they considered that some students with a greater knowledge of topics might use other methods, as later shown regarding Conjecture 3. Besides, T3 shows knowledge of the sequencing of topics by saying which procedures students should know from a curricular point of view (KMLS3). T2 and T3 also mentioned that students have developed a strength in using tabulation as a tool for graphing and, implicitly, T5 agreed with them (KFLM2). However, such procedure has some limitations, which were expressed by T3, showing knowledge of this way of proceeding (KoT1).

Next, participants were shown several images that corroborated their conjecture and were asked to discuss the production shown in Figure 3, in which some students tabulated the constant function and then graphed it.

T4	I think here [in the previous discussion] we all assumed that the constant function would not be a problem. In fact, we didn't even discuss it. But, on further reflection, I think this function is very complex for the students, because it doesn't have the independent variable written down and that can cause confusion
T3	I think it has to do with the examples we use to present functions. We almost always use expressions where *x* is involved and we make them as complex as we want, because we are interested in this idea that the function is a machine that transforms, but here [in the constant function] nothing is being transformed. It may be that our own discourse leaves these reflections hidden from the students.

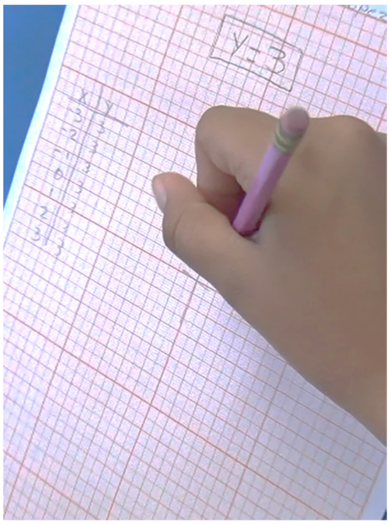

Figure 3. Student's production: graph of the function y = 3 using tabulation. Note: On the right-hand side of the image, the reader can see a table in which values are assigned for the independent variable and the value of 3 is replicated in the dependent variable.

The discussion about this student production allowed the teachers to discuss the origin of this error and, rather than focusing on a distraction, they expressed the complexity of the exercise with respect to the definition and exemplification that is usually used to introduce the topic of functions. Therefore, teachers used knowledge of examples and their potential for teaching this topic (KMT3).

Conjecture 2. *Tabulation will make students ignore the graphing interval proposed in the activity.*

When assuming the assigned role, T1 made the graph of the function representing a parabola based on the tabulation in the interval [−3, 3]. T5 did not detect the change in this interval with respect to the interval [−2, 5] requested in the activity. When they showed what they had done to the whole group, this error was discussed.

T1 *	In this last function, we got a solid of revolution with this shape [the teacher shows the solid generated when rotating the function $y = x^2 + 1$ around the x-axis and within the interval between -3 and 3, both included].
T4 *	It looks very similar to ours, but not so symmetrical. It must have more on the right side and less on the left side [the teacher refers to the solid generated when rotating the function $y = x^2 + 1$ around the x-axis and within the internal between -2 and 5, both included].
T1	In fact, I drew it this way because I considered that our students, at least mine, when they tabulate, they always consider whole numbers and numbers between -3 and 3.
T2	It's something they have learned. My students also do tabulation like that.

This dialogue reveals teachers' deep knowledge of how students interact with the tabulation (KFLM3), not only anticipating its use, but also how it will be used (considering only integers) and the spectrum it will cover (the interval [−3, 3]). This was corroborated by the work done by various students in almost all the graphical representations (see, for example, Figure 4), but the discussion did not provide new information about the knowledge that the teachers employed to elaborate this conjecture.

