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Solving Nonlinear Second-Order Differential Equations through the Attached Flow Method

Carmen Ionescu and Radu Constantinescu *

Department of Physics, University of Craiova, 13 A.I.Cuza, 200585 Craiova, Romania
* Correspondence: rconsta@central.ucv.ro

Abstract: The paper considers a simple and well-known method for reducing the differentiability order of an ordinary differential equation, defining the first derivative as a function that will become the new variable. Practically, we attach to the initial equation a supplementary one, very similar to the flow equation from the dynamical systems. This is why we name it as the "attached flow equation". Despite its apparent simplicity, the approach asks for a closer investigation because the reduced equation in the flow variable could be difficult to integrate. To overcome this difficulty, the paper considers a class of second-order differential equations, proposing a decomposition of the free term in two parts and formulating rules, based on a specific balancing procedure, on how to choose the flow. These are the main novelties of the approach that will be illustrated by solving important equations from the theory of solitons as those arising in the Chafee–Infante, Fisher, or Benjamin–Bona–Mahony models.

Keywords: nonlinear differential equations; attached flow; Chafee–Infante equation; Fisher equation; Benjamin–Bona–Mahony equation

MSC: 34A34; 34C20; 35B08; 35C08; 35G20

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1. Introduction

The nonlinear evolutionary phenomena in the real world are commonly modeled either through nonlinear ordinary differential equations (NODEs), when we speak about one-dimensional dynamics, or through nonlinear partial differential equations (NPDEs), in the case of multi-dimensional processes. Many approaches for dealing with NPDEs, based on analytical or numerical methods [1–4], were proposed and are currently used by specialists. Some of them, such as the inverse scattering method [5,6], Lax operators [7], Hirota biliniarization [8–10], Lie symmetry theory [11–13], or ghost fields method [14–16], are general approaches for investigating and solving NPDEs. There are also more direct approaches, suitable for investigating specific classes of solutions, such as solitary and traveling wave solutions. From this last category belong the methods that look for solutions expressed by specific functions, such as sine–cosine [17], hyperbolic [18], or elliptic functions [19], as well as more elaborate methods, such as G'/G [20], Kudryashov [21], functional expansion [22], and so on [23]. All these methods have, as common features, the reduction of NPDEs to NODEs, by switching to a single coordinate—the wave variable. So, at the end, we arrive at the key issue of solving NODEs.

How to solve NODEs is not a trivial problem. This type of equation can accept many classes of solutions and there is not a clear algorithm on how to obtain them. Many alternative approaches have been proposed, and, mainly, these approaches can be divided into two categories: direct solving methods, based on reduction procedures, and approaches looking for solutions in the form of various expansions, either in terms of predefined functions, or, more generally, of known solutions of an auxiliary equation.

This paper refers to one of the direct solving methods suggested in the math textbooks [24,25] for the autonomous NODEs in which the independent variable does not appear explicitly. It is quite a simple prescription for reducing the differentiability order that consists of attaching to the first derivative of the dependent variable a sort of "flow", a function that will be seen as a new variable. Finding the flow allows us to find the initial unknown variable by integrating the flow equation. It is an approach similar but still different from what is performed in the "first integrals" approach, when the "generalized velocities" become independent variables and the initial equation transforms into a system of two differential equations of decreased orders. Despite its simplicity, the recipe is not easy to be applied, because it is not always clear how to integrate the reduced equation in the flow variable. This paper makes an exhaustive analysis of how these difficulties can be overcome for a quite general class of second-order NODEs with polynomial coefficients. The main idea is related to how the flow should be attached so that the flow equation could be integrated. Based on a forced decomposition of the free term and on balancing requests, we will be able to formulate general rules and algorithms, as well as to effectively solve some equations belonging to the considered class. These two elements, the decomposition of the free term and a balancing analyze, represent in fact the original proposals of our approach.

The paper is organized as follows: after the introduction, we point out the class of equations on which we focus on. The difficulties in solving it and our prescriptions to overcome them will be underlined in Section 3. The Section 4 illustrates how these prescriptions work effectively on some specific equations, very important in the soliton theory. Models such as Benjamin–Bona–Mahony, Chafee–Infante, Fisher, or Dodd–Bullough are considered. Some concluding remarks and comments on the advantages and also on the limits of the attached flow method ends the paper.

