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# Hermite–Hadamard and Pachpatte Type Inequalities for Coordinated Preinvex Fuzzy-Interval-Valued Functions Pertaining to a Fuzzy-Interval Double Integral Operator

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**Abstract:** Many authors have recently examined the relationship between symmetry and generalized convexity. Generalized convexity and symmetry have become a new area of study in the field of inequalities as a result of this close relationship. In this article, we introduce the idea of preinvex fuzzy-interval-valued functions (preinvex F·I-V·F) on coordinates in a rectangle drawn on a plane and show that these functions have Hermite–Hadamard-type inclusions. We also develop Hermite–Hadamard-type inclusions for the combination of two coordinated preinvex functions with interval values. The weighted Hermite–Hadamard-type inclusions for products of coordinated convex interval-valued functions discussed in a recent publication by Khan et al. in 2022 served as the inspiration for our conclusions. Our proven results expand and generalize several previous findings made in the body of literature. Additionally, we offer appropriate examples to corroborate our theoretical main findings.

**Keywords:** fuzzy-interval-valued function; fuzzy-interval double integral operator; coordinated preinvex fuzzy-interval-valued function; Hermite–Hadamard inequality; Hermite–Hadamard–Fejér inequality

**MSC:** 26A33; 26A51; 26D07; 26D10; 26D15; 26D20

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**Citation:** Santos-García, G.; Khan, M.B.; Alrweili, H.; Alahmadi, A.A.; Ghoneim, S.S.M. Hermite–Hadamard and Pachpatte Type Inequalities for Coordinated Preinvex Fuzzy-Interval-Valued Functions Pertaining to a Fuzzy-Interval Double Integral Operator. *Mathematics* **2022**, *10*, 2756. https://doi.org/10.3390/math10152756

Academic Editor: Michael Voskoglou

Received: 30 June 2022
Accepted: 27 July 2022
Published: 3 August 2022



## 1. Introduction

The H-H inequality has been a potent instrument to obtain a lot of excellent results in integral inequalities and optimization theory because of its crucial role in convex analysis. It has recently been generalized using other convexity types, particularly s-convex functions [1–4], log-convex functions [5–7], harmonic convexity [8], and particularly for h-convex functions [9]. Since 2007, numerous H-H inequalities for h-convex function extensions and generalizations have been established in [10–16].

On the other hand, Archimedes' calculation of the circumference of a circle can be linked to the theory of interval analysis, which has a lengthy history. However, due to a lack of applications to other sciences, it was forgotten for a very long time. Burkill [17] developed several fundamental interval function features in 1924. Kolmogorov's [18] generalization of Burkill's findings from single-valued functions to multi-valued functions came shortly after. Of course, throughout the following 20 years, numerous additional outstanding achievements were also obtained. Please take notice that Moore was the first to realize how interval analysis might be used to calculate the error boundaries of computer numerical solutions. The theoretical and applied research on interval analysis has received a lot of attention and has produced useful discoveries during the past 50 years since Moore [19] published the first monograph on the subject in 1966. In more recent years, Nikodem et al. [20] and, particularly, Budak et al. [21], Chalco–Cano et al. [22,23], Costa et al. [24–26], Román–Flores et al. [27,28], Flores–Franuli et al. [29], and Zhao et al. [30–33] have expanded various well-known inequalities. For more information, see [34–69] and the references are therein.

We introduce the coordinated preinvex functions in fuzzy interval-valued settings, which are inspired by Dragomir [34], Latif and Dragomir [44], and Khan et al. [37,38]. We also talk about how coordinated fuzzy-interval preinvexity and preinvexity relate to one another. The key findings of this study are new fuzzy-interval versions of Hermite–Hadamard-type inequalities that we develop with the help of newly defined coordinated fuzzy-interval preinvexity. Finally, we provide some examples to highlight our key findings. The current findings can also be seen as instruments for further study into topics like inequalities for fuzzy-interval-valued functions, fuzzy interval optimization, and generalized convexity.

## 2. Preliminaries

Let $\mathbb{R}_I$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $\mathcal{Q} \in \mathbb{R}_I$ be defined by

$$\mathcal{Q} = [\mathcal{Q}_*,\ \mathcal{Q}^*] = \{\varkappa \in \mathbb{R} |\ \mathcal{Q}_* \leq \varkappa \leq \mathcal{Q}^*\}, (\mathcal{Q}_*,\ \mathcal{Q}^* \in \mathbb{R}) \tag{1}$$

If $\mathcal{Q}_* = \mathcal{Q}^*$, then $\mathcal{Q}$ is said to be degenerate. In this article, all intervals will be non-degenerate intervals. If $\mathcal{Q}_* \geq 0$, then $[\mathcal{Q}_*,\ \mathcal{Q}^*]$ is called a positive interval. The set of all positive intervals is denoted by $\mathbb{R}_I^+$ and defined as $\mathbb{R}_I^+ = \{[\mathcal{Q}_*,\ \mathcal{Q}^*] : [\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathbb{R}_I \text{ and } \mathcal{Q}_* \geq 0\}$.

Let $\lambda \in \mathbb{R}$ and $\lambda \cdot \mathcal{Q}$ be defined by

$$\lambda \cdot \mathcal{Q} = \begin{cases} [\lambda\mathcal{Q}_*,\ \lambda\mathcal{Q}^*] \text{ if } \lambda > 0, \\ \{0\} \qquad\qquad \text{ if } \lambda = 0, \\ [\lambda\mathcal{Q}^*, \lambda\mathcal{Q}_*] \text{ if } \lambda < 0. \end{cases} \tag{2}$$

Then, the Minkowski difference $\mathcal{Z} - \mathcal{Q}$, addition $\mathcal{Q} + \mathcal{Z}$, and $\mathcal{Q} \times \mathcal{Z}$ for $\mathcal{Q}, \mathcal{Z} \in \mathbb{R}_I$ are defined by

$$[\mathcal{Z}_*,\ \mathcal{Z}^*] + [\mathcal{Q}_*,\ \mathcal{Q}^*] = [\mathcal{Z}_* + \mathcal{Q}_*,\ \ \mathcal{Z}^* + \mathcal{Q}^*], \tag{3}$$

$$\begin{aligned} &[\mathcal{Z}_*,\ \mathcal{Z}^*] \times [\mathcal{Q}_*,\ \mathcal{Q}^*] \\ &= [\min\{\mathcal{Z}_*\mathcal{Q}_*,\ \mathcal{Z}^*\mathcal{Q}_*,\ \mathcal{Z}_*\mathcal{Q}^*,\ \mathcal{Z}^*\mathcal{Q}^*\},\ \max\{\mathcal{Z}_*\mathcal{Q}_*,\ \mathcal{Z}^*\mathcal{Q}_*,\ \mathcal{Z}_*\mathcal{Q}^*,\ \mathcal{Z}^*\mathcal{Q}^*\}] \end{aligned} \tag{4}$$

$$[\mathcal{Z}_*,\ \mathcal{Z}^*] - [\mathcal{Q}_*,\ \mathcal{Q}^*] = [\mathcal{Z}_* - \mathcal{Q}^*,\ \ \mathcal{Z}^* - \mathcal{Q}_*], \tag{5}$$

**Remark 1.** *(i) For given* $[\mathcal{Z}_*,\ \mathcal{Z}^*],\ \ [\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathbb{R}_I$*, the relation "*$\supseteq_I$*" defined on* $\mathbb{R}_I$ *by* $[\mathcal{Q}_*,\ \mathcal{Q}^*] \supseteq_I [\mathcal{Z}_*,\ \mathcal{Z}^*]$ *if and only if*

$$\mathcal{Q}_* \leq \mathcal{Z}_*,\ \ \mathcal{Z}^* \leq \mathcal{Q}^*, \tag{6}$$

*for all* $[\mathcal{Z}_*,\ \mathcal{Z}^*],\ \ [\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathbb{R}_I$*, it is a partial interval inclusion relation. The relation* $[\mathcal{Q}_*,\ \mathcal{Q}^*] \supseteq_I [\mathcal{Z}_*,\ \mathcal{Z}^*]$ *coincides with* $[\mathcal{Q}_*,\ \mathcal{Q}^*] \supseteq [\mathcal{Z}_*,\ \mathcal{Z}^*]$ *on* $\mathbb{R}_I$*. It can be easily seen that "*$\supseteq_I$*" looks like "up and down" on the real line* $\mathbb{R}$*, so we call "*$\supseteq_I$*" as "up and down" (or "UD" order, in short) [40].*

*(ii) For given* $[\mathcal{Z}_*,\ \mathcal{Z}^*],\ \ [\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathbb{R}_I$*, we say that* $[\mathcal{Z}_*,\ \mathcal{Z}^*] \leq_I [\mathcal{Q}_*,\ \mathcal{Q}^*]$ *if and only if*

$$\mathcal{Z}_* \leq \mathcal{Q}_*,\ \ \mathcal{Z}^* \leq \mathcal{Q}^* \text{ or } \mathcal{Z}_* \leq \mathcal{Q}_*,\ \ \mathcal{Z}^* < \mathcal{Q}^* \tag{7}$$

*it is a partial interval order relation. The relation* $[\mathcal{Z}_*,\ \mathcal{Z}^*] \leq_I [\mathcal{Q}_*,\ \mathcal{Q}^*]$ *is coincident to* $[\mathcal{Z}_*,\ \mathcal{Z}^*] \leq [\mathcal{Q}_*,\ \mathcal{Q}^*]$ *on* $\mathbb{R}_I$*. It can be easily seen that "*$\leq_I$*" looks like "left and right" on the real line* $\mathbb{R}$*, so we call "*$\leq_I$*" as "left and right" (or "LR" order, in short) [39,40].*

For $[\mathcal{Z}_*,\ \mathcal{Z}^*], [\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathbb{R}_I$, the Hausdorff–Pompeiu distance between intervals $[\mathcal{Z}_*,\ \mathcal{Z}^*]$ and $[\mathcal{Q}_*,\ \mathcal{Q}^*]$ is defined by

$$d_H([\mathcal{Z}_*,\ \mathcal{Z}^*],\ [\mathcal{Q}_*,\ \mathcal{Q}^*]) = max\{|\mathcal{Z}_* - \mathcal{Q}_*|,\ |\mathcal{Z}^* - \mathcal{Q}^*|\}. \tag{8}$$

It is a familiar fact that $(\mathbb{R}_I, d_H)$ is a complete metric space [42–44].

**Definition 1** ([40,41]). *A fuzzy subset L of $\mathbb{R}$ is distinguished by mapping $\widetilde{\psi} : \mathbb{R} \to [0,\ 1]$ called the membership mapping of L. That is, a fuzzy subset L of $\mathbb{R}$ is mapping $\widetilde{\psi} : \mathbb{R} \to [0,\ 1]$. So, for further study, we have chosen this notation.We appoint $\mathbb{F}$ to denote the set ofall fuzzy subsets of $\mathbb{R}$.*

*Let $\widetilde{\psi} \in \mathbb{F}$. Then, $\widetilde{\psi}$ is known as a fuzzy number or fuzzy interval if the following properties are satisfied by $\widetilde{\psi}$:*

*(1) $\widetilde{\psi}$ should be normal if there exists $\varkappa \in \mathbb{R}$ and $\widetilde{\psi}(\varkappa) = 1$;*

*(2) $\widetilde{\psi}$ should be upper semi continuous on $\mathbb{R}$ if for given $\varkappa \in \mathbb{R}$, there exist $\varepsilon > 0$ or there exist $\delta > 0$ such that $\widetilde{\psi}(\varkappa) - \widetilde{\psi}(\omega) < \varepsilon$ for all $\omega \in \mathbb{R}$ with $|\varkappa - \omega| < \delta$;*

*(3) $\widetilde{\psi}$ should be fuzzy convex, that is $\widetilde{\psi}((1-\varphi)x + \varphi\omega) \geq min\big(\widetilde{\psi}(x),\ \widetilde{\psi}(\omega)\big)$, for all $x, \omega \in \mathbb{R}$ and $\varphi \in [0,1]$*

*(4) $\widetilde{\psi}$ should be compactly supported, that is $cl\big\{u \in \mathbb{R}\big|\ \widetilde{\psi}(\varkappa) > 0\big\}$ is compact.*

*We appoint $\mathbb{F}_I$ to denote the set ofall fuzzy intervals or fuzzy numbers of $\mathbb{R}$.*

**Definition 2** ([40,41]). *Given $\widetilde{\psi} \in \mathbb{F}_I$, the level sets or cut sets are given by $\big[\widetilde{\psi}\big]^\lambda = \big\{\varkappa \in \mathbb{R}\big|\ \widetilde{\psi}(\varkappa) > \lambda\big\}$ for all $\lambda \in [0,\ 1]$ and by $\big[\widetilde{\psi}\big]^\lambda = \big\{\varkappa \in \mathbb{R}\big|\ \widetilde{\psi}(\varkappa) > 0\big\}$. These sets are known as $\lambda$-level sets or $\lambda$-cut sets of $\widetilde{\psi}$.*

**Proposition 1** ([39]). *Let $\widetilde{\psi}, \widetilde{\varpi} \in \mathbb{F}_I$. Then relation "$\preccurlyeq$" given on $\mathbb{F}_I$ by $\widetilde{\psi} \preccurlyeq \widetilde{\varpi}$ when and only when, $\big[\widetilde{\psi}\big]^\lambda \leq_I [\widetilde{\varpi}]^\lambda$, for every $\lambda \in [0,\ 1]$, it is left and right order relation.*

Remember the approaching notions, which are offered in literature. If $\widetilde{\psi}, \widetilde{\varpi} \in \mathbb{F}_I$ and $\lambda \in \mathbb{R}$, then, for every $\lambda \in [0,\ 1]$, the arithmetic operations are defined by

$$\big[\widetilde{\psi} \widetilde{+} \widetilde{\varpi}\big]^\lambda = \big[\widetilde{\psi}\big]^\lambda + [\widetilde{\varpi}]^\lambda, \tag{9}$$

$$\big[\widetilde{\psi} \widetilde{\times} \widetilde{\varpi}\big]^\lambda = \big[\widetilde{\psi}\big]^\lambda \times [\,\widetilde{\varpi}]^\lambda, \tag{10}$$

$$\big[\lambda \cdot \widetilde{\psi}\big]^\lambda = \lambda \cdot \big[\widetilde{\psi}\big]^\lambda \tag{11}$$

These operations follow directly from the Equations (2)–(5), respectively.

