

Article

Solution Bounds and Numerical Methods of the Unified Algebraic Lyapunov Equation

Juan Zhang ^{1,*}, Shifeng Li ² and Xiangyang Gan ¹¹ Key Laboratory of Intelligent Computing and Information Processing of Ministry of Education, Xiangtan University, Xiangtan 411105, China² Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Xiangtan 411105, China

* Correspondence: zhangjuan@xtu.edu.cn; Tel.: +86-13107224973

Abstract: In this paper, applying some properties of matrix inequality and Schur complement, we give new upper and lower bounds of the solution for the unified algebraic Lyapunov equation that generalize the forms of discrete and continuous Lyapunov matrix equations. We show that its positive definite solution exists and is unique under certain conditions. Meanwhile, we present three numerical algorithms, including fixed point iterative method, the acceleration fixed point method and the alternating direction implicit method, to solve the unified algebraic Lyapunov equation. The convergence analysis of these algorithms is discussed. Finally, some numerical examples are presented to verify the feasibility of the derived upper and lower bounds, and numerical algorithms.

Keywords: Lyapunov equation; solution bounds; fixed point method; alternating iteration algorithm

MSC: 15A24; 65F45



Citation: Zhang, J.; Li, S.; Gan, X. Solution Bounds and Numerical Methods of the Unified Algebraic Lyapunov Equation. *Mathematics* **2022**, *10*, 2858. <https://doi.org/10.3390/math10162858>

Academic Editors: Xiangmin Jiao and Mihai Postolache

Received: 22 June 2022

Accepted: 9 August 2022

Published: 11 August 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Consider the unified algebraic Lyapunov equation (UALE) [1]

$$PA + A^T P + \theta A^T P A + Q = 0, \quad (1)$$

where $Q \in \mathbb{R}^{n \times n}$ is a positive definite matrix and $P \in \mathbb{R}^{n \times n}$ is the unknown matrix, $\theta > 0$ is the sampling period. It can be seen that the continuous algebraic Lyapunov equations (CALE) and discrete algebraic Lyapunov equations (DALE) are two special cases of UALE (1) when $\theta = 0$ and $\theta = 1$, respectively. In addition, $A > B$ ($A \geq B$) means that matrix $A - B$ is positive definite (semidefinite).

UALE is used widely, e.g., for stability analysis of control systems, high speed signal processing, sampled-data control, and so on [2]. Next, we consider the application of UALE to the linear continuity systems based on the δ -operator. In order to analyze the quality of the design system, the stability and energy control are all important evaluation criteria for analyzing the control system. The discussion of these problems can often be equivalent into the discussion and study of the solution of the Lyapunov equation. The δ -operator (Euler operator) proposed by Australian scholars Middleton and Goodwin in the 1980s [1,3] is defined by

$$\delta x(t) = \begin{cases} dx(t)/dt, & \theta = 0, \\ (x(t + \theta) - x(t))/\theta, & \theta \neq 0, \end{cases}$$

where $\delta = \frac{q-1}{\theta}$, q is the forward shift operator, $qx(k) = x(k + \theta)$ and θ is the sampling period. Considered the linear continuity free systems

$$\dot{x} = Ax, \quad (2)$$

where x is the state vector and A is the coefficient matrix of the corresponding dimension. By discretizing the model (2) using the shift operator q , we obtain the sampling results of the above continuous system in the δ -domain [4]

$$qx(t) = A_q x(t), \quad A_q = e^{At}. \quad (3)$$

By discretizing the model (2) using the δ -operator, we obtain the sampling results of the above continuous system in the δ -domain for the discrete system [4] as

$$\delta x(t) = A_\delta x(t), \quad A_\delta = \frac{A_q - I}{\theta}. \quad (4)$$

Therefore, we have

$$\lim_{\theta \rightarrow 0} A_q = \lim_{\theta \rightarrow 0} e^{At} = I, \quad \lim_{\theta \rightarrow 0} A_\delta = \lim_{\theta \rightarrow 0} \frac{e^{At} - I}{\theta} = A.$$

It can be seen that the results obtained from the δ -domain discretization model tend to the corresponding continuous case when $\theta \rightarrow 0$. By the second Lyapunov method, as long as the appropriate scalar function $V(x)$ is found and $dV(x)/dx < 0$, we can determine that the system is asymptotic stable. It is well-known that the judgment of its gradual stability can often be transformed into the existence of positive definite solutions to UALE on the δ -domain. Consider the Lyapunov function [2]

$$V(x(t)) = x(t)^T P x(t),$$

where V is the unknown positive definite matrix, then the difference of $V(x(t))$ is

$$\theta V(x(t)) = x(t + \theta)^T P x(t + \theta) - x(t)^T P x(t). \quad (5)$$

Applying the definition of δ -operator, it follows that

$$x(t + \theta) = x(t) + \theta \delta x(t). \quad (6)$$

Combining (4) and substituting (6) into (5), we have

$$\theta V(x(t)) = x(t)^T [\theta P A_\delta + \theta A_\delta^T P + \theta^2 A_\delta^T P A_\delta] x(t).$$

By the second Lyapunov method, the system (2) is stable when $\theta V(x(t)) < 0$, that is $PA_\delta + A_\delta^T P + \theta A_\delta^T P A_\delta < 0$. Let

$$PA_\delta + A_\delta^T P + \theta A_\delta^T P A_\delta = -Q, \quad (7)$$

where $Q > 0$ and the determination problem of the discrete system (4) on the δ -domain can be transformed into the existence of the positive definite solution of UALE (7).

In many cases, the bound of the solution can be applied to solve many control problems, such as system stability analysis under perturbation or time delay. Meanwhile, the bounds of the exact solutions of the matrix equations reduce the computational burden. Therefore, the estimation of the bound of the equation solution has become the focus of the scholars. At present, the solutions of continuous and discrete algebraic Lyapunov matrix equations have been systematically studied, but UALE still needs further study. The study of UALE's solution on δ -domain was first given by Zhang in 2004 [5]. Using the matrix inequalities and the eigenvalue inequalities, Lee in [6] propose a unified approach to solve the estimation problem for the solution of UALE. Zhang et al. in [7] extended the upper and lower

bound of the solution of UALE on δ -domain. Besides, UALE can be transformed into a quasi-standard form of the DALE by using a bilinear transformation proposed in [8]. Thus, by extending this approach for CALE associated with linear algebraic techniques, Lee present several upper and lower matrix bounds of the solution of UALE in [9]. In 2013, Zhang et al. expand the solution bounds of UALE through some appropriate deformation in [10].

Lyapunov equations play a fundamental role in many areas of applications, such as control theory, model reduction and signal processing [11,12]. The Bartels–Stewart [13] method as modified by Hammarling [14] is the standard direct method for the solution of Lyapunov equations of small to moderate size. For effective methods to solve large-scale Lyapunov equations, see, e.g., Ref. [15] for the sign-function method, Refs. [16,17] for the alternating direction implicit (ADI) iteration method, Refs. [18,19] for the Smith method and its variants, Refs. [20,21] for Krylov-based methods, or the recent survey [12] and the references therein for an overview of developments and methods. In this paper, we give the upper and lower matrix bounds of UALE's solution by applying some properties of matrix inequality and Schur complement. Meanwhile, this paper propose three numerical algorithms, including fixed point algorithm (FP), accelerated fixed point algorithm (AFP) and the alternate direction implicate method (ADI), to solve UALE.

The paper is organised as follows. In Section 2, a new upper and lower bound of the positive definite solution of UALE is given by using the properties of matrix inequality and Schur complement. Meanwhile, we further give the existence and uniqueness theorem of the positive definite solution of UALE. Section 3 presents three numerical methods to solve positive definite solution of UALE and gives the corresponding convergence analysis. In Section 4, we give some numerical examples to illustrate that the solution bounds and numerical methods are feasible and effective. Finally, we conclude this paper with some remarks in Section 5.

Throughout this paper, we shall adopt the following notations. $\sigma_i(A)$ and $\lambda_i(A)$ ($i = 1, 2, \dots, n$) represent the singular values and eigenvalues of the matrix A , respectively. $\rho(A) = \max_{1 \leq i \leq n} |\lambda_i(A)|$ is the spectral radius of the matrix A . For $\sigma_i(A)$ and $\lambda_i(A)$ of the Hermitian matrix A , respectively, are arranged in the non-increasing order (i.e., $\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_n(A)$ and $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$). $\|A\|$ represents the spectral norm of the matrix X , i.e., $\|A\|^2 = \lambda_1(A^*A)$, where A^* is the conjugate transpose of A . Denote $X(\gamma, \beta)$ is the matrix X with γ as a row indicator set and β as a submatrix of the column index set, where $X \in \mathbb{R}^{n \times n}$. In addition, X is called a square root of the Hermitian matrix A if X satisfies $X^2 = A$ and denoted as $X = A^{1/2}$.

2. Solution Bounds of UALE

According to the properties of Schur complement and the upper bound of UALE's solution given in [5], we give a new upper and lower solution bounds of UALE in this section. Meanwhile, this section also analyzes the existence and uniqueness of UALE's solutions under certain conditions.

Firstly, let us apply a simple transformation to UALE (1). Since

$$PA + A^T P + \theta A^T P A = (\theta A + I)^T P (\theta A + I) / \theta - P / \theta,$$

then UALE (1) can be rewritten as

$$(\theta A + I)^T P (\theta A + I) + \theta Q = P. \quad (8)$$

2.1. Preliminaries

Next, we review an important lemma for matrix inequalities.

Lemma 1 ([22]). For the Hermitian matrices Y and Z ,

(1) if $Y \geq Z > 0$, then $Z^{-1} \geq Y^{-1} > 0$;

(2) if $Y > Z > 0$, then $Z^{-1} > Y^{-1} > 0$.

Further, we introduce the definition and some properties of Schur complement.

