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Some Generalized Versions of Chevet–Saphar Tensor Norms

Ju Myung Kim

Department of Mathematics and Statistics, Sejong University, Seoul 05006, Korea; kjm21@sejong.ac.kr

Abstract: The paper is concerned with some generalized versions g_E and w_E of classical tensor norms. We find a Banach space E for which g_E and w_E are finitely generated tensor norms, and show that g_E and w_E are associated with the ideals of some E-nuclear operators. We also initiate the study of some theories of our tensor norms.

Keywords: Schauder basis; vector-valued sequence; tensor norm; operator ideal

MSC: 46B45; 47L20

1. Introduction

One of the important theories in the study of Banach spaces is the theory of tensor norms (see Section 2 for the definition of tensor norm). It provides not only new examples of Banach spaces but also a powerful tool in the study of Banach operator ideals. One may refer to [1–5] and the references therein for various information and content about tensor norms. Throughout this paper, Banach spaces will be denoted by X and Y over $\mathbb{R}$ or $\mathbb{C}$, with dual spaces X^* and Y^*, and the closed unit ball of X will be denoted by B_X. We will denote by $X \otimes Y$ the algebraic tensor product of X and Y. The most classical two tensor norms are the *injective norm* ε and the *projective norm* π, which were systematically investigated by Grothendieck [6,7]. For $u \in X \otimes Y$,

$$\varepsilon(u; X, Y) := \sup\left\{ \Big| \sum_{n=1}^{l} x^*(x_n) y^*(y_n) \Big| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\},$$

where $\sum_{n=1}^{l} x_n \otimes y_n$ is any representation of u, and

$$\pi(u; X, Y) := \inf\left\{ \sum_{n=1}^{l} \|x_n\| \|y_n\| : u = \sum_{n=1}^{l} x_n \otimes y_n, l \in \mathbb{N} \right\}.$$

More recently, the author [8] introduced a tensor norm related with the injective norm. Lapresté [9] introduced the most generalized version $\alpha_{p,q}$ of the projective norm, and its some related topics were studied by Díaz, López-Molina, Rivera [10] and the author [11]. Many of the interesting tensor norms can be obtained from the tensor norm $\alpha_{p,q}$ $(1 \leq p,\, q \leq \infty, 1/p + 1/q \geq 1)$, which is defined as follows. Let $1 \leq r \leq \infty$ with $1/r = 1/p + 1/q - 1$. For $u \in X \otimes Y$, let

$$\alpha_{p,q}(u) := \inf\left\{ \|(\lambda_n)_{n=1}^{l}\|_r \sup_{x^* \in B_{X^*}} \|(x^*(x_n))_{n=1}^{l}\|_{q^*} \sup_{y^* \in B_{Y^*}} \|(y^*(y_n))_{n=1}^{l}\|_{p^*} : u = \sum_{n=1}^{l} \lambda_n x_n \otimes y_n, l \in \mathbb{N} \right\},$$

where p^* is the conjugate index of p and $\|\cdot\|_p$ means the ℓ_p-norm. Then, we see that

$$g_p(u) := \inf\left\{ \|(\|x_n\|)_{n=1}^{l}\|_p \sup_{y^* \in B_{Y^*}} \|(y^*(y_n))_{n=1}^{l}\|_{p^*} : u = \sum_{n=1}^{l} x_n \otimes y_n, l \in \mathbb{N} \right\} = \alpha_{p,1}(u),$$

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$$w_p(u) := \inf\Big\{ \sup_{x^*\in B_{X^*}} \|(x^*(x_n))_{n=1}^l\|_p \sup_{y^*\in B_{Y^*}} \|(y^*(y_n))_{n=1}^l\|_{p^*} : u = \textstyle\sum_{n=1}^l x_n \otimes y_n, l \in \mathbb{N}\Big\} = \alpha_{p,p^*}(u)$$

and $\pi(u) = \alpha_{1,1}(u)$. The tensor norms g_p and w_p were introduced and studied by Chevet and Saphar [12,13]; see [10,11,14–24] and the references therein for the investigation on related topics.

