mathematics

Article

Weighted Mean Inactivity Time Function with Applications

Antonio Di Crescenzo [1,*,†] and Abdolsaeed Toomaj [2,*,†]

[1] Dipartimento di Matematica, Università degli Studi di Salerno, Via Giovanni Paolo II n. 132, I-84084 Fisciano, SA, Italy
[2] Department of Mathematics and Statistics, Faculty of Basic Sciences and Engineering, Gonbad Kavous University, Gonbad 4971799151, Iran
* Correspondence: adicrescenzo@unisa.it (A.D.C.); ab.toomaj@gonbad.ac.ir or ab.toomaj@gmail.com (A.T.)
† These authors contributed equally to this work.

Abstract: We consider an extension of the mean inactivity time based on a non-negative weight function. We show various properties of the new notion, and relate it to various functions of interest in reliability theory and information measures, such as the dynamic cumulative entropy, the past entropy, the varentropy, and the weighted cumulative entropy. Moreover, based on the comparison of weighted mean inactivity times, we introduce and study a new stochastic order and compare it with other suitable orders. We also discuss some results about the variance of transformed random variables and the weighted generalized cumulative entropy. Then, we investigate certain connections with the location-independent riskier order. Finally, we pinpoint several characterizations and preservation properties of the new stochastic order under shock models, random maxima, and notions of renewal theory.

Keywords: generalized cumulative entropy; lower record values; mean inactivity time; weighted mean inactivity time function; left spread function; renewal theory; variance

MSC: 60E05; 60E15; 62B10; 62N05; 94A17

check for updates

Citation: Di Crescenzo, A.; Toomaj, A. Weighted Mean Inactivity Time Function with Applications. *Mathematics* **2022**, *10*, 2828. https://doi.org/10.3390/math10162828

Academic Editor: Andreas C. Georgiou

Received: 18 July 2022
Accepted: 6 August 2022
Published: 9 August 2022

1. Introduction and Preliminaries

Over recent decades, various concepts of stochastic orders have been defined and studied in the literature for the sake of their useful applications in reliability and economics and as mathematical tools for proving important results in applied probability. A comprehensive discussion of many stochastic comparisons between random variables is reported and investigated in detail in the monograph given by Shaked and Shanthikumar [1]. The mean inactivity time (MIT) function, also known as the mean past lifetime and the mean waiting time, is a well-known reliability measure which has many applications in various disciplines, such as reliability theory, survival analysis, risk theory, and actuarial studies, among others.

Let X be a non-negative absolutely continuous random variable denoting the lifetime of a system or a component or a living organism, and, henceforth, named random lifetime. We denote by $F(x) = \mathbb{P}(X \le x)$ the cumulative distribution function (CDF) of X, and by $f(x)$ the corresponding probability density function (PDF). Under the condition that the system has been found failed before time t, the inactivity time is defined by $X_{[t]} = [t - X \,|\, X \le t]$. In fact, $X_{[t]}$ denotes a random variable whose distribution is the same as the conditional distribution of $t - X$ given that $X \le t$. It is worth emphasizing that in many realistic situations, the random lifetime can refer to the past. For instance, consider a system whose state is observed only at certain preassigned inspection times. If at time t, the system is inspected for the first time and it is found to be "down", then the failure relies on the past (see, e.g., Kayid and Ahmad [2] and Di Crescenzo and Longobardi [3]). Now, we recall the MIT function of X which is defined by

$$\widetilde{\mu}(t) = \mathbb{E}[t - X | X \leq t] = \frac{1}{F(t)} \int_0^t F(x)\,\mathrm{d}x, \qquad t \in D := \{t \in \mathbb{R}_+ : F(t) > 0\}. \tag{1}$$

An interpretation of the MIT function is as follows: assume that at time t we perform an inspection to a device which started working at time 0, and then we realize that it has already failed. Hence, denoting by X the failure time, the MIT function describes the mean time elapsed between the failure and the inspection time t. The MIT is thus useful to infer on the actual time at which the failure of the device occurred. For further interpretations in survival analysis and mathematical insurance we refer the readers to, e.g., Kayid and Izadkhah [4]. Assuming that $\widetilde{\mu}(t)$ is a differentiable function, from (1) we get

$$\widetilde{\mu}'(t) = 1 - \tau(t)\widetilde{\mu}(t), \qquad t \in D, \tag{2}$$

where

$$\tau(x) = \frac{f(x)}{F(x)}, \qquad x \in D \tag{3}$$

denotes the reversed hazard rate function of X. It is known that the reversed hazard rate and the MIT functions under certain assumptions define uniquely $F(t)$ as follows:

$$F(t) = \exp\left\{ -\int_t^\infty \tau(x)\,\mathrm{d}x \right\} = \exp\left\{ -\int_t^\infty \frac{1 - \widetilde{\mu}'(x)}{\widetilde{\mu}(x)}\,\mathrm{d}x \right\}, \qquad t \in D. \tag{4}$$

As pointed out by Finkelstein [5], relation (4) characterizes distribution functions if the following statements hold: (i) $\widetilde{\mu}(0) = 0$ and $\widetilde{\mu}(x) > 0$ for all $x > 0$; (ii) $\widetilde{\mu}'(x) < 1$; (iii) $\int_0^\infty (1 - \widetilde{\mu}'(x))/\widetilde{\mu}(x)\,\mathrm{d}x = \infty$; and (iv) $\int_t^\infty (1 - \widetilde{\mu}'(x))/\widetilde{\mu}(x)\,\mathrm{d}x < \infty$, for all $t > 0$. It follows from Equation (4) and characterization conditions for $\widetilde{\mu}(x)$ that there is no lifetime distribution with decreasing MIT function. Indeed, $\widetilde{\mu}'(x) < 0$, in this case and condition (iv) does not hold (see Finkelstein [5] for further details).

The MIT function has been the object of several investigations. Kayid and Ahmad [2] (see also Ahmad et al. [6]) studied stochastic comparisons based on the MIT function under the reliability operations of convolution and mixture. Badia and Berrade [7] gave an insight into properties of the MIT in mixtures of distributions. Some further properties of MIT function are widely studied and investigated in Finkelstein [5], Goliforushani and Asadi [8], Kundu and Nanda [9], and the references therein. Moreover, Izadkhah and Kayid [10] used the harmonic mean average of the MIT function to propose a new stochastic order. Recently, Toomaj and Di Crescenzo [11] showed that the variance of a random variable can be represented in terms of the square of the MIT function.

Following the lines of the previous investigations, in the present paper, we aim to define a new version of MIT function, namely the weighted MIT function, and to show some applications of such a measure. In analogy with (1), the weighted MIT function is defined through the expectation $\mathbb{E}[\psi(t) - \psi(X)|X \leq t]$, where ψ is a suitable cumulative weight function. By means of suitable choices of ψ we show that the weighted MIT function can be related to various notions of reliability theory, as well as to several information measures of interest, such as the dynamic cumulative entropy, the past entropy, the varentropy, and the weighted cumulative entropy. In other terms, the introduction of the weight function ψ allows to construct a flexible tool which unifies various notions emerging in different applied fields. Indeed, in Example 3 below we show that by introducing a weight based on the function $\phi(t) = \tau(t)\widetilde{\mu}(t)$ one can recover the dynamic cumulative entropy from the weighted MIT function. Similarly, a suitable choice considered in Example 4 allows to express the weighted MIT function in terms of the past entropy and the reversed hazard rate function. Connections with the varentropy and the weighted cumulative entropy are given in Remark 5 and Section 4.2, respectively. Moreover, the generalized MIT function can be used to extend the MIT stochastic order to the weighted version.

Concerning the the main theoretical contributions of this paper, we refer, first, to the introduction of the weighted mean inactivity time function, which allows to suitably extend the MIT function. For the essential background in this area, see the previously mentioned articles and the recent contribution by Khan et al. [12] and references therein. Secondly, we refer to the introduction of the left spread function, which is analog to the right spread function (also known as excess wealth transform). For the main references on this concept see Unnikrishnan Nair and Vineshkumar [13] and Kochar and Xu [14]. The given function, which is of interest in risk management, is also extended to the weighted version. The latter one is shown to be intimately related to the variance of the weighted random variable and to the weighted generalized cumulative entropy.

Therefore, the rest of this paper is organized as follows: in Section 2, some general properties of the weighted mean inactivity time function are discussed. We provide suitable connections with the Receiver Operating Characteristic (ROC) curve (see Remark 1). We also analyze conditions expressed in terms of the reversed hazard rate function, such that the weighted mean inactivity time function is constant, and also that it is increasing. Section 3 is devoted to introduce the weighted mean inactivity time order. We also analyze its properties and connections to other well-known stochastic orders. In particular, we find additional conditions that allow to relate this order with the reversed hazard rate order. In Section 4, we use the weighted MIT function to obtain expressions and various results for the variance of transformed random variables, as well as for the weighted generalized cumulative entropy. Furthermore, attention is given to the determination of bounds and to the representation of measures in terms of expectations. Section 5 is finalized to investigate some connections of the previous results with the location-independent riskier order. In Section 6, we focus on applications to reliability theory, with special attention to ordering results for a shock model governed by a non-homogeneous Poisson process, and for the maximum of independent and identically distributed (i.i.d.) random variables. Finally, we provide also applications to renewal theory based on the weighted mean inactivity time order, with emphasis on the excess lifetime of a customary renewal process.

Throughout this paper, we deal with non-negative random variables. Additionally, it is assumed that the expectations exist whenever they appear. The mean and the variance of X are denoted, respectively, by $\mathbb{E}(X)$ and $Var(X)$. For simplicity, we write $g^n(x)$ instead of $[g(x)]^n$ for any given function g. Moreover, $g'(x)$ denotes the derivative of $g(x)$, and $G^{-1}(u) = \inf\{x \in \mathbb{R}^+ : G(x) \geq u\}$, $u \in [0,1]$, denotes the left-continuous quantile function of $G(x)$. In addition, "log" denotes the natural logarithm, with the convention $0 \log 0 = 0$, and $\mathbf{1}_B(x)$ is the indicator function, i.e., $\mathbf{1}_B(x) = 1$ when $x \in B$, and $\mathbf{1}_B(x) = 0$ otherwise. The terms increasing and decreasing are used in a non-strict sense.

Finally, given two subsets of the real line $\mathcal{U}$ and $\mathcal{V}$, we say that a non-negative function $K(u,v)$ defined on $\mathcal{U} \times \mathcal{V}$ is *totally positive of order 2*, denoted as TP_2, if $K(u_1,v_1)K(u_2,v_2) \geq K(u_1,v_2)K(u_2,v_1)$ for all $u_1 \leq u_2$ in $\mathcal{U}$ and $v_1 \leq v_2$ in $\mathcal{V}$ (see Karlin [15]).

2. Weighted Mean Inactivity Time Function

The aim of this section is to investigate on the weighted mean inactivity time function by applying *the cumulative weight function*, say. For this aim, we first consider an integrable function $\phi\colon [0,\infty) \to [0,\infty)$. Then, the cumulative weight function is defined as

$$\psi(x) := \int_0^x \phi(u)\,\mathrm{d}u, \qquad x \geq 0. \tag{5}$$

This function plays a pivotal role in achieving our results. Specifically, given the random lifetime X, we will focus on various properties of the transformed random variable $\psi(X)$.

Let $\overline{F}(t) = 1 - F(t)$ be the survival function of X, and let

$$\lambda(x) = \frac{f(x)}{\overline{F}(x)}, \qquad \forall\, x \geq 0 : \overline{F}(x) > 0 \tag{6}$$

denote the hazard rate function of X. For example, if we consider $\phi(x) = \lambda(x)$, then (5) gives the cumulative hazard function of X. Due to (5), it is clear that $\psi(x)$ is an absolutely continuous increasing function for $x > 0$, such that $\psi(0) = 0$ and $\psi'(x) = \phi(x) \geq 0$. Additionally, if the weight function $\phi(x)$ is increasing (decreasing) in $x > 0$, then $\psi(x)$ is convex (concave). This function was successfully applied by Toomaj and Di Crescenzo [16] to provide expressions for the variance of cumulative weighted random variable $\psi(X)$ by defining the *weighted mean residual life* (WMRL) function as

$$m_\psi(t) = \mathbb{E}[\psi(X) - \psi(t) | X > t] = \frac{1}{\overline{F}(t)} \int_t^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x, \tag{7}$$

for all $t \geq 0$, such that $\overline{F}(t) > 0$. In analogy with (7), now we can provide the following:

Definition 1. *Given a random lifetime X and a cumulative weight function defined as in (6), the* weighted mean inactivity time *(WMIT) function of X is defined as*

$$\widetilde{\mu}_\psi(t) = \widetilde{\mu}_{\psi(X)}(t) = \mathbb{E}[\psi(t) - \psi(X) | X \leq t] = \frac{1}{F(t)} \int_0^t \phi(x)F(x)\,\mathrm{d}x, \qquad t \in D. \tag{8}$$

We remark that the absolutely continuity of X is not really needed to introduce the WMRL and WMIT functions in Equations (7) and (8), respectively.