Definition 1 ([23]). If matrix $X \in \mathbb{R}^{n \times n}$, the Schur complement of the matrix $X(\gamma)$ is

$$X/\gamma = X/X(\gamma) = X(\gamma^c) - X(\gamma^c, \gamma)[X(\gamma)]^{-1}X(\gamma, \gamma^c),$$

where $X(\gamma)$ represents the matrix $X(\gamma, \gamma)$ and γ^c is the complement of γ .

Lemma 2 ([23]). Let matrix $X \in \mathbb{R}^{(m+n) \times (m+n)}$ be divided into

$$X = \begin{pmatrix} E & F \\ G & H \end{pmatrix},$$

where $E \in \mathbb{R}^{m \times m}$, $F \in \mathbb{R}^{m \times n}$, $G \in \mathbb{R}^{n \times m}$ and $H \in \mathbb{R}^{n \times n}$. If X and E are invertible matrix, then X/E is also invertible matrix and

$$X^{-1} = \begin{pmatrix} E^{-1} + E^{-1}F(X/E)^{-1}GE^{-1} & -E^{-1}F(X/E)^{-1} \\ -(X/E)^{-1}GE^{-1} & (X/E)^{-1} \end{pmatrix},$$

where $X/E = H - GE^{-1}F$.

Lemma 3 ([23]). If matrix X is defined as Lemma 2, then

$$(X/\gamma)^{-1} = X^{-1}(\gamma^c).$$

Lemma 4 ([22]). Let the Hermitian matrix $N \in \mathbb{C}^{(p+q) \times (p+q)}$ be divided into

$$N = \begin{pmatrix} J & Z \\ Z^* & K \end{pmatrix},$$

where $J \in \mathbb{C}^{p \times p}$, $K \in \mathbb{C}^{q \times q}$, then the following proposition is equivalent

- (1) N is positive definite matrix;
- (2) J and K are positive definite, and $\sigma(J^{-\frac{1}{2}}ZK^{-\frac{1}{2}}) < 1$.

Lemma 5 ([22]). Assume that

$$M = \begin{pmatrix} I_p & Z \\ Z^* & I_q \end{pmatrix} \in \mathbb{R}^{p \times q}.$$

Then M is positive definite (semidefinite) if and only if $\sigma_1(Z) < (<=)1$.

Lemma 6 ([24]). Assume that the singular values $\sigma_i^{(1)}, \sigma_i^{(2)}, \dots, \sigma_i^{(m)}$ of $B_1, B_2, \dots, B_m \in \mathbb{C}^{n \times n}$ satisfy $\sigma_1^{(j)} \geq \sigma_2^{(j)} \geq \dots \geq \sigma_n^{(j)}$, $j = 1, 2, \dots, m$, respectively. Then the singular values σ_i ($i = 1, 2, \dots, n$) of the product matrix $B = B_1 B_2 \dots B_m$ satisfy

$$\sum_{i=1}^k \sigma_i \leq \sum_{i=1}^k \prod_{j=1}^m \sigma_i^{(j)} \leq \left\{ \prod_{j=1}^m \sum_{i=1}^k [\sigma_i^{(j)}] \right\}^{\frac{1}{m}}, \quad 1 \leq k \leq n.$$

2.2. The Bound Estimates of UALE's Solutions

Applying the above lemmas, we can further analyze UALE. Note that we always set $\gamma = \{1, 2, \dots, n\}$.

Theorem 1. Consider UALE (8) and define the matrix function

$$F(P) = (\theta A + I)^T P (\theta A + I) + \theta Q, \quad (9)$$

then

$$F(P) = \left[\theta Q^{\frac{1}{2}} (\theta A + I)^T (P^{-1} - (\theta A + I) F(P)^{-1} (\theta A + I)^T)^{-1} \cdot (\theta A + I) Q^{\frac{1}{2}} + \frac{1}{4} \theta^2 Q^2 \right]^{\frac{1}{2}} Q^{\frac{1}{2}} + \frac{1}{2} \theta Q. \quad (10)$$

Proof. Combining the matrix function (9) and the definition of the Schur complement, we have

$$\theta Q = F(P) - (\theta A + I)^T P (\theta A + I) = \left(\begin{array}{cc} F(P) & (\theta A + I)^T \\ \theta A + I & P^{-1} \end{array} \right) / \gamma^c. \quad (11)$$

Applying Lemma 3 to (11), we can obtain

$$\theta Q = \left[\left(\begin{array}{cc} F(P) & (\theta A + I)^T \\ \theta A + I & P^{-1} \end{array} \right)^{-1} (\gamma) \right]^{-1},$$

i.e.,

$$(\theta Q)^{-1} = \left(\begin{array}{cc} F(P) & (\theta A + I)^T \\ \theta A + I & P^{-1} \end{array} \right)^{-1} (\gamma). \quad (12)$$

Applying Lemma 2 to (12), then

$$(\theta Q)^{-1} = F(P)^{-1} + F(P)^{-1} (\theta A + I)^T [P^{-1} - (\theta A + I) \cdot (F(P))^{-1} (\theta A + I)^T]^{-1} (\theta A + I) F(P)^{-1}. \quad (13)$$

Multiplying both sides of (13) by $\theta^{\frac{1}{2}} F(P)$, we have

$$F(P) Q^{-1} F(P) = \theta F(P) + \theta (\theta A + I)^T [P^{-1} - (\theta A + I) \cdot (F(P))^{-1} (\theta A + I)^T]^{-1} (\theta A + I). \quad (14)$$

Multiplying both sides of (14) by $Q^{-\frac{1}{2}}$ and define $\tilde{F}(P) = Q^{-\frac{1}{2}} F(P) Q^{-\frac{1}{2}}$, it follows that

$$\left(\tilde{F}(P) - \frac{1}{2} \theta I \right)^2 = \theta Q^{-\frac{1}{2}} (\theta A + I)^T [P^{-1} - (\theta A + I) F(P)^{-1} (\theta A + I)^T]^{-1} \cdot (\theta A + I) Q^{-\frac{1}{2}} + \frac{1}{4} \theta^2 I. \quad (15)$$

Since $F(P) \geq \theta Q \geq \frac{1}{2} \theta Q$, then $\tilde{F}(P) \geq \frac{1}{2} \theta I$. Therefore, we have

$$\tilde{F}(P) - \frac{1}{2} \theta I = \left[\theta Q^{-\frac{1}{2}} (\theta A + I)^T [P^{-1} - (\theta A + I) (F(P))^{-1} \cdot (\theta A + I)^T]^{-1} (\theta A + I) Q^{-\frac{1}{2}} + \frac{1}{4} \theta^2 I \right]^{\frac{1}{2}},$$

i.e.,

$$F(P) = \left[\theta Q^{\frac{1}{2}} (\theta A + I)^T [P^{-1} - (\theta A + I)(F(P))^{-1} (\theta A + I)^T]^{-1} \cdot (\theta A + I) Q^{\frac{1}{2}} + \frac{1}{4} \theta^2 Q^2 \right]^{\frac{1}{2}} + \frac{1}{2} \theta Q.$$

Thus, the proof is completed. $\square$

In [5], Zhang et al. give the upper bound of the positive definite solution of UALE.

Lemma 7 ([5]). Assume that the positive definite matrix P is the solution of UALE (8) and $\sigma_1(\theta A + I) < 1$, then P has an upper bound

$$P \leq \eta(\theta A + I)^T (\theta A + I) + \theta Q \equiv \Gamma, \quad (16)$$

$$\text{where } \eta = \frac{\lambda_1(\theta Q)}{1 - \sigma_1^2(\theta A + I)}.$$

Theorem 2. Let $A \in \mathbb{R}^{p \times q}$, $\sigma_1(\Gamma^{\frac{1}{2}}) \sigma_1[(\theta A + I)^T] \sigma_1[(\theta Q)^{\frac{1}{2}}] < 1$, where Γ is defined as in Lemma 7. Then

$$\begin{aligned} \Gamma^{-1} - (\theta A + I)(\theta Q)^{-1}(\theta A + I)^T &> 0, \\ P^{-1} - (\theta A + I)P^{-1}(\theta A + I)^T &> 0. \end{aligned}$$

Proof. In Lemma 4, set $J = \Gamma^{-1}$, $Z = (\theta A + I)^T$, $K = (\theta Q)^{-1}$, then

$$N = \begin{pmatrix} \Gamma^{-1} & (\theta A + I)^T \\ \theta A + I & (\theta Q)^{-1} \end{pmatrix}.$$

Since $\sigma_1(\Gamma^{\frac{1}{2}}) \sigma_1[(\theta A + I)^T] \sigma_1[(\theta Q)^{\frac{1}{2}}] < 1$, we can obtain $N > 0$ by Lemma 4. Further, we have $N/\Gamma^{-1} > 0$ by using Lemma 2, i.e.,

$$\Gamma^{-1} - (\theta A + I)(\theta Q)^{-1}(\theta A + I)^T > 0.$$

Due to $\theta Q \leq P \leq \Gamma$, then

$$P^{-1} - (\theta A + I)P^{-1}(\theta A + I)^T \geq \Gamma^{-1} - (\theta A + I)(\theta Q)^{-1}(\theta A + I)^T > 0.$$

Consequently, the proof of this theorem is completed. $\square$

Combined with Theorem 1 and Theorem 2, we can give a lower matrix bound of the positive definite solution P of UALE (8).