In this paper, we consider another generalization of g_p and w_p. These tensor norms are somehow determined by the Banach space ℓ_p. Naturally, one may extend these notions by replacing ℓ_p by a general Banach space with a Schauder basis. Throughout this paper, E is a Banach space having the 1-unconditional Schauder basis $(e_n)_n$, $(e_n^*)_n$ is the sequence of coordinate functionals for $(e_n)_n$ and $E_* := \overline{\text{span}}\{e_n^*\}_{n=1}^\infty$. For a finite subset F of $\mathbb{N}$ and $\{x_n\}_{n\in F} \subset X$, let

$$\|(x_n)_{n\in F}\|_{E(X)} := \Big\| \sum_{n\in F} \|x_n\| e_n \Big\|_E \text{ and } \|(x_n)_{n\in F}\|_{E^w(X)} := \sup_{x^*\in B_{X^*}} \Big\| \sum_{n\in F} x^*(x_n) e_n \Big\|_E.$$

We are now ready to introduce the main notion in this paper.

Definition 1. *For $u \in X \otimes Y$, let*

$$g_E(u; X, Y) := \inf\Big\{ \|(x_n)_{n\in F}\|_{E(X)} \|(y_n)_{n\in F}\|_{E_*^w(Y)} : u = \sum_{n\in F} x_n \otimes y_n, F \subset \mathbb{N} \Big\},$$

$$w_E(u; X, Y) := \inf\Big\{ \|(x_n)_{n\in F}\|_{E^w(X)} \|(y_n)_{n\in F}\|_{E_*^w(Y)} : u = \sum_{n\in F} x_n \otimes y_n, F \subset \mathbb{N} \Big\}.$$

For instance, $g_{\ell_p} = g_p$ and $w_{\ell_p} = w_p$ $(1 \le p < \infty)$, and $g_{c_0} = w_{c_0} = g_\infty = w_\infty$.

Tensor norms are closely related with normed operator ideals. Actually, in view of the monograph of Defant and Floret [2], there is a one-to-one correspondence between maximal Banach operator ideals and finitely generated tensor norms. A tensor norm α is said to *be associated with* a normed operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ if the canonical map from $\mathcal{A}(M, N)$ to $M^* \otimes_\alpha N$ equipped with the norm α is an isometry for every finite-dimensional normed spaces M and N. It is well known that g_p is associated with the ideal of p-nuclear operators. The starting point of this paper comes from [25], where the E-nuclear operators (see Section 2 for the definition of E-nuclear operators) were defined by replacing ℓ_p by E in the notion of p-nuclear operators. The main goal of this paper is to find a Banach space E for which g_E and w_E are tensor norms, and show that g_E and w_E are associated with the ideals of E-nuclear operators. Obtaining some results for g_E and w_E, we provide a base for further investigations of the g_E- and w_E-tensor norms and E-operator ideals. We focus on the Banach space $E = (\sum \ell_q)_p$ $(1 \le p, q \le \infty)$ of infinite ℓ_p direct sum of ℓ_qs, which is a generalization of ℓ_p. For this case, we extend some well known results for g_p and w_p as follows.

In Section 2, for $E = (\sum \ell_q)_p$ $(1 \le p, q \le \infty)$, we prove that g_E and w_E are finitely generated tensor norms, and it is demonstrated that g_E and w_E are associated with the ideals of E-nuclear operators. In Section 3, we prove that g_E is left projective and for every Banach space X, the injective tensor product $X \otimes_\varepsilon E$ is isometric to $X \otimes_{w_E} E$; furthermore, if $(e_n)_n$ is shrinking, then $E^* \otimes_\varepsilon X$ is isometric to $E^* \otimes_{w_E} X$. Additionally, we establish the completions of our E-tensor norms for $E = (\sum \ell_q)_p$, and as an application, we represent E-nuclear operators acting on dual spaces. We refer to the book of Defant and Floret [2] as a reference to the main notions and formulas in the theory of tensor norms and (quasi) normed operator ideals.

2. The g_E- and w_E-Tensor Norms and Their Associated Operator Ideals

Let us recall that a tensor norm α is a norm on $X \otimes Y$ for each pair of Banach spaces X and Y such that

(TN1) $\varepsilon \le \alpha \le \pi$;
(TN2) for all operators $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$,

$$\|T_1 \otimes T_2 : X_1 \otimes_\alpha X_2 \to Y_1 \otimes_\alpha Y_2\| \le \|T_1\|\|T_2\|.$$

A tensor norm α is said to be *finitely generated* if

$$\alpha(u; X, Y) = \inf\{\alpha(u; M, N) : u \in M \otimes N, \dim M, \dim N < \infty\}$$

for every $u \in X \otimes Y$.