In what follows, to ensure the finiteness of $\widetilde{\mu}_\psi(t)$, we implicitly assume that

$$\mathbb{E}[\psi(X)] = \int_0^\infty \psi(x)f(x)\,\mathrm{d}x < \infty. \tag{9}$$

Clearly, if $\psi(t) = t$, and hence $\phi(t) = 1$, then Equation (8) coincides with the MIT function (1), and Equation (7) gives the mean of the residual lifetime of X at age t, i.e.,

$$X_t := [X - t \,|\, X > t], \qquad t \in D. \tag{10}$$

We remark that if ψ is strictly increasing, then the WMIT function of X can be seen as the mean of the residual lifetime of $\psi(X)$ at age $\psi(t)$. Hence, in this case certain properties of the WMIT function of X can be derived from the MIT function of $\psi(X)$. Consequently, in the remainder of the paper we bear in mind that ψ is increasing, but not necessarily in the strict sense. For instance, if $\phi(x) = \mathbf{1}_{[y,+\infty)}(x)$ and thus $\psi(x) = (x-y)\,\mathbf{1}_{[y,+\infty)}(x)$, for a fixed $y > 0$, then recalling Equations (1) and (8) one has that the WMIT function is expressed in terms of the MIT function as follows:

$$\widetilde{\mu}_\psi(t) = \left[\widetilde{\mu}(t) - \frac{F(y)}{F(t)}\,\widetilde{\mu}(y)\right] \cdot \mathbf{1}_{[y,+\infty)}(t), \qquad t \in D.$$

Remark 1. *Let Y be a random variable with PDF $g(t)$ and CDF $G(t)$. Let us consider the cumulative weight function $\psi(x) = G(x)$ and hence $\phi(x) = g(x)$. Clearly, $\psi(X) = G(X)$ takes values in $[0,1]$, with distribution function*

$$F_{\psi(X)}(u) = \mathbb{P}[G(X) \leq u] = F(G^{-1}(u)), \qquad 0 \leq u \leq 1.$$

This function is related to the ROC curve, which was first developed during the Second World War by electrical engineers to analyze radar signals and to study the relation signal/noise, in particular in order to detect correctly enemy objects in battlefields. Recently, this function has been widely studied by Calì and Longobardi [17]. By interchanging the role of F and G in Section 3 of [17], the ROC curve has the following representation

$$\mathrm{ROC}(u) = \overline{F}(G^{-1}(1-u)), \qquad 0 \le u \le 1,$$

so that $\mathrm{ROC}(0) = 0$ *and* $\mathrm{ROC}(1) = 1$. *We also recall the relevant index given by the area under the ROC curve, i.e.,* AUC, *which is defined by (for details, see Section 5 of [17])*

$$\mathrm{AUC} := \int_0^1 \mathrm{ROC}(u)\,\mathrm{d}u = \int_0^\infty g(x)\,\overline{F}(x)\,\mathrm{d}x = \mathbb{E}\big[\overline{F}(Y)\big],$$

where we have set $x = G^{-1}(1-u)$. *Clearly, since* $\psi(x) = G(x)$, *and thus* $\phi(x) = g(x)$, *from (7) one has* $\mathrm{AUC} = m_\psi(0) = m_G(0)$. *On the other hand, from (8) we have also*

$$\mathrm{AUC} = 1 - \int_0^\infty g(x)\,F(x)\,\mathrm{d}x =: 1 - \widetilde{\mu}_\psi(\infty) = 1 - \widetilde{\mu}_G(\infty).$$

Moreover, in this case one has $\psi(\infty) = 1$, *so that applying Equations (8) and (9) the following useful representation is obtained:*

$$\mathrm{AUC} = \mathbb{E}[G(X)].$$

Hereafter, we discuss the WMIT function when the CDF of X is expressed through a distortion of a baseline distribution function, and when the cumulative weight function (5) is the baseline distribution function itself. We recall that an increasing function $\zeta\colon [0,1] \to [0,1]$ such that $\zeta(0) = 0$ and $\zeta(1) = 1$ is called *distortion function*, and is often employed to construct distortion distributions. These functions were introduced in the context of the theory of choice under risk (cf. Yaari [18]). As an example, for some recent applications see Hu et al. [19] (in risk theory), and Navarro [20] and Navarro et al. [21] (in statistics).

Remark 2. *Given an absolutely continuous baseline distribution function* $F_0(x)$ *and a distortion function* ζ, *let the CDF of the random lifetime* X *be a distorted distribution of the form* $F(x) = \zeta[F_0(x)]$.
(i) If the cumulative weight function is given by $\psi(x) = F_0(x)$, *then due to (8) the WMIT function of* X *can be expressed as*

$$\widetilde{\mu}_\psi(t) = \frac{Z[F_0(t)]}{\zeta[F_0(t)]}, \qquad t \in D, \qquad \textit{where } Z[u] = \int_0^u \zeta(y)\,\mathrm{d}y, \quad u \in [0,1].$$

(ii) If, in addition, X *satisfies the proportional reversed hazard model (for details see, for instance, Sankaran and Gleeja [22] and references therein), such that* $\zeta(y) = y^\theta$, $\theta > 0$, *then the WMIT function of* X *becomes*

$$\widetilde{\mu}_\psi(t) = \frac{F_0(t)}{\theta + 1}, \qquad t \in D.$$

As example, it is not hard to see that if X is exponentially distributed, then:

(i) If $\phi(x) = x^r$, $x \ge 0$, $r > 0$, then $\widetilde{\mu}_\psi(t)$ is increasing in $t > 0$ and tends to $+\infty$ as $t \to +\infty$;
(ii) If $\phi(x) = e^{-x}$, $x \ge 0$, then $\widetilde{\mu}_\psi(t)$ is increasing in $t > 0$ and tends to a finite limit as $t \to +\infty$;
(iii) If $\phi(x) = e^{-x^2}$, $x \ge 0$, then $\widetilde{\mu}_\psi(t)$ is not monotonic as $t > 0$ and tends to a finite limit as $t \to +\infty$.

Henceforward, we investigate some further properties of the WMIT function given in (8). To begin with, from Equations (5) and (8) the following lemma is easily obtained.

Lemma 1. *If* X *is an absolutely continuous non-negative random variable, then for all* $t \in D$

$$\widetilde{\mu}'_\psi(t) = \phi(t) - \tau(t)\,\widetilde{\mu}_\psi(t).$$

This result allows to give the condition such that the WMIT function is constant.

Proposition 1. *Let X be an absolutely continuous non-negative random variable. Given a constant $c > 0$, one finds that*

$$\widetilde{\mu}_\psi(t) = c \qquad \text{for all } t > 0,$$

if, and only if,

$$\phi(t) = c\,\tau(t) \qquad \text{for all } t > 0.$$

It is evident from (8) that for a non-negative random variable X, the weighted mean inactivity time function for all $t \in D$ can be rewritten as

$$\widetilde{\mu}_\psi(t) = \psi(t) - \mu_\psi(t), \tag{11}$$

where

$$\mu_\psi(t) = \mathbb{E}[\psi(X)|X \le t] = \frac{1}{F(t)} \int_0^t \psi(x)\,\mathrm{d}F(x), \qquad t \in D$$

is termed as the weighted mean failure time of a system conditioned by a failure before time t, also named *weighted mean past lifetime*. Clearly, if X is absolutely continuous then the derivative of this function is given by

$$\mu'_\psi(t) = \tau(t)[\psi(t) - \mu_\psi(t)], \qquad t \in D. \tag{12}$$

By virtue of (4) and using Lemma 1, the WMIT function under certain assumptions defines uniquely $F(t)$ as follows:

$$F(t) = \exp\left\{-\int_t^\infty \tau(x)\,\mathrm{d}x\right\} = \exp\left\{-\int_t^\infty \frac{\phi(x) - \widetilde{\mu}'_\psi(x)}{\widetilde{\mu}_\psi(x)}\,\mathrm{d}x\right\}, \qquad t \in D. \tag{13}$$

Equation (13) characterizes the distribution function under the following statements:

(i) $\widetilde{\mu}_\psi(0) = 0$ and $\widetilde{\mu}_\psi(x) > 0$ for all $x > 0$;
(ii) $\widetilde{\mu}'_\psi(x) < \phi(x)$;
(iii) $\int_0^\infty (\phi(x) - \widetilde{\mu}'_\psi(x))/\widetilde{\mu}_\psi(x)\,\mathrm{d}x = \infty$, and
(iv) $\int_t^\infty (\phi(x) - \widetilde{\mu}'_\psi(x))/\widetilde{\mu}_\psi(x)\,\mathrm{d}x < \infty$ for all $t > 0$.

Remark 3. *We remark that from (13) and the characterization conditions for $\widetilde{\mu}_\psi(x)$, it follows that there is no lifetime distribution with decreasing WMIT function. Indeed, recalling (13), if $\widetilde{\mu}'_\psi(x) < 0$ and:*

(a) If $C(t) := \int_t^\infty \phi(x)\mathrm{d}x = \infty$ for some $t > 0$, then one has

$$F(t) < \exp\left\{-\int_t^\infty \frac{\phi(x)}{\widetilde{\mu}_\psi(x)}\,\mathrm{d}x\right\} < \exp\left\{-\frac{C(t)}{\widetilde{\mu}_\psi(t)}\right\} = 0;$$

(b) If $C(t) = \int_t^\infty \phi(x)\mathrm{d}x < \infty$ for all $t > 0$, then from condition (i) we have

$$F(0) < \exp\left\{-\frac{C(0)}{\widetilde{\mu}_\psi(0)}\right\} = 0.$$

Hence, in both cases the condition leads to a contradiction, so that there exists no lifetime distribution with decreasing WMIT function.

It is worth to point out that in some situations the conditions (i)–(iv) may be not satisfied, as shown in the following example.

Example 1. *Let X be a non-negative random variable with CDF $F(x)$ and survival function $\overline{F}(x)$. With reference to (5), we consider the weight function $\phi(x) = \overline{F}(x)/F(x)$, also known as odds of survival (see, for instance, Gupta and Peng [23]), so that from (8) we obtain*

$$\widetilde{\mu}_\psi(t) = \frac{1}{F(t)} \int_0^t \overline{F}(x)\,\mathrm{d}x, \qquad t \in D. \tag{14}$$

Clearly, if $\inf(D) = 0$ and if X is absolutely continuous with PDF $f(x)$, such that $0 < f(0) < \infty$, then $\widetilde{\mu}_\psi(0) = 1/f(0) > 0$. In this case, condition (i) above does not hold and hence the distribution function cannot be characterized. The function in the right-hand-side of (14) is known as the mean time to failure of an item that is subject to an age replacement *policy in which a unit is replaced t hours, say, after its installation or at a failure, whichever occurs first (see Section 3.3 of Barlow and Proschan [24] for details). From the latter reference, if X has decreasing (increasing) failure rate, i.e., X is DFR (IFR), then the function $\widetilde{\mu}_\psi(t)$ given in (14) is increasing (decreasing) in t. This conclusion can also be obtained from Point (i) of Theorem 1 below by noting that, due to (6),*

$$\frac{\phi(t)}{\tau(t)} = \frac{1}{\lambda(t)}, \qquad t > 0,$$

which is increasing (decreasing) when X is DFR (IFR).

The following result deals with the WMIT and MIT functions.

Lemma 2. *Let X be a non-negative random variable with weighted mean inactivity time function $\widetilde{\mu}_\psi(t)$ defined as in (8). If $\psi(x)$ is convex (concave) on $[0,\infty)$, then*

$$\widetilde{\mu}_\psi(t) \geq (\leq)\ \psi(\widetilde{\mu}(t)) \qquad \textit{for all } t \in D. \tag{15}$$

Proof. By assumption $\psi(x)$ is increasing convex (concave) on $[0,\infty)$, with $\psi(0) = 0$. Thus, $\psi(x)$ is superadditive (subadditive), i.e., $\psi(z+y) \geq (\leq)\ \psi(z) + \psi(y)$, for $z, y \geq 0$. By substituting $z = x$ and $y = t - x$, with $0 \leq x \leq t$, we obtain $\psi(t) - \psi(x) \geq (\leq)\ \psi(t-x)$ for all $t \geq x \geq 0$. Hence, recalling (8) and (1) we find that

$$\begin{aligned} \widetilde{\mu}_\psi(t) \quad &= \quad \mathbb{E}[\psi(t) - \psi(X) | X \leq t], \\ &\geq (\leq) \quad \mathbb{E}[\psi(t - X) | X \leq t], \\ &\geq (\leq) \quad \psi(\mathbb{E}[t - X | X \leq t]) = \psi(\widetilde{\mu}(t)), \qquad t \in D, \end{aligned}$$

where the last inequality is obtained by using Jensen's inequality. This gives the desired result. □

Lemmas 1 and 2 will be used to derive various results presented in the sequel.

Lemma 3. *Let X be a non-negative random variable with weighted mean inactivity time function $\widetilde{\mu}_\psi(t)$ defined as in (8). Assume that there exist non-negative constants m and M, such that $m \leq \phi(t) \leq M$ for all $t \geq 0$. Then,*

$$m \leq \frac{\widetilde{\mu}_\psi(t)}{\widetilde{\mu}(t)} \leq M \qquad \textit{for all } t \in D. \tag{16}$$

Proof. The proof is immediately obtained by recalling Equations (1) and (8). □

Lemma 3 allows us to obtain ordering relations between the WMIT and MIT functions. Indeed, (i) if $M = 1$, then $\widetilde{\mu}_\psi(t) \leq \widetilde{\mu}(t)$ for all $t \in D$; (ii) if $m = 1$, then $\widetilde{\mu}_\psi(t) \geq \widetilde{\mu}(t)$ for all $t \in D$.

For instance, if $\phi(t) = \overline{F}(t)$ and condition (9) is satisfied, then $M = 1$, and $\widetilde{\mu}_\psi(\infty) = Gini(X)$, where $Gini(X) = \mathbb{E}[|X - X'|]/2$ denotes the Gini mean semi-difference, with X' an independent copy of X.

Hereafter, we focus on a non-parametric class of lifetime distribution based on increasing nature of weighted mean inactivity time function $\widetilde{\mu}_\psi(t)$. As pointed out earlier there is no lifetime distribution with decreasing WMIT function.

Definition 2. *A non-negative random variable X is said to have increasing weighted mean inactivity time function, denoted by IWMIT, if $\widetilde{\mu}_\psi(t)$ is an increasing function of $t \in D$.*

From the results exploited above it follows that the monotonicity properties of $\widetilde{\mu}_\psi(t)$ are based on $\phi(x)$ and $\tau(x)$. This is confirmed in the following theorem, which provides sufficient conditions for the increasingness of $\widetilde{\mu}_\psi(t)$. We recall that X is said to be increasing in the mean inactivity time function, i.e., IMIT, if $\widetilde{\mu}(t)$ is increasing in t.

Theorem 1. *Let X be an absolutely continuous non-negative random variable with reversed hazard rate function $\tau(x)$ defined as in (3). If any of the following conditions hold:*

(i) *$\phi(x)/\tau(x)$ is an increasing function of x;*

(ii) *$\phi(x)$ is increasing in x and X is IMIT;*

(iii) *$\psi(x)\tau(x)/\phi(x)$ is decreasing in $x > 0$;*

then X is $IWMIT$.