Theorem 3. Assume that P is the positive definite solution of UALE (8) and $\sigma_1(\Gamma^{\frac{1}{2}}) \sigma_1(\theta A + I) \sigma_1[(\theta Q)^{\frac{1}{2}}] < 1$, then

$$P \geq \left[\theta Q^{\frac{1}{2}} (\theta A + I)^T \theta Q (\theta A + I) Q^{\frac{1}{2}} + \frac{1}{4} \theta^2 Q^2 \right]^{\frac{1}{2}} Q^{\frac{1}{2}} + \frac{1}{2} \theta Q \equiv \bar{P}_1.$$

Proof. Using the definition of $F(P)$ given in Lemma 1, we have $F(P) = P$ and

$$\begin{aligned} P &= \left[\theta Q^{\frac{1}{2}} (\theta A + I)^T [P^{-1} - (\theta A + I)P^{-1} \right. \\ &\quad \left. \cdot (\theta A + I)^T]^{-1} (\theta A + I) Q^{\frac{1}{2}} + \frac{1}{4} \theta^2 Q^2 \right]^{\frac{1}{2}} + \frac{1}{2} \theta Q. \end{aligned} \quad (17)$$

Since $P \geq \theta Q > 0$, then $P^{-1} \leq (\theta Q)^{-1}$. According to Theorem 2, it follows that

$$\begin{aligned} P &\geq \left[\theta Q^{\frac{1}{2}} (\theta A + I)^T P (\theta A + I) Q^{\frac{1}{2}} + \frac{1}{4} \theta^2 Q^2 \right]^{\frac{1}{2}} Q^{\frac{1}{2}} + \frac{1}{2} \theta Q \\ &\geq \left[\theta Q^{\frac{1}{2}} (\theta A + I)^T \theta Q (\theta A + I) Q^{\frac{1}{2}} + \frac{1}{4} \theta^2 Q^2 \right]^{\frac{1}{2}} Q^{\frac{1}{2}} + \frac{1}{2} \theta Q \equiv \bar{P}_1. \end{aligned}$$

Hence, we complete the proof. $\square$

Next, we give an upper and lower matrix bound of UALE's solution P . Before proving the main result, we can give a sufficient condition that the submatrix of the block matrix is the positive definite matrix by using Lemmas 5 and 6.

Theorem 4. Assume that the Hermitian matrix E is segmented as

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^* & E_{11} \end{pmatrix} \in \mathbb{C}^{2n \times 2n},$$

where $E_{11} \in \mathbb{C}^{n \times n}$ is Hermitian positive definite matrix. If $\sigma_1(E_{12})\lambda_1(E_{11}^{-1}) < 1$, then

$$E/E_{11} = E_{11} - E_{12}^* E_{11}^{-1} E_{12} > 0.$$

Proof. Since E_{11} is the Hermitian positive definite matrix, there exists a unitary matrix U such that

$$U^* E_{11} U = \text{diag}(\lambda_1(E_{11}), \lambda_2(E_{11}), \dots, \lambda_n(E_{11})) = \Lambda,$$

and $\lambda_j(E_{11}) > 0$, $j = 1, 2, \dots, n$. Denote

$$\Lambda_1 = \text{diag}\left(\frac{1}{\sqrt{\lambda_1(E_{11})}}, \frac{1}{\sqrt{\lambda_2(E_{11})}}, \dots, \frac{1}{\sqrt{\lambda_n(E_{11})}}\right),$$

then $V^* E_{11} V = I_n$ with $V = U \Lambda_1$. Let $\bar{V} = \text{diag}(V, V)$, thus

$$\bar{V}^* E \bar{V} = \begin{pmatrix} V^* & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^* & E_{11} \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} I_n & S \\ S^* & I_n \end{pmatrix},$$

where $S = V^* E_{12} V$. Applying Lemma 6, we have

$$\begin{aligned} \sigma_1(S) &= \sigma_1(\Lambda_1^* U^* E_{12} U \Lambda_1) \\ &\leq \sigma_1(\Lambda_1^*) \sigma_1(U^*) \sigma_1(E_{12}) \sigma_1(U) \sigma_1(\Lambda_1) \\ &\leq \sigma_1^2(U) \sigma_1(E_{12}) \sigma_1^2(\Lambda_1) \\ &= \frac{\sigma_1(E_{12})}{\lambda_n(E_{11})} = \sigma_1(E_{12}) \lambda_1(E_{11}^{-1}) < 1. \end{aligned}$$

According to Lemma 5, the matrix E is a positive definite matrix. Therefore, the Schur complement E/E_{11} is a positive definite matrix. $\square$

Combination with Theorems 2 and 4, we can give an upper and lower matrix bound of the positive definite solution P of UALE (8).

Theorem 5. Assume that P is a positive definite solution of UALE (8) and $\sigma_1(\theta A + I) < 1$, $\sigma_1(\theta A + I) \lambda_1(\theta Q) < 1$, $\sigma_1(\Gamma^{\frac{1}{2}}) \sigma_1(\theta A + I) \sigma_1[(\theta Q)^{\frac{1}{2}}] < 1$, where Γ is defined by Lemma 7. Then

$$\tilde{P}_1 \leq P \leq \hat{P}_1,$$

where

$$\begin{aligned}\tilde{P}_1 &\equiv \left[\theta Q^{\frac{1}{2}}(\theta A + I)^T [(\theta Q)^{-1} - (\theta A + I)\Gamma^{-1}(\theta A + I)^T]^{-1} \right. \\ &\quad \cdot (\theta A + I)Q^{\frac{1}{2}} + \frac{1}{4}\theta^2 Q^2 \Big]^{\frac{1}{2}} + \frac{1}{2}\theta Q, \\ \hat{P}_1 &\equiv \left[\theta Q^{\frac{1}{2}}(\theta A + I)^T [\Gamma^{-1} - (\theta A + I)(\theta Q)^{-1}(\theta A + I)^T]^{-1} \right. \\ &\quad \cdot (\theta A + I)Q^{\frac{1}{2}} + \frac{1}{4}\theta^2 Q^2 \Big]^{\frac{1}{2}} + \frac{1}{2}\theta Q.\end{aligned}$$

Proof. Due to $P^{-1} \leq (\theta Q)^{-1}$, we have

$$(\theta Q)^{-1} - (\theta A + I)P^{-1}(\theta A + I)^T \geq (\theta Q)^{-1} - (\theta A + I)(\theta Q)^{-1}(\theta A + I)^T. \quad (18)$$

Meanwhile, according to the definition of Γ , it follows that $\Gamma \geq \theta Q$ and $\Gamma^{-1} \leq (\theta Q)^{-1}$. Thus,

$$(\theta Q)^{-1} - (\theta A + I)\Gamma^{-1}(\theta A + I)^T \geq (\theta Q)^{-1} - (\theta A + I)(\theta Q)^{-1}(\theta A + I)^T. \quad (19)$$

Since $\sigma_1(\theta A + I)\lambda_1(\theta Q) < 1$ and let $E_{11} = (\theta Q)^{-1}$, $E_{12} = (\theta A + I)^T$ in Theorem 4, we can obtain

$$(\theta Q)^{-1} - (\theta A + I)(\theta Q)^{-1}(\theta A + I)^T > 0. \quad (20)$$

Similarly, we can prove that

$$\begin{cases} (\theta Q)^{-1} - (\theta A + I)P^{-1}(\theta A + I)^T > 0, \\ (\theta Q)^{-1} - (\theta A + I)\Gamma^{-1}(\theta A + I)^T > 0. \end{cases} \quad (21)$$

Thus,

$$\begin{aligned}P &\geq \left[\theta Q^{\frac{1}{2}}(\theta A + I)^T [(\theta Q)^{-1} - (\theta A + I)P^{-1}(\theta A + I)^T]^{-1} \right. \\ &\quad \cdot (\theta A + I)Q^{\frac{1}{2}} + \frac{1}{4}\theta^2 Q^2 \Big]^{\frac{1}{2}} + \frac{1}{2}\theta Q \\ &\geq \left[\theta Q^{\frac{1}{2}}(\theta A + I)^T [(\theta Q)^{-1} - (\theta A + I)\Gamma^{-1}(\theta A + I)^T]^{-1} \right. \\ &\quad \cdot (\theta A + I)Q^{\frac{1}{2}} + \frac{1}{4}\theta^2 Q^2 \Big]^{\frac{1}{2}} + \frac{1}{2}\theta Q \equiv \tilde{P}_1,\end{aligned}$$

and

$$\begin{aligned}P &\leq \left[\theta Q^{\frac{1}{2}}(\theta A + I)^T [\Gamma^{-1} - (\theta A + I)P^{-1}(\theta A + I)^T]^{-1} \right. \\ &\quad \cdot (\theta A + I)Q^{\frac{1}{2}} + \frac{1}{4}\theta^2 Q^2 \Big]^{\frac{1}{2}} + \frac{1}{2}\theta Q \\ &\leq \left[\theta Q^{\frac{1}{2}}(\theta A + I)^T [\Gamma^{-1} - (\theta A + I)(\theta Q)^{-1}(\theta A + I)^T]^{-1} \right. \\ &\quad \cdot (\theta A + I)Q^{\frac{1}{2}} + \frac{1}{4}\theta^2 Q^2 \Big]^{\frac{1}{2}} + \frac{1}{2}\theta Q \equiv \hat{P}_1.\end{aligned}$$

□

Remark 1. From the derivation of the previous Lemma, the following inequality clearly holds

$$\begin{aligned}\tilde{P}_1 &= \left[\theta Q^{\frac{1}{2}} (\theta A + I)^T [(\theta Q)^{-1} - (\theta A + I) \Gamma^{-1} (\theta A + I)^T]^{-1} \right. \\ &\quad \cdot \left. (\theta A + I) Q^{\frac{1}{2}} + \frac{1}{4} \theta^2 Q^2 \right]^{\frac{1}{2}} + \frac{1}{2} \theta Q \\ &\geq \left[\theta Q^{\frac{1}{2}} (\theta A + I)^T \theta Q (\theta A + I) Q^{\frac{1}{2}} + \frac{1}{4} \theta^2 Q^2 \right]^{\frac{1}{2}} + \frac{1}{2} \theta Q = \bar{P}_1. \\ \bar{P}_1 &= \left[\theta Q^{\frac{1}{2}} (\theta A + I)^T \theta Q (\theta A + I) Q^{\frac{1}{2}} + \frac{1}{4} \theta^2 Q^2 \right]^{\frac{1}{2}} + \frac{1}{2} \theta Q \\ &\geq \left[0 + \frac{1}{4} \theta^2 Q^2 \right]^{\frac{1}{2}} + \frac{1}{2} \theta Q = \theta Q,\end{aligned}$$

i.e., $\tilde{P}_1 \geq \bar{P}_1 \geq \theta Q$.

