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Generalized Lorentzian Sasakian-Space-Forms with $\mathcal{M}$-Projective Curvature Tensor

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Abstract: In this note, the generalized Lorentzian Sasakian-space-form $M_1^{2n+1}(f_1, f_2, f_3)$ satisfying certain constraints on the $\mathcal{M}$-projective curvature tensor $\mathcal{W}^*$ is considered. Here, we characterize the structure $M_1^{2n+1}(f_1, f_2, f_3)$ when it is, respectively, $\mathcal{M}$-projectively flat, $\mathcal{M}$-projectively semisymmetric, $\mathcal{M}$-projectively pseudosymmetric, and $\varphi - \mathcal{M}$-projectively semisymmetric. Moreover, $M_1^{2n+1}(f_1, f_2, f_3)$ satisfies the conditions $\mathcal{W}^*(\zeta, V_1) \cdot S = 0$, $\mathcal{W}^*(\zeta, V_1) \cdot R = 0$ and $\mathcal{W}^*(\zeta, V_1) \cdot \mathcal{W}^* = 0$ are also examined. Finally, illustrative examples are given for obtained results.

Keywords: contact Lorentzian manifold; generalized Lorentzian Sasakian-space-form; $\mathcal{M}$-projective curvature tensor

MSC: 53D10; 53C50; 53C15

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1. Introduction

Contact Riemannian structures are widely famous and extensively researched in differential geometry, and they have several applications in other fields. The contact Lorentzian structure case (ν, g) has g as a Lorentzian metric and ν is a contact 1-form associated with it. Moreover, it has an exclusive physics relevance [1,2]. Calvaruso and Perrone [3] conducted a thorough investigation of contact pseudo-metric structures. Recently, Calvaruso [4] focused the study on the pertinent contact Lorentzian structures case and discovered some technical apparatus required for future inquiry.

The manifold sectional curvatures assist us in fully determining the curvature tensor R. A real-space-form is a Riemannian manifold with constant sectional curvature d and a curvature tensor that satisfies

$$R(V_1, V_2)V_3 = d\{-g(V_1, V_3)V_2 + g(V_2, V_3)V_1\}.$$

The Euclidean spaces $(d = 0)$, hyperbolic spaces $(d < 0)$, and spheres $(d > 0)$ are models for these spaces.

In contact geometry, for a given almost contact metric manifold M, a φ-section of M at $p \in M$ is a section $\pi \subseteq T_pM$ spanned by a unit vector V_p orthogonal to characteristic vector field ξ_p, and φV_p. The φ-sectional curvature of π is defined by $K(\pi) = R(V_1, \varphi V_1, \varphi V_1, V_1)$ for any vector field $V_1 \in TM$. We recall that Sasakian manifold is a Sasakian-space-form if it has constant φ-sectional curvature. In this context, it is worth mentioning that Alegre et al. [5] proposed and investigated the concept of generalized Sasakian-space-form as an essentially contact metric manifold fitting a similar equation with constant values substituted by functions which are differentiable. That is, a generic Sasakian-space-form is an essentially contact metric manifold in which R is connected to triple smooth functions f_1, f_2, and f_3 defined on the manifold. They also presented different examples of such

spaces. A systematic study of general Sasakian-space-forms concerned with their features and of their related curvature tensors was investigated by many authors (see, [6–9]) and the references therein.

An indefinite Sasakian manifold with constant φ-sectional curvature d is termed as an indefinite Sasakian-space-form. In such a case, its Riemannian curvature tensor is of the form [10,11]:

$$\begin{aligned} R(V_1,V_2)V_3 &= \frac{d+3\epsilon}{4}[-g(V_1,V_3)V_2+g(V_2,V_3)V_1] \\ &+ \frac{d-\epsilon}{4}[g(V_1,\varphi V_3)\varphi V_2-g(V_2,\varphi V_3)\varphi V_1+2g(V_1,\varphi V_2)\varphi V_3] \\ &+ \frac{d-\epsilon}{4}[\epsilon\nu(V_1)\nu(V_3)V_2-\epsilon\nu(V_2)\nu(V_3)V_1-g(V_2,V_3)\nu(V_1)\zeta+g(V_1,V_3)\nu(V_2)\zeta]. \end{aligned}$$

In [12], Alegre and Carriazo described a generic indefinite Sasakian-space-form as an indefinite almost contact metric manifold, obeying a comparable equation with differentiable functions in place of constant values. As a result, we have

$$\begin{aligned} R(V_1,V_2)V_3 &= f_1[-g(V_1,V_3)V_2+g(V_2,V_3)V_1] \\ &+ f_2[g(V_1,\varphi V_3)\varphi V_2-g(V_2,\varphi V_3)\varphi V_1+2g(V_1,\varphi V_2)\varphi V_3] \\ &+ f_3[\epsilon\nu(V_1)\nu(V_3)V_2-\epsilon\nu(V_2)\nu(V_3)V_1+g(V_1,V_3)\nu(V_2)\zeta \\ &- g(V_2,V_3)\nu(V_1)\zeta]. \end{aligned} \tag{1}$$

A smooth manifold equipped with a Lorentzian metric is called a Lorentzian manifold. Afterwards, Riemannian manifolds and Lorentzian manifolds form the most important subclass of pseudo-Riemannian manifolds, and they are especially crucial in general relativity and cosmology applications. In [12], authors were focused on Lorentzian case, along with certain examples of them. In this case: $\epsilon = -1$ and the index of the metric is one. We call such manifold the generalized Lorentzian Sasakian-space-form (briefly, GLSSF), denoted by $M_1^{2n+1}(f_1,f_2,f_3)$.

On the other hand, the $\mathcal{M}$-projective curvature tensor is important curvature tensor from the differential geometric point of view. This curvature tensor bridges the gap between conformal, conharmonic, and concircular curvature tensors on one side and $\mathcal{H}$-projective curvature tensor on the other. The properties of the $\mathcal{M}$-projective curvature tensor have recently come popular and has been studied by many geometers. In particular, studies on contact manifolds with $\mathcal{M}$-projective curvature tensor have contributed significantly to the literature. In [13], the authors proposed the notion of $\mathcal{M}$-projective curvature tensor, and features of this tensor on Lorentzian manifolds and on generalized Sasakian-space-forms were studied earlier in the papers [14–16], respectively.

As a continuation of this study, here we are interested in GLSSF $M_1^{2n+1}(f_1,f_2,f_3)$ satisfying certain conditions on the $\mathcal{M}$-projective curvature tensor. The paper is structured as follow: Section 2 discusses some early findings. In Section 3, we characterize $\mathcal{M}$–projectively flat, $\mathcal{M}$-projectively semisymmetric, $\mathcal{M}$-projectively pseudosymmetric, and $\varphi-\mathcal{M}$-projectively semisymmetric $M_1^{2n+1}(f_1,f_2,f_3)$. Here, we attain necessary and sufficient constraints for $M_1^{2n+1}(f_1,f_2,f_3)$ to be $\mathcal{M}$-projectively flat, $\mathcal{M}$-projectively semisymmetric, and $\varphi-\mathcal{M}$-projectively semisymmetric. In Section 4, we obtain interesting results on $M_1^{2n+1}(f_1,f_2,f_3)$ satisfying the conditions $\mathcal{W}^*(\zeta,V_1)\cdot S=0$, $\mathcal{W}^*(\zeta,V_1)\cdot R=0$ and $\mathcal{W}^*(\zeta,V_1)\cdot\mathcal{W}^*=0$. Finally, some examples are provided to support our findings.