Proposition 1. *Suppose that $(e_n)_n$ is normalized. If g_E and w_E satisfy the triangle inequality, then they are finitely generated tensor norms.*

Proof. We only consider g_E. Let X and Y be Banach spaces. Let $c \in \mathbb{C}$ and let $u = \sum_{n \in F} x_n \otimes y_n$ be an arbitrary representation in $X \otimes Y$. Then

$$g_E(cu; X, Y) \le \|(cx_n)_{n\in F}\|_{E(X)}\|(y_n)_{n\in F}\|_{E_*^w(Y)} = |c|\|(x_n)_{n\in F}\|_{E(X)}\|(y_n)_{n\in F}\|_{E_*^w(Y)}.$$

Thus $g_E(cu; X, Y) \le |c|g_E(u; X, Y)$. Since $g_E(u; X, Y) = g_E((1/c)(cu); X, Y) \le (1/|c|)g_E(cu; X, Y)$, $g_E(cu; X, Y) \ge |c|g_E(u; X, Y)$.

(TN1): Let $u = \sum_{n\in F} x_n \otimes y_n$ be an arbitrary representation in $X \otimes Y$. Let $x^* \in B_{X^*}$ and $y^* \in B_{Y^*}$. Then

$$\Big|\sum_{n\in F} x^*(x_n)y^*(y_n)\Big| = \Big|\Big(\sum_{k\in F} y^*(y_k)e_k^*\Big)\Big(\sum_{n\in F} x^*(x_n)e_n\Big)\Big| \le \|(x_n)_{n\in F}\|_{E(X)}\|(y_n)_{n\in F}\|_{E_*^w(Y)}$$

and

$$g_E(u; X, Y) \le \sum_{n\in F} g_E(x_n \otimes y_n) \le \sum_{n\in F} \|x_n\|\|y_n\|.$$

It follows that $\varepsilon(u; X, Y) \le g_E(u; X, Y) \le \pi(u; X, Y)$, and so

$$g_E(u; X, Y) = 0 \Leftrightarrow u = 0$$

for $u \in X \otimes Y$.

(TN2): Let $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ be operators. Let $u \in X_1 \otimes X_2$ and let $u = \sum_{n\in F} x_n^1 \otimes x_n^2$ be an arbitrary representation. Then

$$\begin{aligned} g_E((T_1 \otimes T_2)(u); Y_1, Y_2) &= g_E\Big(\sum_{n\in F} T_1x_n^1 \otimes T_2x_n^2; Y_1, Y_2\Big) \\ &\le \|(T_1x_n^1)_{n\in F}\|_{E(Y_1)}\|(T_2x_n^2)_{n\in F}\|_{E_*^w(Y_2)} \\ &= \|T_1\|\|T_2\|\|((1/\|T_1\|)T_1x_n^1)_{n\in F}\|_{E(Y_1)}\|((1/\|T_2\|)T_2x_n^2)_{n\in F}\|_{E_*^w(Y_2)} \\ &\le \|T_1\|\|T_2\|\|(x_n^1)_{n\in F}\|_{E(X_1)}\|(x_n^2)_{n\in F}\|_{E_*^w(X_2)}. \end{aligned}$$

Hence

$$g_E((T_1 \otimes T_2)(u); Y_1, Y_2) \le \|T_1\|\|T_2\|g_E(u; X_1, X_2).$$

To show that g_E is finitely generated, let $u \in X \otimes Y$ and let $u = \sum_{n\in F} x_n \otimes y_n$ be an arbitrary representation. Let $M_0 := \operatorname{span}\{x_n\}_{n\in F}$ and $N_0 := \operatorname{span}\{y_n\}_{n\in F}$. Using the Hahn–Banach extension theorem, we have

$$\begin{aligned}
&\inf\{g_E(u; M, N) : u \in M \otimes N, \dim M, \dim N < \infty\} \\
&\leq g_E(u; M_0, N_0) \\
&\leq \|(x_n)_{n\in F}\|_{E(M_0)} \sup_{z^* \in B_{N_0^*}} \Big\| \sum_{n\in F} z^*(y_n) e_n^* \Big\|_{E^*} \\
&= \|(x_n)_{n\in F}\|_{E(X)} \|(y_n)_{n\in F}\|_{E_*^w(Y)}.
\end{aligned}$$

