mathematics

MDPI

Article

A Contribution of Liouville-Type Theorems to the Geometry in the Large of Hadamard Manifolds

Josef Mikeš [1,*], Vladimir Rovenski [2] and Sergey Stepanov [3]

[1] Department of Algebra and Geometry, Palacký University Olomouc, 771 47 Olomouc, Czech Republic
[2] Department of Mathematics, University of Haifa, Mount Carmel, Haifa 3498838, Israel
[3] Department of Mathematics, Finance University, 125468 Moscow, Russia
* Correspondence: josef.mikes@upol.cz

Abstract: A complete, simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. In this article, we prove Liouville-type theorems for isometric and harmonic self-diffeomorphisms of Hadamard manifolds, as well as Liouville-type theorems for Killing–Yano, symmetric Killing and harmonic tensors on Hadamard manifolds.

Keywords: Hadamard manifold; subharmonic, harmonic and convex functions; isometric and harmonic diffeomorphisms; Killing and harmonic tensors

MSC: 53C20

check for updates

Citation: Mikeš, J.; Rovenski, V.; Stepanov, S. A Contribution of Liouville-Type Theorems to the Geometry in the Large of Hadamard Manifolds. *Mathematics* **2022**, *10*, 2880. https://doi.org/10.3390/math10162880

Academic Editor: Sorin V. Sabau

Received: 29 July 2022
Accepted: 9 August 2022
Published: 11 August 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

The most rigid classical concept of curvature, and therefore containing most of the information about a Riemannian manifold (M, g), is the sectional curvature (viewed as a function on the Grassmann bundle of tangent two-planes to M). One aspect that makes the concept of sectional curvature especially important is its connection to the topology of a Riemannian manifold. For example, as a starting point when studying the geometry of Riemannian manifolds of nonpositive sectional curvature, any geometer will first recall the following well-known *Cartan–Hadamard theorem*, which is due to Hadamard in the case of surfaces and by Cartan in the case of arbitrary Riemannian manifolds of nonpositive curvature. Namely, let (M, g) be an n-dimensional ($n \geq 2$) simply connected and complete Riemannian manifold of nonpositive sectional curvature, then (M, g) is diffeomorphic to the Euclidean space $\mathbb{R}^n$. Therefore, a simply connected complete Riemannian manifold of nonpositive curvature is called a Hadamard manifold after the Cartan–Hadamard theorem (see, for example, [1], p. 241 and [2]). Well-known basic examples of such manifolds are the Euclidean space $\mathbb{R}^n$ with zero sectional curvature and the hyperbolic space $\mathbb{H}^n$ with constant negative sectional curvature. At the same time, it is well-known that the flat torus $\mathbb{T}^n$ is a connected complete manifold of zero sectional curvature. However, $\mathbb{T}^n$ is not simply connected and hence is not an example of a Hadamard manifold. Therefore, the requirement of simple connectedness is essential here and, for example, strong enough to distinguish the Euclidean space $\mathbb{R}^n$ from the flat torus $\mathbb{T}^n$.

In addition, we formulate here the obvious consequences of the Cartan–Hadamard theorem. First, from the theorem, we conclude that no compact simply connected manifold admits a metric of nonpositive sectional curvature (see also [1], p. 162). Second, a Hadamard manifold has an infinite volume, which follows from the Cartan–Hadamard theorem (see also [3], p. 4732). Third, a Hadamard manifold of zero scalar curvature is isometric to the Euclidean space of the same dimension. To prove this assertion, recall that the scalar curvature of an n-dimensional Riemannian manifold is defined by the equality $s(x) = \sum_{i \neq j} sec\,(e_i \wedge e_j)$ where $sec\,(e_i \wedge e_j)$ denotes the section curvature of the two-plane spanned by e_i and e_j for any orthonormal basis $\{e_1, \ldots, e_n\}$ of T_xM at any point $x \in M$. In

this case, from the conditions $s \equiv 0$ and $sec \leq 0$, we conclude that $sec \equiv 0$. In this case, the Riemannian manifold is flat. Moreover, if it is simply connected, then it is isometric to a Euclidean space of the same dimension n (see generalizations of this assertion in [2]).

There are many papers on "geometry in the large" of Hadamard manifolds (see the review [2] and other works [3–5]). However, now we are living in the era of geometric analysis and its applications in studying geometric and topological properties on Riemannian manifolds (e.g., [6]). Therefore, we discuss in our paper the global geometry of Hadamard manifolds using a generalized version of the Bochner technique, the purpose of which is to apply geometric analysis to the study of the geometry in the large of Riemannian manifolds (e.g., [7]). In particular, we demonstrate several Liouville-type theorems for isometric and harmonic self-diffeomorphisms of Hadamard manifolds, as well as for Killing and harmonic symmetric tensors on Hadamard manifolds. These theorems supplement similar well-known vanishing theorems for Riemannian manifolds and, in particular, for Hadamard manifolds proved by the classical Bochner technique (see [1], pp. 333–364) and other methods of "geometry in the large" (see, for example, [2]). In turn, the proofs of our theorems use well-known Liouville-type theorems on harmonic (resp. subharmonic and convex) functions on complete Riemannian manifolds, which we partially modified for Hadamard manifolds in the next section of our paper (see [2] for further generalizations of this property of Hadamard manifolds).

Our results were presented at "XXI Geometrical Seminar" (26 June–2 July 2022, University of Belgrade, Serbia).

2. Subharmonic, Superharmonic, Harmonic and Convex Functions

This section is devoted to the relationships between the geometry of a complete Riemannian manifold (in particular, a Hadamard manifold) and the global behavior of its subharmonic and harmonic functions under assumptions on its sectional curvature. For this study, methods of geometric analysis are used. Most of these results are called *Liouville-type theorems* and belong to the *generalized Bochner technique* (e.g., [8], pp. 361–394 and [6,7,9–12]). Recall that the prototype of generalized Bochner technique is the celebrated *classical Bochner technique*, first introduced by Solomon Bochner and dating back some eighty years or so. Its fundamental principle is that some vector fields (e.g., conformal Killing and Killing fields) or differential forms (e.g., harmonic and Killing–Yano forms) must vanish on compact manifolds when conditions are imposed on the curvature sign of those manifolds (e.g., [1], p. 333–364; [8], pp. 322–360 and [13,14]). Careful application of this technique has led to a number of remarkable vanishing theorems, e.g., the famous result of Kodaira in the 1950s (see [8], pp. 351, 361–363).

Below, we consider some facts of geometric analysis that underlie the generalized Bochner technique (e.g., [8], p. 361–394). First, recall that a scalar function $f \in C^2(M)$ is *subharmonic* (see [1], p. 281; [8], p. 35, 85) if it satisfies the differential inequality $\Delta f \geq 0$ for the *Beltrami Laplacian* $\Delta = div \circ grad$ and, in particular, $f \in C^2(M)$ is *harmonic* (see [1], p. 283) if it is a smooth solution of the Laplace equation $\Delta f = 0$.