2. Preliminaries

An almost contact structure (φ,ζ,ν) on M^{2n+1} ($\dim M=2n+1$) is framed by a tensor field φ of kind (1,1), a global vector field ζ, and a 1-form ν fulfilling

$$(i)\quad \varphi(\zeta)=0,\quad \nu\circ\varphi=0, \tag{2}$$

$$(ii)\quad \nu(\zeta)=1,\quad \varphi^2=-Id+\nu\otimes\zeta, \tag{3}$$

and rank $(\varphi) = 2n$. Let us define a metric g on M^{2n+1}; this is said to be compatible with the structure (φ, ζ, ν) if

$$g(V_1, V_2) = g(\varphi V_1, \varphi V_2) - \nu(V_1)\nu(V_2). \tag{4}$$

A smooth manifold M^{2n+1} equipped with the structure (φ, ζ, ν) and g is called an almost contact Lorentzian manifold.

It is noticed that, from (2) and (3), $\nu(V_1) = -g(V_1, \zeta)$. In particular, $g(\zeta, \zeta) = -1$, thus ζ is time-like. Furthermore, (4) implies $g(\varphi V_1, V_2) = -g(V_1, \varphi V_2)$.

Next, if g satisfies

$$g(V_1, \varphi V_2) = (d\nu)(V_1, V_2),$$

then ν is a contact form on M^{2n+1}, ζ the associated Reeb vector field, g an associated metric, and $(M^{2n+1}, \varphi, \zeta, \nu, g)$ is called a contact Lorentzian manifold. If, in addition, ζ is a Killing vector field, then M^{2n+1} is said to be a K-contact Lorentzian manifold. Additionally, the almost contact Lorentzian manifold M^{2n+1} is called normal if $[\varphi, \varphi] + 2d\nu \otimes \zeta = 0$, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ. A normal contact Lorentzian manifold is called Lorentzian Sasakian manifold.

In addition to (1), for a $(2n + 1)$-dimensional GLSSF $M_1^{2n+1}(f_1, f_2, f_3)$ $(n > 1)$, the following relations hold:

$$\begin{aligned}
S(V_1, V_2) &= (2nf_1 - f_3 + 3f_2)g(V_1, V_2) + [(2n-1)f_3 + 3f_2]\nu(V_1)\nu(V_2), && (5)\\
QV_1 &= (2nf_1 - f_3 + 3f_2)V_1 - [(2n-1)f_3 + 3f_2]\nu(V_1)\zeta, && (6)\\
\nu(R(V_1, V_2)V_3) &= -(f_3 - f_1)[g(V_2, V_3)\nu(V_1) - g(V_1, V_3)\nu(V_2)], && (7)\\
R(V_1, V_2)\zeta &= -(f_3 - f_1)(\nu(V_1)V_2 - \nu(V_2)V_1), && (8)\\
R(\zeta, V_1)V_2 &= -(f_3 - f_1)[g(V_1, V_2)\zeta + \nu(V_2)V_1], && (9)\\
r &= -4nf_3 + 6nf_2 + 2n(2n+1)f_1, && (10)
\end{aligned}$$

where r, Q, S, and R respectively denote the scalar curvature, the Ricci operator (related by $S(V_1, V_2) = g(QV_1, V_2)$), the Ricci tensor of type (0, 2), and (1, 3) type curvature tensor.

The $\mathcal{M}$-projective curvature tensor $\mathcal{W}^*$ on a $(2n + 1)$-dimensional GLSSF M_1^{2n+1} (f_1, f_2, f_3) is defined by

$$\begin{aligned}
\mathcal{W}^*(V_1, V_2)V_3 &= R(V_1, V_2)V_3 - \frac{1}{4n}[S(V_2, V_3)V_1 - S(V_1, V_3)V_2\\
&+ g(V_2, V_3)QV_1 - g(V_1, V_3)QV_2]. && (11)
\end{aligned}$$

In view of (5)–(9), one can easily obtain

$$\begin{aligned}
\nu(\mathcal{W}^*(V_1, V_2)V_3) &= g(\mathcal{W}^*(V_1, V_2)\zeta, V_3)\\
&= \frac{[(2n-1)f_3 + 3f_2]}{4n}[g(V_1, V_3)\nu(V_2) - g(V_2, V_3)\nu(V_1)], && (12)\\
\mathcal{W}^*(\zeta, V_2)V_3 &= \frac{-[(2n-1)f_3 + 3f_2]}{4n}(g(V_2, V_3)\zeta + \nu(V_3)V_2), && (13)
\end{aligned}$$

for all vector fields V_1, V_2, V_3 on $M_1^{2n+1}(f_1, f_2, f_3)$.

3. $\mathcal{M}$-Projectively Flat, $\mathcal{M}$-Projectively Semisymmetric, $\mathcal{M}$-Projectively Pseudosymmetric and $\varphi - \mathcal{M}$-Projectively Semisymmetric $M_1^{2n+1}(f_1, f_2, f_3)$

In pseudo-Riemannian geometry, many authors examined the geometry of several kinds of pseudo-Riemannian and Riemannian manifolds with different curvature tensors via flatness and symmetries. A pseudo-Riemannian manifold is said to be flat if its R is always zero and locally symmetric if R is parallel (that is, $\nabla R = 0$), where ∇ stands for the

Levi-Civita connection. The concept of semisymmetric manifolds was introduced as an appropriate extension of locally symmetric manifolds and is as follows

$$R(V_1, V_2) \cdot R = 0,$$

for any V_1, V_2 on M, where $R(V_1, V_2)$ acts on R as a derivation [17]. Every symmetric space is semisymmetric, although generally speaking, the opposite is not true. The notion of semisymmetry has been studied in [18–20] and many authors. A complete intrinsic classification of these spaces was initiated by Szabo [21].