Hence,

$$\inf\{g_E(u; M, N) : u \in M \otimes N, \dim M, \dim N < \infty\} \leq g_E(u; X, Y).$$

□

We can now prove:

Theorem 1. *If $E = (\sum \ell_q)_p$ $(1 \leq p, q < \infty)$, $E = (\sum c_0)_p$ $(1 \leq p < \infty)$ or $E = (\sum \ell_q)_{c_0}$ $(1 \leq q < \infty)$, then g_E and w_E are finitely generated tensor norms.*

Proof. We only consider g_E. Let X and Y be Banach spaces. By Proposition 1, we only need to show the triangle inequality of g_E.

For the he case $E = (\sum \ell_q)_p$ $(1 < p, q < \infty)$, let $u, v \in X \otimes Y$ and let $\delta > 0$ be given. We can find representations

$$u = \sum_{n=1}^{l} \sum_{k=1}^{l} x_{nk}^1 \otimes y_{nk}^1 \text{ and } v = \sum_{n=1}^{l} \sum_{k=1}^{l} x_{nk}^2 \otimes y_{nk}^2$$

such that

$$\|((x_{nk}^1)_{k=1}^l)_{n=1}^l\|_{E(X)} \|((y_{nk}^1)_{k=1}^l)_{n=1}^l\|_{E_*^w(Y)} \leq (1+\delta) g_E(u; X, Y),$$

$$\|((x_{nk}^2)_{k=1}^l)_{n=1}^l\|_{E(X)} \|((y_{nk}^2)_{k=1}^l)_{n=1}^l\|_{E_*^w(Y)} \leq (1+\delta) g_E(v; X, Y).$$

We may assume that

$$\begin{aligned}
&\|((x_{nk}^1)_{k=1}^l)_{n=1}^l\|_{E(X)} \leq ((1+\delta) g_E(u; X, Y))^{1/p}, \\
&\|((y_{nk}^1)_{k=1}^l)_{n=1}^l\|_{E_*^w(Y)} \leq ((1+\delta) g_E(u; X, Y))^{1/p^*}, \\
&\|((x_{nk}^2)_{k=1}^l)_{n=1}^l\|_{E(X)} \leq ((1+\delta) g_E(v; X, Y))^{1/p}, \\
&\|((y_{nk}^2)_{k=1}^l)_{n=1}^l\|_{E_*^w(Y)} \leq ((1+\delta) g_E(v; X, Y))^{1/p^*}.
\end{aligned}$$

Since

$$u + v = \sum_{i=1}^{2} \sum_{n=1}^{l} \sum_{k=1}^{l} x_{nk}^i \otimes y_{nk}^i,$$

$$
\begin{aligned}
&g_E(u+v;X,Y) \\
&\le \Big(\sum_{i=1}^{2}\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}\|x_{nk}^i\|^q\Big)^{p/q}\Big)^{1/p} \sup_{y^*\in B_{Y^*}}\Big(\sum_{i=1}^{2}\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}|y^*(y_{nk}^i)|^{q^*}\Big)^{p^*/q^*}\Big)^{1/p^*} \\
&\le \Big(\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}\|x_{nk}^1\|^q\Big)^{p/q}+\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}\|x_{nk}^2\|^q\Big)^{p/q}\Big)^{1/p} \\
&\Big(\sup_{y^*\in B_{Y^*}}\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}|y^*(y_{nk}^1)|^{q^*}\Big)^{p^*/q^*}+\sup_{y^*\in B_{Y^*}}\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}|y^*(y_{nk}^2)|^{q^*}\Big)^{p^*/q^*}\Big)^{1/p^*} \\
&\le ((1+\delta)(g_E(u;X,Y)+g_E(v;X,Y)))^{1/p}((1+\delta)(g_E(u;X,Y)+g_E(v;X,Y)))^{1/p^*} \\
&= (1+\delta)(g_E(u;X,Y)+g_E(v;X,Y)).
\end{aligned}
$$