In mathematics, subharmonic and harmonic functions are important classes of functions that are widely used in partial differential equations, complex analysis and potential theory and in the geometry of Riemannian manifolds. Simple application of the *Hopf maximum principle* (alternatively the *divergence theorem*) shows that (see [1], p. 73) a compact Riemannian manifold has no subharmonic, superharmonic and harmonic functions except for constant functions. Note that this fact refers to the classical Bochner technique. On the other hand, Huber (see [15]) proved that a complete two-dimensional Riemannian manifold with non-negative curvature does not admit a nonconstant negative subharmonic function. Then, Karp (see [16]) discovered that a complete noncompact Riemannian manifold does not admit a nonconstant negative subharmonic function if it has moderate volume growth. For comparison, recall that the famous Liouville theorem states that a subharmonic function defined over $\mathbb{R}^2$ (or a harmonic function defined over $\mathbb{R}^n$) and bounded from above is constant. Several further results on the properties of subharmonic and harmonic

functions on complete Riemannian manifolds have been obtained by many authors such as Yau, Wu, Li, Schoen and Hamilton et al (see, for example, [6–9]). They ushered in a particularly productive era of applications of geometric analysis in differential geometry that continues to this day. The applications of this new theory are of particular interest (see, e.g., [11,17–19]).

Next, we study the interaction between the geometry of complete Riemannian manifolds of nonpositive sectional curvature, in particular, Hadamard manifolds and some aspects of the theory of functions on these spaces. A well-known question is still very important: which subspaces of subharmonic (and harmonic) functions on complete Riemannian manifolds contain only constant functions and, in particular, identically equal to zero. For this case, the following Yau theorem is well-known (see [9]): if (M, g) is a complete noncompact Riemannian manifold (without boundary) and $f \in C^2(M)$ is a nonnegative subharmonic L^q-function for $1 < q < +\infty$, then f is a constant. By definition, a scalar function f on a Riemannian manifold (M, g) belongs to $L^q(M)$ for some positive number q if $\int_M |f|^q \, d\mathrm{vol}_g < \infty$, where the integral is always understood in terms of the volume form $d\mathrm{vol}_g$ of the metric g.

In addition, here the situation falls into two cases. The first case: when (M, g) has finite volume, then all constant functions are in $L^q(M)$ for any positive number q. Second case: when (M, g) has infinite volume, then among all constant functions, only zero one is in $L^q(M)$. Namely, if the function $f \in L^q(M)$ for some positive number q is a constant function C, then the inequality $\int_M |f|^q \, d\mathrm{vol}_g < \infty$ becomes $|C|^q \cdot \int_M d\mathrm{vol}_g < +\infty$. Moreover, if (M, g) has an infinite volume, then this makes the constant C equal to zero.

To which manifolds do the above definitions and results apply? For example, using the information from the first part of the section on Hadamard manifolds, we conclude that the following lemma is true.

Lemma 1. *The Hadamard manifold (M, g) does not admit a nonzero non-negative subharmonic L^q-function for at least one $q \in (0, +\infty)$.*

Proof. Let (M, g) be a Hadamard manifold, that is, a simply connected complete Riemannian manifold of nonpositive sectional curvature. It is known from [6] that on a complete simply connected Riemannian manifold (M, g) of nonpositive sectional curvature every non-negative subharmonic L^q-function for any $q \in (0, +\infty)$ is a constant function C. Recall that a Hadamard manifold has an infinite volume, therefore, $C = 0$. □

The well-known fundamental fact is that if $f \in C^2(M)$ is a harmonic function, then $|f|^p$ is a non-negative subharmonic function for each $p \geq 1$ (see also [8], p. 373). Therefore, if every non-negative subharmonic L^q-function on (M, g) is constant, then every harmonic L^q-function on (M, g) is also constant (see [6]). Thus, we can formulate a corollary from Lemma 1.

Corollary 1. *A Hadamard manifold (M, g) does not admit a nonzero harmonic L^q-function for at least one $0 < q < +\infty$.*

Remark 1. *First, in the range $1 < q < +\infty$, we can formulate one more classical L^q-Liouville-type result of Yau: Let $f \in C^2(M)$ be a harmonic L^q-function for some $1 < q < +\infty$ on an arbitrary complete Riemannian manifold, then, f is constant. Therefore, our corollary completes Yau's theorem. Second, the effect of curvature on the behavior of harmonic functions is a classical problem. For example, in contrast with Corollary 1, Sullivan proved that there is an abundance of bounded harmonic functions on a strongly negatively curved Hadamard manifold (see [4]).*

In the third part of the section, we consider convex functions. Recall that a function $f \in C^2(M)$ is called *convex* if its Hessian $Hess_g f := \nabla df$ is positive semidefinite at each point $x \in M$, and f is called *strictly convex* if $Hess_g f$ is positive definite at each point $x \in M$

(see [1], p. 281 and [5]). In this case, we have $\Delta f \geq 0$ for a convex function f, hence f is a subharmonic function.

In differential geometry, the existence of convex functions on (M, g) is a long-standing problem. The first solution of this problem can be found in the article [5] by Bishop and O'Neill. They proved that if (M, g) is complete and has finite volume, then it does not possess a nonconstant smooth convex function, thus, nonconstant convex functions exist only on Riemannian manifolds of infinite volume. We supplement their result on the basis of Corollary 1 by the following (see also [12]).

Corollary 2. *A Hadamard manifold does not admit nonzero non-negative convex L^q-functions for at least one $q \in (0, +\infty)$.*

Remark 2. *Busemann functions (see [1], pp. 301–304) play a very important role in studying the topology and geometry of complete Riemannian manifolds. In particular, the Busemann functions are an example of convex functions on a Hadamard manifold (see [20]).*

The fourth part of this section is devoted to Riemannian symmetric spaces. The condition of the parallelism of the curvature tensor $\nabla R = 0$ defines the class of *Riemannian locally symmetric spaces*, which can be equivalently defined as those Riemannian manifolds that are locally reflectively geodesically symmetric around any point $x \in M$ (see [21], pp. 231–234). A Riemannian locally symmetric space (M, g) is called a *Riemannian globally symmetric space* if its locally geodesic symmetries are defined on its entire space. In this case, a Riemannian symmetric space (M, g) is complete (see [21], p. 240). Riemannian globally symmetric spaces can be classified using their isometry groups. The classification distinguishes three basic types of such spaces: spaces of so-called compact type, spaces of so-called noncompact type and spaces of Euclidean type (see, e.g., [21], p. 245). In particular, a Riemannian globally symmetric space (M, g) of noncompact type is simply connected and has nonpositive sectional curvature (see [21], pp. 245–246). Using the above, we assert that a Riemannian globally symmetric space of noncompact type is a prime example of a Hadamard manifold. In particular, a Riemannian globally symmetric space of noncompact type has an infinite volume. Therefore, the following corollary of Lemma 1 holds (see also [12]).