Next, for T: a $(0,k)$-tensor field on M, $k \geq 1$; and A: a symmetric (0,2)-tensor field on M, we describe $R \cdot T$ and $Q(A,T)$: the $(0,k+2)$-tensor fields, respectively, by

$$(R \cdot T)(V_1, ..., V_k; W_1, W_2) = -T(R(W_1, W_2)V_1, V_2, ..., V_k) - ... - T(V_1, ..., V_{k-1}, R(W_1, W_2)V_k)$$

and

$$\begin{aligned} Q(A,T)(V_1, ..., V_K; W_1, W_2) &= -T((W_1 \wedge_A W_2)V_1, V_2, ..., V_k) - ..... \\ &\quad -T(V_1, ..., V_{k-1}, (W_1 \wedge_A W_2)V_k), \end{aligned}$$

where $V_1 \wedge_A V_2$ is the endomorphism given by

$$(V_1 \wedge_A V_2)V_3 = A(V_2, V_3)V_1 - A(V_1, V_3)V_2. \tag{14}$$

A pseudo-Riemannian manifold M is named pseudosymmetric (in the sense of Deszcz [22]) if

$$R \cdot R = L_R Q(g, R)$$

holds on $U_R = \left\{ p \in M : R - \frac{r}{n(n-1)}\mathcal{G} \neq 0 \ \ at \ \ p \right\}$, where $\mathcal{G}$ is the (0,4)-tensor defined by $\mathcal{G}(V_1, V_2, V_3, V_4) = g((V_1 \wedge V_2)V_3, V_4)$ and L_R is some smooth function on U_R.

With this background, in this section we study GLSSF $M_1^{2n+1}(f_1, f_2, f_3)$ satisfying flatness and symmetry conditions on the $\mathcal{M}$-projective curvature tensor $\mathcal{W}^*$. Here, case by case, we characterize, $\mathcal{M}$-projectively flatness, $\mathcal{M}$-projectively semisymmetry, $\mathcal{M}$-projectively pseudosymmetry and $\varphi - \mathcal{M}$-projectively semisymmetry conditions on $M_1^{2n+1}(f_1, f_2, f_3)$. We begin with the following.

3.1. $\mathcal{M}$-Projectively Flat

Definition 1. *A GLSSF $M_1^{2n+1}(f_1, f_2, f_3)$ is called $\mathcal{M}$-projectively flat if the condition $\mathcal{W}^* = 0$ holds on $M_1^{2n+1}(f_1, f_2, f_3)$.*

Let us suppose that $M_1^{2n+1}(f_1, f_2, f_3)$ is a $\mathcal{M}$-projectively flat GLSSF, i.e., $\mathcal{W}^*(V_1, V_2)V_3 = 0$, therefore, from (11) we have

$$R(V_1, V_2)V_3 = \frac{1}{4n}[S(V_2, V_3)V_1 - S(V_1, V_3)V_2 + g(V_2, V_3)QV_1 - g(V_1, V_3)QV_2]. \tag{15}$$

In view of (5) and (6), (15) takes the form

$$\begin{aligned} R(V_1, V_2)V_3 &= \frac{1}{4n}[2(2nf_1 + 3f_2 - f_3)(g(V_2, V_3)V_1 - g(V_1, V_3)V_2) \\ &+ (3f_2 + (2n-1))f_3)(\nu(V_2)\nu(V_3)V_1 - \eta(V_1)\nu(V_3)V_2 \\ &- g(V_2, V_3)\nu(V_1)\zeta + g(V_1, V_3)\nu(V_2)\zeta)]. \end{aligned} \tag{16}$$

Using (1), the Equation (16) reduces to

$$\begin{aligned} & f_1[g(V_2,V_3)V_1 - g(V_1,V_3)V_2] + f_2[g(V_1,\varphi V_3)\varphi V_2 - g(V_2,\varphi V_3)\varphi V_1 + 2g(V_1,\varphi V_2)\varphi V_3] \\ - \; & f_3[\nu(V_1)\nu(V_3)V_2 - \nu(V_2)\nu(V_3)V_1 - g(V_1,V_3)\nu(V_2)\zeta + g(V_2,V_3)\nu(V_1)\zeta] \\ = \; & \frac{1}{4n}[2(2nf_1 + 3f_2 - f_3)(g(V_2,V_3)V_1 - g(V_1,V_3)V_2) \\ + \; & (3f_2 + (2n-1)f_3)(\nu(V_2)\nu(V_3)V_1 - \nu(V_1)\nu(V_3)V_2 \\ - \; & g(V_2,V_3)\nu(V_1)\zeta + g(V_1,V_3)\nu(V_2)\zeta)]. \end{aligned}$$

Putting $V_1 = \zeta$ and $V_3 = \varphi V_3$ in the foregoing equation, we attain

$$\{3f_2 + (2n-1)f_3\}g(V_2,\varphi V_3)\zeta = 0.$$

In consideration of $g(V_2,\varphi V_3) \neq 0$, we have

$$(2n-1)f_3 + 3f_2 = 0.$$

Then, the Equation (5) reduces to

$$S(V_1,V_2) = (2nf_1 - f_3 + 3f_2)g(V_1,V_2) = -2n(f_3 - f_1)g(V_1,V_2).$$

By virtue of the above equation we can write (11) as

$$\begin{aligned} & g(\mathcal{W}^*(V_1,V_2)V_3,V_4) \\ = \; & f_2\{g(V_1,\varphi V_3)g(\varphi V_2,V_4) - g(V_2,\varphi V_3)g(\varphi V_1,V_4) + 2g(V_1,\varphi V_2)g(\varphi V_3,V_4)\} \\ - \; & f_3(g(V_2,V_4)\nu(V_1)\nu(V_3) - g(V_1,V_4)\nu(V_2)\nu(V_3) \\ - \; & g(V_2,V_3)[g(V_1,V_4) + \nu(V_1)\nu(V_4)] \\ + \; & g(V_1,V_3)[g(V_2,V_4) + \nu(V_2)\nu(V_4)]) \\ = \; & 0. \end{aligned}$$

Setting $V_3 = \varphi V_3$ and $V_2 = \varphi V_2$, we have

$$\begin{aligned} & g(\mathcal{W}^*(\varphi V_1,\varphi V_2)V_3,V_4) \\ = \; & f_2\Big\{g(\varphi V_1,\varphi V_3)g(\varphi^2 V_2,V_4) - g(\varphi V_2,\varphi V_3)g(\varphi^2 V_1,V_4) + 2g(\varphi V_1,\varphi^2 V_2)g(\varphi V_3,V_4)\Big\} \\ + \; & f_3\{g(\varphi V_2,V_3)g(\varphi V_1,V_4) - g(\varphi V_1,V_3)g(\varphi V_2,V_4)\} \\ = \; & 0. \end{aligned}$$

Denoting the orthonormal local basis of $TM_1^{2n+1}(f_1,f_2,f_3)$ by $\{e_1,...,e_{2n},e_{2n+1} = \zeta\}$. Obviously, the local basis signature is $\{+,...,+,-\}$ and is denoted by $\{\epsilon_1,...,\epsilon_{2n},\epsilon_{2n+1}\}$. Putting $V_2 = e_i$ and $V_4 = \epsilon_i e_i$ in the last equation and summing over i, we have

$$\{f_3 + (2n+1)f_2\}g(\varphi V_1,\varphi V_3) = 0,$$

since $g(\varphi V_1,\varphi V_3) = \sum_{i=1}^{2n+1}\epsilon_i g(\varphi V_1,e_i)g(\varphi V_3,e_i)$. Because of $g(\varphi V_1,\varphi V_3) \neq 0$, we get

$$f_3 + (2n+1)f_2 = 0.$$

Taking consideration of $(2n-1)f_3 + 3f_2 = 0$ and $n > 1$, we get

$$f_2 = f_3 = 0.$$

Conversely, we suppose that $f_2 = f_3 = 0$. Therefore, we obtain from (1) that

$$R(V_1,V_2)V_3 = -f_1\{g(V_1,V_3)V_2 - g(V_2,V_3)V_1\}. \tag{17}$$

From (5) we have

$$S(V_1, V_2) = 2nf_1 g(V_1, V_2).$$

In view of (1), it follows that $\mathcal{W}^*(V_1, V_2)V_3 = 0$. Hence we can state

Theorem 1. *A GLSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ is $\mathcal{M}$-projectively flat if and only if $f_3 = f_2 = 0$.*

Next, by Schur's Theorem (see [23]) if $M^n(n \geq 3)$ is a connected pseudo-Reimannian manifold, and for each $p \in M$, the sectional curvature $K(p)$ is a constant function on the nondegenerate planes in T_pM, then $K(p)$ is a constant function on $M^n(n \geq 3)$.