Proof. The proof under the conditions (i) and (ii) is similar to those of Theorems 1 and 2 of Toomaj and Di Crescenzo [16], respectively, and hence is omitted. Now, consider case (iii); since $\psi(t) \geq 0$ is increasing in t, it is sufficient to prove that the following function is increasing in $t > 0$:

$$\frac{\widetilde{\mu}_\psi(t)}{\psi(t)} = \frac{\int_0^t \phi(x)F(x)\,\mathrm{d}x}{\psi(t)F(t)} = \frac{\int_0^t \phi(x)F(x)\,\mathrm{d}x}{\int_0^t [\psi(x)f(x) + \phi(x)F(x)]\,\mathrm{d}x}. \tag{17}$$

Define now

$$\Psi(i,t) := \int_0^\infty \nu(i,x)\eta(x,t)\,\mathrm{d}x, \qquad i = 1,2,$$

where

$$\nu(i,x) = \begin{cases} \psi(x)f(x) + \phi(x)F(x), & i = 1 \\ \phi(x)F(x), & i = 2, \end{cases} \qquad \text{and} \qquad \eta(x,t) = \mathbf{1}_{(0,t]}(x).$$

Due to the assumption, $\nu(i,x)$ is TP_2 in $(i,x) \in \{1,2\} \times (0,\infty)$. On the other hand, it is easy to observe that $\eta(x,t)$ is TP_2 in $(x,t) \in (0,\infty)^2$. From the general composition theorem of Karlin [15], it follows that $\Psi(i,t)$ is TP_2 in $(i,t) \in \{1,2\} \times (0,\infty)$. This implies that $\widetilde{\mu}_\psi(t)$ is an increasing function of t, due to (17), and this gives the desired result. □

Remark 4.

(i) *It should be noted that the condition that $\phi(x)/\tau(x)$ is an increasing function of x, given in Point (i) of Theorem 1, is ensured under the assumptions that $\psi(t)$ is convex and X is DRHR, i.e., the reversed hazard rate function $\tau(t)$ is decreasing in t;*

(ii) *We point out that if X is an absolutely continuous non-negative random variable with the reversed hazard rate $\tau(x)$, such that $x\tau(x)$ is decreasing in $x > 0$ and if $\psi(x) = x^r$, $r \geq 1$, then*

$$\frac{\psi(x)\tau(x)}{\phi(x)} = \frac{1}{r}x\tau(x),$$

is a decreasing function of $x > 0$. In this case, thanks to the Point (iii) of Theorem 1, one can conclude that X is $IWMIT$. (See Proposition 13 of Di Crescenzo et al. [25] for a characterization of the property that $x\tau(x)$ is decreasing).

The following examples show the usefulness of Theorem 1.

Example 2. *Let X have Fréchet distribution with CDF $F(x) = \exp\{-cx^{-\gamma}\}$, $x > 0$, for $c, \gamma > 0$. Then, under the conditions considered in Point (ii) of Remark 4 we have that X is IWMIT.*

Example 3. *Assume that $\phi(t) = \tau(t)\widetilde{\mu}(t) = 1 - \widetilde{\mu}'(t)$, where the last equality is due to (2). From (5) we thus have $\psi(t) = \int_0^t \phi(u)\,\mathrm{d}u = t - \widetilde{\mu}(t)$ for all $t > 0$. In this case, from (8) we obtain*

$$\widetilde{\mu}_\psi(t) = \frac{1}{F(t)} \int_0^t f(x)\widetilde{\mu}(x)\,\mathrm{d}x, \qquad t \in D.$$

Hence, making use of Theorem 5.2 of Di Crescenzo and Longobardi [3], we have

$$\widetilde{\mu}_\psi(t) = \mathcal{CE}(X;t), \qquad t > 0, \tag{18}$$

where $\mathcal{CE}(X;t)$ is known as the dynamic cumulative entropy *of X. Recalling Corollary 6.1 of Di Crescenzo and Longobardi [3], we find that if X is IMIT, then $\mathcal{CE}(X;t)$ is increasing in t, and thus from (18) we obtain that X is IWMIT in this case. This conclusion can also be obtained from point (i) of Theorem 1. Further connections with generalized versions of the cumulative entropy can be elaborated on relating the WMIT function to the mixture considered in Equation (11) of Kattumannil et al. [26].*

The following example is dual to Example 2 of Toomaj and Di Crescenzo [16].

Example 4. *Let X be an absolutely continuous non-negative random variable with decreasing and differentiable PDF $f(x)$, with $D = (0, \infty)$ and $0 < f(0) < \infty$. Let*

$$\psi(x) = -\log\frac{f(x)}{f(0)}, \qquad \phi(x) = -\frac{f'(x)}{f(x)} \geq 0, \qquad x > 0.$$

Hence, from (8) and after some calculations, the WMIT function is given by

$$\widetilde{\mu}_\psi(t) = \int_0^t \frac{f(x)}{F(t)} \log\frac{f(x)}{f(t)}\,\mathrm{d}x = -\overline{H}(t) - \log\tau(t), \qquad t > 0, \tag{19}$$

where

$$\overline{H}(t) = -\int_0^t \frac{f(x)}{F(t)} \log\frac{f(x)}{F(t)}\,\mathrm{d}x, \qquad t > 0, \tag{20}$$

is the past entropy of X (cf. Di Crescenzo and Longobardi [27] and Muliere et al. [28]). In this case, due to condition (i) of Theorem 1, if

$$\frac{f'(x)F(x)}{f^2(x)} \quad \text{is decreasing in } x > 0,$$

then X is IWMIT. Equivalently, if $\dfrac{\tau'(x)}{\tau^2(x)}$ is decreasing in $x > 0$ then X is IWMIT.

3. Stochastic Comparisons

In this section, we introduce the weighted mean inactivity time order, and focus our attention on the relations between this one and some well-known stochastic orders. In this regard, we first recall the following notions (see Shaked and Shanthikumar [1], and Kayid and Ahmad [2]).

Definition 3. Let X and Y be two non-negative random variables with cumulative distribution functions $F(t)$ and $G(t)$, and mean inactivity time functions $\widetilde{\mu}_X(t)$ and $\widetilde{\mu}_Y(t)$, respectively. Then:

- X is said to be smaller than Y in the reversed hazard rate (RHR) order, denoted by $X \leq_{rhr} Y$, if, and only if,

$$G(t)/F(t) \qquad \text{is increasing in } t > 0.$$

- X is said to be smaller than Y in the mean inactivity time (MIT) order, denoted by $X \leq_{mit} Y$, if $\widetilde{\mu}_X(t) \geq \widetilde{\mu}_Y(t)$ for all $t > 0$, or equivalently,

$$\int_0^t G(x)\,\mathrm{d}x \Big/ \int_0^t F(x)\,\mathrm{d}x \qquad \text{is increasing in } t > 0.$$

- X is said to be smaller than Y in the dispersive order, denoted by $X \leq_{disp} Y$, if, and only if,

$$G^{-1}(F(t)) - t \qquad \text{is increasing in } t > 0. \tag{21}$$

The MIT order gives a further motivation for studying the MIT function. For instance, if two devices have independent random lifetimes satisfying the ordering $X \leq_{mit} Y$, and if they are both found failed at the inspection time t, then the difference $\widetilde{\mu}_X(t) - \widetilde{\mu}_Y(t)$ gives the mean time elapsed between the failure of the second device and that of the first one.

Now, as already announced, we define a new stochastic order in terms of the WMIT function.

Definition 4. *For a given non-negative weight function ϕ, let X and Y have the weighted mean inactivity time functions $\widetilde{\mu}_{\psi(X)}(t)$ and $\widetilde{\mu}_{\psi(Y)}(t)$, respectively. Then, X is said to be smaller than Y in the weighted mean inactivity time function with respect to weight function $\phi(x)$, denoted by $X \leq^{\phi}_{wmit} Y$, if, and only if,*

$$\widetilde{\mu}_{\psi(X)}(t) \geq \widetilde{\mu}_{\psi(Y)}(t) \qquad \textit{for all } t > 0 \textit{ such that } F(t) > 0 \textit{ and } G(t) > 0.$$

The next theorem provides equivalent conditions for the weighted mean inactivity time order.

Theorem 2. *Let X and Y be two non-negative random variables with CDFs F and G, respectively. Then, for any non-negative weight function ϕ, the following statements are equivalent:*

(i) $X \leq^{\phi}_{wmit} Y$;

(ii) $\frac{\int_0^t \phi(x)G(x)\,\mathrm{d}x}{\int_0^t \phi(x)F(x)\,\mathrm{d}x}$ *is increasing in $t > 0$;*

(iii) $\mathbb{E}[\psi(X)|X \leq t] \leq \mathbb{E}[\psi(Y)|Y \leq t]$ *for all $t > 0$.*

Proof. In this case, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\int_0^t \phi(x)G(x)\,\mathrm{d}x}{\int_0^t \phi(x)F(x)\,\mathrm{d}x} = \frac{\phi(t)\int_0^t \phi(x)[F(x)G(t) - G(x)F(t)]\mathrm{d}x}{\left[\int_0^t \phi(x)F(x)\,\mathrm{d}x\right]^2}, \qquad t > 0.$$

By the definition, one has $X \leq^{\phi}_{wmit} Y$ if, and only if, $\int_0^t \phi(x)[F(x)G(t) - G(x)F(t)]\mathrm{d}x \geq 0$ for all $t > 0$. This proves that (i) and (ii) are equivalent. Finally, the equivalence of statements (i) and (iii) is clear from (8). □

It is worthwhile to mention that by taking $\phi(x) = x$ in Definition 4 one obtains the so-called strong mean inactivity time (SMIT) order studied by Kayid and Izadkhah [4]. As pointed out by the latter authors, the SMIT order lies down between the reversed hazard rate and the MIT orders. So, the WMIT order is a generalization of SMIT order. Hereafter, we illustrate some connections between the WMIT order and two orders recalled in Definition 3.

Theorem 3. *Let X and Y be two non-negative random variables with CDFs $F(t)$ and $G(t)$ and WMIT functions $\widetilde{\mu}_{\psi(X)}(t)$ and $\widetilde{\mu}_{\psi(Y)}(t)$, respectively, where $\psi(\cdot)$ is an increasing non-negative and differentiable function on $(0,\infty)$. Then*

(i) *If $X \leq_{rhr} Y$, then $X \leq^{\phi}_{wmit} Y$;*

(ii) *If $\psi(t)$ is convex on $[0,\infty)$ and $X \leq^{\phi}_{wmit} Y$, then $X \leq_{mit} Y$.*

Proof. (i) Since $X \leq_{rhr} Y$, then $G(x)/F(x)$ is increasing in x, or equivalently

$$\left[\frac{F(x)}{F(t)} - \frac{G(x)}{G(t)}\right] \geq 0, \qquad x \leq t.$$

Since $\phi(x) \geq 0$, from (8) one can conclude that

$$\widetilde{\mu}_{\psi(X)}(t) - \widetilde{\mu}_{\psi(Y)}(t) = \int_0^t \phi(x)\left[\frac{F(x)}{F(t)} - \frac{G(x)}{G(t)}\right] \mathrm{d}x \geq 0,$$

so that $X \leq^{\phi}_{wmit} Y$.
(ii) Let

$$z_t(x) := \phi(x)\left[\frac{F(x)}{F(t)} - \frac{G(x)}{G(t)}\right]\mathbf{1}[x \leq t],$$

and let $\mathrm{d}Z_t(x) = z_t(x)\,\mathrm{d}x$. Then, for all $t > 0$ from (1) we obtain

$$\widetilde{\mu}_Y(t) - \widetilde{\mu}_X(t) = \int_0^\infty \frac{1}{\phi(x)}\,\mathrm{d}Z_t(x) = \int_0^t \frac{1}{\phi(x)}\,\mathrm{d}\left[\int_0^x \phi(u)\left(\frac{F(u)}{F(t)} - \frac{G(u)}{G(t)}\right)\mathrm{d}u\right],$$

where $1/\phi(x)$ is a non-negative decreasing function due to assumption. For all $s > t > 0$, we have

$$\int_0^s \mathrm{d}Z_t(x) = \int_0^t \mathrm{d}Z_t(x) = \int_0^t \phi(x)\left[\frac{F(x)}{F(t)} - \frac{G(x)}{G(t)}\right]\mathrm{d}x \geq 0,$$

where the inequality is obtained by the assumption $X \leq^{\phi}_{wmit} Y$. Let us assume that $t > s > 0$. Due to (8), assumption $X \leq^{\phi}_{wmit} Y$ implies that, for all $t > 0$,

$$\frac{\int_0^t \phi(x)F(x)\,\mathrm{d}x}{\int_0^t \phi(x)G(x)\,\mathrm{d}x} \geq \frac{F(t)}{G(t)}. \tag{22}$$

In addition, $X \leq^{\phi}_{wmit} Y$ implies that $\int_0^x \phi(u)G(u)\,\mathrm{d}u / \int_0^x \phi(u)F(u)\,\mathrm{d}u$, is increasing in x, and then it holds that, for all $t > s > 0$,

$$\frac{\int_0^s \phi(x)F(x)\,\mathrm{d}x}{\int_0^s \phi(x)G(x)\,\mathrm{d}x} \geq \frac{\int_0^t \phi(x)F(x)\,\mathrm{d}x}{\int_0^t \phi(x)G(x)\,\mathrm{d}x}. \tag{23}$$

Combining (22) and (23), one obtains, for all $t > s > 0$,

$$\frac{\int_0^s \phi(x)F(x)\,\mathrm{d}x}{\int_0^s \phi(x)G(x)\,\mathrm{d}x} \geq \frac{F(t)}{G(t)}.$$

which provides that, for all $t > s > 0$,

$$\int_0^s \mathrm{d}Z_t(x) = \int_0^s \phi(x)\left[\frac{G(x)}{G(t)} - \frac{F(x)}{F(t)}\right]\mathrm{d}x \geq 0.$$

Therefore, $X \leq^{\phi}_{wmit} Y$ implies that $\int_0^s \mathrm{d}Z_t(x) \geq 0$, for all $s, t > 0$. Finally, appealing to Lemma 7.1(b) of Barlow and Proschan [29], it is concluded that $\int_0^\infty \frac{1}{\phi(x)} \mathrm{d}Z_t(x) \geq 0$, for all $t > 0$, and hence the proof is completed. □

The instance provided in the following example shows that $X \leq_{mit} Y$ does not imply $X \leq^{\phi}_{wmit} Y$.