2. Class of Equations to Be Investigated

To fix ideas, we consider the case when a dependent variable $u(\xi)$ evolves following an autonomous second order differential equation of the form $\Delta(u,\ u',\ u'') = 0$, where $u' = \frac{du}{d\xi},\ u'' = \frac{d^2u}{d\xi^2}$. Moreover, we suppose that this equation can be solved for the highest derivative and brought to the form:

$$u'' = \Delta(u, u') = b(u)u'^2 + c(u)u' + e(u). \tag{1}$$

The coefficients $b(u),\ c(u),\ e(u)$ are considered as polynomials that could include negative powers of u. For convenience, these powers can be eliminated by bringing to the same denominator, the previous equation becoming:

$$A(u)\ u'' + B(u)\ u'^2 + C(u)\ u' + E(u) = 0. \tag{2}$$

The relation (2) describes in fact the most general form of equation generated when a 2-dimensional nonlinear diffusion equation is reduced to an ODE, using the wave transformation $\xi = x \pm \lambda t$. This is the type of equations we investigate in this paper, considering $A(u),\ B(u),\ C(u)$ and $E(u)$ as arbitrary polynomials that could consist in many monomials of positive degrees in u. Constant polynomials are accepted as "zero degree" terms. Practically, we have:

$$A(u) = \sum_{i=n(A)}^{N(A)} a_i u^i;\ B(u) = \sum_{i=n(B)}^{N(B)} b_i u^i;\ C(u) = \sum_{i=n(C)}^{N(C)} c_i u^i; E(u) = \sum_{i=n(E)}^{N(E)} e_i u^i. \tag{3}$$

In these relations $\{a_i,\ b_i,\ c_i,\ e_i\}$ are real constants, $\{N(A),\ N(B),\ N(C),\ N(E)\}$ denote the maximal degrees in u, while $\{n(A),\ n(B),\ n(C),\ n(E)\}$ denote the minimal degrees in u of the respective polynomials. All of them are considered integer numbers, positive or greater than zero.

Even if, as we mentioned, $A(u),\ B(u),\ C(u)$ and $E(u)$ are considered as polynomials, simpler cases, when part of them are vanishing or take the form of monomials, may occur.

For dealing with a second-order differential equation, we suppose that $A(u) \neq 0$. Our solving method mainly concerns the form of $E(u)$, so we also consider it as a nonvanishing term, $E(u) \neq 0$. In fact, in this paper, we analyze the general case of Equation (2) but we illustrate how our proposed solving method works only on equations for which $E(u)$ is a polynomial of the form given by (3) and $A(u), B(u), C(u)$ are monomials, which are in equations of the generic form:

$$au^{N(A)}\ u'' + bu^{N(B)}\ u'^2 + cu^{N(C)}\ u' + E(u) = 0,\ N(A), N(B), N(C) \geq 0 \qquad a, b, c = const. \tag{4}$$

This last class of equations is of high practical importance, including nonlinear ODEs arising when many important models from the theory of solitons, as for example the Dodd–Bullough–Mikhailov (DBM), Chafee–Infante, Fisher, KdV, Benjamin–Bona–Mahony (BBM), Klein Gordon, etc., are studied. Other equations modeling phenomena from nonlinear optics, hydrodynamics, or plasma physics [26,27] belong to the same category.

Remark 1. *The case $E(u) = 0$ is also interesting, corresponding to models as Hunter–Saxton, Burger, etc. The Equation (2) takes in this case the form:*

$$A(u)\ u'' + B(u)\ u'^2 + C(u)\ u' = 0, \tag{5}$$

and it can be also tackled using a simpler version of the attached flow method.

Remark 2. *Particularly, $E(u)$ could be, it too, a monomial. An interesting example belonging to this case is:*

$$au\ u'' - bu'^2 - cu\ u' - eu^2 = 0. \tag{6}$$

As we will see, it is a completely integrable equation that is used by some authors as an auxiliary equation [28,29].

Remark 3. *Equations for which another function than $E(u)$, that is one of the functions $A(u), B(u)$ or $C(u)$ from (2), are polynomials represent also interesting situations. To this case belongs, for example, the nonlinear oscillator described by the Van der Pol equation:*

$$u'' + a(1 - u^2)u' + u = 0. \tag{7}$$

How the attached flow method could be effectively applied to such equations will be tackled in another paper.

To conclude, we consider, in this paper, equations of the general form (2), investigating when and how they can be directly integrated using the attached flow method. After general considerations on the method, we illustrate how it effectively works by tackling three specific cases appearing in equations of practical interest:

- *Case 1*: equations for which $B(u) = 0, C(u) = 0$, as Benjamin–Bona–Mahony (BBM);
- *Case 2*: equations with $B(u) = 0, C(u) \neq 0$, as Chafee–Infante and Fisher;
- *Case 3*: Equation (6), for which none of the coefficients is vanishing but $E(u)$ becomes a monomial.

As a final note, let us mention that equations with $C(u) = 0$, as for example the Dodd–Bullough–Mikhailov equation, represent a specific case of degenerated equation, that could be directly integrated, leading to solutions in an implicit form.