From Theorem (1), we can get that if a $M_1^{2n+1}(f_1, f_2, f_3)$ is $\mathcal{M}$-projectively flat, then $K(p) = f_1$. Using Schur's Theorem, we have

Proposition 1. *If a GLSSF $M_1^{2n+1}(f_1, f_2, f_3)$ $(n > 1)$ is $\mathcal{M}$-projectively flat, then f_1 is a constant function.*

Further, for an $\mathcal{M}$-projectively flat $M_1^{2n+1}(f_1, f_2, f_3)$ we consider

$$\begin{aligned}(R(V_1, V_2) \cdot R)(V_3, V_4)V_5 &= R(V_1, V_2)R(V_3, V_4)V_5 - R(R(V_1, V_2)V_3, V_4)V_5 \\ &- R(V_3, R(V_1, V_2)V_4)V_5 - R(V_3, V_4)R(V_1, V_2)V_5, \end{aligned} \tag{18}$$

for any vector fields V_1, V_2, V_3, V_4, V_5 on $M_1^{2n+1}(f_1, f_2, f_3)$. Hence, it follows from (17) that

$$\begin{aligned}R(V_1, V_2)R(V_3, V_4)V_5 &= f_1^2[g(V_4, V_5)g(V_2, V_3)V_1 - g(V_3, V_5)g(V_2, V_4)V_1 \\ &- g(V_4, V_5)g(V_1, V_3)V_2 - g(V_3, V_5)g(V_1, V_4)V_2], \end{aligned} \tag{19}$$

$$\begin{aligned}R(R(V_1, V_2)V_3, V_4)V_5 &= f_1^2[g(V_4, V_5)g(V_2, V_3)V_1 - g(V_2, V_3)g(V_1, V_5)V_4 \\ &- g(V_1, V_3)g(V_4, V_5)V_2 + g(V_2, V_5)g(V_1, V_3)V_4], \end{aligned} \tag{20}$$

$$\begin{aligned}R(V_3, R(V_1, V_2)V_4)V_5 &= f_1^2[g(V_4, V_2)g(V_1, V_5)V_3 - g(V_4, V_2)g(V_3, V_5)V_1 \\ &- g(V_4, V_1)g(V_2, V_5)V_3 + g(V_1, V_4)g(V_3, V_5)V_2], \end{aligned} \tag{21}$$

and

$$\begin{aligned}R(V_3, V_4)R(V_1, V_2)V_5 &= f_1^2[g(V_4, V_1))g(V_2, V_5)V_3 - g(V_1, V_3)g(V_2, V_5)V_4 \\ &- g(V_1, V_5)g(V_4, V_2)V_3 + g(V_1, V_5)g(V_3, V_2)V_4]. \end{aligned} \tag{22}$$

By the use of (19)–(22), from (18) it follows that

$$(R(V_1, V_2) \cdot R)(V_3, V_4)V_5 = 0.$$

That is $M_1^{2n+1}(f_1, f_2, f_3)$ is semisymmetric. Thus, we state:

Theorem 2. *Let $M_1^{2n+1}(f_1, f_2, f_3)$ $(n > 1)$ be an $\mathcal{M}$-projectively flat GLSSF. Then, $M_1^{2n+1}(f_1, f_2, f_3)$ is semisymmetric, i.e, $R \cdot R = 0$ holds on $M_1^{2n+1}(f_1, f_2, f_3)$.*

3.2. $\mathcal{M}$-Projectively Semisymmetric

Definition 2. *A GLSSF $M_1^{2n+1}(f_1, f_2, f_3)$ is called $\mathcal{M}$-projectively symmetric if the condition $\nabla\mathcal{W}^* = 0$ holds on $M_1^{2n+1}(f_1, f_2, f_3)$ and it is called $\mathcal{M}$-projectively semisymmetric if*

$$(R(V_1, V_2) \cdot \mathcal{W}^*)(V_3, V_4)V_5 = 0, \tag{23}$$

for any vector fields V_1, V_2, V_3, V_4, V_5 on $M_1^{2n+1}(f_1, f_2, f_3)$.

Let $M_1^{2n+1}(f_1, f_2, f_3)$ be an $\mathcal{M}$-projectively semisymmetric. Then, from (23) we have

$$\begin{aligned} & R(V_1,\zeta)\mathcal{W}^*(V_3,V_4)V_5 - \mathcal{W}^*(R(V_1,\zeta)V_3,V_4)V_5 \\ - \; & \mathcal{W}^*(V_3,R(V_1,\zeta)V_4)V_5 - \mathcal{W}^*(V_3,V_4)R(V_1,\zeta)V_5 = 0. \end{aligned}$$

In view (9), the above equation becomes

$$\begin{aligned} & (f_1-f_3)[-g(\zeta,\mathcal{W}^*(V_3,V_4)V_5)V_1 + g(V_1,\mathcal{W}^*(V_3,V_4)V_5)\zeta + \nu(V_3)\mathcal{W}^*(V_1,V_4)V_5 \\ + \; & g(V_1,V_3)\mathcal{W}^*(\zeta,V_4)V_5 + \nu(V_4)\mathcal{W}^*(V_3,V_1)V_5 + g(V_1,V_4)\mathcal{W}^*(V_3,\zeta)V_5 \\ + \; & \nu(V_5)\mathcal{W}^*(V_3,V_4)V_1 + g(V_1,V_5)\mathcal{W}^*(V_3,V_4)\zeta] = 0. \end{aligned} \tag{24}$$

Putting $V_5 = \zeta$ in (24) then using (12) and (13), we lead to

$$(f_1-f_3)[\mathcal{W}^*(V_3,V_4)V_1 + \frac{(2n-1)f_3+3f_2}{4n}\{g(V_1,V_4)V_3 - g(V_1,V_3)V_4\}] = 0.$$

Here, if $f_1 - f_3 = 0$, then in view of (8) we conclude that $M_1^{2n+1}(f_1, f_2, f_3)$ is ζ-flat. Suppose if $f_1 - f_3 \neq 0$ on some open set $\mathcal{O}$ of $M_1^{2n+1}(f_1, f_2, f_3)$, then we have

$$\mathcal{W}^*(V_3,V_4)V_1 = \frac{-[(2n-1)f_3+3f_2]}{4n}\{g(V_1,V_4)V_3 - g(V_1,V_3)V_4\}.$$