Since $\delta > 0$ was arbitrary,

$$g_E(u+v;X,Y) \le g_E(u;X,Y)+g_E(v;X,Y).$$

For the case $E = (\sum \ell_1)_p$ $(1 < p < \infty)$:

$$
\begin{aligned}
&g_E(u+v;X,Y) \\
&\le \Big(\sum_{i=1}^{2}\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}\|x_{nk}^i\|\Big)^{p}\Big)^{1/p} \sup_{y^*\in B_{Y^*}}\Big(\sum_{i=1}^{2}\sum_{n=1}^{l}(\sup_{1\le k\le l}|y^*(y_{nk}^i)|)^{p^*}\Big)^{1/p^*} \\
&\le \Big(\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}\|x_{nk}^1\|\Big)^{p}+\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}\|x_{nk}^2\|\Big)^{p}\Big)^{1/p} \\
&\Big(\sup_{y^*\in B_{Y^*}}\sum_{n=1}^{l}(\sup_{1\le k\le l}|y^*(y_{nk}^1)|)^{p^*}+\sup_{y^*\in B_{Y^*}}\sum_{n=1}^{l}(\sup_{1\le k\le l}|y^*(y_{nk}^2)|)^{p^*}\Big)^{1/p^*} \\
&\le ((1+\delta)(g_E(u;X,Y)+g_E(v;X,Y)))^{1/p}((1+\delta)(g_E(u;X,Y)+g_E(v;X,Y)))^{1/p^*} \\
&= (1+\delta)(g_E(u;X,Y)+g_E(v;X,Y)).
\end{aligned}
$$

For the case $E = (\sum \ell_q)_1$ $(1 < q < \infty)$: We may assume that

$$\|((x_{nk}^1)_{k=1}^l)_{n=1}^l\|_{E(X)} \le (1+\delta)g_E(u;X,Y), \|((y_{nk}^1)_{k=1}^l)_{n=1}^l\|_{E_*^w(Y)} \le 1,$$

$$\|((x_{nk}^2)_{k=1}^l)_{n=1}^l\|_{E(X)} \le (1+\delta)g_E(v;X,Y), \|((y_{nk}^2)_{k=1}^l)_{n=1}^l\|_{E_*^w(Y)} \le 1.$$

Then

$$
\begin{aligned}
&g_E(u+v;X,Y) \\
&\le \sum_{i=1}^{2}\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}\|x_{nk}^i\|^q\Big)^{1/q} \sup_{y^*\in B_{Y^*}}\sup_{i=1,2,1\le n\le l}\Big(\sum_{k=1}^{l}|y^*(y_{nk}^i)|^{q^*}\Big)^{1/q^*} \\
&\le \sum_{n=1}^{l}\Big(\sum_{k=1}^{l}\|x_{nk}^1\|^q\Big)^{1/q}+\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}\|x_{nk}^2\|^q\Big)^{1/q} \\
&\le (1+\delta)(g_E(u;X,Y)+g_E(v;X,Y)).
\end{aligned}
$$

For the case $E = (\sum c_0)_p$ $(1 < p < \infty)$:

$$
\begin{aligned}
&g_E(u+v;X,Y)\\
&\leq \Big(\sum_{i=1}^{2}\sum_{n=1}^{l}(\sup_{1\leq k\leq l}\|x_{nk}^i\|)^p\Big)^{1/p}\sup_{y^*\in B_{Y^*}}\Big(\sum_{i=1}^{2}\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}|y^*(y_{nk}^i)|\Big)^{p^*}\Big)^{1/p^*}\\
&\leq \Big(\sum_{n=1}^{l}(\sup_{1\leq k\leq l}\|x_{nk}^1\|)^p+\sum_{n=1}^{l}(\sup_{1\leq k\leq l}\|x_{nk}^2\|)^p\Big)^{1/p}\\
&\Big(\sup_{y^*\in B_{Y^*}}\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}|y^*(y_{nk}^1)|\Big)^{p^*}+\sup_{y^*\in B_{Y^*}}\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}|y^*(y_{nk}^2)|\Big)^{p^*}\Big)^{1/p^*}\\
&\leq ((1+\delta)(g_E(u;X,Y)+g_E(v;X,Y)))^{1/p}((1+\delta)(g_E(u;X,Y)+g_E(v;X,Y)))^{1/p^*}\\
&=(1+\delta)(g_E(u;X,Y)+g_E(v;X,Y)).
\end{aligned}
$$