Corollary 3. *A Riemannian globally symmetric space of noncompact type does not admit a nonzero non-negative subharmonic (and harmonic) L^q-function for at least one $q \in (0, +\infty)$.*

Remark 3. *The geometry of a Riemannian symmetric space with nonpositive sectional curvature is described in detail in [22]. In addition to the above, we note that the Hopf maximum principle (see [1], p. 75) shows that a Riemannian globally symmetric space of compact type has no subharmonic, harmonic and convex functions, except for constant functions.*

3. Harmonic Symmetric Tensors

Let $C^\infty(S^pM)$ be the space of C^∞-sections of the bundle $S^pM = S^p(T^*M)$ of covariant symmetric p-tensors $(1 \leq p < \infty)$ on a connected Riemannian manifold (M, g). Then, the following equality is true:

$$dim\ S^p(T^*_xM) = \binom{n+p-1}{p}.$$

for the vector space $S^p(T^*_xM)$ of covariant symmetric p-tensors on T_xM at an arbitrary point $x \in M$.

Now, we define the differential operator $\delta^* : C^\infty S^pM \to C^\infty S^{p+1}M$ of degree one by the formula $\delta^*\varphi = (p+1)\ S^{p+1}(\nabla\ \varphi)$ for an arbitrary $\varphi \in C^\infty(S^pM)$ and the standard pointwise symmetry operator S^{p+1}: $T^*M \otimes S^p(T^*M) \to S^{p+1}(T^*M)$. There exists its formal adjoint operator δ: $C^\infty S^{p+1}M \to C^\infty S^pM$ for δ^*(see [23], pp. 34–35, 434). Using the

above, Sampson determined in [24], p. 147 for an arbitrary Riemannian manifold (M, g) the operator

$$\Delta_S = \delta\,\delta^* - \delta^*\delta\colon\ C^\infty S^p M \to C^\infty S^p M.$$

Moreover, he showed that the operator Δ_S admits the Weitzenböck decomposition (see also [10])

$$\Delta_S = \overline{\Delta} - \Re, \tag{1}$$

where $\Re$ is the *Weitzenböck curvature operator* of the ordinary Lichnerowicz Laplacian $\Delta_L = \overline{\Delta} + \Re$ (see [23], p. 54 and [25], p. 315), which is restricted to covariant symmetric p-tensors. The Weitzenböck curvature operator $\Re$ of Δ_L can be algebraically (even linearly) expressed through the curvature R and Ricci tensors Ric of (M, g). Moreover, it satisfies the identities (see [25], p. 315)

$$g\,(\Re\,(T),\ T^{'}) = g\,(T,\ \Re\,(T^{'}))$$

and

$$trace_g\ \Re\,(T) = \Re\,(trace_g T) \tag{2}$$

for any $T,\ T^{'} \in \otimes^p T^* M$. In particular, from (2) any one concludes that $\Re_x : \otimes^p T_x^* M \to \otimes^p T_x^* M$ is a symmetric endomorphism at any point $x \in M$.

For example, if $p = 1$, then we have the equality (see [10])

$$\Re(\varphi)_i = g^{kl} R_{ki}\ \varphi_l \tag{3}$$

where φ_l and R_{kl} denote the local components of $\varphi \in C^\infty S^1 M$ and the Ricci tensor Ric, respectively. In addition, g^{kl} are the local contravariant components of the metric tensor g. On the other hand, if $p = 2$, then we have the equality (see [23], p. 64; p. 356 and [10])

$$\Re\,(\varphi)_{ij} = -2g^{km} g^{lt} R_{mijt}\ \varphi_{kl} + g^{kl} R_{ki}\varphi_{lj} + g^{kl} R_{kj}\varphi_{li}, \tag{4}$$

where φ_{ij} and R_{ijkl} denote the local components of $\varphi \in C^\infty S^2 M$ and the Riemannian curvature tensor R, respectively. All local components of tensors in (3) and (4) are defined by the following identities: $\varphi_{ij} = \varphi\ (\ e_i,\ e_j)$, $R_{ijkl} = g_{im}\ R^m_{jkl}$ and $R_{kl} = R^i_{kil}$ where $R(e_j,\ e_l)\ e_k = R^i_{kjl}\ e_i$, $g_{im} = g(e_i,\ e_m)$ and $(g^{kl}) = (g_{kl})^{-1}$ for any frame $e_1 \ldots,\ e_n$ of $T_x M$ at an arbitrary point $x \in M$ and for any $i,\ j,\ k, \cdots = 1,\ 2, \ldots, n$.

Remark 4. *The Sampson operator Δ_S is a Laplacian. In particular, the kernel of Δ_S is a finite-dimensional vector space on a compact manifold (M, g). More information about the properties and applications of Δ_S can be found in the following list of articles: [10,11,26,27].*

In accordance with the general theory, we define two vector spaces (see [7], p. 104). First, by the condition

$$\mathbb{H}(S^p M) = \{\varphi \in C^\infty(S^p M) :\ \Delta_S \varphi = 0\}$$

we define the vector space of Δ_S*-harmonic symmetric p-tensors* $\varphi \in C^\infty(S^p M)$. Second, by the condition

$$L^q \mathbb{H}(S^p M) = \{\varphi \in \mathbb{H}(S^p) :\ \|\varphi\| \in L^q(M)\}$$

we define the vector space of Δ_S*-harmonic symmetric L^q-tensors* for some positive q. The norm of $\varphi \in C^\infty(S^p M)$ with respect to the Riemannian metric g is denoted by the symbol $\|\cdot\|$. Using these notations, we conclude from (1) and (2) that if $\varphi \in \mathbb{H}(S^p M)$, then $trace_g\ \varphi \in \mathbb{H}(S^{p-2} M)$. Moreover, we obtain the following.

Theorem 1. *Let (M, g) be a Hadamard manifold, then the are no nonzero Δ_S-harmonic symmetric tensors $\varphi \in C^\infty(S^2 M)$ such that $\|\varphi\|^2$ is a nonzero L^q-function for at least one $q \in (0,\ +\infty)$.*

Proof. We define the non-negative scalar function $f = \|\varphi\|$ for $\varphi \in \mathbb{H}(S^pM)$. In this case, from (1) we deduce the well-known *Bochner–Weitzenböck formula* (see also [10,27])

$$\frac{1}{2}\,\Delta\, f^2 = \|\nabla\varphi\|^2 - g(\Re(\varphi), \varphi) \tag{5}$$

for an arbitrary $\varphi \in \mathbb{H}(S^pM)$. In particular, if $p = 2$, then from (5) in accordance with [23], p. 436, we obtain the formula

$$g\,(\Re(\varphi), \varphi) = \sum_{i<j} sec\,(e_i,\ e_j)\,(\mu_i - \mu_j)^2, \tag{6}$$

where $e_1, e_2, \ldots, e_n$ is the orthonormal basis of T_xM at an arbitrary $x \in M$, such that $\varphi(e_i,\ e_j) = \delta_{ij}\mu_i$. For the sectional curvature $\sec(e_i, e_j)$ in the direction of the two-plane, $\sigma(x) = \operatorname{span}\{e_i, e_j\}$. From (6), we conclude that if the section curvature of (M, g) is nonpositive, then $g(\Re(\varphi), \varphi) \leq 0$ for any $\varphi \in S^2M$. Then, from (4), we conclude that $\Delta\, f^2 \geq 0$, where $f = \|\varphi\|$ for $\varphi \in \mathbb{H}(S^2M)$. Next, we can refer to Lemma 1. □