Contracting V_3 in the above equation, we obtain that

$$(2n+1)S(V_4,V_1) - rg(V_4,V_1) = -[(2n-1)f_3+3f_2]2ng(V_4,V_1).$$

With the help of (5) and (10), the forgoing equation reduces to

$$[3f_2+(2n-1)f_3][g(V_4,V_1)+\nu(V_4)\eta(V_1)] = 0,$$

which gives

$$(2n-1)f_3 + 3f_2 = 0. \tag{25}$$

Now with the help of (25), we obtain from (12) and (13) that

$$\mathcal{W}^*(\zeta,V_2)V_3 = 0 \tag{26}$$

and

$$\mathcal{W}^*(V_1,V_2)\zeta = 0. \tag{27}$$

Therefore, by using (26) and (27) in (24), we get

$$\mathcal{W}^*(V_1,V_2)V_3 = 0,$$

i.e., $M_1^{2n+1}(f_1, f_2, f_3)$ is $\mathcal{M}$-projectively flat. In this situation, $f_2 = f_3 = 0$ holds on $M_1^{2n+1}(f_1, f_2, f_3)$.

Theorem 3. *If a GLSSF $M_1^{2n+1}(f_1, f_2, f_3)$ $(n > 1)$ is $\mathcal{M}$-projectively semisymmetric, then either $M_1^{2n+1}(f_1, f_2, f_3)$ is ζ-flat or $M_1^{2n+1}(f_1, f_2, f_3)$ is $\mathcal{M}$-projectively flat on an open set $\mathcal{O}$ of $M_1^{2n+1}(f_1, f_2, f_3)$.*

On the other hand, suppose that $M_1^{2n+1}(f_1, f_2, f_3)$ with $f_1 \neq f_3$ is $\mathcal{M}$-projectively flat. That is, $f_2 = f_3 = 0$, then we have $\mathcal{W}^* = 0$, and hence $R(V_1, V_2)\mathcal{W}^* = 0$. Thus, we state

Theorem 4. *A GLSSF $M_1^{2n+1}(f_1, f_2, f_3)(n > 1)$ with $f_1 \neq f_3$ is $\mathcal{M}$-projectively semisymmetric if and only if it is $\mathcal{M}$-projectively flat.*

Remark 1. *A pseudo-Riemannian manifold is said to be $\mathcal{M}$-projectively recurrent if $\nabla\mathcal{W}^* = A \otimes \mathcal{W}^*$, where $A(\neq 0)$ is a 1-form. It can be easily shown that an $\mathcal{M}$-projectively recurrent manifold satisfies $R \cdot \mathcal{W}^* = 0$. Hence we immediately get:*

Proposition 2. *If a GLSSF $M_1^{2n+1}(f_1, f_2, f_3)$ $(n > 1)$ is $\mathcal{M}$-projectively recurrent, then either $M_1^{2n+1}(f_1, f_2, f_3)$ is ζ-flat or $M_1^{2n+1}(f_1, f_2, f_3)$ is $\mathcal{M}$-projectively flat on an open set $\mathcal{O}$ of $M_1^{2n+1}(f_1, f_2, f_3)$.*

On account of Theorems 2 and 4, we are able to state the following:

Proposition 3. *Let $M_1^{2n+1}(f_1, f_2, f_3)$ $(n > 1)$ be a GLSSF. If $M_1^{2n+1}(f_1, f_2, f_3)$ is $\mathcal{M}$-projectively semisymmetric with $f_1 \neq f_3$, then $M_1^{2n+1}(f_1, f_2, f_3)$ is semisymmetric.*

3.3. $\mathcal{M}$-Projectively Pseudosymmetric

Definition 3. *A pseudo-Riemannian manifold M is called $\mathcal{M}$-projectively pseudosymmetric if*

$$R \cdot \mathcal{W}^* = \mathcal{L}_M Q(g, \mathcal{W}^*) \tag{28}$$

holds on the set $\{\mathcal{U}_M = p \in M : \mathcal{W}^* \neq 0 \ \ at \ \ p\}$, where $\mathcal{L}_M$ is some function on $\mathcal{U}_M$.

Let $M_1^{2n+1}(f_1, f_2, f_3)$ be a $\mathcal{M}$-projectively pseudosymmeytric. Then, from (28), we have

$$(R(V_1, \zeta) \cdot \mathcal{W}^*)(V_3, V_4)V_5 = \mathcal{L}_M[((V_1 \wedge_g \zeta).\mathcal{W}^*)(V_3, V_4)V_5]. \tag{29}$$

If $M_1^{2n+1}(f_1, f_2, f_3)$ $(n > 1)$ be a GLSSF, from (9) and (14) we get

$$R(\zeta, V_1)V_2 = (f_1 - f_3)(\zeta \wedge V_1)V_2. \tag{30}$$

In view of (29) and (30), it is easy to see that

$$\mathcal{L}_M = (f_1 - f_3).$$

Hence, on account of previous calculations and discussions, we conclude:

Theorem 5. *Let $M_1^{2n+1}(f_1, f_2, f_3)$ $(n > 1)$ be a GLSSF. If $M_1^{2n+1}(f_1, f_2, f_3)$ is $\mathcal{M}$-projectively pseudosymmetric, then $M_1^{2n+1}(f_1, f_2, f_3)$ is either $\mathcal{M}$-projectively flat, in this case $f_2 = f_3 = 0$, or $\mathcal{L}_M = f_1 - f_3$ holds on $M_1^{2n+1}(f_1, f_2, f_3)$.*

However, $\mathcal{L}_M$ need not be zero (in general) and hence there exists $\mathcal{M}$-projectively pseudosymmetric manifolds which are not $\mathcal{M}$-projectively semisymmetric. Thus, the class of $\mathcal{M}$-projectively pseudosymmetric manifolds is a natural extension of the class of $\mathcal{M}$-projectively semisymmetric manifolds. Thus, if $\mathcal{L}_M \neq 0$, then it can be easily seen that $R \cdot \mathcal{W}^* = (f_1 - f_3)Q(g, \mathcal{W}^*)$, which implies that $\mathcal{L}_M = f_1 - f_3$. Therefore, we state

Theorem 6. *Every GLSSF $M_1^{2n+1}(f_1, f_2, f_3)$ is $\mathcal{M}$-projectively pseudosymmetric of the form $R \cdot \mathcal{W}^* = (f_1 - f_3)Q(g, \mathcal{W}^*)$.*

3.4. $\varphi - \mathcal{M}$-Projectively Semisymmetric

Definition 4. *A GLSSF $M_1^{2n+1}(f_1, f_2, f_3)$ $(n > 1)$ is said to be $\varphi - \mathcal{M}$ projectively semisymmetric if it satisfies the condition $\mathcal{W}^*(V_1, V_2) \cdot \varphi = 0$.*