For the case $E=(\sum\ell_q)_{c_0}$ $(1<q<\infty)$: We may assume that

$$\|((x_{nk}^1)_{k=1}^l)_{n=1}^l\|_{E(X)}\leq 1, \|((y_{nk}^1)_{k=1}^l)_{n=1}^l\|_{E_*^w(Y)}\leq(1+\delta)g_E(u;X,Y),$$

$$\|((x_{nk}^2)_{k=1}^l)_{n=1}^l\|_{E(X)}\leq 1, \|((y_{nk}^2)_{k=1}^l)_{n=1}^l\|_{E_*^w(Y)}\leq(1+\delta)g_E(v;X,Y).$$

Then

$$
\begin{aligned}
&g_E(u+v;X,Y)\\
&\leq \sup_{i=1,2,1\leq n\leq l}\Big(\sum_{k=1}^{l}\|x_{nk}^i\|^q\Big)^{1/q}\sup_{y^*\in B_{Y^*}}\sum_{i=1}^{2}\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}|y^*(y_{nk}^i)|^{q^*}\Big)^{1/q^*}\\
&\leq \sup_{y^*\in B_{Y^*}}\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}|y^*(y_{nk}^1)|^{q^*}\Big)^{1/q^*}+\sup_{y^*\in B_{Y^*}}\sum_{n=1}^{l}\Big(\sum_{k=1}^{l}|y^*(y_{nk}^2)|^{q^*}\Big)^{1/q^*}\\
&\leq (1+\delta)(g_E(u;X,Y)+g_E(v;X,Y)).
\end{aligned}
$$

The cases $E=(\sum\ell_1)_{c_0}$ and $E=(\sum c_0)_1$ also follow from similar proofs. □

Throughout the remainder of this paper, we will assume that g_E and w_E are finitely generated tensor norms. For a Banach space X, let us consider the Banach spaces

$$E(X):=\Big\{(x_n)_n \text{ in } X:\sum_{n=1}^{\infty}\|x_n\|e_n \text{ converges in } E\Big\}$$

equipped with the norm $\|(x_n)_n\|_{E(X)}:=\|\sum_{n=1}^{\infty}\|x_n\|e_n\|_E$,

$$E^w(X):=\Big\{(x_n)_n \text{ in } X:\sum_{n=1}^{\infty}x^*(x_n)e_n \text{ converges in } E \text{ for each } x^*\in X^*\Big\}$$

equipped with the norm $\|(x_n)_n\|_{E^w(X)}:=\sup_{x^*\in B_{X^*}}\|\sum_{n=1}^{\infty}x^*(x_n)e_n\|_E$ and

$$E^u(X):=\Big\{(x_n)_n \text{ in } X:\lim_{l\to\infty}\sup_{x^*\in B_{X^*}}\Big\|\sum_{n\geq l}x^*(x_n)e_n\Big\|_E=0\Big\}$$

equipped with the norm $\|(x_n)_n\|_{E^w(X)}$.

Let X and Y be Banach spaces, and let $T:X\to Y$ be an operator such that

$$T=\sum_{n=1}^{\infty}x_n^*\otimes y_n,$$

where $x_n^*\otimes y_n(x)=x_n^*(x)y_n$. The following operators were introduced in [25]. We say that T is *E-nuclear* (respectively, *dual E-nuclear*) if $(x_n^*)_n\in E(X^*)$ (respectively, $E_*^w(X^*)$) and $(y_n)_n\in E_*^w(Y)$ (respectively, $E(Y)$). The collection of all E-nuclear (respectively, dual