Referring to the proof of the existence and uniqueness of the solution of the Lyapunov equation in [10], this paper can similarly provide the analysis of the solution of UALE (see Theorem 5). Next, we prove the existence and uniqueness of the positive definite solution of UALE (8) based on its upper and lower matrix bound given in Theorem 5.

Theorem 6. Consider UALE (8) and assume that $\sigma_1(\theta A + I) < 1$, $\sigma_1(\theta A + I) \lambda_1(\theta Q) < 1$, $\sigma_1(\Gamma^{\frac{1}{2}}) \sigma_1(\theta A + I) \sigma_1[(\theta Q)^{\frac{1}{2}}] < 1$, $(\theta A + I)^T \hat{P}_1 (\theta A + I) + \theta Q \leq \Gamma$ and $\|(\theta A + I)\|^2 \cdot \|\tilde{P}_1^{-1}\|^2 \cdot \|\hat{P}_1^{-1}\|^{-2} < 1$, where the definition of Γ given in Lemma 7. Then UALE (8) has a unique positive definite solution P_+ and

$$\tilde{P}_1 \leq P_+ \leq \hat{P}_1,$$

where $\tilde{P}_1$ and $\hat{P}_1$ are defined as in Theorem 5.

Proof. The proof is similar to [10], and we will not give the detailed proof here. $\square$

3. Numerical Algorithms

As far as we know, the analytic solution of the Lyapunov equation is not easy to solve. Thus, the numerical solution of UALE is a research hotspot and formidable task. However, there are few numerical methods for solving UALE. Numerical methods for solving equations mainly include direct method and iterative method, and the latter is mainly studied in our paper. Hence, based on an equivalent transformation, three numerical algorithms are proposed to solve the positive definite solution of UALE (1).

In this section, we firstly propose FP to solve UALE. Furthermore, we present two iterative methods, including AFP and ADI, to find the positive definite solution of UALE. Meanwhile, we give the convergence analysis of these three algorithms.

3.1. Fixed Point Algorithm

For any given an initial iteration matrix $P_0 \in \Omega = \{P \mid \tilde{P}_1 \leq P \leq \hat{P}_1\}$, the iterative scheme of FP is

$$P_{k+1} = (\theta A + I)^T P_k (\theta A + I) + \theta Q, \quad k = 0, 1, \dots \quad (22)$$

Next, we prove that the matrix sequence $\{P_k\}$ generated by (22) converges to the positive definite solution of UALE (8).

Theorem 7. Assume that the conditions given in the Theorem 6 are satisfied, the sequence $\{P_k\}$ generated by the iterative scheme (22) converges, and converges to the unique positive definite solution P_+ of UALE (8).

Based on FP, we introduce a relaxation parameter ω to speed up its convergence, which we call the acceleration fixed point iteration algorithm (AFP). Its basic idea is to make a weighted average between the $k + 1$ step approximate solution and the k step approximate solution for FP. Then, we can obtain a new approximate solution, that is

$$P_{k+1} = (1 - \omega)P_k + \omega[(\theta A + I)^T P_k (\theta A + I) + \theta Q], \quad (23)$$

where ω is the relaxation parameter.

3.2. Alternating Direction Implicit Algorithm

3.2.1. Iterative Scheme

This subsection mainly introduces ADI to solve UALE (8) and gives its convergence analysis. An iterative scheme of ADI to solve UALE (8) is

$$\begin{cases} \alpha P_{k+\frac{1}{2}} + (I + \frac{1}{2}\theta A^T)P_{k+\frac{1}{2}}A = \alpha P_k - A^T P_k (I + \frac{1}{2}\theta A) - Q, \\ \alpha P_{k+1} + A^T P_{k+1} (I + \frac{1}{2}\theta A) = \alpha P_{k+\frac{1}{2}} - (I + \frac{1}{2}\theta A^T)P_{k+\frac{1}{2}}A - Q, \end{cases} \quad (24)$$

where $k = 0, 1, \dots$ and α is the iteration parameter. Next, we briefly review the relevant definitions and properties. For given matrices $A = (a_{ij}) \in \mathbb{C}^{n_A \times m_A}$ and $B \in \mathbb{C}^{n_B \times m_B}$, the Kronecker product is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m_A}B \\ a_{21}B & a_{22}B & \cdots & a_{2m_A}B \\ \vdots & \vdots & & \vdots \\ a_{n_A1}B & a_{n_A2}B & \cdots & a_{n_Am_A}B \end{pmatrix} \in \mathbb{C}^{(n_A n_B) \times (m_A m_B)},$$

and the vectorization operator $\text{vec} : \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{mn}$:

$$\text{vec}(X) = (x_1^T, x_2^T, \dots, x_n^T)^T \in \mathbb{C}^{mn}, \text{ with } X = (x_1, x_2, \dots, x_n) \in \mathbb{C}^{m \times n}.$$

Some properties related to Kronecker products can be found in [25] and we will not detail them here.

Applying the operator vec to the matrix iteration scheme (24), we can obtain the linear iteration format of the obtained Equation (8)

$$\begin{cases} (\alpha I + A^T \otimes (I + \frac{1}{2}\theta A^T))\text{vec}(P_{k+\frac{1}{2}}) = (\alpha I - (I + \frac{1}{2}\theta A^T) \otimes A^T)\text{vec}(P_k) + \text{vec}(-Q), \\ (\alpha I + (I + \frac{1}{2}\theta A^T) \otimes A^T)\text{vec}(P_{k+1}) = (\alpha I - A^T \otimes (I + \frac{1}{2}\theta A^T))\text{vec}(P_{k+\frac{1}{2}}) + \text{vec}(-Q), \end{cases} \quad (25)$$

where $k = 0, 1, \dots$. Therefore, the following iteration scheme

$$\text{vec}(P_{k+1}) = G \text{vec}(P_k) + H, \quad (26)$$

where

$$\begin{aligned} G &= [\alpha I + (I + \frac{1}{2}\theta A^T) \otimes A^T]^{-1} [\alpha I - A^T \otimes (I + \frac{1}{2}\theta A^T)] \\ &\quad \cdot [\alpha I + A^T \otimes (I + \frac{1}{2}\theta A^T)]^{-1} [\alpha I - (I + \frac{1}{2}\theta A^T) \otimes A^T], \\ H &= [\alpha I + (I + \frac{1}{2}\theta A^T) \otimes A^T]^{-1} [\alpha I - A^T \otimes (I + \frac{1}{2}\theta A^T)] \\ &\quad \cdot [\alpha I + A^T \otimes (I + \frac{1}{2}\theta A^T)]^{-1} \text{vec}(-Q) + [\alpha I + (I + \frac{1}{2}\theta A^T) \otimes A^T]^{-1} \text{vec}(-Q), \end{aligned}$$

and G is called the iterative matrix.

3.2.2. Convergence Analysis of ADI

To prove the convergence of ADI, we give the following definition and lemma.

Definition 2. If $A \in \mathbb{R}^{n \times n}$ satisfy

$$\lambda(A) \in \{z \in \mathbb{C} : \operatorname{Re} z > 0\},$$

then the matrix A is called the N -stable matrix.

Lemma 8 ([22]). Assume that the eigenvalue of the n order Hermitian matrix A satisfied $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$. Then

$$(1) \lambda_n(A) \leq R(x) \leq \lambda_1(A),$$

$$(2) \lambda_n(A) = \min_{x \neq 0} R(x), \lambda_1(A) = \max_{x \neq 0} R(x),$$

where $R(x) = \frac{x^*Ax}{x^*x}$ represents the Rayleigh quotient of A .

Theorem 8. Assume that the conditions given in Theorem 6 are satisfied and $\alpha[A^T \otimes (I + \frac{1}{2}\theta A^T)]$ is N -stable matrix. Then the iterative scheme (24) converges to the positive definite solution of UALE (8).

Proof. Obviously, we need prove that $\rho(G) < 1$ is true and

$$\rho(G) = \rho\{[\alpha I + (I + \frac{1}{2}\theta A^T) \otimes A^T]^{-1}[\alpha I - A^T \otimes (I + \frac{1}{2}\theta A^T)] \\ \cdot [\alpha I + A^T \otimes (I + \frac{1}{2}\theta A^T)]^{-1}[\alpha I - (I + \frac{1}{2}\theta A^T) \otimes A^T]\}.$$

Denote

$$\hat{G} = [\alpha I - A^T \otimes (I + \frac{1}{2}\theta A^T)][\alpha I + A^T \otimes (I + \frac{1}{2}\theta A^T)]^{-1} \\ \cdot [\alpha I - (I + \frac{1}{2}\theta A^T) \otimes A^T][\alpha I + (I + \frac{1}{2}\theta A^T) \otimes A^T]^{-1}.$$

We can easily see that

$$G = [\alpha I + (I + \frac{1}{2}\theta A^T) \otimes A^T]^{-1} \hat{G} [\alpha I + (I + \frac{1}{2}\theta A^T) \otimes A^T].$$

Therefore, we can obtain

$$\rho(G) = \rho(\hat{G}) = \rho(\hat{G}_1 \hat{G}_2) = \|\hat{G}_1 \hat{G}_2\| \leq \|\hat{G}_1\| \cdot \|\hat{G}_2\|,$$

where

$$\begin{cases} \hat{G}_1 = [\alpha I - A^T \otimes (I + \frac{1}{2}\theta A^T)][\alpha I + A^T \otimes (I + \frac{1}{2}\theta A^T)]^{-1}, \\ \hat{G}_2 = [\alpha I - (I + \frac{1}{2}\theta A^T) \otimes A^T][\alpha I + (I + \frac{1}{2}\theta A^T) \otimes A^T]^{-1}. \end{cases}$$