As an application of Theorem 1, consider the Ricci tensor of the $\mathcal{A}$-space. Recall that a Riemannian manifold (M, g) is called Einstein-like of type $\mathcal{A}$ if its Ricci tensor Ric is cyclically parallel, i.e., if $(\nabla_X Ric)(X, X) = 0$ for all $X \in TM$ (see [23], pp. 450–451). In particular, if (M, g) is a compact (without boundary) $\mathcal{A}$-space with nonpositive sectional curvature, then $\nabla Ric = 0$. If, in addition, there exists a point in M at which the sectional curvature of each two-plane is strictly negative, then (M, g) is Einstein, i.e., its Ricci tensor satisfies the condition $Ric = \lambda\, g$ for some constant λ (see [23], p. 451). As the same time, the following generalized theorem holds.

Theorem 2. *The Ricci tensor* Ric *of an n-dimensional $\mathcal{A}$-space (M, g) belongs to the vector space $\mathbb{H}(S^2M)$. However, if (M, g) is a Hadamard $\mathcal{A}$-space such that the square of the norm of its Ricci tensor is a L^q-function for at least one $q \in (0, \infty)$, then (M, g) is isometric to Euclidean space $\mathbb{R}^n$.*

Proof. Now let (M, g) be an n-dimensional $\mathcal{A}$-space then its Ricci tensor Ric satisfies the equations $\delta^* Ric = 0$ and has a constant trace, i.e., the scalar curvature $s = \operatorname{trace}_g Ric$ is constant. This also means that $\delta\ Ric = 0$. In this case, we conclude that $Ric \in \mathbb{H}(S^2M)$. Moreover, if (M, g) is a Hadamard manifold and $\|Ric\|^2 \in L^q(M)$ for at least one $q \in (0, +\infty)$, then $Ric \equiv 0$ by Theorem 1. In this case, from the conditions $Ric \equiv 0$ and $sec \leq 0$, we obtain $sec \equiv 0$; thus, (M, g) is a flat manifold. Again, (M, g) is simply connected, which implies that (M, g) is isometric to the Euclidean space $\mathbb{R}^n$. □

Let $S_0^p(T_x^*M) \subset S^p(T_x^*M)$ be a space of covariant symmetric p-tensors which are totally traceless, that is, traceless on any pair of indices at an arbitrary point $x \in M$. Then,

$$dim\ S_0^p(T_xM) = \binom{n+p-1}{p} - \binom{n+p-3}{n-1}.$$

The Sampson Laplacian Δ_S maps S_0^pM to itself for the bundle S_0^pM of traceless symmetric p-tensors on (M, g), i.e., $\Delta_S :\ C^\infty S_0^pM \to C^\infty S_0^pM$. This property is a corollary of the identities (1) and (2). Then, in particular, the following theorem holds.

Theorem 3. *The Hadamard manifold (M, g) does not admit a nonzero symmetric Δ_S-harmonic p-tensor $\varphi \in C^\infty(S_0^pM)$, such that the square of its norm is a L^q-function for at least one $q \in (0, \infty)$.*

Proof. Consider a nonzero symmetric Δ_S-harmonic p-tensor $\varphi \in C^\infty(S_0^pM)$ on a Hadamard manifold (M, g). Thus, (4) holds. In [28,29], it was proved that the inequality $g(\Re(\varphi_x), \varphi_x) \leq 0$ holds at any point x of a manifold (M, g) of nonpositive sectional curvature. Then, we deduce from (4) that $\Delta\|\varphi\|^2 \geq 0$ holds for a symmetric tensor $\varphi \in C^\infty(S_0^pM)$; thus, $\|\varphi\|^2$ is

a subharmonic function. Moreover, if $\|\varphi\|^2 \in L^q(M)$ is valid for at least one $q > 0$, then $\varphi \equiv 0$ by Lemma 1. □

4. Harmonic Self-Diffeomorphisms

The concept of harmonic mappings is an extension of one of the subharmonic functions. Therefore, it is natural to expect that Liouville-type theorems are also valid for harmonic mappings and, in particular, for harmonic mappings onto Hadamard manifolds (see [19,30,31]). We take a closer look at this topic below.

Suppose we are given a smooth C^∞-mapping $f\colon (M,g) \to (\overline{M},\overline{g})$ between connected Riemannian manifolds (M,g) and $(\overline{M},\overline{g})$. The norm of the differential $df = f_* : TM \to T\overline{M}$ determines the *energy density* of f, which is calculated by $e\,(f) = 1/2\; G(f_*,\; f_*)$ for the metric G determined by g and $\overline{g}$ on the bundle $T^*M \otimes f^*T\overline{M}$ with a fiber $T^*_xM \otimes T_{f(x)}\overline{M}$ over each point $x \in M$. A harmonic map $f:\; (M,g) \to (\overline{M},\overline{g})$ is defined using an extremum of the *energy functional* $E(f) = \int_M e(f)\, d\mathrm{vol}_g$ of f for any open set Ω in M with respect to compactly supported variations of f on Ω. A map $f:\; (M,g) \to (\overline{M},\overline{g})$ is harmonic if and only if it satisfies the Euler–Lagrange equation

$$\mathrm{trace}_g(Df_*) = 0 \tag{7}$$

(see [32]) for the connection D induced by the Levi–Civita connections ∇ and $\overline{\nabla}$ of metrics g and $\overline{g}$ on the bundle $T^*M \otimes f^*T\overline{M}$. Yau and Schoen proved the following celebrated theorem: *If (M,g) is a complete manifold with non-negative Ricci curvature and $(\overline{M},\overline{g})$ is a compact manifold with nonpositive sectional curvature, then an arbitrary harmonic map $f\colon (M,g) \to (\overline{M},\overline{g})$ with finite energy $E(f)$ is a constant map* (see [8], pp. 467–468).

Remark 5. *Since a complete (noncompact) manifold (M,g) of non-negative Ricci curvature has infinite volume (see [9]), a harmonic map onto a compact manifold, as suggested by Yau and Schoen's theorem, cannot be a diffeomorphism.*

In turn, we consider a connected smooth manifold M with two Riemannian metrics g and $\overline{g}$. If $f\colon (M,\; g) \to (M,\; \overline{g})$ is a harmonic self-diffeomorphism, then it is called a *harmonic transformation* of the manifold M. In this case, the Euler–Lagrange equation (7) takes the form $trace_g T = 0$ for the deformation tensor $T = \overline{\nabla} - \nabla$ of certain connections $\overline{\nabla}$ and ∇ (see [26]). Based on this result, we proved in [20] that $\mathrm{id}\colon (M,g) \to (M,\overline{g})$ is a harmonic self-diffeomorphism if and only if $\mathrm{div}\; \bar{g} = -1/2\, d(trace_g \overline{g})$. At the same time, it is known that $\mathrm{div}\; Ric = -1/2\, d(trace_g Ric)$ for the Ricci tensor Ric of a Riemannian manifold (M,g). Therefore, if (M,g) has negative Ricci curvature Ric, then $\mathrm{id}\colon (M,g) \to (M,\bar{g})$, where $\bar{g} = -Ric$ is a harmonic self-diffeomorphism (see also [33], p. 86). We can formulate the following statement on the first example of harmonic self-diffeomorphisms.