**Theorem 1** ([40]). *The space $\mathbb{F}_I$ dealing with a supremum metric, i.e., for $\widetilde{\psi},\ \widetilde{\varpi} \in \mathbb{F}_I$*

$$d_\infty(\widetilde{\psi},\ \widetilde{\varpi}) = \sup_{0 \leq \lambda \leq 1} d_H\Big(\big[\widetilde{\psi}\big]^\lambda,\ [\widetilde{\varpi}]^\lambda\Big), \tag{12}$$

*Is a complete metric space, where H denote the well-known Hausdorff metric on the space of intervals.*

**Definition 3** ([40]). *The F·I-V·F $\widetilde{\mathfrak{S}} : [u, v] \to \mathbb{F}_I$ is said to be convex F·I-V·F on $[u, v]$ if*

$$\widetilde{\mathfrak{S}}(\sigma x + (1-\sigma)\omega) \preccurlyeq \sigma \widetilde{\mathfrak{S}}(x)\ \widetilde{+}\ (1-\sigma)\widetilde{\mathfrak{S}}(\omega), \tag{13}$$

*for all $x,\ \omega \in [u, v]$, $\sigma \in [0,\ 1]$, where $\mathfrak{S}(x) \succcurlyeq \widetilde{0}$. If $\widetilde{\mathfrak{S}}$ is concave F·I-V·F on $[u, v]$, then inequality (14) is reversed.*

**Definition 4** ([37]). *Let $h_1,\ h_2 : [0,\ 1] \subseteq [u,\nu] \to \mathbb{R}^+$ such that $h_1,\ h_2 \not\equiv 0$. Then, F·I-V·F $\widetilde{\mathfrak{S}} : [u,\nu] \to \mathbb{F}_I$ is said to be $(h_1,\ h_2)$-preinvex F·I-V·F on $[u,\nu]$ if*

$$\widetilde{\mathfrak{S}}(x + (1-\sigma)\varphi(\omega, x)) \preccurlyeq \eta_1(\sigma)\eta_2(1-\sigma)\widetilde{\mathfrak{S}}(x)\widetilde{+}\eta_1(1-\sigma)\eta_2(\sigma)\widetilde{\mathfrak{S}}(\omega), \tag{14}$$

*for all $x,\ \omega \in [u,\nu]$, $\sigma \in [0,\ 1]$, where $\widetilde{\mathfrak{S}}(x) \succcurlyeq \widetilde{0}$ and $\varphi : [u,\nu] \times [u,\nu] \to [u,\nu]$. If $\widetilde{\mathfrak{S}}$ is $(\eta_1,\ \eta_2)$-concave on $[u,\nu]$, then inequality (15) is reversed.*

**Remark 2** ([37]). *If $\eta_2(\sigma) \equiv 1$, then $(\eta_1,\ \eta_2)$-preinvex F·I-V·F becomes $\eta_1$-preinvex F·I-V·F, that is*

$$\widetilde{\mathfrak{S}}(x + (1-\sigma)\varphi(\omega, x)) \preccurlyeq \eta_1(\sigma)\widetilde{\mathfrak{S}}(x)\widetilde{+}\eta_1(1-\sigma)\widetilde{\mathfrak{S}}(\omega),\ \forall\ x,\ \omega \in [u,\nu],\ \sigma \in [0,\ 1]. \tag{15}$$

*If $\eta_1(\sigma) = \sigma, \eta_2(\sigma) \equiv 1$, then $(\eta_1,\ \eta_2)$-preinvex F·I-V·F becomes preinvex F·I-V·F, that is*

$$\widetilde{\mathfrak{S}}(x + (1-\sigma)\varphi(\omega, x)) \preccurlyeq \sigma\widetilde{\mathfrak{S}}(x)\widetilde{+}(1-\sigma)\widetilde{\mathfrak{S}}(\omega),\ \forall\ x,\ \omega \in [u,\nu],\ \sigma \in [0,\ 1]. \tag{16}$$

*If $\eta_1(\sigma) = \eta_2(\sigma) \equiv 1$, then $(\eta_1,\ \eta_2)$-preinvex F·I-V·F becomes P F·I-V·F, that is*

$$\widetilde{\mathfrak{S}}(x + (1-\sigma)\varphi(\omega, x)) \preccurlyeq \widetilde{\mathfrak{S}}(x)\widetilde{+}\widetilde{\mathfrak{S}}(\omega),\ \forall\ x,\ \omega \in [u,\nu],\ \sigma \in [0,\ 1]. \tag{17}$$

**Condition 1** (see [46]). *Let K be an invex set with respect to $\theta$. For any $x,\ \omega \in K$ and $\xi \in [0,\ 1]$,*

$$\theta(x, x + \xi\theta(\omega, x)) = -\xi\theta(\omega, x),$$

$$\theta(\omega, x + \xi\theta(\omega, x)) = (1-\xi)\theta(\omega, x).$$

*Clearly for $\xi = 0$, we have $\theta(\omega, x) = 0$ if and only if, $\omega = x$, for all $x,\ \omega \in K$. For the applications of Condition 1, see [46–48].*

**Theorem 2** ([19]). *If $\mathfrak{S} : [u,\nu] \subset \mathbb{R} \to \mathbb{R}_I$ is an I-V·F given by $(x)\ [\mathfrak{S}_*(x),\ \mathfrak{S}^*(x)]$, then $\mathfrak{S}$ is Riemann integrable over $[u,\nu]$ if and only if, $\mathfrak{S}_*$ and $\mathfrak{S}^*$ both are Riemann integrable over $[u,\nu]$ such that*

$$(IR)\int_u^\nu \mathfrak{S}(x)dx = [(R)\int_u^\nu \mathfrak{S}_*(x)dx,\ (R)\int_u^\nu \mathfrak{S}^*(x)dx] \tag{18}$$

*The collection of all Riemann integrable real-valued functions and Riemann integrable I-V·F is denoted by $\mathcal{R}_{[u,\ \nu]}$ and $\mathfrak{T}\mathcal{R}_{[u,\ \nu]}$, respectively.*

**Definition 5** ([45]). *Let $\mathfrak{S} : [\tau,\ \varsigma] \subset \mathbb{R} \to \mathbb{F}_I$ is fuzzy-number valued mapping. The fuzzy Riemann integral ($(FR)$-integral) of $\mathfrak{S}$ over $[\tau,\ \varsigma]$, denoted by $(FR)\int_\tau^\varsigma \mathfrak{S}(\varkappa)d\varkappa$, is defined level-wise by*

$$\left[(FR)\int_\tau^\varsigma \mathfrak{S}(\varkappa)d\varkappa\right]^\lambda = (IR)\int_\tau^\varsigma \mathfrak{S}_\lambda(\varkappa)d\varkappa = \left\{\int_\tau^\varsigma \mathfrak{S}(\varkappa,\lambda)d\varkappa : \mathfrak{S}(\varkappa,\lambda) \in S(\mathfrak{S}_\lambda)\right\}, \tag{19}$$

*where $S(\mathfrak{S}_\lambda) = \{\mathfrak{S}(.,\lambda) \to \mathbb{R} : \mathfrak{S}(.,\lambda)$ is integrable and $\mathfrak{S}(\varkappa,\lambda) = \mathfrak{S}_\lambda(\varkappa)\}$, for every $\lambda \in [0,\ 1]$. $\mathfrak{S}$ is $(FR)$-integrable over $[\tau,\ \varsigma]$ if $(FR)\int_\tau^\varsigma \mathfrak{S}(\varkappa)d\varkappa \in \mathbb{R}_I$.*

Note that, the Theorem 2 is also true for interval double integrals. The collection of all double integrable I-V·F is denoted $\mathfrak{T}\mathfrak{O}_\Delta$, respectively.

**Theorem 3** ([32]). *Let $\Delta = [a,\ b] \times [u,\ \nu]$. If $\mathfrak{S} : \Delta \to \mathbb{R}_I$ is ID-integrable on $\Delta$, then we have*

$$(ID)\int_a^b\int_u^\nu \mathfrak{S}(x,\omega)d\omega dx = (IR)\int_a^b(IR)\int_u^\nu \mathfrak{S}(x,\omega)d\omega dx \tag{20}$$

**Definition 6** ([38]). *A fuzzy-interval-valued map $\widetilde{\mathfrak{S}} : \Delta = [a,\ b] \times [u,\ \nu] \to \mathbb{F}_I$ is called F·I-V·F on coordinates. Then, from $\lambda$-levels, we get the collection of I-V·Fs $\mathfrak{S}_\lambda : \Delta \subset \mathbb{R}^2 \to \mathbb{R}_I$ on coordinates given by $\mathfrak{S}_\lambda(x,\omega) = [\mathfrak{S}_*((x,\omega),\lambda),\ \mathfrak{S}^*((x,\omega),\lambda)]$ for all $(x,\omega) \in \Delta$, where $\mathfrak{S}_*(.,\lambda),\ \mathfrak{S}^*(.,\lambda) : (x,\omega) \to \mathbb{R}$ are called lower and upper functions of $\mathfrak{S}_\lambda$.*

**Definition 7** ([38]). *Let $\widetilde{\mathfrak{S}} : \Delta = [a,\ b] \times [u,\ \nu] \subset \mathbb{R}^2 \to \mathbb{F}_I$ be a coordinated F·I-V·F. Then, $\widetilde{\mathfrak{S}}(x,\omega)$ is said to be continuous at $(x,\omega) \in \Delta = [a,\ b] \times [u,\ \nu]$, if for each $\lambda \in [0,\ 1]$, both end point functions $\mathfrak{S}_*((x,\omega),\lambda)$ and $\mathfrak{S}^*((x,\omega),\lambda)$ are continuous at $(x,\omega) \in \Delta$.*

**Definition 8** ([38]). *Let $\widetilde{\mathfrak{S}} : \Delta = [a,\ b] \times [u,\ \nu] \subset \mathbb{R}^2 \to \mathbb{F}_I$ be a F·I-V·F on coordinates. Then, fuzzy double integral of $\widetilde{\mathfrak{S}}$ over $\Delta = [a,\ b] \times [u,\ \nu]$, denoted by $(FD)\int_a^b \int_u^\nu \widetilde{\mathfrak{S}}(x,\omega)d\omega dx$, it is defined level-wise by*

$$\left[(FD)\int_a^b \int_u^\nu \widetilde{\mathfrak{S}}(x,\omega)d\omega dx\right]^\lambda = (ID)\int_a^b \int_u^\nu \mathfrak{S}_\lambda(x,\omega)d\omega dx(IR)\int_a^b = (IR)\int_u^\nu \mathfrak{S}_\lambda(x,\omega)d\omega dx, \tag{21}$$

*for all $\lambda \in [0,\ 1]$, $\widetilde{\mathfrak{S}}$ is FD-integrable over $\Delta$ if $(FD)\int_a^b \int_u^\nu \widetilde{\mathfrak{S}}(x,\omega)d\omega dx \in \mathbb{F}_I$. Note that, if end-point functions are Lebesgue-integrable, then $\widetilde{\mathfrak{S}}$ is a fuzzy double Aumann-integrable function over $\Delta$.*

**Theorem 4** ([38]). *Let $\widetilde{\mathfrak{S}} : \Delta \subset \mathbb{R}^2 \to \mathbb{F}_I$ be a F·I-V·F on coordinates. Then, from $\lambda$-levels, we get the collection of I-V·Fs $\mathfrak{S}_\lambda : \Delta \subset \mathbb{R}^2 \to \mathbb{R}_I$ are given by $\mathfrak{S}_\lambda(x,\omega) = [\mathfrak{S}_*((x,\omega),\lambda),\ \mathfrak{S}^*((x,\omega),\lambda)]$ for all $(x,\omega) \in \Delta = [a,\ b] \times [u,\ \nu]$ and for all $\lambda \in [0,\ 1]$. Then, $\widetilde{\mathfrak{S}}$ is FD-integrable over $\Delta$ if and only if, $\mathfrak{S}_*((x,\omega),\lambda)$ and $\mathfrak{S}^*((x,\omega),\lambda)$ both are D-integrable over $\Delta$. Moreover, if $\widetilde{\mathfrak{S}}$ is FD-integrable over $\Delta$, then*

$$\begin{aligned}&\left[(FD)\int_a^b \int_u^\nu \widetilde{\mathfrak{S}}(x,\omega)d\omega dx\right]^\lambda = \left[(FR)\int_a^b (FR)\int_u^\nu \widetilde{\mathfrak{S}}(x,\omega)d\omega dx\right]^\lambda = \\ &(IR)\int_a^b (IR)\int_u^\nu \mathfrak{S}_\lambda(x,\omega)d\omega dx = (ID)\int_a^b \int_u^\nu \mathfrak{S}_\lambda(x,\omega)d\omega dx,\end{aligned} \tag{22}$$

*for all $\lambda \in [0,\ 1]$.*

The family of all $FD$-integrable F·I-V·Fs over coordinates is denoted by $\mathcal{F}\mathfrak{O}_\Delta$ for all $\lambda \in [0,\ 1]$.

**Theorem 5** ([38]). *Let $\varrho \in \mathbb{R}$, and $\widetilde{\mathfrak{S}},\ \widetilde{\mathcal{J}} \in \mathcal{F}\mathfrak{O}_\Delta$. Then,*

*(1) $\varrho\widetilde{\mathfrak{S}} \in \mathcal{F}\mathfrak{O}_\Delta$ and*

$$(FD)\iint_\Delta \varrho\widetilde{\mathfrak{S}}dA = \varrho(FD)\iint_\Delta \widetilde{\mathfrak{S}}dA \tag{23}$$

*(2) $\widetilde{\mathfrak{S}}\widetilde{+}\ \widetilde{\mathcal{J}} \in \mathcal{F}\mathfrak{O}_\Delta$, and*

$$(FD)\iint_\Delta \left(\widetilde{\mathfrak{S}}\widetilde{+}\widetilde{\mathcal{J}}\right)dA = (FD)\iint_\Delta \widetilde{\mathfrak{S}}dA\widetilde{+}(FD)\iint_\Delta \widetilde{\mathcal{J}}dA \tag{24}$$

*(3) suppose that $\Delta_1$ and $\Delta_2$ are non-overlapping, then*

$$(FD)\iint_{\Delta_1\cup\ \Delta_2} \widetilde{\mathfrak{S}}dA = (FD)\iint_{\Delta_1} \widetilde{\mathfrak{S}}dA + (FD)\iint_{\Delta_2} \widetilde{\mathfrak{S}}dA \tag{25}$$

**Theorem 6** ([37]). *Let $\widetilde{\mathfrak{S}}, \widetilde{\mathcal{J}}\ : [u,\ u+\varphi(\nu,u)] \to \mathbb{F}_I$ be two $(\eta_1,\ \eta_2)$-preinvex F·I-V·Fs with $\eta_1,\ \eta_2 : [0,\ 1] \to \mathbb{R}^+$ and $\eta_1\left(\frac{1}{2}\right)\eta_2\left(\frac{1}{2}\right) \neq 0$. Then, from $\lambda$-levels, we get the collection of I-V·Fs $\mathfrak{S}_\lambda,\ \mathcal{J}_\lambda : [u,\ u+\varphi(\nu,u)] \subset \mathbb{R} \to \mathbb{R}_I^+$ are given by $\mathfrak{S}_\lambda(x) = [\mathfrak{S}_*(x,\lambda),\ \mathfrak{S}^*(x,\lambda)]$ and*

$\mathcal{J}_\lambda(x) = [\mathcal{J}_*(x,\lambda),\ \mathcal{J}^*(x,\lambda)]$ *for all* $x \in [u,\ u+\varphi(\nu,u)]$ *and for all* $\lambda \in [0,\ 1]$. *If* $\widetilde{\mathfrak{S}}\widetilde{\times}\widetilde{\mathcal{J}}$ *is fuzzy Riemann integrable, then*

$$\begin{aligned}&\tfrac{1}{\varphi(\nu,u)}\ (FR)\int_u^{u+\varphi(\nu,u)}\widetilde{\mathfrak{S}}(x)\widetilde{\times}\widetilde{\mathcal{J}}(x)dx\\ &\preccurlyeq \widetilde{\beta}(u,\nu)\int_0^1[\eta_1(\sigma)\eta_2(1-\sigma)]^2 d\sigma\widetilde{+}\widetilde{\gamma}(u,\nu)\int_0^1\eta_1(\sigma)\eta_2(\sigma)\eta_1(1-\sigma)\eta_2(1-\sigma)d\sigma,\end{aligned} \tag{26}$$

*and,*

$$\begin{aligned}&\tfrac{1}{2\left[\eta_1\left(\frac{1}{2}\right)\eta_2\left(\frac{1}{2}\right)\right]^2}\ \widetilde{\mathfrak{S}}\left(\tfrac{2u+\varphi(\nu,u)}{2}\right)\widetilde{\mathcal{J}}\left(\tfrac{2u+\varphi(\nu,u)}{2}\right)\\ &\preccurlyeq \tfrac{1}{\varphi(\nu,u)}\ (FR)\int_u^{u+\varphi(\nu,u)}\widetilde{\mathfrak{S}}(x)\widetilde{\mathcal{J}}(x)dx\ \widetilde{+}\ \widetilde{\gamma}(u,\nu)\int_0^1[\eta_1(\sigma)\eta_2(1-\sigma)]^2 d\sigma\\ &\widetilde{+}\widetilde{\beta}(u,\nu)\int_0^1\eta_1(\sigma)\eta_2(\sigma)\eta_1(1-\sigma)\eta_2(1-\sigma)d\sigma\end{aligned} \tag{27}$$

*where* $\widetilde{\beta}(u,\nu) = \widetilde{\mathfrak{S}}(u)\widetilde{\times}\widetilde{\mathcal{J}}(u)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(\nu)\widetilde{\times}\widetilde{\mathcal{J}}(\nu)$, $\widetilde{\gamma}(u,\nu) = \widetilde{\mathfrak{S}}(u)\widetilde{\times}\widetilde{\mathcal{J}}(\nu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(\nu)\widetilde{\times}\widetilde{\mathcal{J}}(u)$, *and* $\beta_\lambda(u,\nu) = [\beta_*((u,\nu),\ \lambda),\ \beta^*((u,\nu),\ \lambda)]$ *and* $\gamma_\lambda(u,\nu) = [\gamma_*((u,\nu),\ \lambda),\ \gamma^*((u,\nu),\ \lambda)]$.