Thus

$$\begin{aligned} \|\hat{G}_1\|^2 &= \|[\alpha I - A^T \otimes (I + \frac{1}{2}\theta A^T)][\alpha I + A^T \otimes (I + \frac{1}{2}\theta A^T)]^{-1}\|^2 \\ &= \max_{\|x\|=1} \frac{x^*[\alpha I - A^T \otimes (I + \frac{1}{2}\theta A^T)]^*[\alpha I - A^T \otimes (I + \frac{1}{2}\theta A^T)]x}{x^*[\alpha I + A^T \otimes (I + \frac{1}{2}\theta A^T)]^*[\alpha I + A^T \otimes (I + \frac{1}{2}\theta A^T)]x} \\ &= \max_{\|x\|=1} \frac{\alpha^2 - \alpha x^* \left[[A^T \otimes (I + \frac{1}{2}\theta A^T)]^* + A^T \otimes (I + \frac{1}{2}\theta A^T) \right] x + \|[A^T \otimes (I + \frac{1}{2}\theta A^T)]x\|^2}{\alpha^2 + \alpha x^* \left[[A^T \otimes (I + \frac{1}{2}\theta A^T)]^* + A^T \otimes (I + \frac{1}{2}\theta A^T) \right] x + \|[A^T \otimes (I + \frac{1}{2}\theta A^T)]x\|^2}. \end{aligned} \quad (27)$$

Since $A \in \mathbb{R}^{n \times n}$ and $\theta > 0$, it follows that

$$[A^T \otimes (I + \frac{1}{2}\theta A^T)]^* = [A^T \otimes (I + \frac{1}{2}\theta A^T)]^T = A \otimes (I + \frac{1}{2}\theta A). \quad (28)$$

Meanwhile, set y_i is the eigenvector of the matrix $A^T \otimes (I + \frac{1}{2}\theta A^T)$ corresponding to the eigenvalue $\lambda_i[A^T \otimes (I + \frac{1}{2}\theta A^T)]$ and $\|y_i\| = 1$, then

$$\begin{aligned} & y_i^* \{ [A^T \otimes (I + \frac{1}{2}\theta A^T)]^* + A^T \otimes (I + \frac{1}{2}\theta A^T) \} y_i \\ &= \lambda_i [A^T \otimes (I + \frac{1}{2}\theta A^T)] + \bar{\lambda}_i [A^T \otimes (I + \frac{1}{2}\theta A^T)] \\ &= 2\operatorname{Re}\{\lambda_i [A^T \otimes (I + \frac{1}{2}\theta A^T)]\}. \end{aligned}$$

For $\forall x \in \mathbb{C}^n$, $\|x\| = 1$, using Lemma 8, we have

$$\begin{aligned} & x^* \{ [A^T \otimes (I + \frac{1}{2}\theta A^T)]^* + A^T \otimes (I + \frac{1}{2}\theta A^T) \} x \\ &\geq \min y_i^* \{ [A^T \otimes (I + \frac{1}{2}\theta A^T)]^* + [A^T \otimes (I + \frac{1}{2}\theta A^T)] \} y_i \\ &= \min 2\operatorname{Re}\{\lambda_i [A^T \otimes (I + \frac{1}{2}\theta A^T)]\}. \end{aligned} \quad (29)$$

Applying the equalities (28) and (29), it follows that

$$\begin{aligned} & \max_{\|x\|=1} \frac{\alpha^2 - \alpha x^* [A \otimes (I + \frac{1}{2}\theta A) + A^T \otimes (I + \frac{1}{2}\theta A^T)] x + \|[A^T \otimes (I + \frac{1}{2}\theta A^T)] x\|^2}{\alpha^2 + \alpha x^* [A \otimes (I + \frac{1}{2}\theta A) + A^T \otimes (I + \frac{1}{2}\theta A^T)] x + \|[A^T \otimes (I + \frac{1}{2}\theta A^T)] x\|^2} \\ &\leq \max_{\|x\|=1} \frac{\alpha^2 - \alpha x^* [A \otimes (I + \frac{1}{2}\theta A) + A^T \otimes (I + \frac{1}{2}\theta A^T)] x + \|A^T \otimes (I + \frac{1}{2}\theta A^T)\|^2}{\alpha^2 + \alpha x^* [A \otimes (I + \frac{1}{2}\theta A) + A^T \otimes (I + \frac{1}{2}\theta A^T)] x + \|A^T \otimes (I + \frac{1}{2}\theta A^T)\|^2} \\ &\leq \frac{\alpha^2 - 2\alpha \min_{1 \leq i \leq n} \operatorname{Re}\{\lambda_i [A^T \otimes (I + \frac{1}{2}\theta A^T)]\} + \|A^T \otimes (I + \frac{1}{2}\theta A^T)\|^2}{\alpha^2 + 2\alpha \min_{1 \leq i \leq n} \operatorname{Re}\{\lambda_i [A^T \otimes (I + \frac{1}{2}\theta A^T)]\} + \|A^T \otimes (I + \frac{1}{2}\theta A^T)\|^2}. \end{aligned} \quad (30)$$

Therefore, $\alpha \min_{1 \leq i \leq n} \operatorname{Re}\{\lambda_i [A^T \otimes (I + \frac{1}{2}\theta A^T)]\} > 0$ means that the inequality $\rho(\hat{G}_1) < 1$.

Similarly, when $\alpha \min_{1 \leq i \leq n} \operatorname{Re}\{\lambda_i [(I + \frac{1}{2}\theta A^T) \otimes A^T]\} > 0$ means that the inequality $\rho(\hat{G}_2) < 1$.

In fact, since the matrix $A^T \otimes (I + \frac{1}{2}\theta A^T)$ and $(I + \frac{1}{2}\theta A^T) \otimes A^T$ have the same eigenvalues, and the matrix $\alpha [A^T \otimes (I + \frac{1}{2}\theta A^T)]$ is N -stable, thus $\alpha \min_{1 \leq i \leq n} \operatorname{Re}\{\lambda_i [A^T \otimes (I + \frac{1}{2}\theta A^T)]\} > 0$

holds on and the inequality $\rho(\hat{G}) < 1$ is true. $\square$

3.2.3. Selection for the Optimal Parameter

This section discusses the selection of the parameter α . Applying the iteration scheme (26), we can select the parameter α to minimize the spectral radius of the iteration matrix G . Next, we use the following theorem to select the optimal parameter α^* .

Theorem 9. Consider UALE (1) and denote

$$\begin{cases} \omega_1 = \max_{1 \leq i \leq n} |\lambda_i [A^T \otimes (I + \frac{1}{2}\theta A^T)]|, & \nu_1 = \min_{1 \leq i \leq n} \operatorname{Re}\{\lambda_i [A^T \otimes (I + \frac{1}{2}\theta A^T)]\}, \\ \omega_2 = \max_{1 \leq i \leq n} |\lambda_i [(I + \frac{1}{2}\theta A^T) \otimes A^T]|, & \nu_2 = \min_{1 \leq i \leq n} \operatorname{Re}\{\lambda_i [(I + \frac{1}{2}\theta A^T) \otimes A^T]\}. \end{cases}$$

Then, the optimal parameter α^* is

$$\alpha^* = \arg \min_{\alpha} \psi(\alpha) = \arg \min_{\alpha} \frac{\alpha^2 - 2\alpha v_1 + \omega_1^2}{\alpha^2 + 2\alpha v_1 + \omega_1^2} = \omega_1. \quad (31)$$

Proof. Obviously, $\omega_1 = \omega_2$, $v_1 = v_2$. From the proof of Theorem 8, we have

$$\begin{aligned} |\rho(G)|^2 &\leq \|\hat{G}_1\|^2 \|\hat{G}_2\|^2 \\ &\leq \frac{\alpha^2 - 2\alpha \min \operatorname{Re}\{\varphi_i[A^T \otimes (I + \frac{1}{2}\theta A^T)]\} + \|A^T \otimes (I + \frac{1}{2}\theta A^T)\|^2}{\alpha^2 + 2\alpha \min \operatorname{Re}\{\varphi_i[A^T \otimes (I + \frac{1}{2}\theta A^T)]\} + \|A^T \otimes (I + \frac{1}{2}\theta A^T)\|^2} \\ &\quad \cdot \frac{\alpha^2 - 2\alpha \min \operatorname{Re}\{\mu_j[(I + \frac{1}{2}\theta A^T) \otimes A^T]\} + \|(I + \frac{1}{2}\theta A^T) \otimes A^T\|^2}{\alpha^2 + 2\alpha \min \operatorname{Re}\{\mu_j[(I + \frac{1}{2}\theta A^T) \otimes A^T]\} + \|(I + \frac{1}{2}\theta A^T) \otimes A^T\|^2} \\ &= \frac{\alpha^2 - 2\alpha v_1 + \omega_1^2}{\alpha^2 + 2\alpha v_1 + \omega_1^2} \cdot \frac{\alpha^2 - 2\alpha v_2 + \omega_2^2}{\alpha^2 + 2\alpha v_2 + \omega_2^2} = \left(\frac{\alpha^2 - 2\alpha v_1 + \omega_1^2}{\alpha^2 + 2\alpha v_1 + \omega_1^2} \right)^2, \end{aligned}$$

that is

$$\rho(G) \leq \frac{\alpha^2 - 2\alpha v_1 + \omega_1^2}{\alpha^2 + 2\alpha v_1 + \omega_1^2}.$$

Defined function

$$\psi(\alpha) = \frac{\alpha^2 - 2\alpha v_1 + \omega_1^2}{\alpha^2 + 2\alpha v_1 + \omega_1^2} = 1 - \frac{4\alpha v_1}{\alpha^2 + 2\alpha v_1 + \omega_1^2},$$

and it follows that

$$\frac{d\psi(\alpha)}{d\alpha} = \frac{4v_1(\alpha^2 - \omega_1^2)}{(\alpha^2 + 2\alpha v_1 + \omega_1^2)^2}. \quad (32)$$

Therefore, the function $\psi(\alpha)$ minimum point is $\alpha = \omega_1$ and

$$\alpha^* = \arg \min_{\alpha} \psi(\alpha) = \arg \min_{\alpha} \frac{\alpha^2 - 2\alpha v_1 + \omega_1^2}{\alpha^2 + 2\alpha v_1 + \omega_1^2} = \omega_1.$$

□

4. Numerical Results

The purpose of this section is to verify the validity of the upper and lower matrix bounds for the positive definite solutions of UALE (1) in Section 2 and compare with these results given in [5,7,10] by two numerical examples. Furthermore, this section implements the three algorithms, including FP, AFP and ADI, proposed in Section 3. The whole process is performed on a computer with Intel Core 3.20 GHz CPU, 4.00 GB RAM and MATLAB R2017a. For the sake of convenience, we use “IT” to represent the number of iteration steps and denote Y is the full 1 matrix. Set the iteration accuracy $\varepsilon = 1 \times 10^{-6}$, the maximum number of iteration steps $k_{max} = 100$, and define the normalized remaining

$$NRes(P_k) = \frac{\|A^T P_k + P_k A - \theta P_k N P_k + Q\|}{\|A^T P\| + \|P_k A\| + \|\theta P_k N P_k\| + \|Q\|},$$

where P_k is the k th iteration matrix.