Example 5. *Let X be uniformly distributed in $(0,1)$ and let Y have the following CDF:*

$$G(t) = \frac{t}{2}\mathbf{1}_{[0,\frac{1}{2})}(t) + \left(\frac{1}{2} + \frac{t}{2}\right)\mathbf{1}_{[\frac{1}{2},1)}(t) + \mathbf{1}_{[1,+\infty)}(t), \qquad t \in \mathbb{R}.$$

Since the CDF of X is $F(t) = t$ for $0 \leq t \leq 1$, it is not hard to check that $G(t)/F(t)$ is not monotonic in t, so that there is no RHR order between X and Y. However, one can see that $\int_0^t G(x)\,\mathrm{d}x / \int_0^t F(x)\,\mathrm{d}x$ is increasing in $t > 0$, so that $X \leq_{mit} Y$. Moreover, by taking $\phi(x) = x\,\mathbf{1}_{[0,\frac{1}{2})}(x) + (2-x)\,\mathbf{1}_{[\frac{1}{2},1]}(x)$ for $x \in [0,1]$, one can easily find that $\frac{\int_0^t \phi(x)G(x)\,\mathrm{d}x}{\int_0^t \phi(x)F(x)\,\mathrm{d}x}$ is not monotonic in $t \in (0,1)$. Hence, recalling Theorem 2, in this case $X \leq_{mit} Y$ does not imply $X \leq^{\phi}_{wmit} Y$.

In the context of Theorem 3 (i), Counterexample 1 of Kayid and Izadkhah [4] shows that $X \leq^{\phi}_{wmit} Y$ does not imply $X \leq_{rhr} Y$ for an increasing non-negative and differentiable function $\psi(x) = x^2/2$.

Finally, we remark that further connections between the $\leq_{mit}$-order and the $\leq^{\phi}_{wmit}$-order can be found by following the lines adopted in Section 5 of Belzunce et al. [30] and Section 4 of Belzunce and Martínez-Riquelme [31].

4. Weighted Generalized Cumulative Entropy and Variance

In the following two subsections, we discuss some relevant applications of the weighted mean inactivity time function to supply expressions for the variance of a transformed random variable and the weighted generalized cumulative entropy.

4.1. Variance of Transformed Random Variable

Recently, Toomaj and Di Crescenzo [11] showed that the variance of a random variable X can be represented in terms of MIT function as follows:

$$Var(X) = \mathbb{E}[\widetilde{\mu}^2(X)], \tag{24}$$

provided that the expectation exists. In what follows, we extend the result (24) to the case of the transformed random variable $\psi(X)$, where $\psi(x)$ is the cumulative weight function defined in (5). Indeed, in the following theorem we express the variance of $\psi(X)$ in terms of the WMIT function (8).

Theorem 4. *Let X be an absolutely continuous non-negative random variable with WMIT function $\widetilde{\mu}_\psi(t)$, and having finite second moment $\mathbb{E}[\psi^2(X)]$. Then*

$$Var[\psi(X)] = \mathbb{E}[\widetilde{\mu}^2_\psi(X)]. \tag{25}$$

Proof. Let us set

$$w_\psi(x) := \mu_\psi(x)F(x) = \int_0^x \psi(t)f(t)\,\mathrm{d}t, \quad x > 0.$$

Using (11), we obtain

$$
\begin{aligned}
\mathbb{E}[\widetilde{\mu}_{\psi}^{2}(X)] &= \int_0^\infty [\psi(x) - \mu_\psi(x)]^2 f(x)\,\mathrm{d}x \\
&= \mathbb{E}(\psi^2(X)) + \int_0^\infty \mu_\psi^2(x) f(x)\,\mathrm{d}x - 2\int_0^\infty \psi(x)\mu_\psi(x) f(x)\,\mathrm{d}x. \qquad (26)
\end{aligned}
$$

Recalling (12), it holds that

$$
\begin{aligned}
\int_0^\infty \mu_\psi^2(x) f(x)\,\mathrm{d}x &= \int_0^\infty \mu_\psi(x)\tau(x) w_\psi(x)\,\mathrm{d}x \\
&= \int_0^\infty \psi(x)\tau(x) w_\psi(x)\,\mathrm{d}x - \int_0^\infty \mu_\psi'(x) w_\psi(x)\,\mathrm{d}x \\
&= \int_0^\infty \psi(x)\mu_\psi(x) f(x)\,\mathrm{d}x - \int_0^\infty \mu_\psi'(x) w_\psi(x)\,\mathrm{d}x.
\end{aligned}
$$

Integrating by parts gives

$$
\int_0^\infty \mu_\psi'(x) w_\psi(x)\,\mathrm{d}x = [\mathbb{E}(\psi(X))]^2 - \int_0^\infty \psi(x)\mu_\psi(x) f(x)\,\mathrm{d}x,
$$

which implies

$$
\int_0^\infty \mu_\psi^2(x) f(x)\,\mathrm{d}x = 2\int_0^\infty \psi(x)\mu_\psi(x) f(x)\,\mathrm{d}x - [\mathbb{E}(\psi(X))]^2. \qquad (27)
$$

By substituting Equation (27) into (26), we have

$$
\mathbb{E}[\widetilde{\mu}_{\psi}^{2}(X)] = \mathbb{E}[\psi^2(X)] - [\mathbb{E}(\psi(X))]^2 = Var[\psi(X)].
$$

The proof is thus completed. □

We remark that the result expressed in Theorem 4 is analogous to Theorem 3 of Toomaj and Di Crescenzo [16], where the variance of $\psi(X)$ is expressed as the expectation of the squared weighted mean residual life function of X.

In the proof of Theorem 4, recalling (5), we used relation $\psi(0) = 0$. However, by using similar arguments it is not hard to see that for every increasing and differentiable function g, even with $g(0) \neq 0$, the variance of $g(X)$ can be expressed as

$$
Var[g(X)] = \mathbb{E}[\widetilde{\mu}_{g}^{2}(X)],
$$

where

$$
\widetilde{\mu}_g(t) = \frac{1}{F(t)}\int_0^t g'(x) F(x)\,\mathrm{d}x, \qquad t > 0.
$$

As an application of Equation (25), let us consider the following example.

Example 6. *Consider a parallel system composed by m units having lifetimes $X_1, \dots, X_m$, which are i.i.d. absolutely continuous random variables with CDF $F(x)$ and PDF $f(x)$. The system lifetime is thus $X_{m:m} = \max\{X_1, \dots, X_m\}$, whose CDF is given by $F_{m:m}(x) := \mathbb{P}(X_{m:m} \le x) = [F(x)]^m$, $x \ge 0$. Setting $\psi(t) = F(t)$, and thus $\phi(t) = f(t)$, from (8) we obtain, for $t > 0$,*

$$
\widetilde{\mu}_{\psi(X_{m:m})}(t) = \frac{1}{F_{m:m}(t)}\int_0^t f(x) F_{m:m}(x)\,\mathrm{d}x = \frac{1}{[F(t)]^m}\int_0^t f(x)[F(x)]^m\,\mathrm{d}x = \frac{F(t)}{m+1}.
$$

Thanks to the use of Equation (24) and Theorem 4, thus the variance of the probability integral transformation $F(X_{m:m})$ can be obtained as

$$
Var[F(X_{m:m})] = m\int_0^\infty f(x)[F(x)]^{m-1}\left[\frac{F(x)}{m+1}\right]^2 \mathrm{d}x = \frac{m}{(m+1)^2(m+2)}.
$$

We remark that the expression of the variance of $F_{m:m}(x)$ follows also from the fact that $F_{m:m}(x)$ has a Beta distribution with parameters $\alpha = m$ and $\beta = 1$.

Another useful application of Theorem 4 involves the so-called varentropy. If X is an absolutely continuous non-negative random variable with PDF $f(x)$, the (random) information content of X is defined by

$$IC(X) = -\log f(X).$$

It is worth pointing out that $IC(X)$ is the natural counterpart of the number of bits needed to represent X in the discrete case by a coding scheme that minimizes the average code length. It is well known that

$$H_X = \mathbb{E}[IC(X)] = -\int_0^\infty f(x)\log f(x)\,\mathrm{d}x \tag{28}$$

denotes the differential entropy of X. The varentropy of X is defined as (see Di Crescenzo and Paolillo [32] and references therein)

$$\begin{aligned} V(X) := Var(IC(X)) &= \mathbb{E}[(-\log f(X))^2] - [H_X]^2 \\ &= \int_0^\infty [-\log f(x)]^2 f(x)\,\mathrm{d}x - \left(\int_0^\infty [-\log f(x)] f(x)\,\mathrm{d}x\right)^2, \end{aligned} \tag{29}$$

so that it measures the variability of the information content of X. The relevance of this measure has been pointed out in various investigations, especially in Fradelizi et al. [33] where an optimal varentropy bound for log-concave distributions is obtained.

Remark 5. *Let X be an absolutely continuous non-negative random variable with decreasing and differentiable PDF $f(x)$ over the support $(0,\infty)$ and $0 < f(0) < \infty$, and let*

$$\psi(x) = -\log\frac{f(x)}{f(0)}, \qquad \phi(x) = -\frac{f'(x)}{f(x)} \geq 0, \qquad x > 0.$$

Hence, we have

$$V(X) = Var(\psi(X)) = Var(-\log f(X)),$$

so that, recalling Example 4, due to Equations (19) and (25), we obtain another representation of the varentropy in terms of the past entropy (20) and the reversed hazard rate (3) of X as follows:

$$V(X) = Var(-\log f(X)) = \mathbb{E}\left\{[\overline{H}(X) + \log\tau(X)]^2\right\}.$$

On the other hand, recalling Example 2 of Toomaj and Di Crescenzo [16], a further expression for the varentropy can be given as

$$V(X) = Var(-\log f(X)) = \mathbb{E}\left\{[H(X) + \log\lambda(X)]^2\right\},$$

where $\lambda(x)$ is the hazard rate function (6), and where (cf. Ebrahimi [34])

$$H(t) := -\int_t^\infty \frac{f(x)}{\overline{F}(t)}\log\frac{f(x)}{\overline{F}(t)}\,\mathrm{d}x, \qquad t > 0,$$

is the residual entropy of X, i.e., the entropy of the residual lifetime (10).

Hereafter, we see that the results stated in Remark 5 and stimulated by Theorem 4 can be proved under more general assumptions.

Theorem 5. *Let X be an absolutely continuous non-negative random variable with PDF $f(x)$, such that $\mathbb{E}[(IC(X))^2] < \infty$. Then:*

(i) $V(X) = \mathbb{E}\{[H(X) + \log \lambda(X)]^2\}$;

(ii) $V(X) = \mathbb{E}\{[\overline{H}(X) + \log \tau(X)]^2\}$.

Proof. Let us set
$$g(x) := -\int_x^\infty f(z) \log f(z)\, \mathrm{d}z, \qquad x > 0,$$
such that the differential entropy (28) is given by $H_X = g(0)$. First note that
$$H(x) + \log \lambda(x) = \frac{g(x)}{\overline{F}(x)} + \log f(x), \qquad x > 0.$$

Hence, one has
$$\begin{aligned} \mathbb{E}\Big\{[H(X) + \log \lambda(X)]^2\Big\} &= \int_0^\infty [\log f(x)]^2 f(x)\, \mathrm{d}x + \int_0^\infty g^2(x) \frac{f(x)}{\overline{F}^2(x)}\, \mathrm{d}x \\ &+ 2\int_0^\infty f(x) \log f(x) \frac{g(x)}{\overline{F}(x)}\, \mathrm{d}x. \end{aligned} \tag{30}$$

By noting that
$$g^2(x) = \left(\int_x^\infty f(z) \log f(z)\, \mathrm{d}z\right)^2,$$
and integrating by parts with $u = g^2(x)$ and $v = 1/\overline{F}(x)$, we have
$$\begin{aligned} \int_0^\infty g^2(x) \frac{f(x)}{\overline{F}^2(x)}\, \mathrm{d}x &= \left.\frac{g^2(x)}{\overline{F}(x)}\right]_0^\infty - 2\int_0^\infty f(x) \log f(x) \frac{g(x)}{\overline{F}(x)}\, \mathrm{d}x \\ &= -[H_X]^2 - 2\int_0^\infty f(x) \log f(x) \frac{g(x)}{\overline{F}(x)}\, \mathrm{d}x, \end{aligned} \tag{31}$$

since $\lim_{x\to\infty} \frac{g^2(x)}{\overline{F}(x)} = 0$. By substituting Equation (31) into (30), we have
$$\mathbb{E}\Big\{[H(X) + \log \lambda(X)]^2\Big\} = \mathbb{E}[(-\log f(X))^2] - [H_X]^2 = Var(-\log f(X)),$$
where the last equality is due to (29). The proof of Point (i) is thus completed. The proof of Point (ii) is similar and then is omitted. □

By including a further assumption on f, we obtain the following result.

Proposition 2. *Let the assumptions of Theorem 5 hold.*
(i) If $H(t)$ is decreasing in t and
$$\log \frac{f(x)}{f(t)} \le 1 \quad \textit{for all } x \ge t > 0, \tag{32}$$
then $V(X) \le 1$.
(i) If $\overline{H}(t)$ is increasing in t and
$$\log \frac{f(x)}{f(t)} \le 1 \quad \textit{for all } 0 < x \le t, \tag{33}$$
then $V(X) \le 1$.