**Remark 3.** *If* $\eta_1(\sigma) = \sigma$ *and* $\eta_2(\sigma) \equiv 1$, *then (27) reduces to the result for preinvex F·I-V·F:*

$$\frac{1}{\varphi(\nu,u)}\ (FR)\int_u^{u+\varphi(\nu,u)}\widetilde{\mathfrak{S}}(x)\widetilde{\times}\widetilde{\mathcal{J}}(x)dx \preccurlyeq \frac{1}{3}\widetilde{\beta}(u,\nu)\widetilde{+}\ \frac{1}{6}\widetilde{\gamma}(u,\nu) \tag{28}$$

*If* $\eta_1(\sigma) = \sigma$ *and* $\eta_2(\sigma) \equiv 1$, *then (28) reduces to the result for preinvex F·I-V·F:*

$$\begin{aligned}&2\ \widetilde{\mathfrak{S}}\left(\tfrac{2u+\varphi(\nu,u)}{2}\right)\widetilde{\times}\widetilde{\mathcal{J}}\left(\tfrac{2u+\varphi(\nu,u)}{2}\right)\\ &\preccurlyeq \tfrac{1}{\varphi(\nu,u)}\ (FR)\int_u^{u+\varphi(\nu,u)}\widetilde{\mathfrak{S}}(x)\widetilde{\times}\widetilde{\mathcal{J}}(x)dx\ \widetilde{+}\tfrac{1}{6}\widetilde{\beta}(u,\nu)\widetilde{+}\ \tfrac{1}{3}\widetilde{\gamma}(u,\nu).\end{aligned} \tag{29}$$

**Theorem 7** ([37]). *Let* $\widetilde{\mathfrak{S}} : [u,\ u+\varphi(\nu,u)] \to \mathbb{F}_I$ *be a preinvex F·I-V·F with* $u < u+\varphi(\nu,u)$. *Then, from λ-levels, we get the collection of I-V·Fs* $\mathfrak{S}_\lambda : [u,\ u+\varphi(\nu,u)] \subset \mathbb{R} \to \mathbb{R}_I^+$ *are given by* $\mathfrak{S}_\lambda(x) = [\mathfrak{S}_*(x,\lambda),\ \mathfrak{S}^*(x,\lambda)]$ *for all* $x \in [u,\ u+\varphi(\nu,u)]$ *and for all* $\lambda \in [0,\ 1]$, *and Condition 1 for* $\varphi$ *holds. If* $\widetilde{\mathfrak{S}} \in\ \mathfrak{S}\mathcal{R}_{([u,\ u+\varphi(\nu,u)],\ \lambda)}$ *and* $\psi : [u,\ \nu] \to \mathbb{R},\ \psi(x) \geq 0,$ *symmetric with respect to* $\frac{2\ u+\varphi(\nu,u)}{2}$, *and* $\int_u^{u,\ u+\varphi(\nu,u)}\psi(x)dx > 0$, *then*

$$\widetilde{\mathfrak{S}}\left(\frac{2u+\varphi(\nu,u)}{2}\right) \preccurlyeq \frac{1}{\int_u^{u+\varphi(\nu,u)}\psi(x)dx}\ (FR)\int_u^{u+\varphi(\nu,u)}\widetilde{\mathfrak{S}}(x)\psi(x)dx \preccurlyeq \frac{\widetilde{\mathfrak{S}}(u)\widetilde{+}\widetilde{\mathfrak{S}}(\nu)}{2}. \tag{30}$$

*If* $\mathfrak{S}$ *is preincave F·I-V·F, then inequality (31) is reversed.*
*Note that if* $\psi(x) = 1$, *then we acquire the following inequality:*

$$\widetilde{\mathfrak{S}}\left(\frac{2u+\varphi(\nu,u)}{2}\right) \preccurlyeq \frac{1}{(\nu,u)}\ (FR)\int_u^{u+\varphi(\nu,u)}\widetilde{\mathfrak{S}}(x)dx \preccurlyeq \frac{\widetilde{\mathfrak{S}}(u)\widetilde{+}\widetilde{\mathfrak{S}}(\nu)}{2}. \tag{31}$$

**Coordinated preinvex fuzzy-interval-valued functions**

**Definition 9.** *The F·I-V·F* $\widetilde{\mathfrak{S}} : \Delta \to \mathbb{F}_I$ *is said to be a coordinated preinvex F·I-V·F on* $\Delta$ *if*

$$\begin{aligned}&\widetilde{\mathfrak{S}}(a+(1-\sigma)\varphi_1(b,a),\ u+(1-s)\varphi_2(\nu,u))\\ &\preccurlyeq \sigma s\widetilde{\mathfrak{S}}(a,u)\widetilde{+}\sigma(1-s)\widetilde{\mathfrak{S}}(a,\nu)\widetilde{+}(1-\sigma)s\widetilde{\mathfrak{S}}(b,u)\widetilde{+}(1-\sigma)(1-s)\widetilde{\mathfrak{S}}(b,\nu),\end{aligned} \tag{32}$$

*for all* $(a,\ b),\ (u,\nu) \in \Delta$, *and* $\sigma, s \in [0,\ 1]$, *where* $\widetilde{\mathfrak{S}}(x) \succcurlyeq \widetilde{0}$. *If inequality (33) is reversed, then* $\widetilde{\mathfrak{S}}$ *is called coordinated concave F·I-V·F on* $\Delta$.

The proof that Lemma 1 is straightforward will be omitted here.

**Lemma 1.** *Let $\widetilde{\mathfrak{S}} : \Delta \to \mathbb{F}_I$ be a coordinated F·I-V·F on $\Delta$. Then, $\widetilde{\mathfrak{S}}$ is a coordinated preinvex F·I-V·F on $\Delta$, if and only if there exist two coordinated preinvex F·I-V·Fs $\widetilde{\mathfrak{S}}_x : [u, \nu] \to \mathbb{F}_I$, $\widetilde{\mathfrak{S}}_x(w) = \widetilde{\mathfrak{S}}(x, w)$ and $\widetilde{\mathfrak{S}}_\omega : [a, b] \to \mathbb{F}_I$, $\widetilde{\mathfrak{S}}_\omega(q) = \widetilde{\mathfrak{S}}(q, \omega)$.*

**Proof.** From the Definition 9 of coordinated preinvex F·I-V·F, it can be easily proved. □

From Lemma 1, we can easily note each preinvex F·I-V·F is a coordinated preinvex F·I-V·F. However, the converse is not true, see Example 1.

**Theorem 8.** *Let $\widetilde{\mathfrak{S}} : \Delta \to \mathbb{F}_I$ be a F·I-V·F on $\Delta$. Then, from $\lambda$-levels, we get the collection of I-V·Fs $\mathfrak{S}_\lambda : \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ which are given by*

$$\mathfrak{S}_\lambda(x, \omega) = [\mathfrak{S}_*((x, \omega), \lambda),\ \mathfrak{S}^*((x, \omega), \lambda)], \tag{33}$$

*For all $(x, \omega) \in \Delta$ and for all $\lambda \in [0,\ 1]$. Then, $\widetilde{\mathfrak{S}}$ is coordinated preinvex F·I-V·F on $\Delta$, if and only if, for all $\lambda \in [0,\ 1]$, $\mathfrak{S}_*((x, \omega),\ \lambda)$ and $\mathfrak{S}^*((x, \omega),\ \lambda)$ are coordinated preinvex functions.*

**Proof.** Assume that for each $\lambda \in [0,\ 1]$,$\mathfrak{S}_*(x,\ \lambda)$ and $\mathfrak{S}^*(x,\ \lambda)$ are coordinated preinvex on $\Delta$. Then, from (33), for all $(a,\ b),\ (u, \nu) \in \Delta$, $\sigma$ and $s \in [0,\ 1]$, we have

$$\begin{aligned}
&\mathfrak{S}_*((a + (1-\sigma)\varphi_1(b, a),\ u + (1-s)\varphi_2(\nu, u)),\ \lambda)\\
&\leq \sigma s \mathfrak{S}_*((a, u),\ \lambda) + t(1-s)\mathfrak{S}_*((a, \nu),\ \lambda)\\
&+ s(1-t)\mathfrak{S}_*((a, u),\ \lambda) + (1-\sigma)(1-s)\mathfrak{S}_*((a, \nu),\ \lambda)
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{S}^*((a + (1-\sigma)\varphi_1(b, a),\ u + (1-s)\varphi_2(\nu, u)),\ \lambda)\\
&\leq \sigma s \mathfrak{S}_*((a, u),\ \lambda) + t(1-s)\mathfrak{S}^*((a, \nu),\ \lambda)\\
&+ s(1-t)\mathfrak{S}^*((a, u),\ \lambda) + (1-\sigma)(1-s)\mathfrak{S}^*((a, \nu),\ \lambda)
\end{aligned}$$

Then, by (33), (10), and (12), we obtain

$$\begin{aligned}
&\mathfrak{S}_\lambda((a + (1-\sigma)\varphi_1(b, a),\ u + (1-s)\varphi_2(\nu, u)))\\
&= [\mathfrak{S}_*((a + (1-\sigma)\varphi_1(b, a),\ u + (1-s)\varphi_2(\nu, u)),\ \lambda),\ \mathfrak{S}^*((a + (1-\sigma)\varphi_1(b, a),\ u + (1-s)\varphi_2(\nu, u)),\ \lambda)]\\
&\leq_I \sigma s[\mathfrak{S}_*((a, u),\ \lambda),\ \mathfrak{S}^*((a, u),\ \lambda)] + t(1-s)[\mathfrak{S}_*((a, \nu),\ \lambda),\ \mathfrak{S}^*((a, \nu),\ \lambda)]\\
&+ s(1-\sigma)[\mathfrak{S}_*((a, u),\ \lambda),\ \mathfrak{S}^*((a, u),\ \lambda)] + (1-\sigma)(1-s)[\mathfrak{S}_*((a, \nu),\ \lambda),\ \mathfrak{S}^*((a, \nu),\ \lambda)]
\end{aligned}$$

That is

$$\begin{aligned}
&\widetilde{\mathfrak{S}}(a + (1-\sigma)\varphi_1(b, a),\ u + (1-s)\varphi_2(\nu, u))\\
&\preccurlyeq \sigma s \widetilde{\mathfrak{S}}(a, u) \widetilde{+} \sigma(1-s)\widetilde{\mathfrak{S}}(a, \nu) \widetilde{+} (1-\sigma)s\widetilde{\mathfrak{S}}(b, u) \widetilde{+} (1-\sigma)(1-s)\widetilde{\mathfrak{S}}(b, \nu),
\end{aligned}$$

hence, $\mathfrak{S}$ is a coordinated preinvex F·I-V·F on $\Delta$

Conversely, let $\mathfrak{S}$ be a coordinated preinvex F·I-V·F on $\Delta$. Then, for all $(a,\ b),\ (u, \nu) \in \Delta$, $\sigma$ and $s \in [0,\ 1]$, we have

$$\begin{aligned}
&\widetilde{\mathfrak{S}}(a + (1-\sigma)\varphi_1(b, a),\ u + (1-s)\varphi_2(\nu, u))\\
&\preccurlyeq \sigma s \widetilde{\mathfrak{S}}(a, u) \widetilde{+} \sigma(1-s)\widetilde{\mathfrak{S}}(a, \nu) \widetilde{+} (1-\sigma)s\widetilde{\mathfrak{S}}(b, u) \widetilde{+} (1-\sigma)(1-s)\widetilde{\mathfrak{S}}(b, \nu)
\end{aligned}$$

Therefore, again from (34), for each $\lambda \in [0,\ 1]$, we have

$$\begin{aligned}
&\mathfrak{S}_\lambda((a + (1-\sigma)\varphi_1(b, a),\ u + (1-s)\varphi_2(\nu, u)))\\
&= [\mathfrak{S}_*((a + (1-\sigma)\varphi_1(b, a),\ u + (1-s)\varphi_2(\nu, u)),\ \lambda),\ \mathfrak{S}^*((a + (1-\sigma)\varphi_1(b, a),\ u + (1-s)\varphi_2(\nu, u)),\ \lambda)]
\end{aligned}$$

Again, (10) and (12), we obtain

$$\begin{aligned}&\sigma s\mathfrak{S}_\lambda(a,u)+\sigma(1-s)\mathfrak{S}_\lambda(a,\nu)+(1-\sigma)s\mathfrak{S}_\lambda(b,u)+(1-\sigma)(1-s)\mathfrak{S}_\lambda(b,\nu)\\&=\sigma s[\mathfrak{S}_*((a,u),\ \lambda),\ \mathfrak{S}^*((a,u),\ \lambda)]\\&+t(1-s)[\mathfrak{S}_*((a,\nu),\ \lambda),\ \mathfrak{S}^*((a,\nu),\ \lambda)]\\&+s(1-\sigma)[\mathfrak{S}_*((a,u),\ \lambda),\ \mathfrak{S}^*((a,u),\ \lambda)]\\&+(1-\sigma)(1-s)[\mathfrak{S}_*((a,\nu),\ \lambda),\ \mathfrak{S}^*((a,\nu),\ \lambda)],\end{aligned}$$

for all $x,\omega\in\Delta$ and $\sigma\in[0,\ 1]$. Then, by coordinated preinvexity of $\mathfrak{S}$, we have for all $x,\omega\in\Delta$ and $\sigma\in[0,\ 1]$ such that

$$\begin{aligned}&\mathfrak{S}_*((a+(1-\sigma)\varphi_1(b,a),\ u+(1-s)\varphi_2(\nu,u)),\ \lambda)\\&\leq\sigma s\mathfrak{S}_*(a,u)+\sigma(1-s)\mathfrak{S}_*(a,\nu)+(1-\sigma)s\mathfrak{S}_*(b,u)\\&+(1-\sigma)(1-s)\mathfrak{S}_*(b,\nu),\end{aligned}$$

and

$$\begin{aligned}&\mathfrak{S}^*((a+(1-\sigma)\varphi_1(b,a),\ u+(1-s)\varphi_2(\nu,u)),\ \lambda)\\&\leq\sigma s\mathfrak{S}^*(a,u)+\sigma(1-s)\mathfrak{S}^*(a,\nu)+(1-\sigma)s\mathfrak{S}^*(b,u)\\&+(1-\sigma)(1-s)\mathfrak{S}^*(b,\nu),\end{aligned}$$

for each $\lambda\in[0,\ 1]$. Hence, the result follows. □

**Remark 4.** *If one takes $\varphi_1(b,a)=b-a$ and $\varphi_2(\nu,u)=\nu-u$, then $\widetilde{\mathfrak{S}}$ is known as aconvex F·I-V·F on coordinates if $\widetilde{\mathfrak{S}}$ satisfies the following inequality:*

$$\begin{aligned}&\widetilde{\mathfrak{S}}(\sigma a+(1-\sigma)b,\ su+(1-s)\nu)\\&\preccurlyeq\sigma s\widetilde{\mathfrak{S}}(a,u)\tilde{+}\sigma(1-s)\widetilde{\mathfrak{S}}(a,\nu)\tilde{+}(1-\sigma)s\widetilde{\mathfrak{S}}(b,u)\tilde{+}(1-\sigma)(1-s)\widetilde{\mathfrak{S}}(b,\nu),\end{aligned}\tag{34}$$

*which is valid defined by Khan et al. [38].*

*If one takes $\mathfrak{S}_*(x,\omega)=\mathfrak{S}^*(x,\omega)$ with $\lambda=1$, then $\mathfrak{S}$ is known as a preinvex function on coordinates if $\mathfrak{S}$ satisfies the following inequality*

$$\begin{aligned}&\mathfrak{S}(a+(1-\sigma)\varphi_1(b,a),\ u+(1-s)\varphi_1(\nu,u))\\&\leq\sigma s\mathfrak{S}(a,u)+\sigma(1-s)\mathfrak{S}(a,\nu)+(1-\sigma)s\mathfrak{S}(b,u)\\&+(1-\sigma)(1-s)\mathfrak{S}(b,\nu),\end{aligned}\tag{35}$$

*which is defined by Latif and Dragomir [44].*

*If one takes $\mathfrak{S}_*(x,\omega)=\mathfrak{S}^*(x,\omega)$ with $\lambda=1$, then $\mathfrak{S}$ is known as a convex function on coordinates if $\mathfrak{S}$ satisfies the following inequality*

$$\begin{aligned}&\mathfrak{S}(a\sigma a+(1-\sigma)b,\ su+(1-s)\nu)\\&\leq\sigma s\mathfrak{S}(a,u)+\sigma(1-s)\mathfrak{S}(a,\nu)+(1-\sigma)s\mathfrak{S}(b,u)\\&+(1-\sigma)(1-s)\mathfrak{S}(b,\nu),\end{aligned}\tag{36}$$

*is valid, then $\mathfrak{S}$ is named as IVFon coordinates, which is defined by Dragomir [34].*

**Example 1.** *We consider the F·I-V·Fs $\widetilde{\mathfrak{S}}:[0,\ 1]\times[0,\ 1]\to\mathbb{F}_I$ defined by,*

$$\mathfrak{S}(x,\ \omega)(\sigma)=\begin{cases}\frac{\sigma}{\sigma\omega} & \sigma\in[0,\ \sigma\omega]\\ \frac{2\sigma\omega-\sigma}{\sigma\omega} & \sigma\in(\sigma\omega,\ 2\sigma\omega]\\ 0 & \text{otherwise,}\end{cases}$$

*Then, for each $\lambda\in[0,\ 1]$, we have $\mathfrak{S}_\lambda(x)=[\lambda x\omega,(2-\lambda)x\omega]$. End-point functions $\mathfrak{S}_*((x,\omega),\lambda),\mathfrak{S}^*((x,\omega),\lambda)$ are coordinated concave functions with respect to $\varphi_1(b,a)=b-a$ and $\varphi_2(\nu,u)=\nu-u$ for each $\lambda\in[0,\ 1]$. Hence, $\widetilde{\mathfrak{S}}(x,\omega)$ is a coordinated concave F·I-V·F.*

From Example 1, it can be easily seen that each coordinated preinvex F·I-V·F is not a preinvex F·I-V·F.