Thus, we get

$$(\mathcal{W}^*(V_1, V_2) \cdot \varphi)V_3 = \mathcal{W}^*(V_1, V_2)\varphi V_3 - \varphi\mathcal{W}^*(V_1, V_2)V_3 = 0, \tag{31}$$

for all vector fields V_1, V_2, V_3 on $M_1^{2n+1}(f_1, f_2, f_3)$. Now, by virtue of (11), we have

$$\begin{aligned}\mathcal{W}^*(V_1, V_2)\varphi V_3 &= R(V_1, V_2)\varphi V_3 - \frac{1}{4n}[S(V_2, \varphi V_3)V_1 - S(V_1, \varphi V_3)V_2 \\ &+ g(V_2, \varphi V_3)QV_1 - g(V_1, \varphi V_3)V_2]. \end{aligned} \tag{32}$$

By the use of (1), (5), and (6) in (32), we obtain

$$\begin{aligned}\mathcal{W}^*(V_1, V_2)\varphi V_3 &= -\frac{1}{2n}(3f_2 - f_3)[g(V_2, \varphi V_3)X_1 - g(V_1, \varphi V_3)V_2] \\ &+ \frac{[3f_2 - (2n+1)f_3]}{4n}[g(V_2, \varphi V_3)\nu(V_1)\zeta - g(V_1, \varphi V_3)\nu(V_2)\zeta] \\ &+ f_2[(g(V_2, V_3) + \nu(V_2)\nu(V_3))\varphi V_1 - (g(V_1, V_3) + \nu(V_1)\nu(V_3))\varphi V_2 \\ &+ 2g(V_1, \varphi V_2)(-V_3 + \nu(V_3)\zeta)]. \end{aligned} \tag{33}$$

Similarly,

$$\begin{aligned}\varphi\mathcal{W}^*(V_1, V_2)V_3 &= -\frac{1}{2n}(3f_2 - f_3)[g(V_2, V_3)\varphi V_1 - g(V_1, V_3)\varphi V_2] \\ &+ \frac{[3f_2 - (2n+1)f_3]}{4n}[\nu(V_1)\nu(V_3)\varphi V_2 - \nu(V_2)\nu(V_3)\varphi V_1] \\ &+ f_2[g(V_1, \varphi V_3)(-V_2 + \nu(V_2)\zeta) - g(V_2, \varphi V_3)(-V_1 + \nu(V_1)\zeta) \\ &+ 2g(V_1, \varphi V_2)(-V_3 + \nu(V_3)\zeta)]. \end{aligned} \tag{34}$$

Substituting (33) and (34) in (31), we obtain

$$\begin{aligned}&- \frac{[(2n+3)f_2 - f_3]}{2n}[g(V_2, \varphi V_3)V_1 - g(V_1, \varphi V_3)V_2 - g(V_2, V_3)\varphi V_1 + g(V_1, V_3)\varphi V_2] \\ &+ \frac{[(4n+3)f_2 - (2n+1)f_3]}{4n}[g(V_2, \varphi V_3)\nu(V_1)\zeta - g(V_1, \varphi V_3)\nu(V_2)\zeta - \nu(V_1)\nu(V_3)\varphi V_2 \\ &+ \nu(V_2)\nu(V_3)\varphi V_1] = 0. \end{aligned} \tag{35}$$

Setting $V_2 = \zeta$ in (35), we get

$$\frac{[3f_2 + (2n-1)f_3]}{4n}[-g(V_1, \varphi V_3)\zeta + \nu(V_3)\varphi V_1] = 0. \tag{36}$$

Replacing V_3 by φV_3 in (36) and then taking the inner product with ζ, we lead to

$$[(2n-1)f_3 + 3f_2][g(V_1, V_3) + \nu(V_1)\nu(V_3)] = 0.$$

Since $g(V_1, V_3) + \nu(V_1)\nu(V_3) \neq 0$, we must have

$$f_3 = \frac{-3f_2}{2n-1}. \tag{37}$$

By the above equation, we can write (35) as

$$\begin{aligned}&\frac{2(n+1)}{2n-1}f_2[g(V_2, \varphi V_3)V_1 - g(V_1, \varphi V_3)V_2 - g(V_2, V_3)\varphi V_1 + g(V_1, V_3)\varphi V_2 \\ &- g(V_2, \varphi V_3)\nu(V_1)\zeta + g(V_1, \varphi V_3)\nu(V_2)\zeta + \nu(V_1)\nu(V_3)\varphi V_2 - \nu(V_2)\nu(V_3)\varphi V_1] = 0. \end{aligned}$$

The inner product of the foregoing equation with V_4 and putting $V_2 = e_i$, $V_4 = \epsilon_i e_i$ then summing over i we have

$$f_2 g(V_1, \varphi V_3) = 0.$$

Since $g(V_1, \varphi V_3) \neq 0$, we have $f_2 = 0$. By taking this into consideration, we obtain from (37) that $f_3 = 0$. That is, $f_2 = f_3 = 0$. Hence, $M_1^{2n+1}(f_1, f_2, f_3)$ is $\mathcal{M}$-projectively flat.

Conversely, suppose that $M_1^{2n+1}(f_1, f_2, f_3)$ is $\mathcal{M}$-projectively flat, i.e., $f_2 = f_3 = 0$. Then we have $\mathcal{W}^* = 0$, and hence $\mathcal{W}^*(V_1, V_2) \cdot \varphi = 0$. Thus, we state the following:

Theorem 7. *A GLSSF $M_1^{2n+1}(f_1, f_2, f_3)$, $(n > 1)$ is $\varphi - \mathcal{M}$-projectively semisymmetric if and only if it is $\mathcal{M}$-projectively flat.*

Finally, by combining the results stated in Theorems 1 and 7, we state the following:

Proposition 4. *Let $M_1^{2n+1}(f_1, f_2, f_3)$, $(n > 1)$ be a GLSSF. Then the following statemets are equivalent:*
(i) $M_1^{2n+1}(f_1, f_2, f_3)$ is $\mathcal{M}$-projectively flat;
(ii) $M_1^{2n+1}(f_1, f_2, f_3)$ is $\varphi - \mathcal{M}$-projectively semisymmetric;
(iii) $f_2 = f_3 = 0$ holds on $M_1^{2n+1}(f_1, f_2, f_3)$.