Proof. (i) First, we recall that

$$H(t) + \log \lambda(t) = -\int_t^\infty \frac{f(x)}{\overline{F}(t)} \log \frac{f(x)}{f(t)} \, dx, \qquad t > 0.$$

Hence, by the assumption (32) we have $H(t) + \log \lambda(t) \geq -1$, $t > 0$. On the other hand, if $H(t)$ is decreasing in t, then $H(t) + \log \lambda(t) \leq 1$, $t > 0$ (cf. Theorem 3.2 of Ebrahimi [34]). Therefore, we obtain $|H(t) + \log \lambda(t)| \leq 1$, so that from Theorem 5 we have $V(X) \leq 1$. The proof of Point (i) is thus completed. In the case (ii), one similarly has

$$\overline{H}(t) + \log \tau(t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{f(t)} \, dx, \qquad t > 0,$$

so that from assumption (33) we obtain $\overline{H}(t) + \log \tau(t) \geq -1$, $t > 0$. Moreover, if $\overline{H}(t)$ is increasing in t, then $\overline{H}(t) + \log \lambda(t) \leq 1$, $t > 0$ (cf. Proposition 2.3 of Di Crescenzo and Longobardi [27]). Thus, it follows $|\overline{H}(t) + \log \lambda(t)| \leq 1$, and finally from Theorem 5 we obtain $V(X) \leq 1$. □

Clearly, if $f(x)$ is decreasing in $x > 0$, then the condition (32) holds. However, such a relation can be fulfilled even for non-decreasing densities. For instance, if X has PDF $f(x) = \frac{1}{3}(1 + 2x)e^{-x}$, $x > 0$, then (32) is satisfied. Moreover, if $f(x)$ is increasing in x on a bounded support, then the condition (33) holds. On the other hand, (33) cannot be fulfilled if $f(t)$ is close to 0, for instance for large t when $f(x)$ has support $(0, \infty)$. However, relation (33) can be satisfied if X has a bounded support, for instance when it is uniform on (a, b), $a < b$.

In the next theorem, we state that when the weight function is bounded between two real numbers, the ratio of standard deviation of the transformed random variable with respect to the standard deviation of the associated random variable also lies down between the same bounds.

In the following, we denote by $\sigma(X) = \sqrt{Var(X)}$ the standard deviation of X.

Theorem 6. *Under the conditions of Lemma 3, it holds that*

$$m \leq \frac{\sigma[\psi(X)]}{\sigma(X)} \leq M.$$

In particular, (i) if $m = 0$ and $M = 1$, then $\sigma[\psi(X)] \leq \sigma(X)$ and, (ii) if $m = 1$ and $M < \infty$, then $\sigma[\psi(X)] \geq \sigma(X)$.

Proof. The proof is immediately obtained from (16) and recalling (24) and (25). □

Now, let us consider an application in the following example.

Example 7. *Assume that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random lifetimes with the common CDF $F(x)$ and PDF $f(x)$. The ith smallest value is usually called the ith order statistic, and is denoted by $X_{i:n}$, $i = 1, 2, \ldots, n$. Let us set $\psi(x) = F(x)$ and thus $\phi(x) = f(x)$. If S is the support of f, then*

$$\inf_{x \in S} f(x) =: m \leq f(x) \leq M := \sup_{x \in S} f(x).$$

It is known that the probability integral transform $V_i = F(X_{i:n})$ has a beta distribution with parameters i and $n - i + 1$, respectively. Since

$$Var[V_i] = Var[F(X_{i:n})] = \frac{i(n - i + 1)}{(n + 1)^2(n + 2)}, \qquad i = 1, 2, \ldots, n,$$

from Theorem 6 we have

$$\frac{i(n-i+1)}{M^2(n+1)^2(n+2)} \leq Var[X_{i:n}] \leq \frac{i(n-i+1)}{m^2(n+1)^2(n+2)}, \qquad i = 1, 2, \ldots, n$$

provided that $0 < m \leq M < \infty$.

In the next theorem, we provide a connection between the variance of the weighted random variable $\psi(X)$ and the cumulative entropy. For a non-negative random variable X with CDF $F(x)$ and support $(0, \infty)$, the cumulative entropy (CE), defined by (see Di Crescenzo and Longobardi [35])

$$\mathcal{CE}(X) = -\int_0^\infty F(x) \log F(x) \, \mathrm{d}x = \int_0^\infty F(x)\, T(x) \, \mathrm{d}x, \tag{34}$$

where

$$T(x) = -\log F(x) = \int_x^\infty \tau(u) \, \mathrm{d}u, \qquad x > 0 \tag{35}$$

denotes the cumulative reversed hazard function. Another useful representation of $\mathcal{CE}(X)$ is given in terms of the MIT function as follows:

$$\mathcal{CE}(X) = \mathbb{E}[\widetilde{\mu}(X)] = \int_0^\infty \widetilde{\mu}(x) f(x) \, \mathrm{d}x.$$

Several properties of CE in (34) as well as its dynamic version are widely discussed in Di Crescenzo and Longobardi [35] and Navarro et al. [36] and references therein.

Theorem 7. *If* $\psi(x)$ *is an increasing convex and differentiable function, then,*

$$\sigma[\psi(X)] \geq \psi(\mathcal{CE}(X)).$$

Proof. The proof is based on Jensen's inequality, and is similar to that of Theorem 6 of [16]. □

4.2. Weighted Generalized Cumulative Entropy

As noted in (28), for an absolutely continuous non-negative random variable X having PDF f, the differential entropy is given by $H_X = -\mathbb{E}[\log f(X)]$. It assigns equal importance (or weights) to the occurrence of every event of the form $\{X = x\}$. However, in certain situations they have different qualitative characteristic usually known as utility of an outcome. This motivated us to define the *weighted entropy* of X as (cf. Di Crescenzo and Longobardi [35])

$$H^w(X) = -\mathbb{E}[X \log f(X)] = -\int_0^\infty x f(x) \log f(x) \, \mathrm{d}x. \tag{36}$$

In analogy with (36), Misagh et al. [37] proposed an alternative weighted measure called *weighted cumulative entropy* (WCE) and based on the distribution function $F(x)$ instead of the PDF $f(x)$ in (36), defined by

$$\mathcal{CE}^w(X) = -\int_0^\infty x F(x) \log F(x) \, \mathrm{d}x = \int_0^\infty x F(x) T(x) \, \mathrm{d}x, \tag{37}$$

with $T(x)$ defined in (35). Recently, the WCE was extended by Tahmasebi et al. [38] to the weighted generalized cumulative entropy (WGCE) given by

$$\mathcal{CE}_n^\phi(X) = \int_0^\infty \phi(x) \frac{T^n(x)}{n!} F(x) \, \mathrm{d}x, \tag{38}$$

for all $n \in \mathbb{N} := \{1, 2, \ldots\}$, and for any non-negative weight function $\phi(x)$. In particular by taking $\phi(x) \equiv 1$ in (38), we immediately derive the generalized cumulative entropy (GCE) introduced by Kayal [39]. Several results on weighted entropies are investigated and discussed in Mirali and Baratpour [40], Misagh et al. [37], Suhov and Sekeh [41] and Tahmasebi [38]. Despite the various investigations of these measures, the analysis of their exact meaning and interpretation can still be improved.

Suppose that $\{Y_n, n \in \mathbb{N}\}$ is a sequence of non-negative i.i.d. random variables having the common CDF $F(x)$. We say that Y_i is a lower record value of this sequence if $Y_i < \min\{Y_1, Y_2, \ldots, Y_{i-1}\}$, with $i > 1$, and by definition Y_1 is a lower record value. Let $L(1) = 1$ and $L(n+1) = \min\{j : j > L(n), Y_j < Y_{L(n)}\}$ for $n \in \mathbb{N}$, so that $L(n)$ denotes the index where the nth lower record value occurs. The random variables $X_{n+1} = Y_{L(n+1)}$, $n \in \mathbb{N}_0 := \{0, 1, \ldots\}$, are said to be the lower records, such that $Y_{L(1)} \stackrel{d}{=} X$. Denoting by $F_{n+1}(x)$ the cumulative distribution function of X_{n+1}, $n \in \mathbb{N}_0$, it follows that

$$F_{n+1}(x) = F(x) \sum_{k=0}^{n} \frac{T^k(x)}{k!}, \qquad x \geq 0, \tag{39}$$

so that the PDF of X_{n+1} is given by

$$f_{n+1}(x) = f(x) \frac{T^n(x)}{n!}, \qquad x \geq 0, \tag{40}$$

where $T(x)$ is the cumulative reversed hazard function defined in (35). We recall that the GCE of order n of X is given by (see Kayal [39], and Di Crescenzo and Toomaj [42])

$$\mathcal{CE}_n(X) = \int_0^\infty F(x) \frac{T^n(x)}{n!} \, \mathrm{d}x = \int_0^\infty F(x) \frac{[-\log F(x)]^n}{n!} \, \mathrm{d}x, \tag{41}$$

for all $n \in \mathbb{N}$. Thus, the GCE of order n corresponds to the expected spacings of lower record values. A fractional version of the GCE has been investigated in Di Crescenzo et al. [43]. Let us now provide a suitable extension of $\mathcal{CE}_n(X)$. For all increasing non-negative and differentiable function $\psi(x)$, the weighted GCE of X is expressed as follows:

$$\begin{aligned} \mathcal{CE}_{\psi,n}(X) &= \mathbb{E}[\psi(X_n) - \psi(X_{n+1})] = \int_0^\infty \phi(x)[F_{n+1}(x) - F_n(x)] \mathrm{d}x, \\ &= \int_0^\infty \phi(x) \frac{T^n(x)}{n!} F(x) \, \mathrm{d}x = \mathbb{E}\left[\frac{\phi(X_{n+1})}{\tau(X_{n+1})}\right], \qquad n \in \mathbb{N}. \end{aligned} \tag{42}$$

Note that for $n = 0$ one has $\mathcal{CE}_{\phi,0}(X) = \int_0^\infty \psi(x) F(x) \, \mathrm{d}x$, which may be divergent. Hence, the function $\mathcal{CE}_{\psi,n}(X)$ can be identified with the WGCE introduced in (38). This measure extends the GCE through a suitable ψ. For example, if we take $\psi(t) = t$, then the WGCE coincides with the GCE introduced by Kayal [39], see also Di Crescenzo and Toomaj [42] and Toomaj and Di Crescenzo [11]. Moreover, if we take $\psi(t) = \frac{t^2}{2}$, it concurs with the weighted GCE introduced by Kayal and Moharana [44]. We note that $\mathcal{CE}_{\psi,n}(X)$ can be viewed as the area of the region between the functions $F_{\psi(X_n)}(x)$ and $F_{\psi(X_{n+1})}(x)$, since from (42) we have

$$\mathcal{CE}_{\psi,n}(X) = \mathbb{E}[\psi(X_n) - \psi(X_{n+1})] = \int_0^\infty \left[F_{\psi(X_{n+1})}(x) - F_{\psi(X_n)}(x)\right] \mathrm{d}x, \qquad n \in \mathbb{N}.$$

Proceeding similarly as in the proof of Proposition 1 of Toomaj and Di Crescenzo [11], from (42) one can see that the weighted GCE of X is equivalent to the GCE of a cumulative weighted random variable $\psi(X)$, i.e., $\mathcal{CE}_{\psi,n}(X) = \mathcal{CE}_n(\psi(X))$ for all $n \in \mathbb{N}$.

With reference to the GCE, defined in (41), in the following theorem we obtain a result analogous to Theorem 7 of Toomaj and Di Crescenzo [16]. The proof is omitted, being similar to that theorem by virtue of the following relation

$$\int_t^\infty \frac{T^{n-1}(x)}{(n-1)!}\,\tau(x)\,\mathrm{d}x = \frac{T^n(t)}{n!}, \qquad t \ge 0, \quad n \in \mathbb{N}.$$

Theorem 8. *Let X be an absolutely continuous non-negative random variable with weighted mean inactivity time function $\widetilde{\mu}_\psi(t)$. Then, for all $n \in \mathbb{N}$ one has*

$$\mathcal{CE}_{\psi,n}(X) = \mathbb{E}[\widetilde{\mu}_\psi(X_n)]. \tag{43}$$

In the following theorem, we determine two recurrent expressions for the GCE analogous to those given in Theorems 4 and 5 of Toomaj and Di Crescenzo [11] and, thus, the proof is omitted.

Theorem 9. *Under the assumption of Theorem 8, for all $n \in \mathbb{N}$, we have*

(i)

$$\mathcal{CE}_{\psi,n}(X) = \mathcal{CE}_{\psi,n-1}(X) - \frac{1}{(n-1)!}\,\mathbb{E}[\widetilde{h}_{\psi,n}(X)],$$

where

$$\widetilde{h}_{\psi,n}(t) := \int_0^t \widetilde{\mu}'_\psi(x)\,T^{n-1}(x)\,\mathrm{d}x.$$

(ii)

$$\mathcal{CE}_{\psi,n}(X) = \mathcal{CE}_{\psi,n-1}(X)\Big\{1 - \mathbb{E}[\widetilde{\mu}'_\psi(\widetilde{Z})]\Big\},$$

where $\widetilde{Z}$ is an absolutely continuous non-negative random variable having PDF

$$f_{\widetilde{Z}}(x) = \frac{F(x)}{\mathcal{CE}_{\psi,n-1}(X)}\,\frac{T^{n-1}(x)}{(n-1)!}, \qquad x > 0.$$

It is worthwhile to mention that when X is IWMIT, since $\widetilde{\mu}'_\psi(x) \ge 0$, as an immediate consequence of Theorem 9 we have

$$\mathcal{CE}_{\psi,n}(X) \ge \mathcal{CE}_{\psi,n-1}(X), \qquad \text{for all } n \in \mathbb{N}.$$

Hereafter, we obtain an upper bound for the WGCE in terms of the expected value of the squared weighted mean inactivity time. The proof is omitted being similar to Theorem 6 of Toomaj and Di Crescenzo [11].

Theorem 10. *Let X be an absolutely continuous non-negative random variable and let $\psi(x)$ denote a non-negative weight function. Then, for all $n \in \mathbb{N}$,*

$$\mathcal{CE}_{\psi,n}(X) \le \frac{\sqrt{[2(n-1)]!}}{(n-1)!}\,\sigma[\psi(X)].$$

Remark 6. *We note that, due to Remark 6 of Toomaj and Di Crescenzo [16], we have $H[\psi(X)] = H(X) + \mathbb{E}[\log\phi(X)]$. Hence, by making use of Remark 6 of Toomaj and Di Crescenzo [16] and Proposition 5 of Tahmasebi et al. [38], the following lower bound can be immediately obtained:*

$$\mathcal{CE}_{\psi,n}(X) \ge \frac{1}{n!} C_n \exp\{H(\psi(X))\}, \qquad n \in \mathbb{N},$$

where $C_n = \exp\{\int_0^1 \log(u[-\log u]^n)\,\mathrm{d}u\}$.

Further useful results are given below.