3. The Attached Flow Method

3.1. Setting of the Problem

Let us come back to Equation (2) and consider that, in the most general case, none of the polynomial coefficients $A(u),\ B(u),\ C(u),\ E(u)$ is vanishing. They will be assumed as having the form (3), with the highest polynomial degrees $\{N(A),\ N(B),\ N(C),\ N(E)\}$, and, respectively, with the minimal degrees $\{n(A),\ n(B),\ n(C),\ n(E)\}$ greater or equal to zero. A simple and classical approach for solving (2) supposes its reduction to two first-order differential equations. As it is an autonomous equation, the reduction can be made by defining:

$$u' = f(u). \tag{8}$$

From (8), we have that:

$$u'' = \frac{df}{du}u' = \frac{df}{du}f\ . \tag{9}$$

with (9), Equation (2) become a first-order differential equation:

$$A(u)f(u)\,\frac{df}{du} + B(u)f(u)^2 + C(u)f(u) + E(u) = 0. \tag{10}$$

Practically, we reduce the solving of (2) to solving (10) with the constraint (8). The reduction was generated by a change of variable in which the dependent variable $u(\xi)$ from (2) takes the role of independent one, the new dependent variables becoming the function $f(u)$ attached to the derivative $u'(\xi)$. This reduction procedure has not a specific name, and, as formally (8) is similar to the equations that describe the evolution of 1-dimensional dynamical systems, we call it the *attached flow equation*. The quantity $f(u)$ is called *flow* and the main aim of our study is to give some clear prescriptions for finding it. The solving method as a whole is called the *"attached flow method"*. In a more general context, Equation (8) can be seen as a specific example of projective dynamical systems that can be described using the mathematical concept of differential inclusion [30,31].

Remark 4. *The reduction of the second-order differential Equation (2) to two first-order equations is also achieved in another classical way, the first integral method [32], when we replace this equation with the system:*

$$\begin{cases} u'(\xi) = v(\xi) \\ v'(\xi) = b(u)v(\xi)^2 + c(u)v(\xi) + e(u), \end{cases} \tag{11}$$

where $b(u) = -\frac{B(u)}{A(u)}, c(u) = -\frac{C(u)}{A(u)}, e(u) = -\frac{E(u)}{A(u)}$.

The problem of solving (11) is clearly different from solving the system (8) and (10). In (11) the two variables u and v depend of ξ, while in (10) we have u as independent and $f(u)$ as dependent variables. For differential equations of order higher than two, the method leads to systems with more than two equations, as appear in [33].

Let us come back now to our problem, of solving the system (8) and (10). We note that Equation (10) has the form of an Abel equation of the second kind and it is not integrable for arbitrary coefficients [34]. There are some exceptions when implicit solutions can be obtained and much fewer cases where the solutions can be written explicitly, for example if $C(u) = 0$, when we have a degenerate equation. This is the case analyzed in [35], where (8) is combined with a change of variable and u' is given in terms of two arbitrary functions, f and g, as:

$$u' = f(v) \text{ with } v = g(u). \tag{12}$$

With these choices, the DBM Equation was solved, but a general rule for retrieving $f(u)$ has not been formulated.

3.2. A Decomposition Approach

In the usual approach of solving (2) through the attached flow method, we have to find $f(u)$ solution of (10) and then to use it in (8); however, as we also mentioned, (10) has the form of an Abel equation, with only few cases of integrability. In what we are proposing, we will try to avoid the direct integration of (10), by transforming it into a system with two unknown functions.

Proposition 1. *For nontrivial flows $f(u)$, solving Equation (2) can be reduced to solving the following system in two unknown functions $f(u)$ and $h(u)$:*

$$E(u) = f(u)h(u); \tag{13}$$

$$A(u)\frac{df}{du} + B(u)\,f(u) + C(u) + h(u) = 0. \tag{14}$$

Proof. Let us suppose that the "flow" $f(u)$ introduced in (8) can be correlated to the coefficient $E(u)$ from (10), by introducing a new function, $h(u)$, through the relation (13). Then, Equation (10) becomes:

$$A(u)f(u)\frac{df}{du} + B(u)\,f(u)^2 + C(u)f(u) + f(u)\,h(u) = 0. \tag{15}$$

For a nontrivial flow, $f(u) \neq 0$, we simply obtain linear Equation (14). The problem of finding $u(\xi)$ solution of Equation (2) is practically replaced now by that of finding $f(u)$ and $h(u)$, solutions of (13) and (14), and, then, of integrating (8). □

How and when $E(u)$ can be split in the two parts as in (13) and how the flow $f(u)$ can be chosen become the key issues of our problem.