4. $M_1^{2n+1}(f_1, f_2, f_3)$ Satisfying the Conditions $\mathcal{W}^*(\zeta, V_1) \cdot S = 0$, $\mathcal{W}^*(\zeta, V_1) \cdot R = 0$ and $\mathcal{W}^*(\zeta, V_1) \cdot \mathcal{W}^* = 0$

Case (i): $M_1^{2n+1}(f_1, f_2, f_3)$ **satisfying** $\mathcal{W}^*(\zeta, V_1) \cdot S = 0$.
The condition $\mathcal{W}^*(\zeta, V_1) \cdot S = 0$ is equivalent to

$$S(\mathcal{W}^*(\zeta, V_1)V_2, \zeta) + S(V_2, \mathcal{W}^*(\zeta, V_1)\zeta) = 0. \tag{38}$$

For a $(2n + 1)$-dimensional $M_1^{2n+1}(f_1, f_2, f_3)$, it is well known that

$$S(V_1, \zeta) = -2n(f_1 - f_3)\nu(V_1). \tag{39}$$

Using the Equations (13) and (39), we get

$$S(\mathcal{W}^*(\zeta, V_1)V_2, \zeta) = -2n(f_1 - f_3)\nu(\mathcal{W}^*(\zeta, V_1)V_2),$$

which, by using (13) takes the form

$$S(\mathcal{W}^*(\zeta, V_1)V_2, \zeta) = 2n(f_1 - f_3)\frac{[(2n-1)f_3 + 3f_2]}{4n}[g(V_1, V_2) + \nu(V_1)\nu(V_2)]. \tag{40}$$

Again, in view of (13), we have

$$S(V_2, \mathcal{W}^*(\zeta, V_1)\zeta) = \frac{-[(2n-1)f_3 + 3f_2]}{4n}[2n(f_1 - f_3)\nu(V_1)\nu(V_2) + S(V_1, V_2)]. \tag{41}$$

Substituting (40) and (41) in (38) followed by a simple calculation gives

$$\frac{[(2n-1)f_3 + 3f_2]}{4n}[S(V_1, V_2) - 2n(f_1 - f_3)g(V_1, V_2)] = 0.$$

By making use of (5) in the foregoing equation, we obtain

$$[(2n-1)f_3 + 3f_2][g(V_1, V_2) + \nu(V_1)\nu(V_2)] = 0.$$

Since $g(V_1, V_2) + \nu(V_1)\nu(V_2) \neq 0$, we have $f_3 = \frac{-3f_2}{2n-1}$.
Conversely, if $f_3 = \frac{-3f_2}{2n-1}$ holds on $M_1^{2n+1}(f_1, f_2, f_3)$, then by direct calculation,

$$\begin{aligned} \mathcal{W}^*(\zeta, V_1) \cdot S &= -S(\mathcal{W}^*(\zeta, V_1)V_2, \zeta) - S(V_2, \mathcal{W}^*(\zeta, V_1)\zeta) \\ &= 0. \end{aligned}$$

The above discussion can be summarized as follows:

Theorem 8. *Let $M_1^{2n+1}(f_1, f_2, f_3)$ $(n > 1)$ be a GLSSF. Then $M_1^{2n+1}(f_1, f_2, f_3)$ satisfies $\mathcal{W}^*(\zeta, V_1) \cdot S = 0$ if and only if $f_3 = \frac{-3f_2}{2n-1}$ holds on $M_1^{2n+1}(f_1, f_2, f_3)$.*

Case (ii): $M_1^{2n+1}(f_1, f_2, f_3)$ satisfying $\mathcal{W}^*(\zeta, V_1) \cdot R = 0$.
The condition $\mathcal{W}^*(\zeta, V_1) \cdot R = 0$ is expressed as

$$\begin{aligned} & \mathcal{W}^*(\zeta, V_4)R(V_1, V_2)V_3 - R(\mathcal{W}^*(\zeta, V_4)V_1, V_2)V_3 \\ - \; & R(V_1, \mathcal{W}^*(\zeta, V_4)V_2)V_3 - R(V_1, V_2)\mathcal{W}^*(\zeta, V_4)V_3 = 0, \end{aligned}$$

which in view of (13) provides

$$\begin{aligned} [(2n-1)f_3 + 3f_2] \; \times \; & [g(V_4, (V_1, V_2)V_3)\zeta + \nu(R(V_1, V_2)V_3)V_4 - g(V_4, V_1)R(\zeta, V_2)V_3 \\ - \; & \nu(V_1)R(V_4, V_2)V_3 - g(V_4, V_2)R(V_1, \zeta)V_3 - \nu(V_2)R(V_1, V_4)V_3 \\ - \; & g(V_4, V_3)R(V_1, V_2)\zeta - \nu(V_3)R(V_1, V_2)V_4] \\ = \; & 0. \end{aligned} \tag{42}$$

Setting $V_3 = \zeta$ in (42) and using (8), we get

$$[(2n-1)f_3 + 3f_2] \times [(f_1 - f_3) \times (g(V_2, V_4)V_1 - g(V_1, V_4)V_2) + R(V_1, V_2)V_4] = 0.$$

If $(2n-1)f_3 + 3f_2 = 0$ holds on $M_1^{2n+1}(f_1, f_2, f_3)$, then in view of (5) we have $S(V_1, V_2) = 2n(f_1 - f_3)g(V_1, V_2)$, that is, $M_1^{2n+1}(f_1, f_2, f_3)$ is Einstein manifold.
Suppose if $(2n-1)f_3 + 3f_2 \neq 0$ on some open set $\mathcal{O}$ of $M_1^{2n+1}(f_1, f_2, f_3)$, then we have

$$R(V_1, V_2)V_4 = -(f_1 - f_3)[g(V_2, V_4)V_1 - g(V_1, V_4)V_2]. \tag{43}$$

Thus, we state the following:

Theorem 9. *Let $M_1^{2n+1}(f_1, f_2, f_3)$ $(n > 1)$ be a GLSSF. If $M_1^{2n+1}(f_1, f_2, f_3)$ satisfies $\mathcal{W}^*(\zeta, V_1) \cdot R = 0$, then either $M_1^{2n+1}(f_1, f_2, f_3)$ is Einstein or the curvature tensor of $M_1^{2n+1}(f_1, f_2, f_3)$ is of the form (43) on an open set $\mathcal{O}$ of $M_1^{2n+1}(f_1, f_2, f_3)$.*

Case (iii): $M_1^{2n+1}(f_1, f_2, f_3)$ satisfying $\mathcal{W}^*(\zeta, V_1) \cdot \mathcal{M} = 0$.
The condition $\mathcal{W}^*(\zeta, V_1) \cdot \mathcal{M} = 0$ gives

$$\begin{aligned} & \mathcal{W}^*(\zeta, V_1)\mathcal{W}^*(V_2, V_3)V_4 - \mathcal{W}^*(\mathcal{W}^*(\zeta, V_1)V_2, V_3)V_4 \\ - \; & \mathcal{W}^*(V_2, \mathcal{W}^*(\zeta, V_1)V_3)V_4 - \mathcal{W}^*(V_2, V_3)\mathcal{W}^*(\zeta, V_1)V_4 = 0. \end{aligned}$$

In view of (13) the above equation provides

$$\begin{aligned} [(2n-1)f_3 + 3f_2] \; \times \; & [g(V_1, \mathcal{W}^*(V_2, V_3)V_4)\zeta + \nu(\mathcal{W}^*(V_2, V_3)V_4)V_1 - g(V_1, V_2)\mathcal{W}^*(\zeta, V_3)V_4 \\ - \; & \nu(V_2)\mathcal{W}^*(V_1, V_3)V_4 - g(V_1, V_3)\mathcal{W}^*(V_2, \zeta)V_4 - \nu(V_3)\mathcal{W}^*(V_2, V_1)V_4 \\ - \; & g(V_1, V_4)\mathcal{W}^*(V_2, V_3)\zeta - \zeta(V_4)\mathcal{W}^*(V_2, V_3)V_1] \\ = \; & 0. \end{aligned} \tag{44}$$