**Theorem 9.** *Let $\Delta$ be a coordinated preinvex set, and let $\widetilde{\mathfrak{S}} : \Delta \to \mathbb{F}_I$ be a F·I-V·F. Then, from $\lambda$-levels, we obtain the collection of I-V·Fs $\mathfrak{S}_\lambda : \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ are given by*

$$\mathfrak{S}_\lambda(x,\omega) = [\mathfrak{S}_*((x,\omega),\lambda),\ \mathfrak{S}^*((x,\omega),\lambda)], \tag{37}$$

*for all $(x,\omega) \in \Delta$ and for all $\lambda \in [0,\ 1]$. Then, $\widetilde{\mathfrak{S}}$ is a coordinated preinvex F·I-V·F on $\Delta$, if and only if, for all $\lambda \in [0,\ 1]$, $\mathfrak{S}_*((x,\omega),\ \lambda)$ and $\mathfrak{S}^*((x,\omega),\ \lambda)$ are coordinated preinvex functions.*

**Proof.** The proof of Theorem 9 is similar to that of Theorem 8. □

**Example 2.** *We consider the F·I-V·Fs $\widetilde{\mathfrak{S}} : [0,\ 1] \times [0,\ 1] \to \mathbb{F}_I$ defined by,*

$$\widetilde{\mathfrak{S}}(x)(\sigma) = \begin{cases} \frac{\sigma}{2(6-e^x)(6-e^\omega)}, & \sigma \in [0,\ 2(6-e^x)(6-e^\omega)] \\ \frac{4(6-e^x)(6-e^\omega)-\sigma}{2(6-e^x)(6-e^\omega)}, & \sigma \in (2(6-e^x)(6-e^\omega),\ 4(6-e^x)(6-e^\omega)] \\ 0, & \text{otherwise.} \end{cases}$$

*Then, for each $\lambda \in [0,\ 1]$, we have $\mathfrak{S}_\lambda(x) = [2\lambda(6-e^x)(6-e^\omega), (4-2\lambda)(6-e^x)(6-e^\omega)\ ]$. End-point functions $\mathfrak{S}_*((x,\omega),\lambda)$, $\mathfrak{S}^*((x,\omega),\lambda)$ are coordinated preincave functions with respect to $\varphi_1(b,a) = b-a$ and $\varphi_2(\nu,u) = \nu - u$ for each $\lambda \in [0,\ 1]$. Hence, $\widetilde{\mathfrak{S}}(x,\omega)$ is a coordinated preincave F·I-V·F.*

In the next results, to avoid confusion, we will not include the symbols $(R)$, $(IR)$, $(FR)$, $(ID)$, and $(FD)$ before the integral sign.

## 3. Fuzzy-Interval Hermite-Hadamard Inequalities

In this section, we propose *HH*- and *HH*–Fejér inequalities for coordinated preinvex F·I-V·Fs, and verify with the help of some nontrivial example.

**Theorem 10.** *Let $\widetilde{\mathfrak{S}} : \Delta = [a,\ a+\varphi_1(b,a)] \times [u,\ u+\varphi_2(\nu,u)] \to \mathbb{F}_I$ be a coordinated preinvex F·I-V·F on $\Delta$. Then, from $\lambda$-levels, we get the collection of I-V·Fs $\mathfrak{S}_\lambda : \Delta \to \mathbb{R}_I^+$ are given by $\mathfrak{S}_\lambda(x,\omega) = [\mathfrak{S}_*((x,\omega),\lambda),\ \mathfrak{S}^*((x,\omega),\lambda)]$ for all $(x,\omega) \in \Delta$ and for all $\lambda \in [0,\ 1]$, and Condition 1 for $\varphi_1$ and $\varphi_2$ holds. Then, the following inequality holds:*

$$\begin{aligned} &\widetilde{\mathfrak{S}}\left(\frac{2a+\varphi_1(b,a)}{2},\ \frac{2u+\varphi_2(\nu,u)}{2}\right) \\ &\preccurlyeq \frac{1}{2}\left[\frac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)dx \widetilde{+} \frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}\left(\frac{2a+\varphi_1(b,a)}{2},\omega\right)d\omega\right] \\ &\preccurlyeq \frac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(x,\omega)d\omega dx \\ &\preccurlyeq \frac{1}{4\varphi_1(b,a)}\left[\int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}(x,u)dx \widetilde{+} \int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}(x,\nu)dx\right] \\ &\widetilde{+} \frac{1}{4\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(a,\omega)d\omega \widetilde{+} \int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(b,\omega)d\omega\right] \preccurlyeq \frac{\widetilde{\mathfrak{S}}(a,u) \widetilde{+} \widetilde{\mathfrak{S}}(b,u) \widetilde{+} \widetilde{\mathfrak{S}}(a,\nu) \widetilde{+} \widetilde{\mathfrak{S}}(b,\nu)}{4}. \end{aligned} \tag{38}$$

*If $\widetilde{\mathfrak{S}}(x)$ preincave F·I-V·F, then inequality (38) is reversed such that,*

$$\begin{aligned}
&\widetilde{\mathfrak{S}}\left(\tfrac{2a+\varphi_1(b,a)}{2},\ \tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\succcurlyeq \frac{1}{2}\left[\tfrac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\widetilde{\mathfrak{S}}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx\,\widetilde{+}\,\tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\widetilde{\mathfrak{S}}\left(\tfrac{2a+\varphi_1(b,a)}{2},\omega\right)d\omega\right]\\
&\succcurlyeq \tfrac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)}\widetilde{\mathfrak{S}}(x,\omega)d\omega dx\\
&\succcurlyeq \frac{1}{4\varphi_1(b,a)}\left[\int_a^{a+\varphi_1(b,a)}\widetilde{\mathfrak{S}}(x,u)dx\,\widetilde{+}\,\int_a^{a+\varphi_1(b,a)}\widetilde{\mathfrak{S}}(x,\nu)dx\right]+\frac{1}{4\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)}\widetilde{\mathfrak{S}}(a,\omega)d\omega\,\widetilde{+}\,\int_u^{u+\varphi_2(\nu,u)}\widetilde{\mathfrak{S}}(b,\omega)d\omega\right]\\
&\succcurlyeq \tfrac{\widetilde{\mathfrak{S}}(a,u)\widetilde{+}\widetilde{\mathfrak{S}}(b,u)\widetilde{+}\widetilde{\mathfrak{S}}(a,\nu)\widetilde{+}\widetilde{\mathfrak{S}}(b,\nu)}{4}
\end{aligned} \tag{39}$$

**Proof.** Let $\widetilde{\mathfrak{S}}:[a,\ a+\varphi_1(b,a)]\to\mathbb{F}_I$ be a coordinated preinvex F·I-V·F. Then, by hypotheses, we have

$$\begin{aligned}
&4\widetilde{\mathfrak{S}}\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\preccurlyeq \widetilde{\mathfrak{S}}(a+(1-\sigma)\varphi_1(b,a),\ u+(1-s)\varphi_2(\nu,u))\widetilde{+}\widetilde{\mathfrak{S}}(b+\sigma\varphi_1(b,a),\ \nu+s\varphi_2(\nu,u)).
\end{aligned}$$

By using Theorem 10, for every $\lambda\in[0,\ 1]$, we have

$$\begin{aligned}
&4\mathfrak{S}_*\left(\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right),\ \lambda\right)\\
&\le \mathfrak{S}_*((a+(1-\sigma)\varphi_1(b,a),\ u+(1-s)\varphi_2(\nu,u)),\ \lambda)\\
&+\mathfrak{S}_*((b+\sigma\varphi_1(b,a),\ \nu+s\varphi_2(\nu,u)),\ \lambda),\\
&4\mathfrak{S}^*\left(\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right),\ \lambda\right)\\
&\le \mathfrak{S}^*((a+(1-\sigma)\varphi_1(b,a),\ u+(1-s)\varphi_2(\nu,u)),\ \lambda)\\
&+\mathfrak{S}^*((b+\sigma\varphi_1(b,a),\ \nu+s\varphi_2(\nu,u)),\lambda).
\end{aligned}$$

By using Lemma 1, we have

$$\begin{aligned}
&2\mathfrak{S}_*\left(\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right),\ \lambda\right)\le \mathfrak{S}_*((x,\ u+(1-s)\varphi_2(\nu,u)),\ \lambda)+\mathfrak{S}_*((x,\ \nu+s\varphi_2(\nu,u)),\ \lambda),\\
&2\mathfrak{S}^*\left(\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right),\ \lambda\right)\le \mathfrak{S}^*((x,\ u+(1-s)\varphi_2(\nu,u)),\ \lambda)+\mathfrak{S}^*((x,\ \nu+s\varphi_2(\nu,u)),\lambda),
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
&2\mathfrak{S}_*\left(\left(\tfrac{2a+\varphi_1(b,a)}{2},\omega\right),\ \lambda\right)\le \mathfrak{S}_*((a+(1-\sigma)\varphi_1(b,a),\ \omega),\ \lambda)+\mathfrak{S}_*((\nu+s\varphi_2(\nu,u),\ \omega),\ \lambda),\\
&2\mathfrak{S}^*\left(\left(\tfrac{2a+\varphi_1(b,a)}{2},\omega\right),\ \lambda\right)\le \mathfrak{S}^*((a+(1-\sigma)\varphi_1(b,a),\ \omega),\ \lambda)+\mathfrak{S}^*((\nu+s\varphi_2(\nu,u),\ \omega),\lambda).
\end{aligned} \tag{41}$$

From (41) and (42), we have

$$\begin{aligned}
&2\left[\mathfrak{S}_*\left(\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right),\ \lambda\right),\mathfrak{S}^*\left(\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right),\ \lambda\right)\right]\\
&\le_I [\mathfrak{S}_*((x,\ u+(1-s)\varphi_2(\nu,u)),\ \lambda),\mathfrak{S}^*((x,\ u+(1-s)\varphi_2(\nu,u)),\ \lambda)]\\
&+[\mathfrak{S}_*((x,\ \nu+s\varphi_2(\nu,u)),\ \lambda),\mathfrak{S}^*((x,\ \nu+s\varphi_2(\nu,u)),\lambda)],
\end{aligned}$$

and

$$\begin{aligned}
&2\left[\mathfrak{S}_*\left(\left(\tfrac{2a+\varphi_1(b,a)}{2},\omega\right),\ \lambda\right),\mathfrak{S}^*\left(\left(\tfrac{2a+\varphi_1(b,a)}{2},\omega\right),\ \lambda\right)\right]\\
&\le_I [\mathfrak{S}_*((a+(1-\sigma)\varphi_1(b,a),\ \omega),\ \lambda),\mathfrak{S}^*((a+(1-\sigma)\varphi_1(b,a),\ \omega),\ \lambda)]\\
&+[\mathfrak{S}_*((a+(1-\sigma)\varphi_1(b,a),\ \omega),\ \lambda),\mathfrak{S}^*((a+(1-\sigma)\varphi_1(b,a),\ \omega),\lambda)],
\end{aligned}$$

It follows that

$$\mathfrak{S}_\lambda\left(x,\frac{2u+\varphi_2(\nu,u)}{2}\right)\le_I \mathfrak{S}_\lambda(x,\ u+(1-s)\varphi_2(\nu,u))+\mathfrak{S}_\lambda(x,\ \nu+s\varphi_2(\nu,u)) \tag{42}$$

and

$$\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\omega\right) \leq_I \mathfrak{S}_\lambda(a+(1-\sigma)\varphi_1(b,a),\ \omega) + \mathfrak{S}_\lambda(b+\sigma\varphi_1(b,a),\ \omega) \quad (43)$$

Since $\mathfrak{S}_\lambda(x,.)$ And $\mathfrak{S}_\lambda(.,\omega)$, both are coordinated preinvex-I-V·Fs, then from inequality (32), for every $\lambda \in [0,\ 1]$, inequality (42) and (43), we have

$$\mathfrak{S}_\lambda\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_I \frac{1}{\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(x,\omega)d\omega \leq_I \frac{\mathfrak{S}_\lambda(x,u)+\mathfrak{S}_\lambda(x,\nu)}{2}. \quad (44)$$

and

$$\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\omega\right) \leq_I \frac{1}{\varphi_1(b,a)} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,\omega)dx \leq_I \frac{\mathfrak{S}_\lambda(a,\omega)+\mathfrak{S}_\lambda(b,\omega)}{2}. \quad (45)$$

Dividing double inequality (44) by $\varphi_1(b,a)$, and integrating with respect to $x$ over $[a,\ a+\varphi_1(b,a)]$, we have

$$\begin{aligned} &\frac{1}{\varphi_1(b,a)} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)dx \\ &\leq_I \frac{1}{\varphi_1(b,a)\varphi_2(\nu,u)} \int_a^{a+\varphi_1(b,a)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(x,\omega)d\omega dx \\ &\leq_I \frac{1}{2\varphi_1(b,a)}\left[\int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,u)dx + \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,\nu)dx.\right] \end{aligned} \quad (46)$$

Similarly, dividing double inequality (46) by $\varphi_2(\nu,u)$, and integrating with respect to $x$ over $[u,\ u+\varphi_2(\nu,u)]$, we have

$$\begin{aligned} &\frac{1}{\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda\left(\frac{2a\ +\ \varphi_1(b,a)}{2},\omega\right)d\omega \leq_I \frac{1}{\varphi_1(b,a)\varphi_2(\nu,u)} \int_a^{a+\varphi_1(b,a)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(x,\omega)d\omega dx \\ &\leq_I \frac{1}{2\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(a,\omega)d\omega + \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(b,\omega)d\omega\right]. \end{aligned} \quad (47)$$

By adding (46) and (47), we have

$$\begin{aligned} &\frac{1}{2}\left[\frac{1}{\varphi_1(b,a)} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda\left(x, \frac{2u\ +\ \varphi_2(\nu,u)}{2}\right)dx + \frac{1}{\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\omega\right)d\omega\right] \\ &\leq_I \frac{1}{\varphi_1(b,a)\varphi_2(\nu,u)} \int_a^{a+\varphi_1(b,a)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(x,\omega)d\omega dx \\ &\leq_I \frac{1}{4\varphi_1(b,a)}\left[\int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,u)dx + \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,\nu)dx\right] + \frac{1}{4\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(a,\omega)d\omega + \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(b,\omega)d\omega\right] \end{aligned} \quad (48)$$