Remark 5. *First of all, let us mention that there are situations of direct integrability for Equation (14). It is in fact of the general form*

$$\alpha(u,f)\frac{df}{du} = \beta(u,f), \tag{16}$$

and can be simply solved when it is an exact differential equation, that is when:

$$\frac{\partial\alpha}{\partial u} = \frac{\partial\beta}{\partial f}. \tag{17}$$

In this case, the solution is given by (see, for example, exercise 1.7.1.26 from [25]):

$$\int_{f_0}^{f} \alpha(u,z)dz + \int_{u_0}^{u} \beta(z,f)dz = C. \tag{18}$$

In our case, of Equation (14), condition (17) would ask that:

$$A'_u(u) = B(u). \tag{19}$$

What is happening if (19) is not fulfilled? These situations are investigated in the next subsection following some specific balancing rules.

3.3. Balancing Rules

Theorem 1. *Let us consider Equation (2) where $A(u)$, $B(u)$, $C(u)$ and $E(u)$ are polynomials of the form (3), with degrees in $u(\xi)$ expressed through positive and integer numbers:*

$$N(A) \geq n(A) \geq 0; N(B) \geq n(B) \geq 0;\; N(C) \geq n(C) \geq 0; N(E) \geq n(E) \geq 0. \tag{20}$$

Let us make the supplementary supposition that the two functions $f(u)$ and $h(u)$ from (14) have the form of power series, with degrees running from unit to unit, between $\{n(f), N(f)\}$, respectively, $\{n(h), N(h)\}$:

$$f(u) = \sum_{i=n(f)}^{N(f)} \phi_i u^i,\; h(u) = \sum_{k=n(h)}^{N(h)} \eta_k u^k. \tag{21}$$

The parameters ϕ_i, η_k from (21) are constants that can be determined, and, depending on the initial equation, the summation limits for the flow are given by one of the following relations.

(1) *for the maximum value $N(f)$*, we can have:

$$\begin{aligned}
N(f) &= N(C)-N(A)+1 \text{ if } N(B) \leq N(A)-1 \leq 2N(C)-N(E); \\
N(f) &= \frac{N(E)\text{-}N(A)+1}{2} \text{ if } \begin{cases} N(B) \leq N(A)-1 \\ 2N(C)-N(E) \leq N(A)-1 \end{cases}; \\
N(f) &= N(C)-N(B) \text{ if } N(A)-1 \leq N(B) \leq 2N(C)-N(E); \\
N(f) &= \frac{N(E)\text{-}N(B)}{2} \text{ if } \begin{cases} N(A)-1 \leq N(B) \\ 2N(C)-N(E) \leq N(B) \end{cases}; \\
N(f) &= N(E)-N(C) \text{ if } \begin{cases} N(A)-1 \leq 2N(C)-N(E) \\ N(B) \geq N(E)-2N(C) \end{cases}.
\end{aligned} \tag{22}$$

(2) the *minimal degree* $n(f)$ could belong to the following set:

$$n(f) \in \{n(C)-n(A)+1, \frac{n(E)-n(A)+1}{2}, n(C)-n(B), \frac{n(E)-n(B)}{2}, n(E)-n(C)\}. \tag{23}$$

Proof. It is clear that the decomposition (13) will be not unique, and it will impose, as necessary but not sufficient conditions, the requirements:

$$N(f)+N(h) = N(E); n(f)+n(h) = n(E). \tag{24}$$

On the other side, Equation (14) can be fulfilled if a balancing among the various powers of u is accomplished, both at the maximal and at the minimal values. The requirement (24) and the balancing of the terms from (14) will help us in finding the maximal and the minimal degrees in u, which could appear in the functions $f(u)$ and $h(u)$. Let us note that the maximal degrees of the terms from (14) are, respectively, $N(A)+N(f)-1$, $N(B)+N(f)$, $N(C)$, $N(h) = N(E)-N(f)$. We have to ask that at least two terms from (14) contain the same extreme powers in u, so that they can compensate each other. Depending on the relations among the maximal degrees $N(A)$, $N(B)$, $N(C)$ and $N(E)$, the requirement of balancing in pairs will generate as values of $N(f)$ one of the possibilities from (22). For example, when the maximal powers in u belong to the terms in $A(u)$ and $C(u)$, by equating the corresponding degrees, we obtain $N(f) = N(C)-N(A)+1$. The other cases from (22) can be easily obtained. Similar reasoning leads to the conclusion that the minimal degree $n(f)$ could belong to the set (23). □

Lemma 1. *From Relation (22), we obtain the following conclusion on the flow degrees:*

- $N(f) > 0$ if $N(E) > N(A)-1$ or $N(E) > N(B)$ or $N(C) > N(B)$;
- $N(f) < 0$ if $N(E) < N(A)-1$ and $N(E) < N(B)$ and $N(E) < N(C)$;
- $n(f) < 0$ if $n(E) < n(A)-1$ or $n(E) < n(B)$ or $n(E) < n(C)$;
- $n(f) > 0$ if $n(E) > n(A)-1$ and $n(E) > n(B)$ and $n(E) > n(C)$.