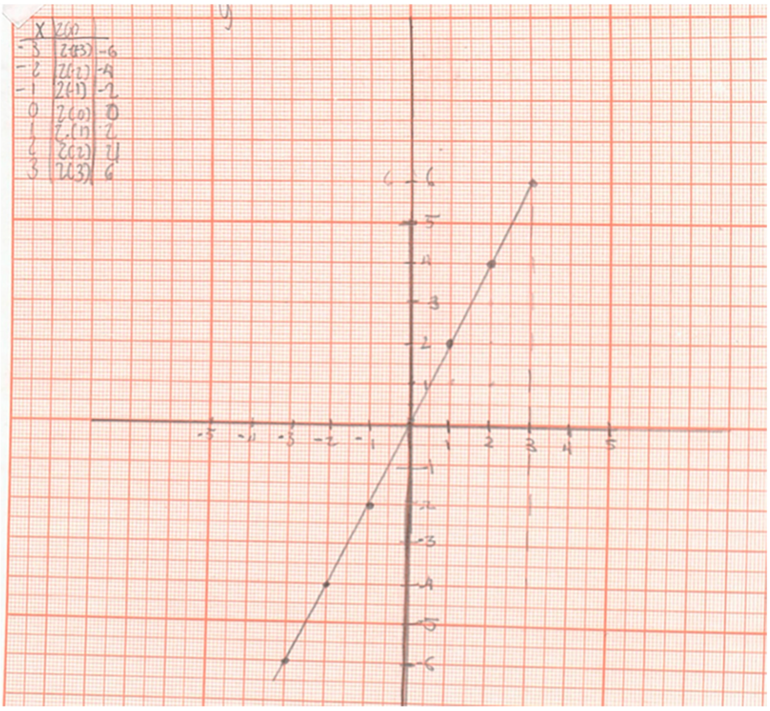

Figure 4. Student's production: tabulation without respecting the proposed interval. Note: The activity requested the graphical representation to be limited to the interval [0, 3]. In the upper-left part of the image, the reader can see that the interval [−3, 3] was tabulated considering only integers.

Conjecture 3. *High-achieving students will transform functions into known conics.*

All functions were selected to be drawn based on the recognition of the parameters and their translation into graphical language. In particular, the third function was discussed based on the work of T4.

T4 *	I observed that, if I cleared this root [by squaring both sides of the equality] and passed this x^2 to this side [by adding x^2 on both sides of the equality], I got the equation of a circle. So, by limiting it to the interval, I only drew a quarter of a circle.
R1	Let's pause here. T4, would your students work on this function in this way?
T4	Yes, not all of them, but those with high performance levels would
R1	What do you think, T3, would any of your students do it.
T3	Yes, my students would understand that this is a circle of radius 6.
T2	I think the question is different. My students would also understand that the new expression is a circle. But I think that what we are presenting is whether students would genuinely proceed in this way. I say no.
T5	I don't think they would proceed like that either. Even those with high performance levels would determine the domain, the range and plot it. I don't think they would notice that it is a perfect circle.

In this discussion, knowledge of the equation of the circumference, the meaning of its elements (KoT2) and its transformation to a graphical register (KoT3) is evident. Furthermore, based on the previous dialogues, exhibited in Conjectures 1 and 2, teachers knew that students have worked with the equation of the circumference (KMLS3) and that this procedure would, in some cases, be adopted by students with high performance levels (KFLM3).

Regarding the activity carried out with students, it is possible to observe that the teacher was influenced by this behavior because she asked in open discussion what the function represented. One student mentioned that it was a circle with center at the origin and radius 6. Figure 5 shows a graph drawn based on tabulation and justified with the values mentioned by the student. For this reason, the fulfilment of the conjecture was not discussed among the participating teachers.

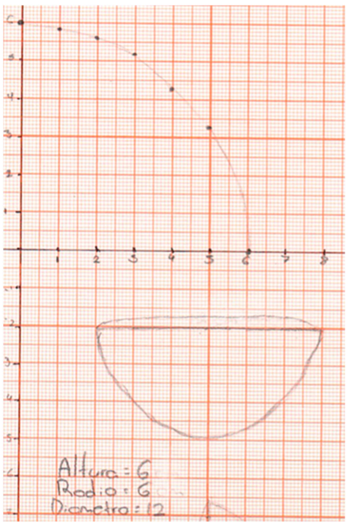

Figure 5. Student production: graph of the function representing a quarter of a circle. Note: At the bottom of the figure, we read height = 6, which corresponds to the cut with the ordinate axis, radius = 6 and diameter = 12, which we assume was used to draw the solid of revolution shown between the graph of the function and the legend.

Conjecture 4. *Students could draw circumferences without perspective on the cuts of the solids of revolution.*

From the second instruction in the activity, T2, who was playing the role of a low performance level student, questioned the drawing made by T3, pointing out that he could not see any circumference in his line. It should be noted that in the activities done by the students prior to this one, it had already been discussed with tangible material that circumferences could be observed in the cross-sections of the solids of revolution. This issue was addressed in the group discussion.