4.1. Numerical Examples

Example 1. Consider UALE (1), set $\omega = 1.2$, $\theta = 0.1$,

$$A = \begin{pmatrix} -6.5 & 0.5 & 0.1 \\ -0.3 & -7.5 & 0.3 \\ 0.1 & 0.2 & -12.2 \end{pmatrix}, Q = \begin{pmatrix} 1.69 & 0.2 & -0.05 \\ 0.19 & 1.99 & 1.2 \\ 0.4 & 0.99 & 1.95 \end{pmatrix}.$$

We can calculate

$$\begin{aligned} \sigma_1(\Gamma^{\frac{1}{2}})\sigma_1(\theta A + I)\sigma_1((\theta Q)^{\frac{1}{2}}) &= 0.1065 < 1, \quad \sigma_1(\theta A + I)\lambda_1(\theta Q) = 0.1027 < 1, \\ \lambda_1((\theta A + I)^T \hat{P}_1(\theta A + I) + \theta Q - \Gamma) &= -0.0016 < 0, \quad \sigma_1(\theta A + I) = 0.3318 < 1, \\ \|\theta A + I\|^2 \|\hat{P}_1^{-1}\|^2 \|\tilde{P}_1^{-1}\|^{-2} &= 0.1213 \in [0, 1]. \end{aligned}$$

Thus, the conditions given in Theorem 6 are satisfied and the positive definite solution of UALE (1) can be calculated by FP, AFP or ADI

$$P_+ = \begin{pmatrix} 0.1923 & 0.0240 & -0.0037 \\ 0.0233 & 0.2146 & 0.1147 \\ 0.0380 & 0.0944 & 0.2036 \end{pmatrix}.$$

Futhermore, the upper bound $\hat{P}_1$ and the lower bound $\tilde{P}_1$ given by Theorem 5 can be calculated

$$\hat{P}_1 = \begin{pmatrix} 0.1952 & 0.0243 & -0.0033 \\ 0.0237 & 0.2154 & 0.1142 \\ 0.0379 & 0.0940 & 0.2046 \end{pmatrix}, \quad \tilde{P}_1 = \begin{pmatrix} 0.1893 & 0.0233 & -0.0036 \\ 0.0224 & 0.2132 & 0.1144 \\ 0.0380 & 0.0942 & 0.2035 \end{pmatrix}.$$

Meanwhile, the upper bound $\hat{P}_2$ and the lower bound $\tilde{P}_2$ given in [7] can be calculated

$$\hat{P}_2 = \begin{pmatrix} 8.7915 & -0.2700 & -0.4346 \\ -0.2800 & 12.0715 & -0.0547 \\ 0.0154 & -0.2647 & 33.3609 \end{pmatrix}, \quad \tilde{P}_2 = \begin{pmatrix} -8.8390 & 0.3599 & 0.4220 \\ 0.3499 & -11.9025 & 2.2231 \\ 0.8720 & 2.0131 & -33.3169 \end{pmatrix}.$$

The upper bound of $\hat{P}_3$ and the lower bound $\tilde{P}_3$ given in [5]

$$\hat{P}_3 = \begin{pmatrix} 0.2127 & 0.0236 & -0.0049 \\ 0.0226 & 0.2221 & 0.1213 \\ 0.0401 & 0.1003 & 0.2125 \end{pmatrix}, \quad \tilde{P}_3 = \begin{pmatrix} 0.1810 & 0.0210 & -0.0050 \\ 0.0200 & 0.2053 & 0.1203 \\ 0.0400 & 0.0993 & 0.1998 \end{pmatrix}.$$

The upper bound of $\hat{P}_4$ and the lower bound $\tilde{P}_4$ given in [10]

$$\hat{P}_4 = \begin{pmatrix} 0.2953 & 0.0893 & -0.0009 \\ 0.0883 & 0.2590 & 0.1211 \\ 0.0441 & 0.1001 & 0.2127 \end{pmatrix}, \quad \tilde{P}_4 = \begin{pmatrix} 0.1812 & 0.0267 & -0.0046 \\ 0.0257 & 0.2048 & 0.1201 \\ 0.0404 & 0.0991 & 0.1967 \end{pmatrix}.$$

Therefore, we can obtain

$$\begin{aligned} \lambda_1(\hat{P}_2 - \hat{P}_1) &= -8.9644, \quad \lambda_3(\tilde{P}_2 - \tilde{P}_1) = 8.5681, \quad \lambda_1(\hat{P}_3 - \hat{P}_1) = -3.0162 \times 10^{-5}, \\ \lambda_3(\tilde{P}_3 - \tilde{P}_1) &= 7.1830 \times 10^{-4}, \quad \lambda_1(\hat{P}_4 - \hat{P}_1) = -0.0013, \quad \lambda_3(\tilde{P}_4 - \tilde{P}_1) = 1.6200 \times 10^{-4}, \end{aligned}$$

i.e.,

$$\hat{P}_2 > \hat{P}_1, \quad \hat{P}_3 > \hat{P}_1, \quad \hat{P}_4 > \hat{P}_1, \quad \tilde{P}_2 < \tilde{P}_1, \quad \tilde{P}_3 < \tilde{P}_1, \quad \tilde{P}_4 < \tilde{P}_1,$$

that is, the upper and lower bounds for the solution of UALE (1) have higher accuracy than the upper and lower bounds given in [5,7,10].

When the initial matrix P_0 is selected as Y , the upper bound $\hat{P}_1$ and the lower bound $\tilde{P}_1$, respectively, IT and iteration precision required by these three algorithms to solve UALE are shown in Table 1, respectively. From the numerical results in Table 1, it is not difficult to find that the convergence speed of these numerical algorithms can be accelerated when the upper and lower bounds are used as the initial matrix. Meanwhile, it follows from Table 1 that ADI performs at their best.

Table 1. The numerical results of FP, AFP and ADI when $\theta = 0.1$.

Initial Value	Iterative Algorithm	IT	NRes
$P_0 = Y$	ADI	7	1.9645×10^{-7}
	AFP	8	1.1513×10^{-7}
	FP	13	3.0842×10^{-7}
$P_0 = \hat{P}_1$	ADI	4	6.0366×10^{-7}
	AFP	5	4.4932×10^{-7}
	FP	6	8.2206×10^{-7}
$P_0 = \tilde{P}_1$	ADI	4	7.9131×10^{-7}
	AFP	5	4.6128×10^{-7}
	FP	6	3.3257×10^{-7}

Figure 1, Figure 2 and Figure 3 respectively show the convergence process of the three algorithms to solve UALE (1) by selecting different initial matrix.

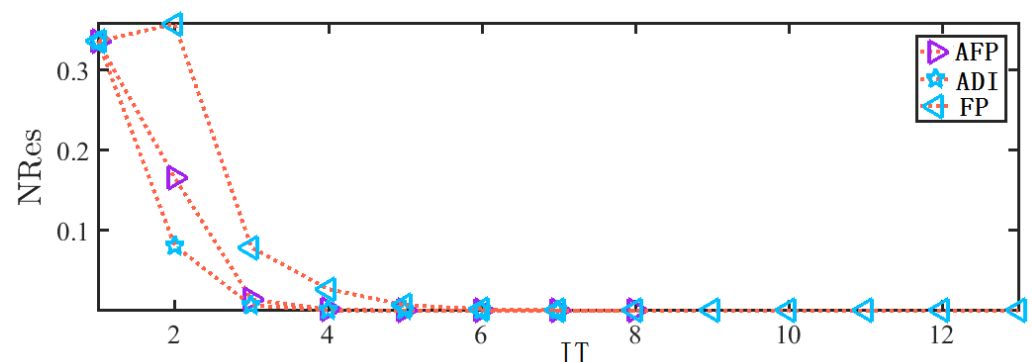


Figure 1. The relation of IT and NRes for the three algorithms when $P_0 = Y$.

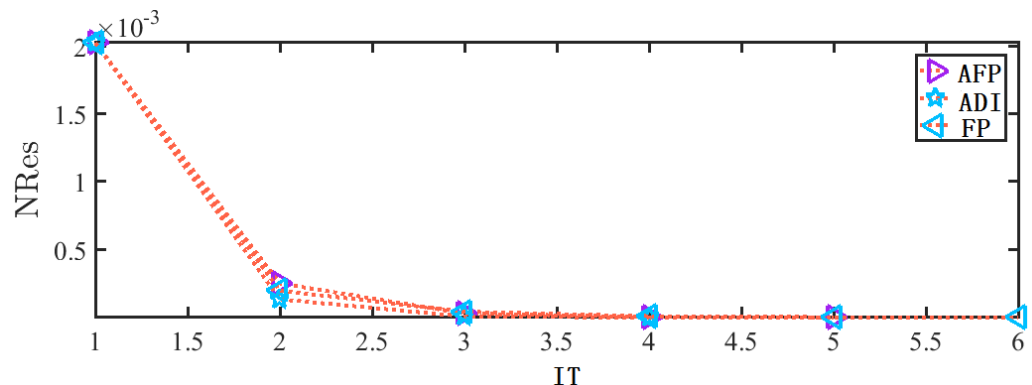


Figure 2. The relation of IT and NRes for the three algorithms $P_0 = \hat{P}_1$.