Since $\mathfrak{S}$ is F·I-V·F, then inequality (48), we have

$$\begin{aligned} &\frac{1}{2}\left[\frac{1}{\varphi_1(b,a)} \int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)dx \widetilde{+} \frac{1}{\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}\left(\frac{2a+\varphi_1(b,a)}{2},\omega\right)d\omega\right] \\ &\preccurlyeq \frac{1}{\varphi_1(b,a)\varphi_2(\nu,u)} \int_a^{a+\varphi_1(b,a)} \int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(x,\omega)d\omega dx \\ &\preccurlyeq \frac{1}{4\varphi_1(b,a)}\left[\int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}(x,u)dx \widetilde{+} \int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}(x,\nu)dx\right] \widetilde{+} \frac{1}{4\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(a,\omega)d\omega \widetilde{+} \int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(b,\omega)d\omega\right] \end{aligned} \quad (49)$$

From the left side of inequality (32), for each $\lambda \in [0,\ 1]$, we have

$$\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\ \frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_I \frac{1}{\varphi_1(b,a)} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)dx, \quad (50)$$

$$\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\ \frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_I \frac{1}{\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\omega\right)d\omega. \quad (51)$$

Taking addition of inequality (50) with inequality (51), we have

$$\mathfrak{S}_{\lambda}\left(\frac{2a+\varphi_1(b,a)}{2},\ \frac{2u+\varphi_2(\nu,u)}{2}\right)$$
$$\leq_I \frac{1}{2}\left[\frac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}\left(x,\frac{2u+\varphi_2(\nu,u)}{2}\right)dx+\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}\left(\frac{2a+\varphi_1(b,a)}{2},\omega\right)d\omega\right] \tag{52}$$

Since $\widetilde{\mathfrak{S}}$ is a F·I-V·F, then it follows that

$$\widetilde{\mathfrak{S}}\left(\frac{2a+\varphi_1(b,a)}{2},\ \frac{2u+\varphi_2(\nu,u)}{2}\right)$$
$$\preccurlyeq \frac{1}{2}\left[\frac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\widetilde{\mathfrak{S}}\left(x,\frac{2u+\varphi_2(\nu,u)}{2}\right)dx\ \widetilde{+}\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\widetilde{\mathfrak{S}}\left(\frac{2a+\varphi_1(b,a)}{2},\omega\right)d\omega\right] \tag{53}$$

Now from right side of inequality (32), for every $\lambda \in [0,\ 1]$, we have

$$\frac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,u)dx \leq_I \frac{\mathfrak{S}_{\lambda}(a,u)+\mathfrak{S}_{\lambda}(b,u)}{2} \tag{54}$$

$$\frac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,\nu)dx \leq_I \frac{\mathfrak{S}_{\lambda}(a,\nu)+\mathfrak{S}_{\lambda}(b,\nu)}{2} \tag{55}$$

$$\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(a,\omega)d\omega \leq_I \frac{\mathfrak{S}_{\lambda}(a,\nu)+\mathfrak{S}_{\lambda}(a,u)}{2} \tag{56}$$

$$\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(b,\omega)d\omega \leq_I \frac{\mathfrak{S}_{\lambda}(b,\nu)+\mathfrak{S}_{\lambda}(b,u)}{2} \tag{57}$$

By adding inequalities (54)–(57), we have

$$\frac{1}{4\varphi_1(b,a)}\left[\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,u)dx+\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,\nu)dx\right]+\frac{1}{4\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(a,\omega)d\omega+\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(b,\omega)d\omega\right]$$
$$\leq_I \frac{\mathfrak{S}_{\lambda}(a,u)+\mathfrak{S}_{\lambda}(b,u)+\mathfrak{S}_{\lambda}(a,\nu)+\mathfrak{S}_{\lambda}(b,\nu)}{4}$$

Since $\mathfrak{S}$ is a F·I-V·F, then it follows that

$$\frac{1}{4\varphi_1(b,a)}\left[\int_a^{a+\varphi_1(b,a)}\widetilde{\mathfrak{S}}(x,u)dx\widetilde{+}\ \int_a^{a+\varphi_1(b,a)}\widetilde{\mathfrak{S}}(x,\nu)dx\right]+\frac{1}{4\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)}\widetilde{\mathfrak{S}}(a,\omega)d\omega\ \widetilde{+}\int_u^{u+\varphi_2(\nu,u)}\widetilde{\mathfrak{S}}(b,\omega)d\omega\right]$$
$$\preccurlyeq \frac{\widetilde{\mathfrak{S}}(a,u)\widetilde{+}\widetilde{\mathfrak{S}}(b,u)\widetilde{+}\widetilde{\mathfrak{S}}(a,\nu)\widetilde{+}\widetilde{\mathfrak{S}}(b,\nu)}{4} \tag{58}$$

By combining inequalities (50), (53), and (58), we get the desired result. □

**Remark 5.** *If one takes $\varphi_1(b,a) = b-a$ and $\varphi_2(\nu,u) = \nu-u$, then from (39), we acquire the following inequality, see [38]:*

$$\widetilde{\mathfrak{S}}\left(\frac{a+b}{2},\ \frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b\widetilde{\mathfrak{S}}\left(x,\frac{u+\nu}{2}\right)dx\widetilde{+}\frac{1}{\nu-u}\int_u^\nu\widetilde{\mathfrak{S}}\left(\frac{a+b}{2},\omega\right)d\omega\right] \preccurlyeq \frac{1}{(b-a)(\nu-u)}\int_a^b\int_u^\nu\widetilde{\mathfrak{S}}(x,\omega)d\omega dx$$
$$\preccurlyeq \frac{1}{4(b-a)}\left[\int_a^b\widetilde{\mathfrak{S}}(x,u)dx\widetilde{+}\int_a^b\widetilde{\mathfrak{S}}(x,\nu)dx\right]\widetilde{+}\frac{1}{4(\nu-u)}\left[\int_u^\nu\widetilde{\mathfrak{S}}(a,\omega)d\omega\widetilde{+}\int_u^\nu\widetilde{\mathfrak{S}}(b,\omega)d\omega\right] \preccurlyeq \frac{\widetilde{\mathfrak{S}}(a,u)\widetilde{+}\widetilde{\mathfrak{S}}(b,u)\widetilde{+}\widetilde{\mathfrak{S}}(a,\nu)\widetilde{+}\widetilde{\mathfrak{S}}(b,\nu)}{4}. \tag{59}$$

*If $\mathfrak{S}_*(x,\omega) = \mathfrak{S}^*(x,\omega)$ with $\lambda = 1$, then from (39), we acquire the following inequality, see [44]:*

$$\begin{aligned}
&\mathfrak{S}\left(\tfrac{2a+\varphi_1(b,a)}{2}, \tfrac{2u+\varphi_2(\nu,u)}{2}\right) \\
&\leq \frac{1}{2}\left[\tfrac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)} \mathfrak{S}\left(x, \tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx + \tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}\left(\tfrac{2a+\varphi_1(b,a)}{2}, \omega\right)d\omega\right] \\
&\leq \tfrac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}(x,\omega)d\omega dx \\
&\leq \frac{1}{4\varphi_1(b,a)}\left[\int_a^{a+\varphi_1(b,a)} \mathfrak{S}(x,u)dx + \int_a^{a+\varphi_1(b,a)} \mathfrak{S}(x,\nu)dx\right] \\
&+\frac{1}{4\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}(a,\omega)d\omega + \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}(b,\omega)d\omega\right] \\
&\leq \tfrac{\mathfrak{S}(a,u)+\mathfrak{S}(b,u)+\mathfrak{S}(a,\nu)+\mathfrak{S}(b,\nu)}{4}
\end{aligned} \tag{60}$$

*If $\mathfrak{S}_*(x,\omega) = \mathfrak{S}^*(x,\omega)$ with $\lambda = 1$ and, $\varphi_1(b,a) = b-a$ and $\varphi_2(\nu,u) = \nu - u$, then from (39), we acquire the following inequality, see [34]:*

$$\begin{aligned}
&\mathfrak{S}\left(\tfrac{a+b}{2}, \tfrac{u+\nu}{2}\right) \leq \frac{1}{2}\left[\tfrac{1}{b-a}\int_a^b \mathfrak{S}\left(x, \tfrac{u+\nu}{2}\right)dx + \tfrac{1}{\nu-u}\int_u^\nu \mathfrak{S}\left(\tfrac{a+b}{2}, \omega\right)d\omega\right] \\
&\leq \tfrac{1}{(b-a)(\nu-u)}\int_a^b\int_u^\nu \mathfrak{S}(x,\omega)d\omega dx \\
&\leq \frac{1}{4(b-a)}\left[\int_a^b \mathfrak{S}(x,u)dx + \int_a^b \mathfrak{S}(x,\nu)dx\right] + \frac{1}{4(\nu-u)}\left[\int_u^\nu \mathfrak{S}(a,\omega)d\omega + \int_u^\nu \mathfrak{S}(b,\omega)d\omega\right] \\
&\leq \tfrac{\mathfrak{S}(a,u)+\mathfrak{S}(b,u)+\mathfrak{S}(a,\nu)+\mathfrak{S}(b,\nu)}{4}.
\end{aligned} \tag{61}$$

**Example 3.** *We consider the F·I-V·Fs $\widetilde{\mathfrak{S}} : [0,\ 1] \times [0,\ 1] \to \mathbb{F}_I$ defined by,*

$$\mathfrak{S}(x)(\sigma) = \begin{cases} \frac{\sigma}{2(6+e^x)(6+e^\omega)}, & \sigma \in [0,\ 2(6+e^x)(6+e^\omega)] \\ \frac{4(6+e^x)(6+e^\omega)-\sigma}{2(6+e^x)(6+e^\omega)}, & \sigma \in (2(6+e^x)(6+e^\omega),\ 4(6+e^x)(6+e^\omega)] \\ 0\ , & \text{otherwise}, \end{cases}$$

*Then, for each $\lambda \in [0,\ 1]$, we have $\mathfrak{S}_\lambda(x) = [2\lambda(6+e^x)(6+e^\omega), (4+2\lambda)(6+e^x)(6+e^\omega)\ ]$. End-point functions $\mathfrak{S}_*((x,\omega),\lambda)$, $\mathfrak{S}^*((x,\omega),\lambda)$ are coordinated preinvex functions with respect to $\varphi_1(b,a) = b-a$ and $\varphi_2(\nu,u) = \nu-u$ for each $\lambda \in [0,\ 1]$. Hence, $\widetilde{\mathfrak{S}}(x,\omega)$. is a coordinated preinvex F·I-V·F.*

$$\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right) = \left[2\lambda\left(5+e^{\frac{1}{2}}\right)^2, 2(2+\lambda)\left(6+e^{\frac{1}{2}}\right)^2\right]$$

$$\begin{aligned}
&\frac{1}{2}\left[\tfrac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda\left(x, \tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx + \frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2}, \omega\right)d\omega\right] \\
&= \left[4\lambda\left(6+e^{\frac{1}{2}}\right)(5+e), 4(2+\lambda)\left(6+e^{\frac{1}{2}}\right)(5+e)\ \right]
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(x,\omega)d\omega dx = \left[2\lambda(5+e)^2, 2(2+\lambda)(5+e)^2\ \right] \\
&\tfrac{1}{4\varphi_1(b,a)}\left[\int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,u)dx + \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,\nu)dx\right] \\
&+\tfrac{1}{4\varphi_2(\nu,u)}\left[\int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(a,\omega)d\omega + \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(b,\omega)d\omega\right] \\
&= [\lambda(5+e)(13+e), (2+\lambda)(5+e)(13+e)]
\end{aligned}$$

$$\frac{\mathfrak{S}_\lambda(a,u)+\mathfrak{S}_\lambda(b,u)+\mathfrak{S}_\lambda(a,\nu)+\mathfrak{S}_\lambda(b,\nu)}{4} = \left[\lambda\frac{(6+e)(20+e)+49}{2}, 2(2+\lambda)\frac{(6+e)(20+e)+49}{2}\right]$$

*That is*

$$\begin{aligned}&\left[2\lambda\left(5+e^{\frac{1}{2}}\right)^2, 2(2+\lambda)\left(6+e^{\frac{1}{2}}\right)^2\right]\\ &\leq_I \left[4\lambda\left(6+e^{\frac{1}{2}}\right)(5+e), 4(2+\lambda)\left(6+e^{\frac{1}{2}}\right)(5+e)\right]\\ &\leq_I \left[2\lambda(5+e)^2, 2(2+\lambda)(5+e)^2\right]\\ &\leq_I [\lambda(5+e)(13+e), (2+\lambda)(5+e)(13+e)]\\ &\leq_I \left[\lambda\frac{(6+e)(20+e)+49}{2}, 2(2+\lambda)\frac{(6+e)(20+e)+49}{2}\right]\end{aligned}$$

*Hence, Theorem 10 has been verified.*

We now obtain some *HH*-inequalities for the product of coordinated preinvex F·I-V·Fs which are known as Pachpatte Type inequalities. These inequalities are refinements of some known inequalities; see [34,37,38,44].

**Theorem 11.** *Let $\widetilde{\mathfrak{S}}, \widetilde{\mathcal{J}} : \Delta = [a,\ a+\varphi_1(b,a)] \times [u,\ u+\varphi_2(\nu,u)] \subset \mathbb{R}^2 \to \mathbb{F}_I$ be two coordinated preinvex F·I-V·Fs on $\Delta$, whose $\lambda$-levels $\mathfrak{S}_\lambda,\ \mathcal{J}_\lambda : [a,\ a+\varphi_1(b,a)] \times [u,\ u+\varphi_2(\nu,u)] \to \mathbb{R}_I^+$ are defined by $\mathfrak{S}_\lambda(x,\omega) = [\mathfrak{S}_*((x,\omega),\lambda),\ \mathfrak{S}^*((x,\omega),\lambda)]$ and $\mathcal{J}_\lambda(x,\omega) = [\mathcal{J}_*((x,\omega),\lambda),\ \mathcal{J}^*((x,\omega),\lambda)]$ for all $(x,\omega) \in \Delta$ and for all $\lambda \in [0,\ 1]$. If Condition 1 for $\varphi_1$ and $\varphi_2$ is fulfilled, then following inequality hold:*

$$\begin{aligned}\frac{1}{\varphi_1(b,a)\varphi_2(\nu,u)} &\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(x,\omega)\widetilde{\times}\widetilde{\mathcal{J}}(x,\omega)d\omega dx\\ &\preccurlyeq \frac{1}{9}\widetilde{\alpha}(a,b,u,\nu)\widetilde{+}\frac{1}{18}\widetilde{\beta}(a,b,u,\nu)\widetilde{+}\frac{1}{36}\widetilde{\gamma}(a,b,u,\nu),\end{aligned} \tag{62}$$

*where*

$$\begin{aligned}&\widetilde{\alpha}(a,b,u,\nu)\\ &= \widetilde{\mathfrak{S}}(a,u)\widetilde{\times}\widetilde{\mathcal{J}}(a,u)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(a,\nu)\widetilde{\times}\widetilde{\mathcal{J}}(a,\nu)\widetilde{+}\widetilde{\mathfrak{S}}(b,u)\widetilde{\times}\widetilde{\mathcal{J}}(b,u)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(b,\nu)\widetilde{\times}\widetilde{\mathcal{J}}(b,\nu),\end{aligned}$$

$$\begin{aligned}&\widetilde{\beta}(a,b,u,\nu)\\ &= \widetilde{\mathfrak{S}}(a,u)\widetilde{\times}\widetilde{\mathcal{J}}(a,\nu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(a,\nu)\widetilde{\times}\widetilde{\mathcal{J}}(a,u)\widetilde{+}\widetilde{\mathfrak{S}}(b,u)\widetilde{\times}\widetilde{\mathcal{J}}(b,\nu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(b,\nu)\widetilde{\times}\widetilde{\mathcal{J}}(b,u),\\ &\widetilde{+}\widetilde{\mathfrak{S}}(a,u)\widetilde{\times}\widetilde{\mathcal{J}}(b,u)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(b,\nu)\widetilde{\times}\widetilde{\mathcal{J}}(a,\nu)\widetilde{+}\widetilde{\mathfrak{S}}(b,u)\widetilde{\times}\widetilde{\mathcal{J}}(a,u)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(a,\nu)\widetilde{\times}\widetilde{\mathcal{J}}(b,\nu)\end{aligned}$$

$$\begin{aligned}&\widetilde{\gamma}(a,b,u,\nu)\\ &= \widetilde{\mathfrak{S}}(a,u)\widetilde{\times}\widetilde{\mathcal{J}}(b,\nu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(b,u)\widetilde{\times}\widetilde{\mathcal{J}}(a,\nu)\widetilde{+}\ \widetilde{\mathfrak{S}}(b,\nu)\widetilde{\times}\widetilde{\mathcal{J}}(a,u)\widetilde{+}\widetilde{\mathfrak{S}}(b,u)\widetilde{\times}\widetilde{\mathcal{J}}(a,\nu)\end{aligned}$$