T2	I would like to focus on one aspect that I discussed with T3. The activity asks to measure the radius of the circles that would be generated at different cuts in the solid. I understand that the intention is that we observe that this radius coincides with the image of the function at that point. However, it strikes me that, when working with a drawing in two dimensions, the circles will not appear or will deform the image of the solid.
R1	Could you expand a little more on your idea?
T2	Solids are three-dimensional. To represent them on our sheet, we have to give them perspective. Circles will not comply with the definition that the distance from the center to any point is the same. That's what I mean when I say that the circles are not going to be drawn. Although I also think that some students will intentionally draw them even though the drawing no longer looks like a solid of revolution, because they read that there must be circles.
R1	What do you [the other teachers] think about this?
T1	I think this activity, without the context of the previous ones, might seem poorly written in terms of circles. I think students may have problems with the tracing of three-dimensional figures. I have seen this in other topics. They lack perspective in their drawings. Perhaps this part of the activity could be transferred to GeoGebra here, as it has a 3D tool that allows you to see the figure from different angles.

T2 explains possible difficulties that students may have when contrasting the notion that they have so far of a solid of revolution and its properties with the drawing they make of solids in two dimensions and with the meaning of circumference (KFLM2). Although T1 does not fully cover the elements that make up the exposed difficulty, he shows his knowledge of a software that would allow the visualization of the solid of revolution without the problem of perspective (KMT2).

Although this conjecture did not have a decisive impact on the development of the activity, during the implementation it could be observed that several students had difficulties in representing the solids of revolution. During the discussion with the teachers, the graph shown in Figure 6 was considered.

T2	This is what I was referring to. In the graph you can see that it joins the ends of the segments with a straight line, but it does not allow you to see a circumference.
T5	I think it's a problem of the size of the drawing. In fact, if we observe, on the right he [the student] drew an arc, but he didn't put his symmetrical one, but maybe it's because he felt that the drawing was overlapping.
T3	I see that the arc drawn on the right does not correspond to the perspective of a well-drawn cone, but I dare to say that they [the students] understood that there is a circumference there.
T1	I was surprised by the fact that he took the drawing out of the Cartesian plane to draw it properly. It reminds me of an idea I read some time ago about students who don't recognize that the square is a rhombus because it is not rotated. I feel that in this graph they felt the need to put the cone as they have always drawn it, with its vertical axis, to convince themselves about the drawing.

With this interpretation, T1 connects, in an informal way, the notion of prototypical schemas with the drawing made by the students in Figure 6. Although the evidence is not strong enough to categorize it as KFLM1, it allows us to see the influence that concrete research results have on teachers' knowledge. In this case, we consider that this influence falls on the knowledge of a typical way of relating to solids (KFML3).

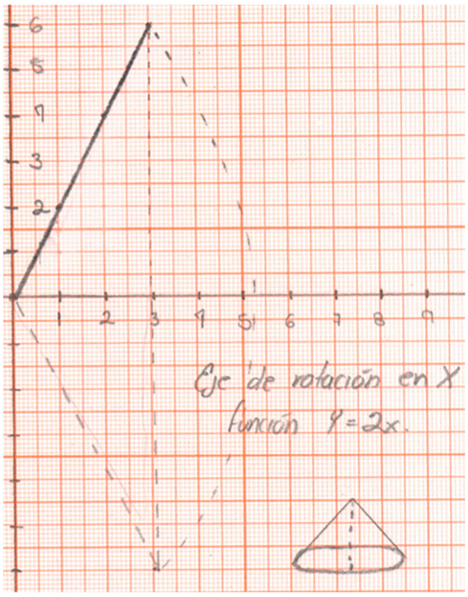

Figure 6. Student production: drawing of a solid of revolution from the graph of a segment. Note: Legend says: "Axis of rotation at x, function y = 2x".

Conjecture 5. *To identify the value of the radius of the circumferences in the cross sections, students will measure or approximate by counting.*

One discussion that emerge from the dialogue that led to Conjecture 4 focused on how determining the radius of circumferences would be approached.