Corollary 4. *Let (M,g) be an n-dimensional $(n \geq 2)$ Hadamard manifold with negative Ricci curvature, then* $\mathrm{id}\colon (M,g) \to (M,-Ric)$ *is a harmonic self-diffeomorphism.*

Remark 6. *The assumption of Corollary 4 holds when (M,g) is a symmetric space of noncompact type of rank one. In this case, it is a Hadamard manifold of negative sectional and Ricci curvature (see [34]).*

The second example of a harmonic self-diffeomorphism $f\colon (M,g) \to (M,\overline{g})$ we constructed in [26] as a composition of conformal and projective transformations, under which the deformation tensor is $T = d\sigma \otimes \mathrm{id}_{TM} + \mathrm{id}_{TM} \otimes d\sigma - 2/n\; d\sigma^\# \otimes g$, where $d\sigma = (n^2+n-2)^{-1} d\ln(\det\; \overline{g}/det\; g)$ and $(\mathrm{d}\sigma^\#)$ is the vector field given by the identity $g(d\sigma^\#, X) = d\sigma(X)$ for all $X \in TM$. In turn, our first Liouville-type theorem of this section is as follows:

Theorem 4. *Let $(M, \bar{g})$ be an n-dimensional $(n \geq 2)$ Hadamard manifold and g be another complete Riemannian metric on M such that its Ricci tensor is non-negative. Then any harmonic transformation $f\colon (M, g) \to (M, \overline{g})$ is a constant map if its energy density $e(f)$ is a L^q-function for at least one $q \in (0, +\infty)$.*

Proof. Consider a simply connected smooth manifold M with two complete Riemannian metrics g and $\overline{g}$. Let $f\colon (M, g) \to (M, \overline{g})$ be a harmonic transformation, then a standard calculation gives (see also [32], p. 123):

$$\Delta e(f) = Q(f) + G(Df_*, Df_*), \tag{8}$$

where

$$Q(f) = g(Ric, f^*\overline{g}) \; - \; \mathrm{trace}_g\,(\mathrm{trace}_g(f^*\overline{R})).$$

for the Riemannian curvature tensor $\overline{R}$ of $(M, \overline{g})$ and the Ricci tensor Ric of (M, g). From (8) we conclude that $Q(f) \geq 0$ holds if the Ricci curvature of g is non-negative and the sectional curvature of $\overline{g}$ is nonpositive (see also [32]). In this case, from (8), we obtain that $\Delta e(f) \geq 0$ under the above curvature assumptions, hence $e(f)$ is a subharmonic function on (M, g). At the same time, recall that every non-negative subharmonic L^q-function for $q \in (0, +\infty)$ is constant on a complete Riemannian manifold (M, g) with non-negative Ricci curvature (see [6]). In turn, this constant must be equal to zero, since the volume of such manifold (M, g) is infinite (see [9]). To conclude the proof, we note that a simply connected manifold M with a complete Riemannian metric $\overline{g}$ of nonpositive sectional curvature is a Hadamard manifold. □

A vector field V on a complete Riemannian manifold (M, g) is called an *infinitesimal harmonic transformation* (see [26]) if V generates a flow which is a local one-parameter group of harmonic self-diffeomorphisms (in other words, harmonic diffeomorphisms of (M, g) onto itself). Analytic characteristic of such vector field V has the form $\mathrm{trace}_g(L_V \nabla) = 0$ for the Lie derivative L_V (see [26,27]). In addition, we have proved in [27] that a vector field V is an infinitesimal harmonic transformation if and only if

$$\Delta_S\, \varphi = 0, \tag{9}$$

where φ is the one-form defined by the identity $\varphi(X) = g(X, V)$ for all $X \in TM$. Then, according to (3) and (9), the *energy density function* $e(V) = 1/2\, \|V\|^2$ of an infinitesimal harmonic transformation V satisfies the equation (see also [26])

$$\Delta e(V) = \|\nabla V\|^2 - Ric(V, V). \tag{10}$$

In this case, using our Lemma 1, we can formulate the following:

Theorem 5. *An n-dimensional Hadamard manifold does not admit a nonzero infinitesimal harmonic transformation if its energy density function is an L^q-function for at least one $q \in (0, +\infty)$.*

Self-similar solutions to *Hamilton's Ricci flow* are called *Ricci solitons*; they play an important role in the study of singularities of the flow. For the past two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians (e.g., [17]). Namely, let M be a connected smooth manifold, then a triplet (g, V, λ) is a Ricci soliton if and only if g is a complete Riemannian metric and V is a smooth vector field, both defined on M, such that

$$Ric = \lambda\, g - \frac{1}{2}\, L_V g,$$

for some real constant $\lambda \in \mathbb{R}$. In turn, we proved (see [11,27]) that the vector field V of a Ricci soliton (g, V, λ) is an infinitesimal harmonic transformation.

The space of local solutions (g, V, λ) of (8) is quite large, but little is known about its global properties. In contrast, the following statement is an obvious consequence of Theorem 5.

Corollary 5. *The metric g of the Hadamard manifold (M, g) cannot be the metric of a Ricci soliton (g, V, λ), such that λ is an arbitrary real constant and the energy density function $e(V)$ is a L^q-function for at least one $q \in (0, +\infty)$.*

5. Isometric Self-Diffeomorphisms

Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$. In this case, a diffeomorphism $F\colon M \to M$ of a Riemannian manifold (M, g) onto itself is called an *isometric transformation* (or in other words an *isometric self-diffeomorphism*) if it preserves the distance d, i.e., $d(x, y) = d(F(x), F(y))$ for all $x, y \in M$ (see also [1], p. 202 and [35], p. 39). In addition, any isometric self-diffeomorphism of (M, g) preserves the metric tensor g, i.e. $Fg = g$. The converse is also true (see [35], p. 39). Therefore, an isometric transformation is an example of harmonic transformations.