Theorem 11. *Let X be an absolutely continuous non-negative random variable and let $\psi(x)$ denote a non-negative weight function. Let $n \in \mathbb{N}$.*

(i) *If $\psi(x)$ is an increasing convex (concave) function on $(0, \infty)$, then*
- $\dfrac{\mathcal{CE}_{\psi,n}(X)}{\mathcal{CE}_n(X)}$ *is decreasing (increasing) in $n \in \mathbb{N}$;*
- $\mathcal{CE}_{\psi,n}(X) \geq (\leq)\, \psi(\mathcal{CE}_n(X))$ *for all $n \in \mathbb{N}$.*

(ii) *Under the condition of Lemma 3, it holds that*

$$m \leq \frac{\mathcal{CE}_{\psi,n}(X)}{\mathcal{CE}_n(X)} \leq M.$$

In particular, if $M = 1$ then $\mathcal{CE}_{\psi,n}(X) \leq \mathcal{CE}_n(X)$, whereas if $m = 1$ then $\mathcal{CE}_{\psi,n}(X) \geq \mathcal{CE}_n(X)$.

Proof. (i) The proofs are analogue to Theorems 8 and 10 of Toomaj and Di Crescenzo [16], respectively. (ii) The proof is immediately obtained from (16) and recalling (43). □

In the next corollary, we provide different probabilistic expressions for the WGCE. The second one involves the covariance of the transformation of the n-th epoch time and the random variable $T(X_n)$. The proof is similar to that of Theorem 13 of Toomaj and Di Crescenzo [16], and, thus, is omitted.

Corollary 1. *For all $n \in \mathbb{N}$, it holds that*

(i) $\dfrac{1}{n}\,\mathbb{E}\left[\dfrac{\phi(X_n)T(X_n)}{\tau(X_n)}\right] = \mathcal{CE}_{\psi,n}(X)$;

(ii) $\dfrac{1}{n}\,\mathrm{Cov}[\psi(X_n), T(X_n)] = -\mathcal{CE}_{\psi,n}(X)$.

We can now prove the following theorem, which allows to compare the WGCE of two random variables under the dispersive ordering.

Theorem 12. *Let X and Y be absolutely continuous non-negative random variables, and let ψ be a cumulative weight function defined as in (5). If $\psi(X) \leq_{disp} \psi(Y)$, then $\mathcal{CE}_{\psi,n}(X) \leq \mathcal{CE}_{\psi,n}(Y)$ for all $n \in \mathbb{N}$.*

Proof. Let us consider the cumulative weighted random variables $\psi(X)$ and $\psi(Y)$ with CDFs H and Q, respectively. It is easy to see that

$$\psi(Y) \stackrel{d}{=} Q^{-1}H(\psi(X)), \tag{44}$$

where $Q^{-1}H$ is an increasing function. Since $\psi(X) \leq_{disp} \psi(Y)$, by the Definition 3 it holds that $Q^{-1}H(x) - x$ is increasing in $x > 0$. Taking into account that $\mathcal{CE}_{\varphi,n}(X) = \mathcal{CE}_n(\varphi(X))$ for an increasing function φ, by taking $\varphi(x) = Q^{-1}H(x)$, Point (ii) of Theorem 11 implies that $\mathcal{CE}_n(Q^{-1}H(\psi(X))) \geq \mathcal{CE}_n(\psi(X))$ for all $n \in \mathbb{N}$. From (44), we immediately obtain that $\mathcal{CE}_n(\psi(Y)) \geq \mathcal{CE}_n(\psi(X))$, which yields $\mathcal{CE}_{\psi,n}(Y) \geq \mathcal{CE}_{\psi,n}(X)$ for all $n \in \mathbb{N}$. □

In the following theorem, we can show that if two random variables X and Y are ordered with respect to their reversed failure rate functions, then their corresponding variance and WGCE will be ordered too, provided that a weighted MIT is increasing, and the cumulative weight functions are increasing. We recall that if X is greater than Y in the usual stochastic order, denoted by $X \geq_{st} Y$, then

$$\mathbb{E}[h(X)] \geq \mathbb{E}[h(Y)], \tag{45}$$

for all increasing functions h.

Theorem 13. *Let X and Y be absolutely continuous non-negative random variables with weighted mean inactivity time functions $\widetilde{\mu}_{\psi(X)}(t)$ and $\widetilde{\mu}_{\psi(Y)}(t)$, respectively, such that $X \geq_{st} Y$. If $X \leq_{wmit}^{\phi} Y$ and either X or Y is IWMIT, then*

(i) $Var[\psi(X)] \geq Var[\psi(Y)]$;

(ii) $\mathcal{CE}_{\psi,n}(X) \geq \mathcal{CE}_{\psi,n}(Y)$, *for all* $n \in \mathbb{N}$.

Proof. (i) Let X be IWMIT. From (25), we obtain

$$Var[\psi(X)] = \mathbb{E}[\widetilde{\mu}^2_{\psi(X)}(X)] \geq \mathbb{E}[\widetilde{\mu}^2_{\psi(X)}(Y)] \geq \mathbb{E}[\widetilde{\mu}^2_{\psi(Y)}(Y)] = Var[\psi(Y)].$$

The first inequality is obtained by noting that X is IWMIT, so that $\widetilde{\mu}^2_{\psi(X)}(t)$ is increasing, and by virtue of (45). The last inequality is obtained by the fact that $X \leq_{wmit}^{\phi} Y$ implies $\widetilde{\mu}_{\psi(X)}(t) \geq \widetilde{\mu}_{\psi(Y)}(t)$, $t > 0$, due to Definition 4. When Y is IWMIT, the proof is similar.
(ii) Let X be IWMIT. From Theorem 8, for all $n \in \mathbb{N}$, we obtain

$$\mathcal{CE}_{\psi,n}(X) = \mathbb{E}[\widetilde{\mu}_{\psi(X)}(X_n)] \geq \mathbb{E}[\widetilde{\mu}_{\psi(X)}(Y_n)] \geq \mathbb{E}[\widetilde{\mu}_{\psi(Y)}(Y_n)] = \mathcal{CE}_{\psi,n}(Y).$$

The first inequality is obtained as follows: it is not hard to find that $X \geq_{st} Y$ implies $X_n \geq_{st} Y_n$ for all $n \in \mathbb{N}$, and, hence, the first inequality is concluded by virtue of (45) since $\widetilde{\mu}_{\psi(X)}(t)$ is increasing. The second inequality is obtained noting that assumption $X \leq_{wmit}^{\phi} Y$ implies $\widetilde{\mu}_{\psi(X)}(t) \geq \widetilde{\mu}_{\psi(Y)}(t)$, $t > 0$, from Definition 4. When Y is IWMIT, the proof is similar. $\square$

5. Connection with the Location-Independent Riskier Order

In recent decades, the attention of scholars on quantiles of probability distributions has increased continuously, since they have an immediate interpretation in terms of over/or undershoot probabilities. Several applications of quantiles have been oriented to current problems of risk management involving the concept of Value at Risk (VaR). For a random variable X with CDF F, the VaR or left-continuous inverse (quantile function) is defined by

$$F^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}, \quad \text{for } p \in (0,1).$$

In today's financial world, VaR has become the benchmark risk measure: its importance is unquestionable since regulators accept this model as the basis for setting capital requirements for market risk exposure; see, e.g., Denuit et al. [45]. The excess wealth transform (or right spread function) of a random variable X with distribution function F and with a finite mean, is defined by (see Fernández-Ponce et al. [46])

$$W_X(p) = \mathbb{E}[(X - F^{-1}(p))^+] = \int_{F^{-1}(p)}^{\infty} \overline{F}(x)\,\mathrm{d}x = \int_p^1 (F^{-1}(q) - F^{-1}(p))\,\mathrm{d}q, \tag{46}$$

for $p \in (0,1)$. For any real number a, we denote by a^+ its positive part, that is, $a^+ = a$ if $a > 0$, and $a^+ = 0$ if $a \leq 0$. We remark that it is not necessary for the random variable X to be non-negative in order for $W_X(p)$ to be well defined. Indeed, it is only required that X has a finite mean. Based on this concept, the excess wealth order (or the right spread order) is introduced in Fernández-Ponce et al. [46], by expressing that the expected shortfall risk measure (for the positive tail) is comparable, that is, $\mathbb{E}[(X - F^{-1}(p))^+] \leq \mathbb{E}[(Y - G^{-1}(p))^+]$ for all $p \in (0,1)$. Some applications of this function and the excess wealth order are considered in Toomaj and Di Crescenzo [11,16]. Hereafter, we define the left spread function, which is dual to the right spread function given in (46).

Definition 5. *Let X be a random variable having CDF $F(x)$ and with finite mean. The left spread function of X, for $0 < p < 1$ is defined by*

$$\widetilde{W}_X(p) = \mathbb{E}[(F^{-1}(p) - X)^+] = \int_0^{F^{-1}(p)} F(x)\,\mathrm{d}x = \int_0^p (F^{-1}(p) - F^{-1}(q))\,\mathrm{d}q.$$

The left spread function is an increasing function of p. Moreover, it is closely related to the MIT function given in (1) by the following relation, if X is non-negative:

$$\widetilde{\mu}(F^{-1}(p)) = \mathbb{E}[F^{-1}(p) - X | X \le F^{-1}(p)] = \frac{\widetilde{W}_X(p)}{p}, \quad 0 < p < 1.$$

Thanks to the previous identity, in the next theorem we show that the variance and the GCE of a random variable can be expressed in terms of the left spread function. The results follow from Theorems 19 and 21 of Toomaj and Di Crescenzo [11] and, thus, the proof is omitted.

Theorem 14. *Let X denote an absolutely continuous non-negative random variable with CDF F. Then, it holds that*

(i) $Var(X) = \int_0^1 \left[\widetilde{\mu}(F^{-1}(p))\right]^2 \mathrm{d}p;$

(ii) $\mathcal{CE}_n(X) = \frac{1}{(n-1)!} \int_0^1 \widetilde{\mu}(F^{-1}(p))(-\log p)^{n-1}\,\mathrm{d}p,$ *for all* $n \in \mathbb{N}$.

Let us consider the following example.

Example 8. *If X is uniformly distributed in $[0, b]$, then*

$$\widetilde{\mu}(F^{-1}(p)) = \frac{bp}{2}.$$

Recalling Theorem 14, we obtain

$$Var(X) = \int_0^1 \left[\widetilde{\mu}(F^{-1}(p))\right]^2 \mathrm{d}p = \frac{b^2}{12}.$$

On the other hand, for any $n \in \mathbb{N}$ we obtain

$$\mathcal{CE}_n(X) = \frac{b}{2(n-1)!} \int_0^1 p\,(-\log p)^{n-1}\,\mathrm{d}p = \frac{b}{2^{n+1}}.$$

In economics, many stochastic orders are built to compare the risks of two random assets. To keep the comparison independent of locations, Jewitt [47] proposes the following concept. A non-negative random asset Y is said to be *location independent riskier* than another non-negative random asset X, denoted by $X \le_{lir} Y$, if, and only if,

$$\int_0^{F^{-1}(p)} F(x)\,\mathrm{d}x \le \int_0^{G^{-1}(p)} G(x)\,\mathrm{d}x, \quad \text{for all } p \in (0,1),$$

or equivalently

$$\widetilde{\mu}_X(F^{-1}(p)) \le \widetilde{\mu}_Y(G^{-1}(p)), \quad \text{for all } p \in (0,1), \tag{47}$$

where $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$ denote the MIT functions of X and Y, respectively. Roughly speaking, if the inequality (47) holds then Y has more weight in the lower tail than X. Intuitively, having a great weight in the lower tail is something which should be avoided by risk averters. One advantage of the above definition is that it is a "choice based" criterion of risk which does not stipulate that the distributions have equal means. The proof of the next theorem is straightforward due to Theorem 14 and applying the equivalent condition (47) for $X \leq_{lir} Y$, and, therefore, it is omitted.

Theorem 15. *Let X and Y be two absolutely continuous non-negative random variables, such that $X \leq_{lir} Y$. Then*

(i) $Var(X) \leq Var(Y)$;

(ii) $\mathcal{CE}_n(X) \leq \mathcal{CE}_n(Y)$ *for all* $n \in \mathbb{N}$.

Based on Theorem 15, it is worth pointing out that if Y is more risky than X both in the variance and GCE, then it has a larger variance and GCE. Hereafter, we obtain expressions for the transformed random variable and weighted GCE in terms of transformed excess wealth function. For an absolutely continuous non-negative random variable X with CDF $F(x)$, assume that $\psi(\cdot)$ is an increasing non-negative function defined by (5). The *transformed (or weighted) left spread function*, for all $0 < p < 1$, is defined by

$$\begin{aligned} \widetilde{W}_{\psi(X)}(p) &= \mathbb{E}[(\psi(F^{-1}(p)) - \psi(X))^+] = \int_0^{F^{-1}(p)} \phi(x)F(x)\,\mathrm{d}x \\ &= \int_0^p \left[\psi(F^{-1}(p)) - \psi(F^{-1}(q))\right] \mathrm{d}q. \end{aligned} \tag{48}$$

When $\psi(t) = t$, then from (48) we have that $\widetilde{W}_{\psi(X)}(p)$ is equal to the left spread function introduced in Definition 5. Moreover, this function is related to the weighted mean inactivity time function by the following relation:

$$\widetilde{\mu}_{\psi(X)}(F^{-1}(p)) = \frac{\widetilde{W}_{\psi(X)}(p)}{p}, \qquad 0 < p < 1. \tag{49}$$

Now, in the following theorem, we provide expressions for both the variance of a transformed random variable and the weighted GCE in terms of (49).

Theorem 16. *Let X denote an absolutely continuous random variable with CDF F. Then, it holds that*

(i) $Var[\psi(X)] = \int_0^1 \left[\widetilde{\mu}_{\psi(X)}(F^{-1}(p))\right]^2 \mathrm{d}p$;

(ii) $\mathcal{CE}_{\psi,n}(X) = \dfrac{1}{(n-1)!} \int_0^1 \widetilde{\mu}_{\psi(X)}(F^{-1}(p))(-\log p)^{n-1}\,\mathrm{d}p$, *for all* $n \in \mathbb{N}$.