*and for each $\lambda \in [0,\ 1]$, $\widetilde{\alpha}(a,b,u,\nu)$, $\widetilde{\beta}(a,b,u,\nu)$ and $\widetilde{\gamma}(a,b,u,\nu)$ are defined as follows:*

$$\alpha_\lambda(a,b,u,\nu) = [\alpha_*((a,b,u,\nu),\ \lambda),\ \alpha^*((a,b,u,\nu),\ \lambda)]$$

$$\beta_\lambda(a,b,u,\nu) = [\beta_*((a,b,u,\nu),\ \lambda),\ \beta^*((a,b,u,\nu),\ \lambda)]$$

$$\gamma_\lambda(a,b,u,\nu) = [\gamma_*((a,b,u,\nu),\ \lambda),\ \gamma^*((a,b,u,\nu),\ \lambda)].$$

**Proof.** Let $\widetilde{\mathfrak{S}}$ and $\widetilde{\mathcal{J}}$ both are coordinated preinvex F·I-V·Fs on $[a,\ a+\varphi_1(b,a)] \times [u,\ u+\varphi_2(\nu,u)]$. Then

$$\widetilde{\mathfrak{S}}(a+(1-\sigma)\varphi_1(b,a),\ u+(1-s)\varphi_2(\nu,u)) \preccurlyeq \sigma s\widetilde{\mathfrak{S}}(a,u)\widetilde{+}\sigma(1-s)\widetilde{\mathfrak{S}}(a,\nu)\widetilde{+}(1-\sigma)s\widetilde{\mathfrak{S}}(b,u)\widetilde{+}(1-\sigma)(1-s)\widetilde{\mathfrak{S}}(b,\nu),$$

and

$$\widetilde{\mathcal{J}}(a+(1-\sigma)\varphi_1(b,a),\ u+(1-s)\varphi_2(\nu,u)) \preccurlyeq \sigma s\widetilde{\mathcal{J}}(a,u)\widetilde{+}\sigma(1-s)\widetilde{\mathcal{J}}(a,\nu)\widetilde{+}(1-\sigma)s\widetilde{\mathcal{J}}(b,u)\widetilde{+}(1-\sigma)(1-s)\widetilde{\mathcal{J}}(b,\nu).$$

Since $\widetilde{\mathfrak{S}}$ and $\widetilde{\mathcal{J}}$ both are coordinated preinvex F·I-V·Fs, then by Lemma 1, there exist

$$\widetilde{\mathfrak{S}}_x : [u, \nu] \to \mathbb{F}_I$$
$$\widetilde{\mathfrak{S}}_x(\omega) = \widetilde{\mathfrak{S}}(x, \omega)$$
$$\widetilde{\mathcal{J}}_x : [u, \nu] \to \mathbb{F}_I$$
$$\widetilde{\mathcal{J}}_x(\omega) = \widetilde{\mathcal{J}}(x, \omega)$$

and

$$\widetilde{\mathfrak{S}}_\omega : [a, b] \to \mathbb{F}_I$$
$$\widetilde{\mathfrak{S}}_\omega(x) = \widetilde{\mathfrak{S}}(x, \omega)$$
$$\widetilde{\mathcal{J}}_\omega : [a, b] \to \mathbb{F}_I$$
$$\widetilde{\mathcal{J}}_\omega(x) = \widetilde{\mathcal{J}}(x, \omega)$$

Since $\widetilde{\mathfrak{S}}_x$, $\widetilde{\mathcal{J}}_x$, $\widetilde{\mathfrak{S}}_\omega$ and $\widetilde{\mathcal{J}}_\omega$ are F·I-V·Fs, then by inequality (29), we have

$$\begin{aligned}&\frac{1}{\varphi_1(b,a)} \int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}_\omega(x) \times \widetilde{\mathcal{J}}_\omega(x)dx \\ &\preccurlyeq \frac{1}{3}\Big[\widetilde{\mathfrak{S}}_\omega(a) \times \widetilde{\mathcal{J}}_\omega(a) + \widetilde{\mathfrak{S}}_\omega(b) \times \widetilde{\mathcal{J}}_\omega(b)\Big] + \frac{1}{6}\Big[\widetilde{\mathfrak{S}}_\omega(a) \times \widetilde{\mathcal{J}}_\omega(b) + \widetilde{\mathfrak{S}}_\omega(b) \times \widetilde{\mathcal{J}}_\omega(a)\Big],\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}_x(\omega) \times \widetilde{\mathcal{J}}_x(\omega)d\omega \\ &\preccurlyeq \frac{1}{3}\Big[\widetilde{\mathfrak{S}}_x(u) \times \widetilde{\mathcal{J}}_x(u) + \widetilde{\mathfrak{S}}_x(\nu) \times \widetilde{\mathcal{J}}_x(\nu)\Big] + \frac{1}{6}\Big[\widetilde{\mathfrak{S}}_x(u) \times \widetilde{\mathcal{J}}_x(\nu) + \widetilde{\mathfrak{S}}_x(u) \times \widetilde{\mathcal{J}}_x(\nu)\Big].\end{aligned}$$

For each $\lambda \in [0,\ 1]$, we have

$$\begin{aligned}&\frac{1}{\varphi_1(b,a)} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_{\lambda\omega}(x) \times \mathcal{J}_{\lambda\omega}(x)dx \\ &\leq_I \frac{1}{3}[\mathfrak{S}_{\lambda\omega}(a) \times \mathcal{J}_{\lambda\omega}(a) + \mathfrak{S}_{\lambda\omega}(b) \times \mathcal{J}_{\lambda\omega}(b)] + \frac{1}{6}[\mathfrak{S}_{\lambda\omega}(a) \times \mathcal{J}_{\lambda\omega}(b) + \mathfrak{S}_{\lambda\omega}(b) \times \mathcal{J}_{\lambda\omega}(a)],\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_{\lambda x}(\omega) \times \mathcal{J}_{\lambda x}(\omega)d\omega \\ &\leq_I \frac{1}{3}[\mathfrak{S}_{\lambda x}(u) \times \mathcal{J}_{\lambda x}(u) + \mathfrak{S}_{\lambda x}(\nu) \times \mathcal{J}_{\lambda x}(\nu)] + \frac{1}{6}[\mathfrak{S}_{\lambda x}(u) \times \mathcal{J}_{\lambda x}(\nu) + \mathfrak{S}_{\lambda x}(u) \times \mathcal{J}_{\lambda x}(\nu)]\end{aligned}$$

The above inequalities can be written as

$$\begin{aligned}&\frac{1}{\varphi_1(b,a)} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,\omega) \times \mathcal{J}_\lambda(x,\omega)dx \\ &\leq_I \frac{1}{3}[\mathfrak{S}_\lambda(a,\omega) \times \mathcal{J}_\lambda(a,\omega) + \mathfrak{S}_\lambda(b,\omega) \times \mathcal{J}_\lambda(b,\omega)] + \frac{1}{6}[\mathfrak{S}_\lambda(a,\omega) \times \mathcal{J}_\lambda(b,\omega) + \mathfrak{S}_\lambda(b,\omega) \times \mathcal{J}_\lambda(a,\omega)],\end{aligned} \tag{63}$$

and

$$\begin{aligned}&\frac{1}{\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(x,\omega) \times \mathcal{J}_\lambda(x,\omega)d\omega \\ &\leq_I \frac{1}{3}[\mathfrak{S}_\lambda(x,u) \times \mathcal{J}_\lambda(x,u) + \mathfrak{S}_\lambda(x,\nu) \times \mathcal{J}_\lambda(x,\nu)] + \frac{1}{6}[\mathfrak{S}_\lambda(x,u) \times \mathcal{J}_\lambda(x,u) + \mathfrak{S}_\lambda(x,\nu) \times \mathcal{J}_\lambda(x,\nu)].\end{aligned} \tag{64}$$

Firstly, we solve inequality (63), taking integration on the both sides of inequality with respect to $\omega$ over interval $[u,\ u + \varphi_2(\nu,u)]$ and dividing both sides by $\varphi_2(\nu,u)$, we have

$$\begin{aligned}&\frac{1}{\varphi_1(b,a)\varphi_2(\nu,u)} \int_a^{a+\varphi_1(b,a)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(x,\omega) \times \mathcal{J}_\lambda(x,\omega)d\omega dx \\ &\leq_I \frac{1}{3\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_\lambda(a,\omega) \times \mathcal{J}_\lambda(a,\omega) + \mathfrak{S}_\lambda(b,\omega) \times \mathcal{J}_\lambda(b,\omega)]d\omega \\ &+ \frac{1}{6\varphi_2(\nu,u)} \int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_\lambda(a,\omega) \times \mathcal{J}_\lambda(b,\omega) + \mathfrak{S}_\lambda(b,\omega) \times \mathcal{J}_\lambda(a,\omega)]d\omega.\end{aligned} \tag{65}$$

Now again by inequality (29), for each $\lambda \in [0,\ 1]$, we have

$$\begin{aligned}&\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(a,\omega)\times\mathcal{J}_\lambda(a,\omega)d\omega\\&\leq_I \frac{1}{3}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(a,u)+\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(a,\nu)]d\omega+\frac{1}{6}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(a,\nu)+\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(a,\nu)]d\omega.\end{aligned}\tag{66}$$

$$\begin{aligned}&\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(b,\omega)\times\mathcal{J}_\lambda(b,\omega)d\omega\\&\leq_I \frac{1}{3}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(b,u)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(b,\nu)]d\omega+\frac{1}{6}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(b,\nu)+\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(a,\nu)]d\omega\end{aligned}\tag{67}$$

$$\begin{aligned}&\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(a,\omega)\times\mathcal{J}_\lambda(b,\omega)d\omega\\&\leq_I \frac{1}{3}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(b,u)+\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(b,\nu)]d\omega+\frac{1}{6}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(b,\nu)+\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(b,u)]d\omega.\end{aligned}\tag{68}$$

$$\begin{aligned}&\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(b,\omega)\times\mathcal{J}_\lambda(a,\omega)d\omega\\&\leq_I \frac{1}{3}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(a,u)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(a,\nu)]d\omega+\frac{1}{6}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(a,\nu)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(a,u)]d\omega.\end{aligned}\tag{69}$$

From (66)–(69), inequality (65) we have

$$\frac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(x,\omega)\times\mathcal{J}_\lambda(x,\omega)d\omega dx\leq_I\frac{1}{9}\alpha_\lambda(a,b,u,\nu)+\frac{1}{18}\beta_\lambda(a,b,u,\nu)+\frac{1}{36}\gamma_\lambda(a,b,u,\nu).$$

That is

$$\frac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)}\widetilde{\mathfrak{S}}(x,\omega)\widetilde{\times}\widetilde{\mathcal{J}}(x,\omega)d\omega dx\preccurlyeq\frac{1}{9}\widetilde{\alpha}(a,b,u,\nu)\widetilde{+}\frac{1}{18}\widetilde{\beta}(a,b,u,\nu)\widetilde{+}\frac{1}{36}\widetilde{\gamma}(a,b,u,\nu).$$

Hence, this concludes the proof of theorem. □

**Theorem 12.** *Let $\widetilde{\mathfrak{S}},\widetilde{\mathcal{J}}:\Delta=[a,\ a+\varphi_1(b,a)]\times[u,\ u+\varphi_2(\nu,u)]\subset\mathbb{R}^2\to\mathbb{F}_I$ be two coordinated preinvex F·I-V·Fs. Then, from λ-levels, we get the collection of I-V·Fs $\mathfrak{S}_\lambda,\ \mathcal{J}_\lambda:\Delta\subset\mathbb{R}^2\to\mathbb{R}_I^+$ are given by $\mathfrak{S}_\lambda(x)=[\mathfrak{S}_*((x,\omega),\lambda),\ \mathfrak{S}^*((x,\omega),\lambda)]$ and $\mathcal{J}_\lambda(x)=[\mathcal{J}_*((x,\omega),\lambda),\ \mathcal{J}^*((x,\omega),\lambda)]$ for all $(x,\omega)\in\Delta$ and for all $\lambda\in[0,\ 1]$. If Condition 1 for $\varphi_1$ and $\varphi_2$ is fulfilled, then following inequality hold:*

$$\begin{aligned}&4\,\widetilde{\mathfrak{S}}\left(\frac{2a+\varphi_1(b,a)}{2},\frac{2u+\varphi_2(\nu,u)}{2}\right)\widetilde{\times}\widetilde{\mathcal{J}}\left(\frac{2a+\varphi_1(b,a)}{2},\frac{2u+\varphi_2(\nu,u)}{2}\right)\\&\preccurlyeq\frac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)}\widetilde{\mathfrak{S}}(x,\omega)\widetilde{\times}\widetilde{\mathcal{J}}(x,\omega)d\omega dx\widetilde{+}\frac{5}{36}\widetilde{\alpha}(a,b,u,\nu)\widetilde{+}\frac{7}{36}\widetilde{\beta}(a,b,u,\nu)\widetilde{+}\frac{2}{9}\widetilde{\gamma}(a,b,u,\nu),\end{aligned}\tag{70}$$

*where $\widetilde{\alpha}(a,b,u,\nu)$, $\widetilde{\beta}(a,b,u,\nu)$, and $\widetilde{\gamma}(a,b,u,\nu)$ are given in Theorem 11.*

**Proof.** $\widetilde{\mathfrak{S}},\widetilde{\mathcal{J}}:\Delta\to\mathbb{F}_I$ are two coordinated preinvex F·I-V·Fs, and then from inequality (30) and for each $\lambda\in[0,\ 1]$, we have

$$\begin{aligned}&2\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\frac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\frac{2u+\varphi_2(\nu,u)}{2}\right)\\&\leq_I\frac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_\lambda\left(x,\frac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(x,\frac{2u+\varphi_2(\nu,u)}{2}\right)dx\\&+\frac{1}{6}\left[\mathfrak{S}_\lambda\left(a,\frac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(a,\frac{2u+\varphi_2(\nu,u)}{2}\right)+\mathfrak{S}_\lambda\left(b,\frac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(b,\frac{2u+\varphi_2(\nu,u)}{2}\right)\right]\\&+\frac{1}{3}\left[\mathfrak{S}_\lambda\left(a,\frac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(b,\frac{2u+\varphi_2(\nu,u)}{2}\right)\right]+\mathfrak{S}_\lambda\left(b,\frac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(a,\frac{2u+\varphi_2(\nu,u)}{2}\right)\end{aligned}\tag{71}$$

and

$$\begin{aligned}&2\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\frac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\frac{2u+\varphi_2(\nu,u)}{2}\right)\\&\leq_I\frac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\omega\right)\times\mathcal{J}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\omega\right)d\omega\\&+\frac{1}{6}\left[\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},u\right)\times\mathcal{J}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},u\right)+\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\nu\right)\times\mathcal{J}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\nu\right)\right]\\&+\frac{1}{3}\left[\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},u\right)\times\mathcal{J}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\nu\right)+\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},\nu\right)\times\mathcal{J}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2},u\right)\right]\end{aligned}\tag{72}$$

Summing the inequalities (71) and (72), then taking the multiplication of the resultant one by 2, we obtain

$$\begin{aligned}
&8\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_I \tfrac{2}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_\lambda\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx\\
&+\tfrac{2}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\omega\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\omega\right)dx\\
&+\tfrac{1}{6}\left[2\mathfrak{S}_\lambda\left(a,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(a,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)+2\mathfrak{S}_\lambda\left(b,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(b,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\right]\\
&+\tfrac{1}{6}\left[2\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)+2\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\right]\\
&+\tfrac{1}{3}\left[2\mathfrak{S}_\lambda\left(a,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(b,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)+2\mathfrak{S}_\lambda\left(b,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(a,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\right]\\
&+\tfrac{1}{3}\left[2\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)+2\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\right]
\end{aligned}. \quad (73)$$