By putting $V_3 = \zeta$ in (44) and making use of (13), we get

$$[(2n-1)f_3 + 3f_2]\left[-\mathcal{W}^*(V_2, V_1)V_4 + \frac{[3f_2 + (2n-1)f_3]}{4n}[g(V_2, V_4)V_1 - g(V_1, V_4)V_2]\right] = 0.$$

Therefore, either $f_3 = \frac{-3f_2}{2n-1}$ or

$$\mathcal{W}^*(V_2, V_1)V_4 = \frac{[3f_2 + (2n-1)f_3]}{4n}[g(V_2, V_4)V_1 - g(V_1, V_4)V_2].$$

Contracting V_2 in the above equation, we have

$$(2n+1)S(V_1,V_4) - rg(V_1,V_4) = -[(2n-1)f_3 + 3f_2]2ng(V_1,V_4).$$

With the help of (5) and (10), we simplify

$$[3f_2 + (2n-1)f_3][g(V_1,V_4) + \nu(X_1)\nu(V_4)] = 0.$$

Since $[g(V_1,V_4) + \nu(V_1)\nu(V_4)] \neq 0$, we get $f_3 = \frac{-3f_2}{2n-1}$. Thus, in both cases we have $f_3 = \frac{-3f_2}{2n-1}$.

Conversely, if $M_1^{2n+1}(f_1,f_2,f_3)$ satisfies $f_3 = \frac{-3f_2}{2n-1}$, then in view of (13), we have $\mathcal{W}^*(\zeta,V_1)\cdot\mathcal{W}^* = 0$. Hence, we state the following:

Theorem 10. *Let $M_1^{2n+1}(f_1,f_2,f_3)$ $(n > 1)$ be a GLSSF. Then $M_1^{2n+1}(f_1,f_2,f_3)$ satisfies $\mathcal{W}^*(\zeta,V_1)\cdot\mathcal{W}^* = 0$ if and only if $f_3 = \frac{-3f_2}{2n-1}$ holds on $M_1^{2n+1}(f_1,f_2,f_3)$.*

Finally, by combining the results stated in Theorems 8 and 10 we state:

Proposition 5. *Let $M_1^{2n+1}(f_1,f_2,f_3)$, $(n > 1)$ be a GLSSF. Then the following statements are equivalent:*
(i) $M_1^{2n+1}(f_1,f_2,f_3)$ satisfies $\mathcal{W}^*(\zeta,V_1)\cdot S = 0$;
(ii) $M_1^{2n+1}(f_1,f_2,f_3)$ satisfies $\mathcal{W}^*(\zeta,V_1)\cdot\mathcal{W}^* = 0$;
(iii) $f_3 = \frac{-3f_2}{2n-1}$ holds on $M_1^{2n+1}(f_1,f_2,f_3)$.

Example 1. *By using warped product, we can construct GLSSF (see [5,12]). Given an smooth function $f > 0$ and almost Hermitian manifold $(\aleph^{2n}, J, \mathcal{G})$ on $\mathbb{R}$, we deliberate the warped product $\mathcal{M} = \mathbb{R}\times_f \aleph$, with metric specified by*

$$g_f = -\pi^*(g_{\mathbb{R}}) + (f\circ\pi)^2\sigma^*(\mathcal{G}),$$

where π and σ are the projections from $\mathbb{R}\times\aleph$ on $\mathbb{R}$ and $\aleph$, respectively. Here we denote * the horizontal lift with respect to the connection ν, that is, $g(X^*,Y^*) = \mathcal{G}(X,Y)$. Then, for any vector field V_1 we have $\varphi(V_1) = (J\sigma_*V_1)^*$ and $\nu(V_1) = -g_f(V_1,\zeta)$ with $\zeta = \frac{\partial}{\partial t}$, where t denotes the coordinate of $\mathbb{R}$.

Clearly, this is an almost contact Lorentzian structure on the warped product $\mathcal{M}$. This metric is the one used to construct Robertson–Walker spaces (see [23]).

Example 2. *Let $\aleph^{2n}(F_1,F_2)$ be a generalized complex-space-form of dimension $2n$ (see [12,23]). Then, the warped product $M_1^{2n+1}(f_1,f_2,f_3) = \mathbb{R}\times_f\aleph$ endowed with the almost contact Lorentzian structure (φ,ζ,ν,g_f) is a GLSSF, with functions:*

$$f_1 = \frac{(f\circ\pi)'^2 + (F_1\circ\sigma)}{(f\circ\pi)^2},\quad f_2 = \frac{F_2\circ\sigma}{(f\circ\pi)^2},\quad f_3 = \frac{(f\circ\pi)'^2 + (F_1\circ\sigma)}{(f\circ\pi)^2} + \frac{(f\circ\pi)''}{f\circ\pi}.$$

In particular, if $\aleph(a,b)$ is a generalized complex-space-form, we obtain GLSSF $M_1^{2n+1}(f_1,f_2,f_3)$ with functions:

$$f_1 = \frac{(f\circ\pi)'^2 + a}{(f\circ\pi)^2},\quad f_2 = \frac{b}{(f\circ\pi)^2},\quad f_3 = \frac{(f\circ\pi)'^2 + a}{(f\circ\pi)^2} + \frac{(f\circ\pi)''}{(f\circ\pi)},$$

where $(f\circ\pi) = (f\circ\pi)(t)$, $t\in\mathbb{R}$ and $(f\circ\pi)'$ signifies the derivative of $(f\circ\pi)$ with respect to t.

If we choose $n = 2$, $a = 0$, $b = 0$ and $(f \circ \pi)(t) = e^t > 0$ with $t \in \mathbb{R}$, then $f_1 = 1$, $f_2 = 0$ and $f_3 = 0$. Therefore, the condition $f_2 = f_3 = 0$ holds. Hence, from Proposition 4 for a GLSSF $M_1^{2n+1}(f_1, f_2, f_3)$ we have $f_2 = f_3 = 0$. Thus, in dimension 5, that is, for $n = 2$ we have $f_2 = f_3 = 0$. Consequently, all the corresponding constraints of Proposition 4 are tested.

On the other hand, if we elect $n = 2$, $b = -1$, $a = 0$ and $(f \circ \pi)(t) = t$ with $t > 0$, then $f_1 = \frac{1}{t^2}$, $f_2 = \frac{-1}{t^2}$ and $f_3 = \frac{1}{t^2}$. Hence $f_3 = -f_2$. Hence, from Proposition 5 for a GLSSF $M_1^{2n+1}(f_1, f_2, f_3)$ we have $f_2 = -f_3$. Thus, in dimension 5, that is, for $n = 2$ we have $f_2 = -f_3$. Consequently, all the correspondent constraints of Proposition 5 are tested.

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