Remark 6. *Unlike the assumptions from (20), the summation limits $\{n(f), N(f), n(h), N(h)\}$ can take negative and non-integer values. More precisely, we can see that, in some cases, the relations (22) and (23) will ask for half-integer values of $N(f)$ and $n(f)$. This is happening, for example, if $N(f) = \frac{N(E)\text{-}N(A)+1}{2}$ and $N(E)-N(A)$ is even.*

So, there are many possibilities of looking for flows of the form (21). What still needs to be performed is to try to determine the appropriate coefficients $\{\phi_i, \eta_k\}$ for each possible pair $\{N(f), n(f)\}$. There is not a specific algorithm and the analysis has to be made on a case by case basis. The three specific cases mentioned at the end of Section 2 are detailed here and the specific graduation constraints for the flows are formulated in the following corollaries; they are effectively applied in the next section.

Corollary 1. *When $B(u) = 0$, $C(u) = 0$ and $A(u)$, $E(u)$ are polynomials of positive degrees as in (3), the flows may have the following degrees:*

$$N(f) = \frac{N(E) - N(A) + 1}{2}, \; n(f) = \frac{n(E) - n(A) + 1}{2}. \tag{25}$$

Proof. Equation (10) takes now the form

$$A(u)f(u)\,\frac{df}{du} + E(u) = 0.$$

The general balancing rule supposes the compensation among the maximal and minimal powers of u, which appear in the terms generated by $A(u)$ and $E(u)$. These two terms have the maximal degrees $N(A) + 2N(f) - 1$ and, respectively, $N(E)$. By equating these numbers we obtain $N(f)$ from (25). Similarly, the minimal degree $n(f)$ can be obtained. □

Remark 7. *Despite the similarities, (25) cannot be seen as a particular case of (22) and (23), considering, for example, $N(B) = N(C) = n(B) = n(C) = 0$. As we already mentioned, $B = C = 0$ would suppose to put $N(B) = N(C) = -\infty$, so we must consider this case as a case in itself, independent of the general situation described by (22) and (23).*

Corollary 2. *When $B(u) = 0$ only, while $A(u), C(u)$ and $E(u)$ are polynomials as in (3), we obtain:*

$$\begin{aligned} N(f) &\in \{\frac{N(E) - N(A) + 1}{2}, N(C) - N(A) + 1, N(E) - N(C)\}; \\ n(f) &\in \{\frac{n(E) - n(A) + 1}{2}, n(C) - n(A) + 1, n(E) - n(C)\}. \end{aligned} \tag{26}$$

Proof. The first value for $N(f)$ from (26) arises when we consider that the terms with $A(u)$ and $E(u)$ have the maximal powers in u and have to compensate each other: $N(A) + 2N(f) - 1 = N(E)$. The other values appear if we compensate the terms with $A(u)$ and $C(u)$, respectively, $C(u)$ and $E(u)$. Similar reasoning yields the minimal values $n(f)$. □

This second case corresponds to important equations, such as Fisher or Chafee–Infante, and are analyzed below.

Corollary 3. *In the case when all the functions $A(u)$, $B(u)$, $C(u)$ and $E(u)$ are nonvanishing monomials with the unique degrees denoted by $N(A), N(B), N(C), N(E)$, the flow $f(u)$ will be also a monomial that could have the degrees:*

$$N(f) \in \{\frac{N(E) - N(A) + 1}{2}, \frac{N(E) - N(B)}{2}, \; N(C) - N(A) + 1, \; N(E) - N(C)\}$$

Proof. The generic degrees of the terms appearing in (14) are given by:

$$N(A) + N(f) - 1, \; N(B) + N(f), \; N(C), \; N(h) \equiv N(E) - N(f). \tag{27}$$

Two subcases of balancing that would allow us to solve (14) and to effectively find the degree for the flow $f(u)$ may occur: (i) All the terms have the same degree and can be balanced among themselves, that is:

$$N(A) + N(f) - 1 = \; N(B) + N(f) = \; N(C) = \; N(h) \equiv N(E) - N(f);$$

$$N(f) = \frac{N(E) - N(A) + 1}{2} = \frac{N(E) - N(B)}{2} = N(E) - N(C). \tag{28}$$