Now, with the help of integral inequality (30) for each integral on the right-hand side of (73), we have

$$\begin{aligned}
&2\mathfrak{S}_\lambda\left(a,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(a,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_I \tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(a,\omega)\times\mathcal{J}_\lambda(a,\omega)d\omega\\
&+\tfrac{1}{6}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(a,u)+\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(a,\nu)]\\
&+\tfrac{1}{3}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(a,\nu)+\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(a,u)]
\end{aligned} \quad (74)$$

$$\begin{aligned}
&2\mathfrak{S}_\lambda\left(b,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(b,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_I \tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(b,\omega)\times\mathcal{J}_\lambda(b,\omega)d\omega\\
&+\tfrac{1}{6}[\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(b,u)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(b,\nu)]\\
&+\tfrac{1}{3}[\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(b,\nu)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(b,u)]
\end{aligned} \quad (75)$$

$$\begin{aligned}
&2\mathfrak{S}_\lambda\left(a,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(b,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_I \tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(a,\omega)\times\mathcal{J}_\lambda(b,\omega)d\omega\\
&+\tfrac{1}{6}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(b,u)+\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(b,\nu)]\\
&+\tfrac{1}{3}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(b,\nu)+\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(b,u)]
\end{aligned} \quad (76)$$

$$\begin{aligned}
&2\mathfrak{S}_\lambda\left(b,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(a,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_I \tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(b,\omega)\times\mathcal{J}_\lambda(a,\omega)d\omega\\
&+\tfrac{1}{6}[\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(a,u)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(a,\nu)]\\
&+\tfrac{1}{3}[\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(a,\nu)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(a,u)]
\end{aligned} \quad (77)$$

$$\begin{aligned}
&2\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\\
&\leq_I \tfrac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_\lambda(x,u)\times\mathcal{J}_\lambda(x,u)dx\\
&+\tfrac{1}{6}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(a,u)+\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(b,u)]\\
&+\tfrac{1}{3}\left[\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)+\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\right]
\end{aligned} \quad (78)$$

$$\begin{aligned}
&2\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\\
&\leq_I \tfrac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_\lambda(x,\nu)\times\mathcal{J}_\lambda(x,\nu)dx\\
&+\tfrac{1}{6}[\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(a,\nu)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(b,\nu)]\\
&+\tfrac{1}{3}\left[\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)+\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\right]
\end{aligned} \quad (79)$$

$$\begin{aligned}
&2\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\\
&\leq_I \tfrac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_\lambda(x,u)\times\mathcal{J}_\lambda(x,\nu)dx\\
&+\tfrac{1}{6}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(a,\nu)+\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(b,\nu)]\\
&+\tfrac{1}{3}\left[\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)+\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\right]
\end{aligned} \quad (80)$$

$$
\begin{aligned}
&2\mathfrak{S}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\times\mathcal{J}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\\
&\le_I \tfrac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,\nu)\times\mathcal{J}_{\lambda}(x,u)dx\\
&+\tfrac{1}{6}[\mathfrak{S}_{\lambda}(a,\nu)\times\mathcal{J}_{\lambda}(a,u)+\mathfrak{S}_{\lambda}(b,\nu)\times\mathcal{J}_{\lambda}(b,u)]\\
&+\tfrac{1}{3}\left[\mathfrak{S}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\times\mathcal{J}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)+\mathfrak{S}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},\nu\right)\times\mathcal{J}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},u\right)\right]
\end{aligned}
\tag{81}
$$

From (74)–(81), we have

$$
\begin{aligned}
&8\mathfrak{S}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\le_I \tfrac{2}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_{\lambda}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx\\
&+\tfrac{2}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},\omega\right)\times\mathcal{J}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},\omega\right)dx+\tfrac{1}{6\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(a,\omega)\times\mathcal{J}_{\lambda}(a,\omega)d\omega\\
&+\tfrac{1}{6\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(b,\omega)\times\mathcal{J}_{\lambda}(b,\omega)d\omega+\tfrac{1}{6\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,u)\times\mathcal{J}_{\lambda}(x,u)dx\\
&+\tfrac{1}{6\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,\nu)\times\mathcal{J}_{\lambda}(x,\nu)dx+\tfrac{1}{3\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(a,\omega)\times\mathcal{J}_{\lambda}(b,\omega)d\omega\\
&+\tfrac{1}{3\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(b,\omega)\times\mathcal{J}_{\lambda}(a,\omega)d\omega+\tfrac{1}{3\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,u)\times\mathcal{J}_{\lambda}(x,\nu)dx\\
&+\tfrac{1}{3\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,\nu)\times\mathcal{J}_{\lambda}(x,u)dx\ ,+\tfrac{1}{18}\alpha_{\lambda}(a,b,u,\nu)+\tfrac{1}{9}\beta_{\lambda}(a,b,u,\nu)+\tfrac{2}{9}\gamma_{\lambda}(a,b,u,\nu)
\end{aligned}
\tag{82}
$$

Now, again with the help of integral inequality (30) for first two integrals on the right-hand side of (82), we have the following relation:

$$
\begin{aligned}
&\tfrac{2}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_{\lambda}\left(x,\tfrac{2u+\varphi_2(\nu,u)}{2}\right)dx\\
&\le_I \tfrac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(x,\omega)\times\mathcal{J}_{\lambda}(x,\omega)d\omega dx\\
&+\tfrac{1}{3\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}[\mathfrak{S}_{\lambda}(x,u)\times\mathcal{J}_{\lambda}(x,u)+\mathfrak{S}_{\lambda}(x,\nu)\times\mathcal{J}_{\lambda}(x,\nu)]dx\\
&+\tfrac{1}{6\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}[\mathfrak{S}_{\lambda}(x,u)\times\mathcal{J}_{\lambda}(x,\nu)+\mathfrak{S}_{\lambda}(x,\nu)\times\mathcal{J}_{\lambda}(x,u)]dx,
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
&\tfrac{2}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},\omega\right)\times\mathcal{J}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},\omega\right)dx\\
&\le_I \tfrac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(x,\omega)\times\mathcal{J}_{\lambda}(x,\omega)d\omega dx\\
&+\tfrac{1}{3\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_{\lambda}(a,\omega)\times\mathcal{J}_{\lambda}(a,\omega)+\mathfrak{S}_{\lambda}(b,\omega)\times\mathcal{J}_{\lambda}(b,\omega)]d\omega\\
&+\tfrac{1}{6\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_{\lambda}(a,\omega)\times\mathcal{J}_{\lambda}(b,\omega)+\mathfrak{S}_{\lambda}(b,\omega)\times\mathcal{J}_{\lambda}(a,\omega)]d\omega
\end{aligned}
\tag{84}
$$

From (83) and (84), we have

$$
\begin{aligned}
&8\mathfrak{S}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_{\lambda}\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\le_I \tfrac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(x,\omega)\times\mathcal{J}_{\lambda}(x,\omega)d\omega dx\\
&+\tfrac{1}{3\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}[\mathfrak{S}_{\lambda}(x,u)\times\mathcal{J}_{\lambda}(x,u)+\mathfrak{S}_{\lambda}(x,\nu)\times\mathcal{J}_{\lambda}(x,\nu)]dx\\
&+\tfrac{1}{6\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}[\mathfrak{S}_{\lambda}(x,u)\times\mathcal{J}_{\lambda}(x,\nu)+\mathfrak{S}_{\lambda}(x,\nu)\times\mathcal{J}_{\lambda}(x,u)]dx\\
&+\tfrac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(x,\omega)\times\mathcal{J}_{\lambda}(x,\omega)d\omega dx\\
&+\tfrac{1}{3\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_{\lambda}(a,\omega)\times\mathcal{J}_{\lambda}(a,\omega)+\mathfrak{S}_{\lambda}(b,\omega)\times\mathcal{J}_{\lambda}(b,\omega)]d\omega\\
&+\tfrac{1}{6\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_{\lambda}(a,\omega)\times\mathcal{J}_{\lambda}(b,\omega)+\mathfrak{S}_{\lambda}(b,\omega)\times\mathcal{J}_{\lambda}(a,\omega)]d\omega,+\tfrac{1}{6\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(a,\omega)\times\mathcal{J}_{\lambda}(a,\omega)d\omega\\
&+\tfrac{1}{6\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(b,\omega)\times\mathcal{J}_{\lambda}(b,\omega)d\omega\\
&+\tfrac{1}{6\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,u)\times\mathcal{J}_{\lambda}(x,u)dx+\tfrac{1}{6\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,\nu)\times\mathcal{J}_{\lambda}(x,\nu)dx\\
&+\tfrac{1}{3\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(a,\omega)\times\mathcal{J}_{\lambda}(b,\omega)d\omega+\tfrac{1}{3\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_{\lambda}(b,\omega)\times\mathcal{J}_{\lambda}(a,\omega)d\omega\\
&+\tfrac{1}{3\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,u)\times\mathcal{J}_{\lambda}(x,\nu)dx+\tfrac{1}{3\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_{\lambda}(x,\nu)\times\mathcal{J}_{\lambda}(x,u)dx\ ,\\
&+\tfrac{1}{18}\alpha_{\lambda}(a,b,u,\nu)+\tfrac{1}{9}\beta_{\lambda}(a,b,u,\nu)+\tfrac{2}{9}\gamma_{\lambda}(a,b,u,\nu).
\end{aligned}
$$

It follows that

$$\begin{aligned}
&8\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_I \tfrac{2}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(x,\omega)\times\mathcal{J}_\lambda(x,\omega)d\omega dx\\
&+\tfrac{2}{3\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}[\mathfrak{S}_\lambda(x,u)\times\mathcal{J}_\lambda(x,u)+\mathfrak{S}_\lambda(x,\nu)\times\mathcal{J}_\lambda(x,\nu)]dx\\
&+\tfrac{1}{3\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}[\mathfrak{S}_\lambda(x,u)\times\mathcal{J}_\lambda(x,\nu)+\mathfrak{S}_\lambda(x,\nu)\times\mathcal{J}_\lambda(x,u)]dx\\
&+\tfrac{2}{3\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_\lambda(a,\omega)\times\mathcal{J}_\lambda(a,\omega)+\mathfrak{S}_\lambda(b,\omega)\times\mathcal{J}_\lambda(b,\omega)]d\omega\\
&+\tfrac{1}{3\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}[\mathfrak{S}_\lambda(a,\omega)\times\mathcal{J}_\lambda(b,\omega)+\mathfrak{S}_\lambda(b,\omega)\times\mathcal{J}_\lambda(a,\omega)]d\omega\\
&+\tfrac{1}{18}\alpha_\lambda(a,b,u,\nu)+\tfrac{1}{9}\beta_\lambda(a,b,u,\nu)+\tfrac{2}{9}\gamma_\lambda(a,b,u,\nu).
\end{aligned} \tag{85}$$

Now, using integral inequality (25) for integrals on the right-hand side of (85), we have the following relation:

$$\begin{aligned}
&\tfrac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_\lambda(x,u)\times\mathcal{J}_\lambda(x,u)dx\\
&\leq_I \tfrac{1}{3}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(a,u)+\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(b,u)]+\tfrac{1}{6}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(b,u)+\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(a,u)],
\end{aligned} \tag{86}$$

$$\begin{aligned}
&\tfrac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_\lambda(x,\nu)\times\mathcal{J}_\lambda(x,\nu)dx\\
&\leq_I \tfrac{1}{3}[\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(a,\nu)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(b,\nu)]+\tfrac{1}{6}[\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(b,\nu)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(a,\nu)],
\end{aligned} \tag{87}$$

$$\begin{aligned}
&\tfrac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_\lambda(x,u)\times\mathcal{J}_\lambda(x,\nu)dx\\
&\leq_I \tfrac{1}{3}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(a,\nu)+\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(b,\nu)]+\tfrac{1}{6}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(b,\nu)+\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(a,\nu)],
\end{aligned} \tag{88}$$

$$\begin{aligned}
&\tfrac{1}{\varphi_1(b,a)}\int_a^{a+\varphi_1(b,a)}\mathfrak{S}_\lambda(x,\nu)\times\mathcal{J}_\lambda(x,u)dx\\
&\leq_I \tfrac{1}{3}[\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(a,u)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(b,u)]+\tfrac{1}{6}[\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(b,u)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(a,u)],
\end{aligned} \tag{89}$$

$$\begin{aligned}
&\tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(a,\omega)\times\mathcal{J}_\lambda(a,\omega)d\omega\\
&\leq_I \tfrac{1}{3}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(a,u)+\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(a,\nu)]+\tfrac{1}{6}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(a,\nu)+\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(a,u)],
\end{aligned} \tag{90}$$

$$\begin{aligned}
&\tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(b,\omega)\times\mathcal{J}_\lambda(b,\omega)d\omega\\
&\leq_I \tfrac{1}{3}[\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(b,u)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(b,\nu)]+\tfrac{1}{6}[\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(b,\nu)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(b,u)],
\end{aligned} \tag{91}$$

$$\begin{aligned}
&\tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(a,\omega)\times\mathcal{J}_\lambda(b,\omega)d\omega\\
&\leq_I \tfrac{1}{3}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(b,u)+\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(b,\nu)]+\tfrac{1}{6}[\mathfrak{S}_\lambda(a,u)\times\mathcal{J}_\lambda(b,\nu)+\mathfrak{S}_\lambda(a,\nu)\times\mathcal{J}_\lambda(b,u)],
\end{aligned} \tag{92}$$

$$\begin{aligned}
&\tfrac{1}{\varphi_2(\nu,u)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(b,\omega)\times\mathcal{J}_\lambda(a,\omega)d\omega\\
&\leq_I \tfrac{1}{3}[\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(a,u)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(a,\nu)]+\tfrac{1}{6}[\mathfrak{S}_\lambda(b,u)\times\mathcal{J}_\lambda(a,\nu)+\mathfrak{S}_\lambda(b,\nu)\times\mathcal{J}_\lambda(a,u)].
\end{aligned} \tag{93}$$

From (86)–(93), inequality (95), we have

$$\begin{aligned}
&4\,\mathfrak{S}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\times\mathcal{J}_\lambda\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\leq_I \tfrac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)}\mathfrak{S}_\lambda(x,\omega)\times\mathcal{J}_\lambda(x,\omega)d\omega dx\\
&+\tfrac{5}{36}\alpha_\lambda(a,b,u,\nu)+\tfrac{7}{36}\beta_\lambda(a,b,u,\nu)+\tfrac{2}{9}\gamma_\lambda(a,b,u,\nu)
\end{aligned}$$

That is

$$\begin{aligned}
&4\,\widetilde{\mathfrak{S}}\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\widetilde{\times}\widetilde{\mathcal{J}}\left(\tfrac{2a+\varphi_1(b,a)}{2},\tfrac{2u+\varphi_2(\nu,u)}{2}\right)\\
&\preccurlyeq \tfrac{1}{\varphi_1(b,a)\varphi_2(\nu,u)}\int_a^{a+\varphi_1(b,a)}\int_u^{u+\varphi_2(\nu,u)}\widetilde{\mathfrak{S}}(x,\omega)\widetilde{\times}\widetilde{\mathcal{J}}(x,\omega)d\omega dx+\tfrac{5}{36}\widetilde{\alpha}(a,b,u,\nu)\widetilde{+}\tfrac{7}{36}\widetilde{\beta}(a,b,u,\nu)\widetilde{+}\tfrac{2}{9}\widetilde{\gamma}(a,b,u,\nu).
\end{aligned}$$

We now give *HH*-Fejér inequality for coordinated preinvex F·I-V·Fs by means of FOR in the following result. □