Proof. (i) By taking $p = F(x)$, it holds that

$$\int_0^1 \left[\widetilde{\mu}_{\psi(X)}(F^{-1}(p))\right]^2 \mathrm{d}p = \int_0^\infty \left[\widetilde{\mu}_{\psi(X)}(x)\right]^2 \mathrm{d}F(x) = Var[\psi(X)],$$

where the last equality is obtained from Theorem 4. The proof of Point (i) is thus completed. By virtue of (43), Point (ii) can be proved in a similar way. □

6. Applications

In this section, we propound two applications in reliability and renewal theory based on results given in the preceding sections.

6.1. Reliability

Let us consider a one-unit system which has the ability to withstand a random number of shocks. We assume that the shocks arrive according to a non-homogeneous Poisson process, and that the number of shocks and the interarrival (or successive) times of shocks are independent. Let N denote the random number of shocks survived by the system, whereas X_j denotes the random interarrival time between the $(j-1)$-th and j-th shocks. Hence, the lifetime T of the system is given by $T = \sum_{j=1}^{N} X_j$. Moreover, let the interarrivals be independent and identically distributed, and let the renewal process describing the number of shocks have cumulative intensity function $\Lambda(t) = -\log \overline{F}(t) = \int_0^t \lambda(\tau)\,\mathrm{d}\tau$, $t \geq 0$, where $\lambda(\tau)$ is the associated hazard rate (6). Then, the CDF of T can be written as

$$F_T(t) = \sum_{k=0}^{\infty} P(k) \frac{\Lambda^k(t)}{k!} e^{-\Lambda(t)}, \qquad t > 0, \tag{50}$$

where $P(k) = \mathbb{P}(N \leq k)$, $k \in \mathbb{N}$, is the distribution function of the number of shocks survived by the device, with $\overline{P}(0) = 1 - P(0) = 1$. Relation (50) also holds for a repairable system as discussed in Chahkandi et al. [48].

Theorem 17. *Let us consider two devices with random lifetimes T_1 and T_2 subject to shocks arriving according to a non-homogeneous Poisson process, and let $P_1(k)$ and $P_2(k)$ be, respectively, the distribution functions of the number of shocks survived by the two devices. If $N_1 \leq_{rhr} N_2$, then $T_1 \leq_{wmit}^{\phi} T_2$.*

Proof. By making use of (50), where T is replaced by T_i, we have for all $t > 0$,

$$\int_0^t \phi(x) F_{T_i}(x)\,\mathrm{d}x = \sum_{k=0}^{\infty} P_i(k) \int_0^t \phi(x) \frac{\Lambda^k(x)}{k!} \overline{F}(x)\,\mathrm{d}x, \qquad i = 1, 2.$$

From (ii) of Theorem 2, it is sufficient to see that $\int_0^t \phi(x) F_{T_2}(x)\,\mathrm{d}x / \int_0^t \phi(x) F_{T_1}(x)\,\mathrm{d}x$ is an increasing function of t, or, equivalently, $\int_0^t \phi(x) F_{T_i}(x)\,\mathrm{d}x$ is TP_2 in $(i,t) \in \{1,2\} \times \mathbb{R}^+$. Since $N_1 \leq_{rhr} N_2$ by assumption, then $P_i(k)$ is TP_2 in $(i,k) \in \{1,2\} \times \mathbb{N}$. On the other hand, it is not hard to see that

$$\int_0^t \phi(x) \frac{\Lambda^k(x)}{k!} \overline{F}(x)\,\mathrm{d}x,$$

is TP_2 in $(t,k) \in \mathbb{R}^+ \times \mathbb{N}$. Then, the general composition theorem of Karlin [27] provides that $\int_0^t \phi(x) F_{T_i}(x)\,\mathrm{d}x$ is TP_2 in $(i,t) \in \{1,2\} \times \mathbb{R}^+$ and hence the claimed result follows. □

In the special case in which the interarrival times are independent and identically exponentially distributed, one clearly has that $\Lambda^k(t) = (\lambda t)^k$ in the right-hand-side of the distribution function (50). Let us consider the cumulative weight function $\psi(x) = x^r$, i.e., the weight function $\phi(x) = r x^{r-1}$, for $r \in \mathbb{N}$.

Theorem 18. *Let T_1 and T_2 be the random lifetimes of two devices subject to shocks governed by a homogeneous Poisson process having intensity λ, and let N_i, $i = 1, 2$, be the random number of shocks survived by the i-th device, with $P_i(k) = P(N \leq k)$, $k \in \mathbb{N}$. If, for $r \in \mathbb{N}$,*

$$\frac{\sum_{k=0}^{j-r} \binom{r+k-1}{k} P_2(k)}{\sum_{k=0}^{j-r} \binom{r+k-1}{k} P_1(k)} \quad \textit{is increasing in } k \in \mathbb{N}, \tag{51}$$

then $T_1 \leq_{wmit}^{\phi} T_2$, for the cumulative weight function $\psi(x) = x^r$.

Proof. It is known that the distribution function of T_i, $i = 1, 2$, is given by

$$H_{T_i}(x) = \sum_{k=0}^{\infty} P_i(k) \frac{e^{-\lambda x} (\lambda x)^k}{k!}, \qquad x \geq 0. \tag{52}$$

Let us consider the following well-known relation:

$$\int_t^\infty e^{-\lambda x} \frac{\lambda^{k+1} x^k}{k!} \, \mathrm{d}x = \sum_{j=0}^{k} e^{-\lambda t} \frac{(\lambda t)^j}{j!}, \qquad k \in \mathbb{N}_0, \quad t > 0.$$

Recalling (52) and using the aforementioned equation, after some manipulations we obtain, for $r \in \mathbb{N}$ and $i = 1, 2$,

$$\begin{aligned}
\int_0^t r x^{r-1} H_{T_i}(x) \, \mathrm{d}x &= \int_0^t r x^{r-1} \sum_{k=0}^{\infty} P_i(k) \frac{e^{-\lambda x} (\lambda x)^k}{k!} \, \mathrm{d}x \\
&= \frac{r!}{\lambda^r} \sum_{k=0}^{\infty} P_i(k) \binom{r+k-1}{k} \int_0^t e^{-\lambda x} \frac{\lambda^{k+r} x^{k+r-1}}{(k+r-1)!} \, \mathrm{d}x \\
&= \frac{r!}{\lambda^r} \sum_{k=0}^{\infty} P_i(k) \binom{r+k-1}{k} \left[1 - \int_t^\infty e^{-\lambda x} \frac{\lambda^{k+r} x^{k+r-1}}{(k+r-1)!} \, \mathrm{d}x \right] \\
&= \frac{r!}{\lambda^r} \sum_{k=0}^{\infty} P_i(k) \binom{r+k-1}{k} \left[1 - \sum_{j=0}^{k+r-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \right] \\
&= \frac{r!}{\lambda^r} \sum_{k=0}^{\infty} P_i(k) \binom{r+k-1}{k} \sum_{j=k+r}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \\
&= \frac{r!}{\lambda^r} \sum_{j=r}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \sum_{k=0}^{j-r} \binom{r+k-1}{k} P_i(k).
\end{aligned}$$

Since $e^{-\lambda t}(\lambda t)^j / j!$ is TP_2 in $(j, t) \in \mathbb{N} \times \mathbb{R}^+$, and recalling relation (51), the general composition theorem of Karlin [15] implies that $\int_0^t r x^{r-1} H_{T_i}(x) \, \mathrm{d}x$ is TP_2 in $(i, t) \in \{1, 2\} \times \mathbb{R}^+$. This is equivalent to state that $T_1 \leq_{wmit}^{\phi} T_2$ for $\psi(x) = x^r$. $\square$

We remark that the case concerning the weight function $\phi(x) = x$ is considered in Theorem 14 of Kayid and Izadkhah [4].

Let us now consider another application. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, and let N be a positive integer-valued random variable, which is independent of the X_i. Denote by

$$X_{N:N} = \max\{X_1, X_2, \ldots, X_N\}$$

the maximum extreme order statistic in a sample having random size. This random variable arises naturally in reliability theory as the lifetime of a parallel system with the random number of identical components with lifetimes $X_1, X_2, \ldots, X_N$. In life testing, if a random censoring is adopted, then the completely observed data constitute a sample $X_1, X_2, \ldots, X_N$ of random size $N > 0$. Let $X_{N_i:N_i}$ denote the maximum order statistic among $X_1, X_2, \ldots, X_{N_i}$, where N_i is a positive integer-valued random variable which is independent from the sequence of $X_1, X_2, \ldots$ for each $i = 1, 2$. Now, we have the following theorem.

Theorem 19. *Let the weight function $\phi(x)$ be increasing in x. If $N_1 \leq_{hr} N_2$, then $X_{N_1:N_1} \leq_{wmit}^{\phi} X_{N_2:N_2}$.*

Proof. Denote by $H_{N_i:N_i}(t)$ the distribution function of $X_{N_i:N_i}$ given as

$$H_{N_i:N_i}(t) = \sum_{k=1}^{\infty} p_k^i \, F^k(t), \quad \text{for all } t > 0,$$

where $F(t)$ is the common cumulative distribution function of the X_i and $p_k^i = P(N_i = k)$, $k \in \mathbb{N}$, is the probability mass function of N_i, $i = 1, 2$. Clearly, $F^k(t)$ is the CDF of $X_{N:N}$ conditional on $N = k$. It is not hard to see that for all $t > 0$ and for each $i = 1, 2$ one has

$$\varphi(t,i) = \int_0^t \phi(x) H_{N_i:N_i}(x)\,\mathrm{d}x = \sum_{k=1}^{\infty} \eta(t,k)\rho(k,i),$$

where $\eta(t,k) = \int_0^t \phi(x)F^k(x)\,\mathrm{d}x$, and $\rho(k,i) = p_k^i$. Denote $\nu(k,i) = \sum_{j=k}^{\infty} p_j^i$, for each $k \in \mathbb{N}$ and $i = 1,2$. Assumption $N_1 \le_{hr} N_2$ (inequality $\le_{hr}$ stands for the hazard rate order between N_1 and N_2) implies that $\nu(k,i)$ is TP_2 in $(k,i) \in \mathbb{N} \times \{1,2\}$. On the other hand, $\eta(t,k)$ is TP_2 in $(t,k) \in \mathbb{R}^+ \times \mathbb{N}$. Applying Lemma 2.1 in Ortega [49] gives $\varphi(t,i)$ is TP_2 in $(t,i) \in \mathbb{R}^+ \times \{1,2\}$, which is equivalent to say that $X_{N_1:N_1} \le_{wmit}^{\phi} X_{N_2:N_2}$. □

6.2. Renewal Theory

Let us consider a renewal process with i.i.d. non-negative interarrival times $\{X_n\}_{n\in\mathbb{N}}$ having common distribution function $F(t)$ and finite mean $\mu = \mathbb{E}[X_n]$. Let $S_n = \sum_{i=1}^{n} X_i$, $n \in \mathbb{N}$, with $S_0 \equiv 0$, be the time of the kth arrival. We define $N(t) = \max\{n : S_n \le t\}$, which represents the number of renewals during $(0,t]$. The excess lifetime $\gamma(t) = S_{N(t)+1} - t$ at time $t \ge 0$ is the time elapsed from the time t to the first arrival after t. Recall that $\gamma(0)$ has distribution function F, that is, $\gamma(0) \overset{d}{=} X_1$. The expected number of renewals in $(0,t]$ can be obtained as

$$M(t) = \mathbb{E}[N(t)] = F(t) + \int_0^t F(t-u)\,\mathrm{d}M(u). \tag{53}$$

It is well-known that the CDF of $\gamma(t)$ is given as

$$\mathbb{P}[\gamma(t) \le x] = F(t+x) + \int_0^t F(t-u+x)\,\mathrm{d}M(u) - M(t), \tag{54}$$

for all $x,t \ge 0$. In the literature, several results have been given to characterize the stochastic orders by the excess lifetime in a renewal process. For more details on definitions and properties, readers are referred to Barlow and Proschan [29]. Next, we will investigate the behavior of the excess lifetime of a renewal process with WMIT interarrivals. We recall that the CDF of the residual lifetime (10) is given by

$$F_t(x) = P(X - t \le x | X > t) = \frac{F(t+x) - F(t)}{1 - F(t)}, \quad t > 0.$$

Moreover, we say that X is new better than used (NBU) if $X_t \le_{st} X$ for all $t > 0$, where X_t is the residual lifetime defined in (10).