(ii) The degrees of the terms are such that they are compensated in pairs. Many subcases can now appear, as for example:

$$\begin{aligned} N(f) &= N(E) - N(C) \text{ and } N(A) - 1 = N(B); \\ N(f) &= N(C) - N(A) + 1 = \frac{N(E) - N(B)}{2}; \\ N(f) &= N(C) - N(B) = \frac{N(E) - N(A) + 1}{2}. \end{aligned} \tag{29}$$

Any compatible $N(f)$ given separately by each of the previous equations will indicate what degree $N(f)$ the flow could have. It is clear that we could have many solutions, that is, many forms of the flow could be possible. □

4. Examples of Equations Solved through Attached Flows

4.1. Equations with $B = 0$ and $C = 0$. The BBM Model

The BBM equation is well known in physical applications for modeling long waves in a nonlinear dispersive system and accepting soliton-like solutions [36]. In 2-dimensions, the Benjamin–Bona–Mahony (BBM) equation is given as:

$$u_t + u_x - \alpha(u^2)_x - \beta u_{xxt} = 0; \quad \alpha, \beta = const.$$

It can be reduced to an ODE if we pass to the variable $\xi = x + \lambda t$, considering λ as the wave velocity:

$$\lambda u' + u' - \alpha(u^2)' - \lambda\beta u''' = 0.$$

The integration of the previous equation brings the integration constant k as a supplementary parameter. We have:

$$\lambda\beta u'' + \alpha u^2 - (\lambda + 1)u - k = 0, \tag{30}$$

We see that (30) has the form of (2) with $B(u) = C(u) = 0$, $A(u) = \lambda\beta$ - a constant monomial, and $E(u) = \alpha u^2 - (\lambda + 1)u - k$. The degrees in u are $N(A) = 0$, while $E(u)$ is a polynomial with $N(E) = 2$, $n(E) = 0$. We note that, in addition to the minimum and maximum degrees, $E(u)$ also contains a term of the intermediate degree 1, which will not affect the functioning of the method.

The first step in our solving algorithm consists in considering the flow equation, $u' = f(u)$. With it and with the previous identification, the relation (30) leads to the reduced BBM equation of the form:

$$\beta\lambda f(u)\frac{df}{du} + E(u) = 0. \tag{31}$$

Depending on the explicit form of $E(u)$, this last equation could be directly integrated. Alternatively, we can use the precepts we proposed in the attached flow approach. For that, we have to decompose $E(u)$ as in (13). After simplification, (31) takes the reduced form:

$$\beta\lambda\frac{df}{du} + h(u) = 0. \tag{32}$$

The balancing between the two terms from (32) leads to:

$$\begin{aligned} N(f) - 1 &= N(h) = N(E) - N(f) \Rightarrow 2N(f) - 1 = N(E) = 2, \\ n(f) - 1 &= n(h) \Rightarrow 2n(f) - 1 = n(E) = 0, \end{aligned}$$

that is:

$$N(f) = \frac{3}{2},\ n(f) = \frac{1}{2}. \tag{33}$$

It is what (25) also gives. We can choose:

$$\begin{aligned} f(u) &= \phi_2 u^{\frac{3}{2}} + \phi_1 u^{\frac{1}{2}}, &(34)\\ h(u) &= \eta_2 u^{\frac{1}{2}} + \eta_1 u^{-\frac{1}{2}}. &(35) \end{aligned}$$

The coefficients from the previous relations can be determined from the requirements of satisfying (13) and (14). They lead to the relations:

$$\eta_2 = -\frac{3\phi_2\beta\lambda}{2},\ \ \eta_1 = -\frac{\phi_1\beta\lambda}{2}, \tag{36}$$

$$\phi_2\eta_2 = \alpha,\ \phi_2\eta_1 + \phi_1\eta_2 = -(\lambda+1),\ \phi_1\eta_1 = -k\,. \tag{37}$$

By finding η and η' from (36) and (37), we find that the flow $f(u)$ has the expression:

$$f(u) = \sqrt{\frac{1}{8\beta\lambda k}}(\lambda+1)u^{\frac{3}{2}} + \sqrt{\frac{2k}{\beta\lambda}}u^{\frac{1}{2}}, \tag{38}$$

while the flow Equation (8) is:

$$u' = \sqrt{\frac{1}{8\beta\lambda k}}\Big((\lambda+1)u^{\frac{3}{2}} + 4ku^{\frac{1}{2}}\Big). \tag{39}$$

Relation (39) can be analytically integrated and leads to solutions that depend on the values accepted by the parameters k, λ and β. One possible solution, corresponding to $k = -\frac{3(\lambda+1)^2}{16\alpha}$, is:

$$u(\xi) = \frac{4k}{\lambda+1}tg^2\left(\sqrt{\frac{\lambda+1}{8\beta\lambda}}t + C_2\right). \tag{40}$$

If we would choose in (30) the constant of integration $k = 0$, we can check that the solution will have the form:

$$u(\xi) = \frac{1}{\left(\sqrt{\frac{\alpha}{6\beta}}\xi + C_1\right)^2}. \tag{41}$$

Both solutions, (40) and (41), are quite similar with solutions reported in [36], where the G'/G method was applied.