**Theorem 13.** *Let $\widetilde{\mathfrak{S}} : \Delta = [a,\ a+\varphi_1(b,a)] \times [u,\ u+\varphi_2(\nu,u)] \to \mathbb{F}_I$ be a coordinated preinvex F·I-V·F with $a < b$ and $u < \nu$. Then, from $\lambda$-levels, we get the collection of I-V·Fs $\mathfrak{S}_\lambda : \Delta \to \mathbb{R}_I^+$ are given by $\mathfrak{S}_\lambda(x,\omega) = [\mathfrak{S}_*((x,\omega),\lambda),\ \mathfrak{S}^*((x,\omega),\lambda)]$ for all $(x,\omega) \in \Delta$ and for all $\lambda \in [0,\ 1]$. Let $\psi : [a,\ a+\varphi_1(b,a)] \to \mathbb{R}$ with $(x) \geq 0$, $\int_a^{a+\varphi_1(b,a)} \psi(x)dx > 0$, and $\mathcal{K} : [u,\ u+\varphi_2(\nu,u)] \to \mathbb{R}$ with $\mathcal{K}(\omega) \geq 0$, $\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega > 0$, be two symmetric functions with respect to $\frac{2a+\varphi_1(b,a)}{2}$ and $\frac{2u+\varphi_2(\nu,u)}{2}$, respectively. If Condition 1 for $\varphi_1$ and $\varphi_2$ holds, then following inequality hold:*

$$\begin{aligned}
&\widetilde{\mathfrak{S}}\left(\frac{2a+\varphi_1(b,a)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right) \\
&\preccurlyeq \frac{1}{2}\left[\begin{array}{l} \frac{1}{\int_a^b \psi(x)dx} \int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)\psi(x)dx \\ \widetilde{+} \frac{1}{\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}\left(\frac{2a+\varphi_1(b,a)}{2}, \omega\right)\mathcal{K}(\omega)d\omega \end{array}\right] \\
&\preccurlyeq \frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx \int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_a^{a+\varphi_1(b,a)} \int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(x,\omega)\psi(x)\mathcal{K}(\omega)d\omega dx \\
&\preccurlyeq \frac{1}{4\int_a^{a+\varphi_1(b,a)} \psi(x)dx}\left[\int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}(x,u)dx \widetilde{+} \int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}(x,\nu)dx\right] \\
&\widetilde{+} \frac{1}{4\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega}\left[\int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(a,\omega)d\omega \widetilde{+} \int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(b,\omega)d\omega\right] \\
&\preccurlyeq \frac{\widetilde{\mathfrak{S}}(a,u)\widetilde{+}\widetilde{\mathfrak{S}}(b,u)\widetilde{+}\widetilde{\mathfrak{S}}(a,\nu)\widetilde{+}\widetilde{\mathfrak{S}}(b,\nu)}{4}
\end{aligned} \tag{94}$$

**Proof.** Since $\widetilde{\mathfrak{S}}$ both is a coordinated preinvex F·I-V·F on $\Delta$, it follows that for functions, then by Lemma 1, there exist

$$\widetilde{\mathfrak{S}}_x : [u,\nu] \to \mathbb{F}_I,\ \widetilde{\mathfrak{S}}_x(\omega) = \widetilde{\mathfrak{S}}(x,\omega),\ \widetilde{\mathfrak{S}}_\omega : [a,b] \to \mathbb{F}_I,\ \widetilde{\mathfrak{S}}_\omega(x) = \widetilde{\mathfrak{S}}(x,\omega)$$

Thus, from inequality (31), for each $\lambda \in [0,\ 1]$, we have

$$\mathfrak{S}_{\lambda x}\left(\frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_I \frac{1}{\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_{\lambda x}(\omega)\mathcal{K}(\omega)d\omega \leq_I \frac{\mathfrak{S}_{\lambda x}(u) + \mathfrak{S}_{\lambda x}(\nu)}{2},$$

and

$$\mathfrak{S}_{\lambda\omega}\left(\frac{2a+\varphi_1(b,a)}{2}\right) \leq_I \frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_{\lambda\omega}(x)\psi(x)dx \leq_I \frac{\mathfrak{S}_{\lambda\omega}(a) + \mathfrak{S}_{\lambda\omega}(b)}{2}$$

The above inequalities can be written as

$$\mathfrak{S}_\lambda\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_I \frac{1}{\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(x,\omega)\mathcal{K}(\omega)d\omega \leq_I \frac{\mathfrak{S}_\lambda(x,u) + \mathfrak{S}_\lambda(x,\ \nu)}{2}, \tag{95}$$

and

$$\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2}, \omega\right) \leq_I \frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,\omega)\psi(x)dx \leq_I \frac{\mathfrak{S}_\lambda(a,\omega) + \mathfrak{S}_\lambda(b,\omega)}{2} \tag{96}$$

Multiplying (95) by $\psi(x)$ and then integrating the resultant with respect to $x$ over $[a,\ a+\varphi_1(b,a)]$, we have

$$\int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)\psi(x)dx$$
$$\leq_I \frac{1}{\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_a^{a+\varphi_1(b,a)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(x,\omega)\psi(x)\mathcal{K}(\omega)d\omega dx \leq_I \int_a^{a+\varphi_1(b,a)} \frac{\mathfrak{S}_\lambda(x,u)+\mathfrak{S}_\lambda(x,\nu)}{2}\psi(x)dx. \tag{97}$$

Now, multiplying (96) by $\mathcal{K}(\omega)$ and then integrating the resultant with respect to $\omega$ over $[u,\ u+\varphi_2(\nu,u)]$, we have

$$\int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2}, \omega\right)\mathcal{K}(\omega)d\omega$$
$$\leq_I \frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx} \int_a^{a+\varphi_1(b,a)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(x,\omega)\psi(x)\mathcal{K}(\omega)dxd\omega \leq_I \int_a^{a+\varphi_1(b,a)} \frac{\mathfrak{S}_\lambda(a,\omega)+\mathfrak{S}_\lambda(b,\omega)}{2}\mathcal{K}(\omega)d\omega \tag{98}$$

Since $\int_a^{a+\varphi_1(b,a)} \psi(x)dx > 0$ and $\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega > 0$, then dividing (97) and (98) by $\int_a^{a+\varphi_1(b,a)} \psi(x)dx > 0$ and $\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega > 0$, respectively, we get

$$\frac{1}{2}\left[\frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)\psi(x)dx + \frac{1}{\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2}, \omega\right)\mathcal{K}(\omega)d\omega\right]$$
$$\leq_I \frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx \int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_a^{a+\varphi_1(b,a)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(x,\omega)\psi(x)\mathcal{K}(\omega)d\omega dx$$
$$\leq_I \frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx} \int_a^{a+\varphi_1(b,a)} \frac{\mathfrak{S}_\lambda(x,u)+\mathfrak{S}_\lambda(x,\nu)}{4}\psi(x)dx \tag{99}$$
$$+\frac{1}{\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_u^{u+\varphi_2(\nu,u)} \frac{\mathfrak{S}_\lambda(a,\omega)+\mathfrak{S}_\lambda(b,\omega)}{4}\mathcal{K}(\omega)d\omega$$

Now, from the left part of double inequalities (95) and (96), we obtain

$$\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_I \frac{1}{\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2}, \omega\right)\mathcal{K}(\omega)d\omega, \tag{100}$$

and

$$\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_I \frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)\psi(x)dx \tag{101}$$

Summing the inequalities (100) and (101), we get

$$\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_I \frac{1}{2}\left[\begin{array}{c} \frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)\psi(x)dx \\ +\frac{1}{\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2}, \omega\right)\mathcal{K}(\omega)d\omega \end{array}\right] \tag{102}$$

Similarly, from the right part of (101) and (102), we can obtain

$$\frac{1}{\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(a,\omega)\mathcal{K}(\omega)d\omega \leq_I \frac{\mathfrak{S}_\lambda(a,u)+\mathfrak{S}_\lambda(a,\ \nu)}{2}, \tag{103}$$

$$\frac{1}{\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(b,\omega)\mathcal{K}(\omega)d\omega \leq_I \frac{\mathfrak{S}_\lambda(b,u)+\mathfrak{S}_\lambda(b,\ \nu)}{2}, \tag{104}$$

and

$$\frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,u)\psi(x)dx \leq_I \frac{\mathfrak{S}_\lambda(a,u) + \mathfrak{S}_\lambda(b,u)}{2} \tag{105}$$

$$\frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,\nu)\psi(x)dx \leq_I \frac{\mathfrak{S}_\lambda(a,\nu) + \mathfrak{S}_\lambda(b,\nu)}{2} \tag{106}$$

Adding (103)–(106) and dividing by 4, we get

$$\begin{aligned} &\frac{1}{4\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega}\left[\int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(a,\omega)\mathcal{K}(\omega)d\omega + \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(b,\omega)\mathcal{K}(\omega)d\omega\right] \\ &+\frac{1}{4\int_a^{a+\varphi_1(b,a)} \psi(x)dx}\left[\int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,u)\psi(x)dx + \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,\nu)\psi(x)dx\right] \\ &\leq_I \frac{\mathfrak{S}_\lambda(a,u)+\mathfrak{S}_\lambda(a,\nu)+\mathfrak{S}_\lambda(b,u)+\mathfrak{S}_\lambda(b,\nu)}{4} \end{aligned} \tag{107}$$

Combing inequalities (99), (102), and (107), we obtain

$$\begin{aligned} &\mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right) \leq_I \frac{1}{2}\left[\begin{array}{c} \frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx} \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)\psi(x)dx \\ +\frac{1}{\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda\left(\frac{2a+\varphi_1(b,a)}{2}, \omega\right)\mathcal{K}(\omega)d\omega \end{array}\right] \\ &\leq_I \frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx \int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_a^{a+\varphi_1(b,a)} \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}(x,\omega)\psi(x)\mathcal{K}(\omega)d\omega dx \\ &\leq_I \frac{1}{4\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega}\left[\int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(a,\omega)\mathcal{K}(\omega)d\omega + \int_u^{u+\varphi_2(\nu,u)} \mathfrak{S}_\lambda(b,\omega)\mathcal{K}(\omega)d\omega\right] \\ &+\frac{1}{4\int_a^{a+\varphi_1(b,a)} \psi(x)dx}\left[\int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,u)\psi(x)dx + \int_a^{a+\varphi_1(b,a)} \mathfrak{S}_\lambda(x,\nu)\psi(x)dx\right] \\ &\leq_I \frac{\mathfrak{S}_\lambda(a,u)+\mathfrak{S}_\lambda(a,\nu)}{2} + \frac{\mathfrak{S}_\lambda(b,u)+\mathfrak{S}_\lambda(b,\nu)}{2} + \frac{\mathfrak{S}_\lambda(a,u)+\mathfrak{S}_\lambda(b,u)}{2} + \frac{\mathfrak{S}_\lambda(a,\nu)+\mathfrak{S}_\lambda(b,\nu)}{2} \end{aligned}$$

That is

$$\begin{aligned} &\widetilde{\mathfrak{S}}\left(\frac{2a+\varphi_1(b,a)}{2}, \frac{2u+\varphi_2(\nu,u)}{2}\right) \preccurlyeq \frac{1}{2}\left[\begin{array}{c} \frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx} \int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}\left(x, \frac{2u+\varphi_2(\nu,u)}{2}\right)\psi(x)dx \\ \widetilde{+}\frac{1}{\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}\left(\frac{2a+\varphi_1(b,a)}{2}, \omega\right)\mathcal{K}(\omega)d\omega \end{array}\right] \\ &\preccurlyeq \frac{1}{\int_a^{a+\varphi_1(b,a)} \psi(x)dx \int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega} \int_a^{a+\varphi_1(b,a)} \int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(x,\omega)\psi(x)\mathcal{K}(\omega)d\omega dx \\ &\preccurlyeq \frac{1}{4\int_a^{a+\varphi_1(b,a)} \psi(x)dx}\left[\int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}(x,u)dx \widetilde{+} \int_a^{a+\varphi_1(b,a)} \widetilde{\mathfrak{S}}(x,\nu)dx\right] \\ &\widetilde{+}\frac{1}{4\int_u^{u+\varphi_2(\nu,u)} \mathcal{K}(\omega)d\omega}\left[\int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(a,\omega)d\omega \widetilde{+} \int_u^{u+\varphi_2(\nu,u)} \widetilde{\mathfrak{S}}(b,\omega)d\omega\right] \\ &\preccurlyeq \frac{\widetilde{\mathfrak{S}}(a,u)\widetilde{+}\widetilde{\mathfrak{S}}(b,u)\widetilde{+}\widetilde{\mathfrak{S}}(a,\nu)\widetilde{+}\widetilde{\mathfrak{S}}(b,\nu)}{4} \end{aligned}$$

Hence, this concludes the proof. □

**Remark 6.** *If one takes $\mathcal{K}(\omega) = 1 = \psi(x)$, then from (94) we achieve (39).*

*If one takes $\varphi_1(b,a) = b - a$ and $\varphi_2(\nu,u) = \nu - u$, then from (94), we acquire the following inequality, see [38]:*

$$\begin{aligned}
&\widetilde{\mathfrak{S}}\left(\tfrac{a+b}{2},\tfrac{u+v}{2}\right) \preccurlyeq \tfrac{1}{2}\left[\tfrac{1}{\int_a^b \psi(x)dx}\int_a^b \widetilde{\mathfrak{S}}\left(x,\tfrac{u+v}{2}\right)\psi(x)dx \widetilde{+} \tfrac{1}{\int_a^b \mathcal{K}(\omega)d\omega}\int_a^b \widetilde{\mathfrak{S}}\left(\tfrac{a+b}{2},\omega\right)\mathcal{K}(\omega)d\omega\right] \\
&\preccurlyeq \tfrac{1}{\int_a^b \psi(x)dx \int_a^b \mathcal{K}(\omega)d\omega}\int_a^b\int_u^v \widetilde{\mathfrak{S}}(x,\omega)\psi(x)\mathcal{K}(\omega)d\omega dx \\
&\preccurlyeq \frac{1}{4\int_a^b \psi(x)dx}\left[\int_a^b \widetilde{\mathfrak{S}}(x,u)dx \widetilde{+} \int_a^b \widetilde{\mathfrak{S}}(x,v)dx\right]\widetilde{+}\frac{1}{4\int_a^b \mathcal{K}(\omega)d\omega}\left[\int_u^v \widetilde{\mathfrak{S}}(a,\omega)d\omega \widetilde{+} \int_u^v \widetilde{\mathfrak{S}}(b,\omega)d\omega\right] \\
&\preccurlyeq \tfrac{\widetilde{\mathfrak{S}}(a,u)\widetilde{+}\widetilde{\mathfrak{S}}(b,u)\widetilde{+}\widetilde{\mathfrak{S}}(a,v)\widetilde{+}\widetilde{\mathfrak{S}}(b,v)}{4}
\end{aligned} \tag{108}$$

*If one takes $\varphi_1(b,a) = b - a$, $\varphi_2(v,u) = v - u$ and $\mathcal{K}(\omega) = 1 = \psi(x)$, then from (94), we acquire the inequality (59), see [38].*

## 4. Conclusions

As an extension of convex fuzzy-interval-valued functions on coordinates, we have proposed the idea of fuzzy interval-valued preinvex functions in this article. For coordinated preinvex fuzzy interval-valued functions, we have created H–H-type inequalities. The product of two coordinated preinvex fuzzy-interval-valued functions was also examined, which are known as Pachpatte Type inequalities, as well as several new H–H-type inclusions. Other types of interval-valued preinvex functions on the coordinates may be included in the results produced in this study. Future work will explore fuzzy interval-valued fractional integrals on coordinates to study H–H-type and H–H–Fejér-type inequalities with the help of fuzzy-order relation for coordinated preinvex fuzzy interval-valued functions. We hope that the concepts and findings presented in this article will inspire readers to conduct more research.

**Author Contributions:** Conceptualization, M.B.K.; methodology, M.B.K.; validation, H.A., S.S.M.G. and A.A.A.; formal analysis, G.S.-G.; investigation, S.S.M.G.; resources, H.A.; data curation, A.A.A.; writing—original draft preparation, M.B.K., G.S.-G. and A.A.A.; writing—review and editing, M.B.K. and H.A.; visualization, A.A.A.; supervision, M.B.K. and S.S.M.G.; project administration, M.B.K.; funding acquisition, G.S.-G., S.S.M.G. and A.A.A. All authors have read and agreed to the published version of the manuscript.

**Funding:** The research of Santos-García was funded by the Spanish MINECO project TRACES TIN2015–67522–C3–3–R and the authors appreciate Taif University Researchers Supporting Project TURSP 2020/121, Taif University, Taif, Saudi Arabia for supporting this work.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors would like to thank the Rector, COMSATS University Islamabad, Islamabad, Pakistan, for providing excellent research and academic environments. The research of Santos-García was funded by the Spanish MINECO project TRACES TIN2015–67522–C3–3–R and the authors appreciate Taif University Researchers Supporting Project TURSP 2020/121, Taif University, Taif, Saudi Arabia for supporting this work.

**Conflicts of Interest:** The authors declare no conflict of interest.

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