T2	One thing that is not solved by using GeoGebra is the determination of the radius. GeoGebra can give us that value, but more than being interested in the value, we want the students to know what that number means.
T3	I was also thinking about how to continue the activity without using GeoGebra. The students could use millimeter sheets. This would allow them to count because the graphs would have a scale given by the sheet itself.
T1	They could also use the ruler to measure. If they use properly the scale, the data should be accurate.
T2	In straight lines yes, but curved lines will have differences.
R2	Differences between them?
T2	No. Well, yes, also, but I was referring to the difference between the data obtained and the value of the function at that point.
T1	But I think that's the discussion that the teacher could provoke. At this point in the activity, what the students would be able to do is to measure, either by counting or with a ruler. Then you [the teacher] must get them to know that the most accurate way to obtain the data is to evaluate the point on the function. The part where you need to be more careful about is to ensure that they measure towards the function and not towards the sides, where the perspective gives a smaller measurement.

At the beginning and at the end of this dialogue, T2 and T1 complement their knowledge of the task that is being discussed (KMT3). They determine the intentionality of the activity and, in the case of T3 and T1, they show knowledge of ways in which they think students would interact with this part of the activity (KFLM3). The role of GeoGebra is relegated, not because it is not useful from a technical point of view to solve the activity, but because its inclusion would affect the construction of the notion being worked on, i.e., determining the formula to obtain the volume of solids of revolution using integrals. This determination that limits the use of a virtual material based on its didactic-mathematical characteristics shows the kind of knowledge of this software required by the MTSK model (KMT2).

5. Conclusions

This research aimed at understanding the knowledge that allows teachers to anticipate students' mathematical behavior in an introductory activity on calculating the volume of solids of revolution. Using the MTSK model [12] to analyze such knowledge was useful considering its focus on mathematics. The dialogues presented in the results section show a predominance of teachers' KoT, KMT, and KFLM. In particular, to elaborate their conjectures, teachers used their knowledge of students' difficulties and of the way students interact with mathematical content. It was natural that these two categories of knowledge emerged regularly because, according to [23], moments of difficulty draw teachers' attention when analyzing (in our case predicting) students' performance in an activity.

It is interesting to note that tabulation was the graphical representation most chosen by the students. This procedure presents some limitations that in the future may lead to difficulties in other mathematical aspects. Regarding the difficulties associated with visualization, it has been conjectured that some students will draw circles without perspective in the cuts of the solids of revolution as they will not comply with the definition that the distance from the center to any point is the same, in addition to the difficulties associated with 2D and 3D representations mentioned by [19]. In short, the different comments expressed by the teachers are closely linked to the mathematical practices developed when working on graphing functions with students.

Regarding the limitations of the study, the authors are aware of two. Firstly, it is the analysis itself with the MTSK model and the opacity that may be involved in segmenting the specialized knowledge of the mathematics teacher which makes it impossible to state that these results can be re-evidenced in the different subdomains of knowledge. Secondly, we are dealing precisely with evidence of knowledge and not with clues. On some occasions, the researchers' own interpretation of the teachers' comments has been what has determined which subdomain of knowledge could be implied in the conjecture (see for instance the reference to KFLM1 in the fourth conjecture). Although the evidence is not conclusive, we believe that not specifying it would be detrimental to the analysis and understanding of the case study.

Conjectures made by the teachers, being influenced by their experience (both from the point of view of students and from the point of view of teachers), can be limited to the Mexican context as well as to those whose curricular framework is similar to that of this country. Although we do not know what kind of conjectures might arise in curricular frameworks of a different nature, we consider that the knowledge required by the teacher to elaborate such conjectures will not differ from that found in this study, since such knowledge coincides with that found in the review by [24] for the general knowledge required in calculus courses.

The emphasis on KoT, KFLM and KMT allows us to establish two implications. On the one hand, this provides us with information on the practical potential of developing such knowledge in teachers. This is not intended to detract from the importance of the other knowledge that, from the point of view of the MTSK, makes up the specialized knowledge of the mathematics teacher, since other professional competences may require it. On the other hand, the emphasis on these subdomains denotes the absence of evidence of knowledge about the structure of mathematics and about mathematical practices, which may be a product of the methodological decisions of this research but may also be due to an impossibility on the part of teachers to make use of this type of knowledge in professional activities, which would be interesting to explore in a future research study.

Being able to determine the knowledge required by teachers to conjecture students' mathematical behavior influences other professional competences, for example, teachers' competence to interpret students' mathematical productions [25].

This research study has clear theoretical and practical implications. For instance, the results presented here could be used in training processes both at initial and ongoing stages. This idea addresses the needs claimed by various authors regarding the determination of useful knowledge, both for the design of training experiences [26], and for discussing teacher educators' knowledge [27].

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