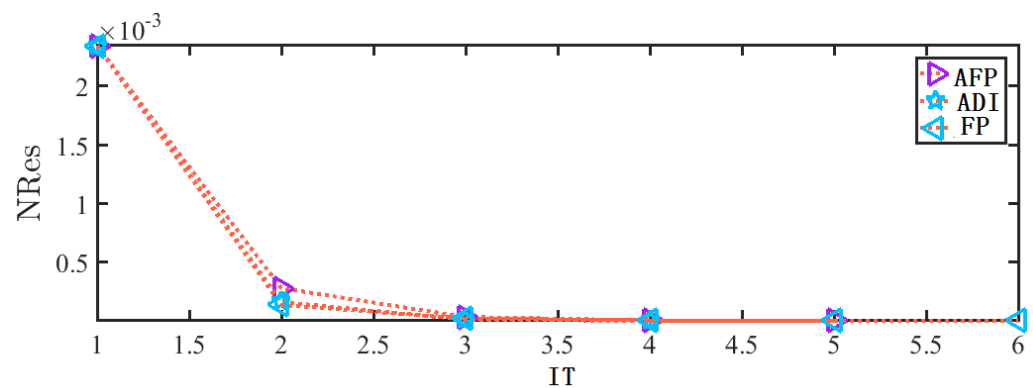


Figure 3. The relation of IT and NRes for the three algorithms $P_0 = \tilde{P}_1$.

Example 2. Consider UALE (1), set $\omega = 1.2$, $\theta = 0.5$,

$$A = \begin{pmatrix} -1.62 & 0.1 & -0.01 & 0.13 \\ 0.035 & -1.51 & 0.35 & -0.025 \\ -0.002 & 0.003 & -1.3 & -0.01 \\ 0.0005 & 0.02 & 0.1 & -1.44 \end{pmatrix}, Q = \begin{pmatrix} 0.95 & 0.75 & 0.05 & 0.18 \\ 0.39 & 0.845 & 0.6 & 0.45 \\ 0.15 & 0.5 & 1.8 & 0.3 \\ 0.25 & 0.1 & 0.915 & 4.21 \end{pmatrix}.$$

We can calculate

$$\begin{aligned} \sigma_1(\Gamma^{\frac{1}{2}})\sigma_1(\theta A + I)\sigma_1((\theta Q)^{\frac{1}{2}}) &= 0.8024 < 1, \quad \sigma_1(\theta A + I)\lambda_1(\theta Q) = 0.7605 < 1, \\ \lambda_1((\theta A + I)^T \hat{P}_1(\theta A + I) + \theta Q - \Gamma) &= -0.0045 < 0, \quad \sigma_1(\theta A + I) = 0.3466 < 1, \\ \|\theta A + I\|^2 \|\hat{P}_1^{-1}\|^2 \|\tilde{P}_1^{-1}\|^{-2} &= 0.1685 \in [0, 1]. \end{aligned}$$

Thus, the conditions given in Theorem 6 are satisfied and the positive definite solution can be calculated by FP, AFP or ADI

$$P_+ = \begin{pmatrix} 0.4950 & 0.4007 & 0.0453 & 0.1020 \\ 0.2121 & 0.4603 & 0.3619 & 0.2532 \\ 0.0919 & 0.3028 & 1.1121 & 0.2155 \\ 0.1386 & 0.0694 & 0.5471 & 2.2883 \end{pmatrix}.$$

Futhermore, the upper bound $\hat{P}_1$ and the lower bound $\tilde{P}_1$ given by Theorem 5 can be calculated

$$\hat{P}_1 = \begin{pmatrix} 0.4998 & 0.4031 & 0.0430 & 0.1031 \\ 0.2160 & 0.4715 & 0.3806 & 0.2585 \\ 0.0948 & 0.3188 & 1.1487 & 0.2103 \\ 0.1432 & 0.0720 & 0.5419 & 2.2997 \end{pmatrix}, \tilde{P}_1 = \begin{pmatrix} 0.4953 & 0.3989 & 0.0395 & 0.1003 \\ 0.2116 & 0.4580 & 0.3556 & 0.2540 \\ 0.0898 & 0.2956 & 1.0646 & 0.2004 \\ 0.1396 & 0.0679 & 0.5316 & 2.2778 \end{pmatrix}.$$

Meanwhile, the upper bound $\hat{P}_2$ and the lower bound $\tilde{P}_2$ given in [7] can be calculated

$$\hat{P}_2 = \begin{pmatrix} 1.8434 & 0.5552 & 0.0562 & 0.0460 \\ 0.1952 & 1.6923 & 0.3578 & 0.5435 \\ 0.1562 & 0.2578 & 1.9006 & 0.2429 \\ 0.1160 & 0.1935 & 0.8579 & 4.5928 \end{pmatrix}, \tilde{P}_2 = \begin{pmatrix} -2.4769 & 1.0234 & 0.0092 & 0.4534 \\ 0.6634 & -2.1406 & 1.2900 & 0.4263 \\ 0.1092 & 1.1900 & -0.6086 & 0.4836 \\ 0.5234 & 0.0763 & 1.0986 & 1.4618 \end{pmatrix}.$$

The upper bound of $\hat{P}_3$ and the lower bound $\tilde{P}_3$ given in [5]

$$\hat{P}_3 = \begin{pmatrix} 0.5718 & 0.4117 & 0.0297 & 0.1224 \\ 0.2317 & 0.5890 & 0.4160 & 0.2329 \\ 0.0797 & 0.3660 & 1.3137 & 0.1759 \\ 0.1574 & 0.0579 & 0.4834 & 2.3251 \end{pmatrix}, \tilde{P}_3 = \begin{pmatrix} 0.4806 & 0.3771 & 0.0253 & 0.0919 \\ 0.1971 & 0.4320 & 0.3067 & 0.2255 \\ 0.0753 & 0.2567 & 0.9237 & 0.1515 \\ 0.1269 & 0.0505 & 0.4590 & 2.1176 \end{pmatrix}.$$

The upper bound of $\hat{P}_3$ and the lower bound $\tilde{P}_3$ given in [10]

$$\hat{P}_4 = \begin{pmatrix} 0.5820 & 0.2036 & -0.4001 & 0.1206 \\ 0.0236 & 1.4627 & 2.5262 & 0.0638 \\ -0.3501 & 2.4762 & 6.5365 & -0.1463 \\ 0.1556 & -0.1112 & 0.1612 & 2.5537 \end{pmatrix}, \tilde{P}_4 = \begin{pmatrix} 0.4761 & 0.3733 & 0.0208 & 0.0903 \\ 0.1933 & 0.4328 & 0.3220 & 0.2234 \\ 0.0708 & 0.2720 & 0.9556 & 0.1471 \\ 0.1253 & 0.0484 & 0.4546 & 2.1094 \end{pmatrix}.$$

Therefore, we can obtain

$$\lambda_1(\hat{P}_2 - \hat{P}_1) = -0.4553, \lambda_4(\tilde{P}_2 - \tilde{P}_1) = 0.7295, \lambda_1(\hat{P}_3 - \hat{P}_1) = -1.7959 \times 10^{-4},$$

$$\lambda_4(\tilde{P}_3 - \tilde{P}_1) = 0.0106, \lambda_1(\hat{P}_4 - \hat{P}_1) = -4.9958 \times 10^{-4}, \lambda_4(\tilde{P}_4 - \tilde{P}_1) = 0.0387,$$

i.e.,

$$\hat{P}_2 > \hat{P}_1, \hat{P}_3 > \hat{P}_1, \hat{P}_4 > \hat{P}_1, \tilde{P}_2 < \tilde{P}_1, \tilde{P}_3 < \tilde{P}_1, \tilde{P}_4 < \tilde{P}_1,$$

that is, the upper and lower bounds for the solution of UALE (1) have higher accuracy than the upper and lower bounds given in [5,7,10].

The corresponding numerical results are shown in the following Figures 4–6 and Table 2. Since the analysis is similar to that of the above example, it will not be described here.

Table 2. The numerical results of FP, AFP and ADI when $\theta = 0.5$.

Initial Value	Iterative Algorithm	IT	NRes
$P_0 = Y$	ADI	4	6.5351×10^{-7}
	AFP	8	4.4291×10^{-7}
	FP	8	2.1450×10^{-7}
$P_0 = \hat{P}_1$	ADI	3	8.3175×10^{-7}
	AFP	6	4.0539×10^{-7}
	FP	5	4.6347×10^{-7}
$P_0 = \tilde{P}_1$	ADI	3	8.3175×10^{-7}
	AFP	6	4.9539×10^{-7}
	FP	5	4.6347×10^{-7}

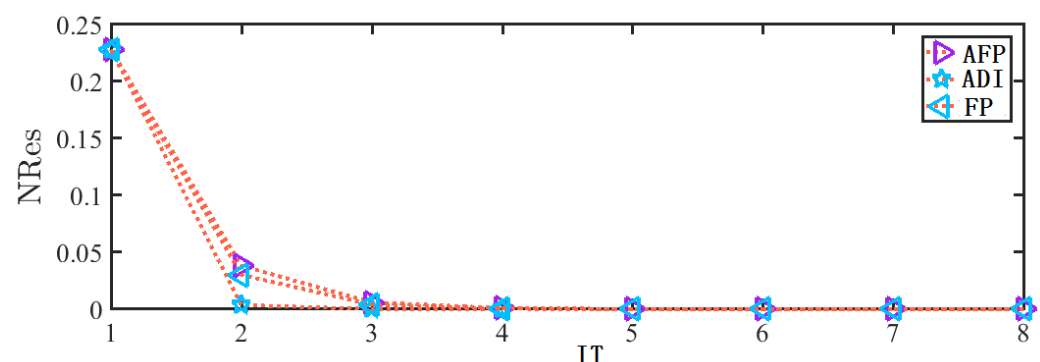


Figure 4. The relation of IT and NRes for the three algorithms when $P_0 = Y$.