4.2. Equations with $B(u) = 0$. The Generalized Fisher Model

We consider now the attached flow method for a model with $B(u) = 0$, described by an equation of the form:

$$u'' + \lambda u' + k(u - u^\alpha) = 0,\ \alpha = 2, 3, \ldots \tag{42}$$

Fisher and Chafee–Infante equations belong to this class, corresponding to $\alpha = 2$ and $\alpha = 3$. They arise starting from well-known diffusion-reaction equations that, in the $(1+1)$-dimensional space $\{x, t\}$, describe nonlinear phenomena from many fields.

From the perspective of our work, Equation (42) has the form (2), with:

$$\begin{aligned} A(u) &= 1, C(u) = \lambda = const., E(u) = k(u - u^\alpha),\\ N(A) &= 0,\ N(C) = 0,\ N(E) = \alpha,\ n(E) = 1. \end{aligned}$$

The decomposition (13) supposes that:

$$E(u) = f(u)h(u) = k(u - u^{\alpha}),$$

while the reduced Equation (14) becomes:

$$\frac{df}{du} + \lambda + h(u) = 0. \tag{43}$$

The balancing analysis (26) gives the following possible maximal and minimal degrees for allowable flows:

$$N(f) \in \{\frac{\alpha+1}{2}, 1, \alpha\}; n(f) = 1. \tag{44}$$

It is important to see that the last two options cannot compensate the three terms from the reduction of Equation (43). So, the only option we have to consider for the flow is:

$$N(f) = \frac{\alpha+1}{2};\ n(f) = 1. \tag{45}$$

Let us see what (45) means for the cases $\alpha = \{2, 3\}$.

4.2.1. The Fisher Equation

The Fisher equation corresponds to $\alpha = 2$ and has the form:

$$u'' + \lambda u' + k(u - u^2) = 0. \tag{46}$$

Exact and explicit solutions for this model were already pointed out many years ago, with a fairly similar approach in [37], but using two interesting tricks: (i) a change of variable given by $v = u^{2/\alpha}$ and (ii) imposing a constraint in the new variable similar to the flow equation. As we show, the attached flow method does not ask for a change of variable. Decomposition (13) imposes in this case:

$$E(u) = f(u)h(u) = ku(1 - u). \tag{47}$$

For $\alpha = 2$ the relation (45) leads to: $N(f) = \frac{3}{2}$ and $n(f) = 1$. Following (21) the flow should have the form:

$$f(u) = \phi_2 u^{\frac{3}{2}} + \phi_1 u. \tag{48}$$

The compatibility with the requirements (13) and (14) leads to:

$$f(u) = \sqrt{\frac{2k}{3}}\left(u^{\frac{3}{2}} \pm u\right). \tag{49}$$

The two expressions from (49) can be used in (8) and, by direct integration, give solutions of the Fisher equation very similar to what Wang obtained:

$$\begin{aligned} u(\xi) &= \frac{1}{\left(1 - C\exp\sqrt{\frac{k}{6}}\xi\right)^2}; \\ u(\xi) &= \frac{C\exp\sqrt{\frac{2k}{3}}\xi}{\left(1 - C\exp\sqrt{\frac{k}{6}}\xi\right)^2}. \end{aligned} \tag{50}$$

with $\xi = x + \lambda t$, we find that in both cases the wave velocity should be $\lambda = 5\sqrt{\frac{k}{6}}$. Depending on the value of k, we could have periodic, hyperbolic, or rational solutions. No need for a supplementary change of variable as in [37].

4.2.2. The Chafee–Infante Equation

Let us pass now to the Chafee–Infante equation, corresponding to $\alpha = 3$ in (42). It is described by the equation:

$$u'' + \lambda u' + ku(u+1)(u-1) = 0. \tag{51}$$

From the perspective of our method, with the conventions made for the polynomial degrees, we can identify:

$$N(A) = 0,\ N(C) = 0,\ N(E) = 3, n(E) = 1.$$

The combination from (45) suggests to choose the functions $f(u)$ and $h(u)$ from the decomposition (13) as:

$$f(u) = \phi_2 u^2 + \phi_1 u, h(u) = \eta_1 u + \eta_0. \tag{52}$$

By direct check, we find that the possible forms of the two functions are:

$$\begin{aligned} f(u) &= \sqrt{\frac{k}{2}}\left(u^2 \pm u\right), \\ h(u) &= \sqrt{2k}(u \mp 1). \end{aligned}$$

The flow Equation (8) will be now:.

$$u' = \sqrt{\frac{k}{2}}\left(u^2 \pm u\right). \tag{53}$$

It is very simple to be integrated and leads to solutions $u(\xi)$ of (51) in the form:

$$u(\xi) = \frac{1}{1 - Ce^{\sqrt{\frac{k}{2}}\xi}}; \tag{54}$$

$$u(\xi) = \frac{Ce^{\sqrt{\frac{k}{2}}\xi}}{1 - Ce^{\sqrt{\frac{k}{2}}\xi}}. \tag{55}$$

Our approach leads to solutions without asking the solution to have a predefined form as in other approaches.