Theorem 20. *Let $X_t \le_{wmit}^{\phi} X$ for all $t > 0$. If X is IWMIT and is NBU, then $\gamma(t) \le_{wmit}^{\phi} \gamma(0)$ for all $t > 0$.*

Proof. Since $X_t \le_{wmit}^{\phi} X$ for all $t > 0$, it follows that

$$\int_0^s \phi(x)[F(t+x) - F(t)]\,\mathrm{d}x \ge [F(t+s) - F(t)] \int_0^s \phi(x) \frac{F(x)}{F(s)}\,\mathrm{d}x,$$

for all $s > 0$. By (53) and (54), we have that

$$
\begin{aligned}
&\int_0^s \phi(x)\mathbb{P}[\gamma(t) \le x]\,\mathrm{d}x \\
= &\int_0^s \phi(x)[F(t+x)-F(t)]\,\mathrm{d}x + \int_0^s\int_0^t \phi(x)[F(t-u+x)-F(t-u)]\,\mathrm{d}M(u)\,\mathrm{d}x \\
= &\int_0^s \phi(x)[F(t+x)-F(t)]\,\mathrm{d}x + \int_0^t\int_0^s \phi(x)[F(t-u+x)-F(t-u)]\,\mathrm{d}x\,\mathrm{d}M(u) \\
\ge &\int_0^s \phi(x)[F(t+x)-F(t)]\,\mathrm{d}x + \int_0^t [F(t-u+s)-F(t-u)]\int_0^s \phi(x)\frac{F(x)}{F(s)}\,\mathrm{d}x\,\mathrm{d}M(u) \\
= &\int_0^s \phi(x)[F(t+x)-F(t)]\,\mathrm{d}x + \int_0^s \phi(x)\frac{F(x)}{F(s)}\,\mathrm{d}x\int_0^t [F(t-u+s)-F(t-u)]\,\mathrm{d}M(u) \\
= &\int_0^s \phi(x)[F(t+x)-F(t)]\,\mathrm{d}x + \int_0^s \phi(x)\frac{F(x)}{F(s)}\,\mathrm{d}x[P(\gamma(t)\le s)-F(t+s)+F(t)] \\
\ge &[F(t+x)-F(t)]\int_0^s \phi(x)\frac{F(x)}{F(s)}\,\mathrm{d}x + \int_0^s \phi(x)\frac{F(x)}{F(s)}\,\mathrm{d}x[P(\gamma(t)\le s)-F(t+s)+F(t)] \\
= &\int_0^s \phi(x)\frac{F(x)}{F(s)}\,\mathrm{d}x\,\mathbb{P}[\gamma(t)\le s].
\end{aligned}
$$

Hence, it holds that for all $t, s \ge 0$,

$$
\int_0^s \phi(x)\frac{\mathbb{P}[\gamma(t)\le x]}{\mathbb{P}[\gamma(t)\le s]}\,\mathrm{d}x \ge \int_0^s \phi(x)\frac{F(x)}{F(s)}\,\mathrm{d}x,
$$

which means that $\gamma(t) \le_{wmit}^{\phi} \gamma(0)$ for all $t > 0$. □

7. Concluding Remarks

It is of interest for the industry to perform systematic studies using reliability concepts in view of economic repercussions and safety issues. Due to the existence of a great number of scenarios, a statistical comparison of reliability measures is desired in several applied contexts, such as reliability engineering and biomedical fields. For this reason, we have introduced a stochastic order based on the MIT function, named weighted mean inactivity time (WMIT) order, which is dual to the weighted mean residual life order. The relationship of this new order with other well-known stochastic orders has been discussed. It was shown that the WMIT order lies in the framework of the RHR and the MIT orders under suitable conditions, and hence it enjoys several useful properties which can be applied in reliability and survival analysis. Moreover, we also discussed its monotonicity properties. Further, we used the WMIT to determine the expressions for the variance of transformed random variable, as well as the weighted GCE. Among the several results on such measures, we provided some characterizations and preservation properties of the new order under shock models, random maxima, and renewal theory. Our results provide new concepts and applications in reliability, statistics, and risk theory.

Further properties and applications of the new stochastic order and the new proposed class will be the object of future investigations. For example, the result of this paper can be extended to the doubly truncated (interval) random variables. Specifically, given the random lifetime X and the cumulative weighted random variable $\psi(X)$, one can consider

$$
[\psi(X)-\psi(t_1)|t_1 \le X \le t_2] \qquad \text{and} \qquad [\psi(t_2)-\psi(X)|t_1 \le X \le t_2]
$$

where $(t_1, t_2) \in D^\star = \{(t_1, t_2) : F(t_1) < F(t_2)\}$. Given that the lifetime having age t_1 will expire before age t_2, the first random variable is related to the remaining lifetime, whereas the second one is related to the inactivity time (see, e.g., Sankaran and Sunoj [50], Khorashadizadeh et al. [51] and references therein).

Moreover, the harmonic mean inactivity time order introduced by Izadkhah and Kayid [10], based on the harmonic mean average of the MIT function, can be extended to a new stochastic order based on the comparison of the harmonic mean average of the WMIT function.

Author Contributions: A.D.C. and A.T. contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: Research partially supported by Gonbad Kavous University, GNCS-INdAM, and MIUR (PRIN 2017, Project "Stochastic Models for Complex Systems", no. 2017JFFHSH).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

AUC	Area under the ROC curve
CDF	Cumulative distribution function
CE	Cumulative entropy
DFR	Decreasing failure rate
DRHR	Decreasing reversed hazard rate
GCE	Generalized cumulative entropy
IFR	Increasing failure rate
i.i.d.	Independent and identically distributed
IWMIT	Increasing weighted mean inactivity time
MIT	Mean inactivity time
NBU	New better than used
PDF	Probability density function
RHR	Reversed hazard rate
ROC	Receiver Operating Characteristic
SMIT	Strong mean inactivity time
TP_2	Totally positive of order 2
VaR	Value at Risk
WCE	Weighted cumulative entropy
WGCE	Weighted generalized cumulative entropy
WMIT	Weighted mean inactivity time
WMRL	Weighted mean residual life

References

1. Shaked, M.; Shanthikumar, J.G. *Stochastic Orders and Their Applications*; Academic Press: San Diego, CA, USA, 2007.
2. Kayid, M.; Ahmad, I.A. On the mean inactivity time ordering with reliability applications. *Probab. Eng. Inf. Sci.* **2004**, *18*, 395–409. [CrossRef]
3. Di Crescenzo, A.; Longobardi, M. On cumulative entropies. *J. Statist. Plan. Inference* **2009**, *139*, 4072–4087. [CrossRef]
4. Kayid, M.; Izadkhah, S. Mean inactivity time function, associated orderings, and classes of life distributions. *IEEE Trans. Reliab.* **2014**, *63*, 593–602. [CrossRef]
5. Finkelstein, M.S. On the reversed hazard rate. *Reliab. Eng. Syst. Saf.* **2002**, *78*, 71–75. [CrossRef]
6. Ahmad, I.A.; Kayid, M.; Pellerey, F. Further results involving the MIT order and the IMIT class. *Probab. Engrg. Inf. Sci.* **2005**, *19*, 377–395. [CrossRef]
7. Badia, F.; Berrade, M. On the reversed hazard rate and mean inactivity time of mixtures. In *Advances in Mathematical Modeling for Reliability*; Bedford, T., Quigley, J., Walls, L., Alkali, B., Daneshkhah, A., Hardman, G., Eds.; Delft University Press: Amsterdam, The Netherlands, 2008; pp. 103–110.
8. Goliforushani, S.; Asadi, M. On the discrete mean past lifetime. *Metrika* **2008**, *68*, 209–217. [CrossRef]
9. Kundu, C.; Nanda, A.K. Some reliability properties of the inactivity time. *Commun. Statist. Theory Meth.* **2010**, *39*, 899–911. [CrossRef]
10. Izadkhah, S.; Kayid, M. Reliability analysis of the harmonic mean inactivity time order. *IEEE Trans. Reliab.* **2013**, *62*, 329–337. [CrossRef]
11. Toomaj, A.; Di Crescenzo, A. Generalized entropies, variance and applications. *Entropy* **2020**, *22*, 709. [CrossRef]

12. Khan, R.A.; Bhattacharyya, D.; Mitra, M. On some properties of the mean inactivity time function. *Stat. Probab. Lett.* **2021**, *170*, 108993. [CrossRef]
13. Unnikrishnan Nair, N.; Vineshkumar, B. Relation between cumulative residual entropy and excess wealth transform with applications to reliability and risk. *Stochastics Qual. Control* **2021**, *36*, 43–57. [CrossRef]
14. Kochar, S.; Xu, M. Excess wealth transform with applications. In *Stochastic Orders in Reliability and Risk. Lecture Notes in Statistics*; Li, H., Li, X., Eds.; Springer: New York, NY, USA, 2021; Volume 208, pp. 273–288.
15. Karlin, S. *Total Positivity*; Stanford University Press: Stanford, CA, USA, 1968.
16. Toomaj, A.; Di Crescenzo, A. Connections between weighted generalized cumulative residual entropy and variance. *Mathematics* **2020**, *8*, 1072. [CrossRef]
17. Calì, C.; Longobardi, M. Some mathematical properties of the ROC curve and their applications. *Ricerche di Matematica* **2015**, *64*, 391–402. [CrossRef]
18. Yaari, M.E. The dual theory of choice under risk. *Econometrica* **1987**, *55*, 95–115. [CrossRef]
19. Hu, W.; Chen, C.; Shi, Y.; Chen, Z. A tail measure with variable risk tolerance: Application in dynamic portfolio insurance strategy. *Meth. Comput. Appl. Probab.* **2022**, *24*, 831–874. [CrossRef]
20. Navarro, J. Prediction of record values by using quantile regression curves and distortion functions. *Metrika* **2022**, *85*, 675–706. [CrossRef]
21. Navarro, J.; Torrado, N.; del Águila, Y. Comparisons between largest order statistics from multiple-outlier models with dependence. *Meth. Comput. Appl. Probab.* **2018**, *20*, 411–433. [CrossRef]
22. Sankaran, P.G.; Gleeja, V.L. Proportional reversed hazard and frailty models. *Metrika* **2008**, *68*, 333–342. [CrossRef]
23. Gupta, R.C.; Peng, C. Estimating reliability in proportional odds ratio models. *Comput. Stat. Data Anal.* **2009**, *53*, 1495–1510. [CrossRef]
24. Barlow, R.E.; Proschan, F. *Mathematical Theory of Reliability*; Wiley: New York, NY, USA, 1965.
25. Di Crescenzo, A.; Martinucci, B.; Mulero, J. A quantile-based probabilistic mean value theorem. *Probab. Eng. Inf. Sci.* **2016**, *30*, 261–280. [CrossRef]
26. Kattumannil, S.K.; Sreedevi, E.P.; Balakrishnan, N. A generalized measure of cumulative residual entropy. *Entropy* **2022**, *24*, 444. [CrossRef] [PubMed]
27. Di Crescenzo, A.; Longobardi, M. Entropy-based measure of uncertainty in past lifetime distributions. *J. Appl. Probab.* **2002**, *39*, 434–440. [CrossRef]
28. Muliere, P.; Parmigiani, G.; Polson, N.G. A note on the residual entropy function. *Probab. Engrg. Inform. Sci.* **1993**, *7*, 413–420. [CrossRef]
29. Barlow, R.E.; Proschan, F. *Statistical Theory of Reliability and Life Testing*; Holt, Rinehart and Winston: New York, NY, USA, 1975.
30. Belzunce, F.; Martínez-Riquelme, C.; Ruiz, J.M. On sufficient conditions for mean residual life and related orders. *Comput. Stat. Data Anal.* **2013**, *61*, 199–210. [CrossRef]
31. Belzunce, F.; Martínez-Riquelme, C. On the unimodality of the likelihood ratio with applications. *Stat. Pap.* **2019**, *60*, 223–237. [CrossRef]
32. Di Crescenzo, A.; Paolillo, L. Analysis and applications of the residual varentropy of random lifetimes. *Probab. Engrgy Inform. Sci.* **2021**, *35*, 680–698. [CrossRef]
33. Fradelizi, M.; Madiman, M.; Wang, L. Optimal concentration of information content for logconcave densities. In *High Dimensional Probability VII. The Cargèse Volume*; Progress in Probability; Houdré, C., Mason, D.M., Reynaud-Bouret, P., Rosiński, J., Eds.; Springer: Cham, Switzerland, 2016; Volume 71, pp. 45–60.
34. Ebrahimi, N. How to measure uncertainty in the residual life time distribution. *Sankhya A* **1996**, *58*, 48–56.
35. Di Crescenzo, A.; Longobardi, M. On weighted residual and past entropies. *Sci. Math. Jpn.* **2006**, *64*, 255–266.
36. Navarro, J.; del Aguila, Y.; Asadi, M.; Some new results on the cumulative residual entropy. *J. Statist. Plan. Inf.* **2010**, *140*, 310–322. [CrossRef]
37. Misagh, F.; Panahi, Y.; Yari, G.H.; Shahi, R. Weighted Cumulative entropy and its estimation. In Proceedings of the 2011 IEEE International Conference on Quality and Reliability, ICQR, Bangkok, Thailand, 14–17 September 2011. [CrossRef]
38. Tahmasebi, S.; Longobardi, M.; Foroghi, F.; Lak, F. An extension of weighted generalized cumulative past measure of information. *Ricerche Mat.* **2020**, *69*, 53–81. [CrossRef]
39. Kayal, S. On generalized cumulative entropies. *Probab. Engrg. Inf. Sci.* **2016**, *30*, 640–662. [CrossRef]
40. Mirali, M.; Baratpour, S. Some results on weighted cumulative entropy. *J. Iran. Stat. Soc.* **2017**, *17*, 21–32.
41. Suhov, Y.; Sekeh, S.Y. Weighted cumulative entropies: An extension of CRE and CE. *arXiv* **2015**, arXiv:1507.07051v1
42. Di Crescenzo, A.; Toomaj, A. Further results on the generalized cumulative entropy. *Kybernetika* **2017**, *53*, 959–982. [CrossRef]
43. Di Crescenzo, A.; Kayal, S.; Meoli, A. Fractional generalized cumulative entropy and its dynamic version. *Commun. Nonlinear Sci. Numer. Simulat.* **2021**, *102*, 105899. [CrossRef]
44. Kayal, S.; Moharana, S.R. A shift-dependent generalized cumulative entropy of order n. *Commun. Statist. Simulation Comput.* **2018**, *48*, 1768–1783. [CrossRef]
45. Denuit, M.; Dhaene, J.; Goovaerts, M.; Kaas, R. *Actuarial Theory for Dependent Risks Measures, Orders and Models*; John Wiley & Sons: Hoboken, NJ, USA, 2006.

46. Fernandez-Ponce, J.M.; Kochar, S.C.; Muñoz-Perez, J. Partial orderings of distributions based on right spread functions. *J. Appl. Probab.* **1998**, *35*, 221–228. [CrossRef]
47. Jewitt, I. Choosing between risky prospects: The characterization of comparative statics results, and location independent risk. *Manag. Sci.* **1989**, *35*, 60–70. [CrossRef]
48. Chahkandi, M.; Ahmadi, J.; Baratpour, S. Some results for repairable systems with minimal repairs. *Appl. Stoch. Models Bus. Ind.* **2014**, *30*, 218–226. [CrossRef]
49. Ortega, E.M. A note on some functional relationships involving the mean inactivity time order. *IEEE Trans. Reliab.* **2009**, *58*, 172–178. [CrossRef]
50. Sankaran, P.G.; Sunoj, S.M. Identification of models using failure rate and mean residual life of doubly truncated random variables. *Stat. Pap.* **2004**, *45*, 97–109. [CrossRef]
51. Khorashadizadeh, M.; Rezaei Roknabadi, A.H.; Mohtashami Borzadaran, G.R. Doubly truncated (interval) cumulative residual andpast entropy. *Stat. Probab. Lett.* **2013**, *83*, 1464–1471. [CrossRef]