Given an isometric self-diffeomorphism $F\colon M \to M$, the function $d_F(x, F(x))$ is called the *displacement function* of F (see [2]). Recall that the displacement function d_F of an isometric self-diffeomorphism $F : M \to M$ of a Hadamard manifold (M, g) is convex and its square d_F^2 is smooth and convex (see [2,5,36]). At the same time, by Corollary 2, a Hadamard manifold does not admit nonzero non-negative convex L^q-functions for some $q \in (0, +\infty)$. Therefore, the following theorem is true:

Theorem 6. *Let d_F be the displacement function of an isometric self-diffeomorphism $F\colon M \to M$ of a Hadamard manifold (M, g). If its square d_F^2 is a L^q-function for at least one $q \in (0, +\infty)$, then F is the identity.*

Recall that a vector field V on a Riemannian manifold (M, g) is an *infinitesimal isometry* if it generates a local one-parameter group of local isometric diffeomorphisms (M, g) onto itself. Moreover, a vector field V is an infinitesimal isometry if and only if $L_V g = 0$, where L_V denotes the Lie derivation with respect to V (see [35], p. 42). In this case, one can obtain (see also [35], p. 56)

$$(Hess_g\, e(V))\,(X, X) = \|\nabla_X V\|^2 - g(R(V, X)\, X, V) \tag{11}$$

for the *energy density function* $e(V) = 1/2\,\|V\|^2$ of infinitesimal isometry V and an arbitrary $X \in TM$. If, in addition, (M, g) is a Hadamard manifold, then $g(R(V, X)X, V) \leq 0$, hence $(Hess_g\, e(V))\,(X, X) \geq 0$ for any $X \in TM$. Therefore, $e(V)$ is a non-negative convex function. Based of Corollary 2, we formulate the following:

Corollary 6. *A Hadamard manifold (M, g) does not admit a nonzero infinitesimal isometry, such that its energy density function is a L^q-function for at least one $q \in (0, \infty)$.*

On the other hand, by [35], p. 56 from (11), we obtain that the Laplacian of energy density function $e(V) = 1/2\,\|V\|^2$ of an infinitesimal isometric V has the form (10).

Based of (10) and Lemma 1, we formulate the following.

Corollary 7. *A Hadamard manifold (M, g) does not admit a nonzero infinitesimal isometry, such that its energy density function is a L^q-function for at least one $q \in (0, +\infty)$.*

Remark 7. *For comparison with our theorem and corollary, we refer to Theorem 1 from [31] on the bounded isometry of a Hadamard manifold (M, g). In particular, from Theorem 1 we conclude that every bounded isometry of a Hadamard manifold (M, g) is trivial if the Euclidean factor in the de Rham decomposition of (M, g) is also trivial.*

6. Symmetric Killing Tensors and Killing Vectors

A tensor field $\varphi \in C^\infty(S^pM)$ satisfying the equation $\delta^*\varphi = 0$ is well-known in general relativity as a symmetric Killing p-tensor (see [37], p. 559 and [38]). It is a natural generalization of a *Killing vector* (see [35], pp. 42–43 and [37], p. 292). Killing vectors and Killing symmetric tensors have many applications. For example, while Killing vectors give linear first integrals of geodesic equations, Killing tensors give quadratic, cubic and higher-order first integrals on Riemannian manifolds.

Dairbekov and Sharafutdinov in [28] and Heil in [39], proved the following: *On a compact Riemannian manifold (M, g) with nonpositive sectional curvature, every Killing tensor is parallel. If there is a point in (M, g) at which the sectional curvature is negative on all two-dimensional planes, then it is proportional to a power of the metric.*

In this section, we consider symmetric Killing tensors on Hadamard manifolds and supplement the above result. Recall that the Ricci tensor Ric of an $\mathcal{A}$-space is a symmetric traceless Killing 2-tensor and, moreover, is a Δ_S-harmonic tensor. We consider a generalization of this concept to the case of a symmetric traceless Killing p-tensor for $p \geq 2$.

Let a smooth symmetric Killing p-tensor φ be traceless, then from the equation $\delta^*\varphi = 0$ we obtain $\delta\ \varphi = 0$. In this case, $\varphi \in \mathbb{H}(S^pM)$ holds. Therefore, if $\varphi \in C^\infty(S^pM)$ is a divergence-free symmetric Killing tensor on a Riemannian manifold (M, g), then it satisfies the following systems of differential equations: $\Delta_S\ \varphi = 0$ and $\delta\ \varphi = 0$. Conversely, if (M, g) is compact and $\varphi \in C^\infty(S^pM)$ satisfies the equations $\Delta_S\ \varphi = 0$ and $\delta\ \varphi = 0$, then φ is a divergence-free symmetric Killing tensor (see [27]). This theorem is a natural generalization of the classical theorem on Killing vectors (see [35], p. 44 and [14], p. 44). The following theorem is also valid.

Theorem 7. *A Hadamard manifold (M, g) does not admit a nonzero symmetric Killing p-tensor $\varphi \in C^\infty(S^p_0M)$, such that the square of its norm belongs to $L^q(M)$ for at least one $q \in (0, +\infty)$.*

Killing vectors are a classical object of Riemannian geometry (e.g., [1], pp. 313–332). Recall that a smooth vector field Von a Riemannian manifold (M, g) is a Killing vector field if the Lie derivative of the metric tensor g with respect to V is zero, i.e., $L_V g = \delta^*\varphi = 0$ for $\varphi = g\ (V, \cdot\ )$. Therefore, if V is a Killing vector field, then it satisfies the following differential equations: $\Delta_S\varphi = 0$ and $\delta\ \varphi = 0$ (see [10]). Conversely, if (M, g) is compact and V satisfies the above equations, then V is a Killing vector. Thus, V is an example of infinitesimal harmonic transformations. In this case, the following is valid.

Corollary 8. *A Hadamard manifold does not admit a nonzero Killing tensor if its norm is an L^q-function for at least one $q \in (1, +\infty)$.*

Remark 8. *Corollary 8 generalizes the classical Bochner theorem on Killing vector fields on compact manifolds (see [1], p. 313 and [14], p. 44) and supplements the following theorem: If (M, g) is a complete Riemannian manifold with nonpositive Ricci curvature, then every Killing vector on (M, g) with finite global norm is parallel (see [40]).*

7. Killing–Yano Tensors on Riemannian Globally Symmetric Spaces of Noncompact Type

The concept of *Killing–Yano tensors* was introduced into physics by Penrose et al. (see [41,42], etc.), and it played an important role in the development of general relativity (e.g., [37], pp. 559–563). On the other hand, these tensors were introduced in differential geometry thanks to K. Yano (see [14], p. 68 and [43]) and were fruitfully studied for a long time in the geometry of Riemannian manifolds (e.g., [25,28,29,44]). In particular, in [44], it was proved that a compact simply connected symmetric space carries a nonparallel Killing–Yano p-tensor $(p \geq 2)$ if and only if it isometric to the Riemannian product $S^k \times N$, where S^k is a round sphere and $k > p$. In turn, here, we give some applications of results

on subharmonic functions to the geometry of Killing–Yano p-tensors on symmetric spaces of noncompact type.

Let (M, g) be an n-dimensional Riemannian manifold. Then, the following equality holds:

$$\dim \Lambda^p(T_x^* M) = \binom{n}{p}$$

for the vector space $\Lambda^p(T_x^* M)$ of covariant skew-symmetric p-tensors ($1 \leq p \leq n-1$) on $T_x M$ at an arbitrary point $x \in M$. A Killing–Yano p-tensor (or, Killing p-form) on (M, g) is a skew-symmetric tensor, whose covariant derivative is totally skew-symmetric, i.e., by definition, if $\omega \in C^\infty(\Lambda^p M)$ is a Killing–Yano tensor, then $\nabla\omega \in C^\infty(\Lambda^{p+1} M)$.