4.3. Equations with all Nonvanishing Coefficients

Let us consider now the third case announced in Section 2, when all the coefficients $A(u), B(u), C(u)$ and $E(u)$ are nonvanishing monomials. To illustrate how our solving procedure works in this case, we consider Equation (6) and the balancing results (28) and (29). The main ingredients of the approach, Equations (13) and (14), take now the form:

$$-eu^2 = f(u)h(u), \tag{56}$$

$$au\,\frac{df}{du}(u) - bf(u) - cu + h(u) = 0. \tag{57}$$

The terms in (57) have the degrees $N(f), N(f),\ 1, N(h)$ and can compensate each other, as in (28), if

$$N(f) = N(h) = 1. \tag{58}$$

We can choose:

$$f(u) = \phi_1 u + \phi_0,\ \ h(u) = \eta_1 u + \eta_0. \tag{59}$$

The compatibility of (59) with (56) and (57) asks for a relation among the coefficients of the form:

$$\phi_0 = 0;\ {\phi_1}^2(a-b) - c\phi_1 - e = 0.$$

When $a = b$ we obtain $\phi_1 = -\frac{e}{c}$ and the flow equation has the solution:

$$u(\xi) = e^{-\frac{e}{c}\xi}.$$

If $a - b \neq 0$:

$$\phi_1 = \frac{c \pm \sqrt{c^2 + 4e(a-b)}}{2(a-b)},$$

and the solution $u(\xi)$ of (6) takes the general form:

$$u(\xi) = u_0 e^{\frac{c \pm \sqrt{c^2+4e(a-b)}}{2(a-b)}\xi},$$

where $u_0 = u(\xi = 0)$. Depending on the relation among the parameters a, b, c, e, we can recover all the solutions of (6)—periodic, hyperbolic, or rational, as mentioned in the literature [28,29].

5. Conclusions

In this paper we proposed a simpler approach for solving ordinary differential equations of the general form (2). The approach consisted of defining the first derivative of the unknown function as a new variable, called flow, and we named the approach the attached flow method. To make the method effective, we proposed, as an original contribution, two supplementary steps: the first one asked for a forced decomposition of the free term $E(u)$ in two parts, as in (13); in the second step, we formulated clear rules, based on balancing requests, on how flow $f(u)$ should look. Without these steps, flow Equation (8) alone leads to an Abel equation of the second kind, which is an equation that can be solved in certain cases and has solutions expressed in implicit form only. The flow is not unique, as each allowable expression generates, by integration of (8), a class of solutions for the initial Equation (2). The advantage of our approach is given exactly by the fact that, in contradiction with what happens with Abel's equations, exact analytical solutions can be generated in explicit form.

To illustrate how the method works, we applied it to solve some particular cases of Equation (2), which are of major practical interest. We supposed that $A(u)$ and $E(u)$ are not vanishing and considered that: (i) $B(u) = C(u) = 0$, the case corresponding to the BBM model; (ii) only $B(u) = 0$, the situation for which a generalized model leading to the Fisher equation and to the Chafee–Infante equation was analyzed; (iii) none of the coefficient functions are canceled, but they are all monomials. In all these cases, we show how the flows could be obtained and we effectively write down possible solutions. No changes of variables or other were tricks used before solving the same equations were required. We have to mention that when $C(u) = 0$, the Abel Equation (15) is degenerated and can be solved directly, leading at least to implicit solutions. The case $E(u) = 0$ could be also of interest, corresponding, as we already mentioned (see Remark 1 in Section 2), to models such as Burger or Hunter–Saxton. Equation (10) becomes quite simple in this last case and leads to solutions in an implicit form.

There are two limits of the procedure we presented here: we exclusively considered equations with polynomial coefficients and we looked for flows exclusively as power series. In fact, we checked that the method works for $A(U)$, $B(u)$, $C(u)$ monomials and for $E(u)$ with the highest degrees in u. In other circumstances, as in the case of the Van der Pol equation, no go situations can appear. How the method could be extended in such cases will be tackled in forthcoming papers.

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