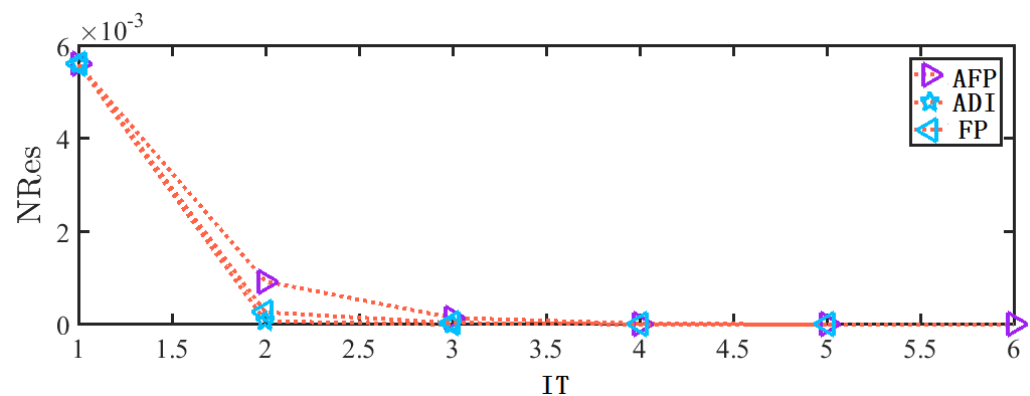


Figure 5. The relation of IT and NRes for the three algorithms when $P_0 = \hat{P}_1$.

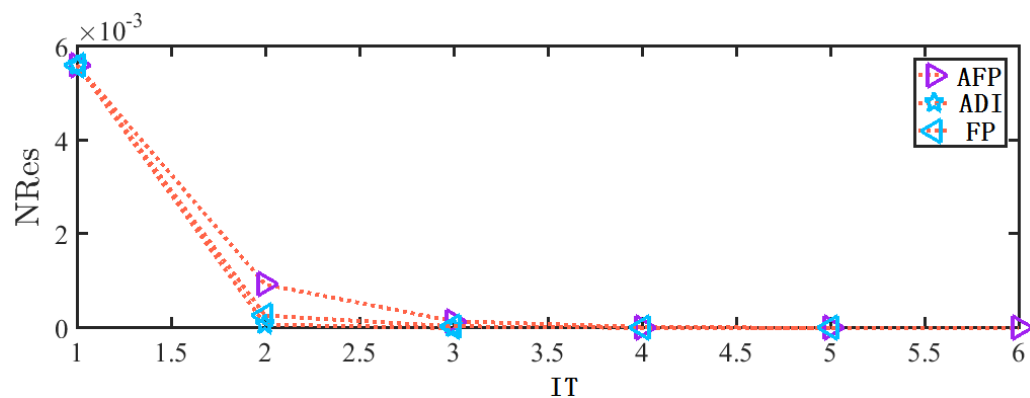


Figure 6. The relation of IT and NRes for the three algorithms when $P_0 = \tilde{P}_1$.

4.2. Discussion of the Numerical Results

Combined with the above numerical results, we can conclude that:

- (i) The iteration format of FP is simple, and each iteration step does not need to solve other unknown equations. But its convergence speed is the slowest among the three algorithms. Therefore, AFP is proposed to accelerate its convergence rate.
- (ii) Although the convergence speed of ADI is the fastest among the three algorithms, its iteration format is relatively complex, in which two unknown Lyapunov equations need to be solved in each iteration step. Meanwhile, its convergence speed is related to the selection of parameters. Therefore, for solving large UALE, the low-rank ADI algorithms and more appropriate parameter selection methods should be considered.
- (iii) This paper only puts forward the simplest acceleration algorithm. How to accelerate the iterative process more effectively is worth our further consideration.

5. Conclusions

UALE studied in this paper has important control system background. Based on the basic matrix inequalities and the properties of Schur complement, we give the upper and lower matrix bounds of positive definite solutions of UALE, and further prove the existence and uniqueness theorem of positive definite solutions of UALE. Meanwhile, we propose three numerical algorithms, including FP, AFP and ADI, to solve the positive definite solution of UALE. Numerical examples are given to demonstrate the good performances of the obtained bounds and the numerical algorithms proposed in this paper.

Author Contributions: Data curation, J.Z., S.L. and X.G.; Funding acquisition, J.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This work was partially supported by the National Natural Science Foundation of China (12171412, 11771370), by the Natural Science Foundation for Distinguished Young Scholars of Hunan Province (2021JJ10037), by the Hunan Youth Science and Technology Innovation Talents Project (2021RC3110), and by the Key Project of Education Department of Hunan Province (19A500, 21A0116).

Conflicts of Interest: We declare that we have no conflict of interest.

References

1. Middleton, R.H.; Goodwin, G.C. *Digital Control and Estimation—A Unified Approach*; Prentice-Hall: Englewood Cliffs, NJ, USA, 1990.
2. Gajic, Z.; Qureshi, M. *Lyapunov Matrix Equation in System Stability and Control*; Mathematics in Science and Engineering Series; Academic Press: San Diego, CA, USA, 1995.
3. Middleton, R.H.; Goodwin, G.C. Improved finite word length characteristics in digital control using delta operators. *IEEE Trans. Automat. Contr.* **1986**, *31*, 1015–1021. [\[CrossRef\]](#)
4. Zhang, D.J.; Yang, C.W. The research and development of feedback control system Delta operator theory. *Control. Theory Appl.* **1998**, *15*, 153–160.
5. Zhang, D.J.; Yang, P.; Wu, J. Matrix bounds for the solution of the unified algebraic Lyapunov equation using Delta operator. *Control. Theory Appl.* **2004**, *21*, 94–96.
6. Lee, C.H. Solution bounds of the continuous and discrete Lyapunov matrix equations. *J. Optim. Theory Appl.* **2004**, *120*, 559–578. [\[CrossRef\]](#)
7. Zhang, J.; Liu, J.Z. New estimates for the solution of the Lyapunov matrix differential equation. *Comput. Technol. Autom.* **2007**, *20*, 6–19. [\[CrossRef\]](#)
8. Lee, C.H.; Chen, C.Y. Matrix solution bounds of the continuous Lyapunov equation by using a bilinear transformation. *J. Frankl. Inst.* **2009**, *346*, 741–751. [\[CrossRef\]](#)
9. Lee, C.H. A unified approach of the measurement of solution bounds of the continuous and discrete algebraic Lyapunov equations. *J. Frankl. Inst.* **2016**, *11*, 2534–2551. [\[CrossRef\]](#)
10. Zhang, J.; Liu, J.Z. Bounds of the solution for the unified algebraic Lyapunov equation using Delta Operator. *Math. Pract. Theory* **2013**, *64*, 377–389.
11. Antoulas, A.C. Approximation of large-scale dynamical systems. In *Advances in Design and Control*; SIAM: Philadelphia, PA, USA, 2005; Volume 6.
12. Benner, P.; Saak, J. Numerical solution of large and sparse continuous time algebraic matrix Riccati and Lyapunov equations: A state of the art survey. *GAMM Mitteilungen* **2013**, *36*, 32–52. [\[CrossRef\]](#)
13. Bartels, R.H.; Stewart, G.W. Solution of the matrix equation $AX + XA = C$: Algorithm 432, Comm. *ACM* **1972**, *15*, 820–826. [\[CrossRef\]](#)
14. Hammarling, S. Numerical solution of the stable, non-negative definite Lyapunov equation. *IMA J. Numer. Anal.* **1982**, *2*, 303–323. [\[CrossRef\]](#)
15. Roberts, J.D. Linear model reduction and solution of the algebraic Riccati equation by use of the sign function, *Internat. J. Control* **1980**, *32*, 677–687. [\[CrossRef\]](#)
16. Benner, P.; Kurschner, P.; Saak, J. Efficient handling of complex shift parameters in the low-rank Cholesky factor ADI method. *Numer. Algorithms* **2013**, *62*, 225–251. [\[CrossRef\]](#)
17. Benner, P.; Li, J.; Penzl, T. Numerical solution of large-scale Lyapunov equations, Riccati equations, and linear-quadratic optimal control problems. *Numer. Linear Algebra Appl.* **2008**, *15*, 755–777. [\[CrossRef\]](#)
18. Gugercin, S.; Sorensen, D.; Antoulas, A. A modified low-rank Smith method for large-scale Lyapunov equations. *Numer. Algorithms* **2003**, *32*, 27–55. [\[CrossRef\]](#)
19. Penzl, T. A cyclic low-rank Smith method for large sparse Lyapunov equations. *SIAM J. Sci. Comput.* **2000**, *21*, 1401–1418. [\[CrossRef\]](#)
20. Druskin, V.; Knizhnerman, L.; Simoncini, V. Analysis of the rational Krylov subspace and ADI methods for solving the Lyapunov equation. *SIAM J. Numer. Anal.* **2011**, *49*, 1875–1898. [\[CrossRef\]](#)
21. Stykel, T.; Simoncini, V. Krylov subspace methods for projected Lyapunov equations. *Appl. Numer. Math.* **2012**, *62*, 35–50. [\[CrossRef\]](#)
22. Horn, R.A.; Johnson, C.R. *Matrix Analysis*; Cambridge University Press: Cambridge, UK, 1985.
23. Zhang, F. *The Schur Complement and Its Applications*; Springer: New York, NY, USA, 2005.
24. Chen, D. Some inequalities for singular values of the matrix. *Math. J.* **1990**, *11A*, 128–132.
25. Golub, G.H.; Loan, C. *Matrix Computations*, 3rd ed.; Johns Hopkins University Press: Baltimore, MD, USA, 1996.