Let d: $C^\infty(\Lambda^p M) \to C^\infty(\Lambda^{p+1} M)$ be the operator of exterior derivative and δ : $C^\infty(\Lambda^p M) \to C^\infty(\Lambda^{p-1} M)$ be the codifferentiation operator, defined as the canonical formal adjoint of d (see [1], pp. 334–335). Using these operators, one constructs the well-known Hodge–de Rham Laplacian $\Delta_H = \delta\, d + d\, \delta$, which admits a Weitzenböck decomposition (see [1], p. 347; [23], pp. 77–79 and [25])

$$\Delta_H\, \omega = \overline{\Delta}\, \omega + \Re(\omega) \tag{12}$$

for any $\omega \in C^\infty(\Lambda^p M)$, and an algebraic symmetric operator $\Re$: $\Lambda^p M \to \Lambda^p M$, i.e., the Weitzenböck curvature operator of the Lichnerowicz Laplacian, which is restricted to skew-symmetric p-tensors.

We define a non-negative scalar function by the equality $f = \|\omega\|$. Then, using (12), we write the well-known Bochner–Weizenböck formula (see also [25])

$$\frac{1}{2}\, \Delta\, f^2 = -g\, (\Delta_H \omega, \omega) + \|\nabla\omega\,\|^2 + g\, (\Re(\omega), \omega). \tag{13}$$

An arbitrary Killing–Yano tensor $\omega \in C^\infty(\Lambda^p M)$ satisfies the equation (see [25])

$$\Delta_H\, \omega = \frac{p+1}{p}\, \Re\, (\omega). \tag{14}$$

In this case, from (13) and (14), we obtain the inequality

$$\frac{1}{2}\, \Delta f^2 \geq -\frac{1}{p}\, g(\Re\, (\omega), \omega). \tag{15}$$

Recall that the Riemannian curvature tensor R of (M, g) defines a symmetric algebraic operator $\overline{R} : \Lambda^2(T_x M) \to \Lambda^2(T_x M)$ on the vector space $\Lambda^2(T_x M)$ of 2-forms over tangent space $T_x M$ at an arbitrary point $x \in M$ (see [1], pp. 82–83 and [23], p. 51). There are many papers on the relationship between the behavior of the curvature operator $\overline{R}$ of a Riemannian manifold (M, g) and some of its global characteristics, such as its homotopy type and topological type. We say that a manifold (M, g) has a *nonpositive curvature operator* $\overline{R}$ if the quadratic form $g(\overline{R}(\theta), \theta) \leq 0$ for all nonzero two-forms $\theta \neq 0$. At the same time, it can be also concluded here that if the curvature operator $\overline{R}$ is nonpositive then the quadratic form $g(\Re(\omega), \omega) \leq 0$ for any $\omega \in \Lambda^p M$ by the formulas from [1], pp. 345–346. Moreover, a Riemannian symmetric space has a nonpositive curvature operator $\overline{R}$ if and only if it has nonpositive sectional curvature (see [22]). In this case, we deduce from (15) that

$$\Delta\, f^2 \geq 0$$

for $f = \|\omega\|$. Hence $f^2 = \|\omega\|^2$ is a subharmonic function. At the same time, it is known that a Riemannian globally symmetric space of noncompact type does not admit a nonzero non-negative subharmonic L^q-function for $q \in (0, \infty)$. Therefore, we formulate the following.

Theorem 8. *An n-dimensional Riemannian globally symmetric space of noncompact type does not admit a nonzero Killing–Yano p-tensor* $(1 \leq p \leq n-1)$ *such that the square of its norm is an* L^q*-function for at least one* $q \in (0, +\infty)$.

For any p with $0 \leq p \leq n$, they define the Hodge operator $*$ to be the unique vector-bundle isomorphism (see [23], p. 33)

$$* : \Lambda^p M \to \Lambda^{n-p} M$$

such that $*^2 = (-1)^{p\,(n-p)}$ and $\omega \wedge (*\omega') = g(\omega, \omega')\, d\mathrm{vol}_g$ for any $\omega,\ \omega' \in \Lambda^p M$ and the volume form $d\mathrm{vol}_g$ of (M, g). Moreover, a p-tensor ω for an arbitrary Killing–Yano $(n-p)$-tensor ω is called *closed conformal Killing–Yano p-tensor* or, *closed conformal Killing p-form* (see [18,45]). In particular, for any closed conformal Killing–Yano tensor $\omega \in C^\infty(\Lambda^p M)$ we have (see [45])

$$\Delta_H\, \omega = \frac{n-p+1}{n-p}\, \Re\,(\omega). \tag{16}$$

By a direct calculation based on (13) and (16), we obtain the inequality

$$\frac{1}{2}\, \Delta\, f^2 \geq -\frac{1}{n-p}\, g\,(\Re\,(\omega), \omega)$$

for a closed conformal Killing–Yano p-tensor ω.

Then, arguments similar to those carried out above for the Killing–Yano p-tensor allow us to formulate the following.

Theorem 9. *An n-dimensional Riemannian globally symmetric space of noncompact type does not admit a nonzero closed conformal Killing–Yano p-tensor* $(1 \leq p \leq n-1)$*, such that the square of its norm is an* L^q*-function for at least one* $q \in (0, +\infty)$.

Remark 9. *The results of this section generalize the classical theorems on Killing–Yano and closed conformal Killing–Yano tensors on compact manifolds (see [14], p. 68–70 and [45]) and supplement results from [18] and also results from [12], where* $q > 1$ *in the corresponding theorems.*

8. Conclusions

In our article, we supplemented the information about Hadamard manifolds using a generalized version of the Bochner technique or, in other words, using modern methods of geometric analysis. Most of our theorems have analogues in the scientific literature, but unlike our results, they are obtained by the methods of the classical Bochner technique or by other methods of geometry in the large under more stringent restrictions, for example, for compact manifolds. This is the importance and difference of our article from other scientific works on Hadamard manifolds.

Author Contributions: Investigation, J.M., V.R. and S.S. All authors have contributed equally in writting the article. All authors have read and agreed to the published version of the manuscript.

Funding: The first author was supported by the Internal Grant Agency of the Faculty of Science of Palacky University, Olomouc (grant No. 2022017 "Mathematical Structures").

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Petersen, P. *Riemannian Geometry*; Springer: Berlin/Heidelberg, Germany, 2016.
2. Kiyoshi, S. Hadamard manifolds, Geometry of geodesics and related topics. *Adv. Stud. Pure Math.* **1984**, *3*, 239–281.
3. Bertrand, J.; Sandeep, K. Sharp Green's function estimates on Hadamard manifolds and Adams inequality. *Int. Math. Res. Notices* **2021**, *6*, 4729–4767. [CrossRef]
4. Sullivan, D. The Dirichlet problem at infinity for a negatively curved manifold. *J. Diff. Geom.* **1983**, *18*, 723–732. [CrossRef]
5. Bishop, R.L.; O'Neil, B. Manifolds of negative curvature. *Trans. Am. Math. Soc.* **1969**, *145*, 1–49. [CrossRef]
6. Li, P.; Schoen, R. L^p and mean value properties of subharmonic functions on Riemannian manifolds. *Acta Math.* **1984**, *153*, 279–301. [CrossRef]
7. Pigola, S.; Rigoli, M.; Setti, A.G. *Vanishing and Finiteness Results in Geometric Analysis. A Generalization of the Bochner Technique*; Birkhäuser: Basel, Switzerland, 2008.
8. Wu, H.-H. The Bochner technique in differential geometry. *Math. Rep.* **1988**, *3*, 289–538. [CrossRef]
9. Yau, S.T. Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry. *Indiana Math. J.* **1976**, *25*, 659–670. [CrossRef]
10. Mikeš, J.; Rovenski, V.; Stepanov, S.E. An example of Lichnerowicz-type Laplacian. *Ann. Global Anal. Geom.* **2020**, *58*, 19–34. [CrossRef]
11. Rovenski, V.; Stepanov, S.; Tsyganok, I. On evolution equations under the Hamilton's Ricci flow. *Results Math.* **2020**, *75*, 166. [CrossRef]
12. Mikeš, J.; Stepanov, S.E. What is the Bochner technique and where is it applied. *Lobachevskii J. Math.* **2022**, *43*, 709–719.
13. Bérard, P.H. From vanishing theorems to estimating theorems: The Bochner technique revisited. *Bull. Am. Math. Soc.* **1988**, *19*, 371–406. [CrossRef]
14. Yano, K. *Integral Formulas in Riemannian Geometry*; Marcel Dekker Inc.: New York, NY, USA, 1970.
15. Huber, A. On subharmonic functions and differential geometry in the large. *Comment. Math. Helv.* **1957**, *32*, 13–72. [CrossRef]
16. Karp, L. Subharmonic functions on real and complex manifolds. *Math. Z.* **1982**, *179*, 535–554. [CrossRef]
17. Morgan, J.; Tian, G. *Ricci Flow and the Poincaré Conjecture*; AMS, Clay Mathematical Institute: Cambridge, UK, 2007.
18. Stepanov, S.E.; Tsyganok, I.I. Conformal Killing forms on complete Riemannian manifolds with non-positive curvature operator. *J. Math. Anal. Appl.* **2018**, *458*, 1–8. [CrossRef]
19. Shen, Y. A Liouville theorem for harmonic maps. *Amer. J. Math.* **1995**, *117*, 773–785. [CrossRef]
20. Stepanov, S.E.; Shandra, I.G. Harmonic diffeomorphisms of manifolds. *St. Petersburg Math. J.* **2005**, *16*, 401–412. [CrossRef]
21. Wolf, J.A. *Spaces of Constant Curvature*; AMS: Cupertino, CA, USA, 2011.
22. Duchesne, B. Infinite dimensional Riemannian symmetric spaces with fixed-sing curvature operator. *Ann. Inst. Fourier Grenoble* **2015**, *65*, 211–244. [CrossRef]
23. Besse, A.L. *Einstein Manifolds*; Springer: Berlin, Germany, 1987.
24. Sampson, J.H. On a theorem of Chern. *Trans. AMS* **1973**, *177*, 141–153. [CrossRef]
25. Lichnerowicz, A. Propaganteurs et commutateurs et relativite generale. *Inst. Hautes Etudes Sci. Publ. Math.* **1961**, *10*, 293–344. [CrossRef]
26. Stepanov, S.E.; Shandra, I.G. Geometry of infinitesimal harmonic transformations. *Ann. Glob. Anal. Geom.* **2003**, *24*, 291–299. [CrossRef]
27. Stepanov, S.E.; Mikeš, J. On the Sampson Laplacian. *Filomat* **2019**, *33*, 1059–1070. [CrossRef]
28. Dolan, P.; Kladouchou, A.; Card, C. On the significance of Killing tensors. *Gen. Relativ. Gravit.* **1989**, *21*, 427–437. [CrossRef]
29. Heil, K. Killing and Conformal Killing Tensors. Ph.D. Thesis, Institute für Geometrie und Topologie der Universität Stugart, Stuttgart, Germany, 2017.
30. Tachikawa, A. Harmonic mappings from into a Hadamard manifold. *J. Math. Soc. Japan* **1990**, *42*, 147–153. [CrossRef]
31. Wolf, J.A. Homogeneity and bounded isometries in manifolds of negative curvature. *Illinois J. Math.* **1964**, *8*, 14–18. [CrossRef]
32. Eells, J.Jr.; Sampson, J.H. Harmonic mappings of Riemannian manifolds. *Am. J. Math.* **1964**, *86*, 109–160. [CrossRef]
33. Show, B.; Knopf, D. *The Ricci Flow: An Introduction*; AMS: Cupertino, CA, USA, 2004.
34. Heintze, E. On homogeneous manifolds of negative curvature. *Math. Ann.* **1974**, *211*, 23–34. [CrossRef]
35. Kobayashi, S. *Transformation Groups in Differential Geometry*; Springer: New York, NY, USA, 1995.
36. Wu, H. A remark on the Bochner technique in differential geometry. *Proc. AMS* **1980**, *78*, 403–408. [CrossRef]
37. Stephani, H.; Kramer, D.; MacCallum, M.; Hoenselaers, C.; Herlt, E. *Exact Solutions of Einstein's Field Equations*; Cambridge University Press: Cambridge, UK, 2003.
38. Collinson, C.D.; Howarth, L. Generalized Killing tensors. *Gen. Relativ. Gravit.* **2000**, *32*, 1767–1776. CrossRef]
39. Dairbekov, N.S.; Sharafutdinov, V.A. Conformal Killing symmetric tensor fields on Riemannian manifolds. *Mat. Tr.* **2010**, *13*, 85–145. [CrossRef]
40. Heil, K.; Moroianu, A.; Semmelmann, U. Killing and conformal Killing tensors. *J. Geom. Phys.* **2016**, *106*, 383–400. [CrossRef]
41. Yorozu, S. Killing vector fields on complete Riemannian manifolds. *Proc. AMS* **1982**, *84*, 115–120. [CrossRef]
42. Penrose, R. et al. On a quadratic first integral for the charged particle orbits in the charged Kerr solution. *Commun. Math. Phys.* **1972**, *27*, 303–308.
43. Penrose, R. Naked singularities. *Ann. N. Y. Acad. Sci.* **1973**, *224*, 125–134. [CrossRef]

44. Yano, K. Some remarks on tensor fields and curvature. *Ann. Math.* **1952**, *55*, 328–347. [CrossRef]
45. Tachibana, S.; Yamaguchi, S. The first proper space of for p-forms in compact Riemannian manifolds of positive curvature operator. *J. Diff. Geom* **1980**, *15*, 51–60. [CrossRef]