E-nuclear) operators from X to Y is denoted by $\mathcal{N}_E(X,Y)$ (respectively, $\mathcal{N}^E(X,Y)$), and for $T \in \mathcal{N}_E(X,Y)$ (respectively, $\mathcal{N}^E(X,Y)$), let

$$\|T\|_{\mathcal{N}_E} := \inf \|(x_n^*)_n\|_{E(X^*)}\|(y_n)_n\|_{E_*^w(Y)}$$

$$(\text{respectively, } \|T\|_{\mathcal{N}^E} := \inf \|(x_n^*)_n\|_{E_*^w(X^*)}\|(y_n)_n\|_{E(Y)}),$$

where the infimum is taken over all such representations. We say that T is *uniform E-nuclear* (respectively, *dual uniform E-nuclear*) if $(x_n^*)_n \in E^u(X^*)$ (respectively, $E_*^w(X^*)$) and $(y_n)_n \in E_*^w(Y)$ (respectively, $E^u(Y)$). The collection of all uniform E-nuclear (respectively, dual uniform E-nuclear) operators from X to Y is denoted by ${}_u\mathcal{N}_E(X,Y)$ (respectively, ${}_u\mathcal{N}^E(X,Y)$) and for $T \in {}_u\mathcal{N}_E(X,Y)$ (respectively, ${}_u\mathcal{N}^E(X,Y)$), let

$$\|T\|_{{}_u\mathcal{N}_E} := \inf \|(x_n^*)_n\|_{E^w(X^*)}\|(y_n)_n\|_{E_*^w(Y)}$$

$$(\text{respectively, } \|T\|_{{}_u\mathcal{N}^E} := \inf \|(x_n^*)_n\|_{E_*^w(X^*)}\|(y_n)_n\|_{E^w(Y)}),$$

where the infimum is taken over all such representations. For instance, $\mathcal{N}_{\ell_p}$ is the ideal of *p-nuclear operators*, and ${}_u\mathcal{N}_{\ell_p}$ is the ideal of *p-compact operators* (cf. [2,15]).

Let $\mathcal{F}$ be the ideal of finite rank operators and let X and Y be Banach spaces. For $T \in \mathcal{F}(X,Y)$, let

$$\|T\|_{\mathcal{N}_E^0} := \inf\left\{\|(x_n^*)_{n\in F}\|_{E(X^*)}\|(y_n)_{n\in F}\|_{E_*^w(Y)} : T = \sum_{n\in F} x_n^* \underline{\otimes} y_n,\ \text{finite } F \subset \mathbb{N}\right\},$$

$$\|T\|_{\mathcal{N}_0^E} := \inf\left\{\|(x_n^*)_{n\in F}\|_{E_*^w(X^*)}\|(y_n)_{n\in F}\|_{E(Y)} : T = \sum_{n\in F} x_n^* \underline{\otimes} y_n,\ \text{finite } F \subset \mathbb{N}\right\},$$

$$\|T\|_{{}_u\mathcal{N}_E^0} := \inf\left\{\|(x_n^*)_{n\in F}\|_{E^w(X^*)}\|(y_n)_{n\in F}\|_{E_*^w(Y)} : T = \sum_{n\in F} x_n^* \underline{\otimes} y_n,\ \text{finite } F \subset \mathbb{N}\right\},$$

$$\|T\|_{{}_u\mathcal{N}_0^E} := \inf\left\{\|(x_n^*)_{n\in F}\|_{E_*^w(X^*)}\|(y_n)_{n\in F}\|_{E^w(Y)} : T = \sum_{n\in F} x_n^* \underline{\otimes} y_n,\ \text{finite } F \subset \mathbb{N}\right\}.$$

Let α^t be the *transposed tensor norm* (see [2]) of a tensor norm α. Let X and Y be Banach spaces. For $T = \sum_{n\in F} x_n^* \underline{\otimes} y_n \in \mathcal{F}(X,Y)$, let $u_T := \sum_{n\in F} x_n^* \otimes y_n \in X^* \otimes Y$. Then we see that

$$\|T\|_{\mathcal{N}_E^0} = g_E(u_T; X^*, Y), \|T\|_{{}_u\mathcal{N}_E^0} = w_E(u_T; X^*, Y),$$

$$\|T\|_{\mathcal{N}_0^E} = g_E^t(u_T; X^*, Y), \|T\|_{{}_u\mathcal{N}_0^E} = w_E^t(u_T; X^*, Y).$$

Proposition 2. *If X or Y is a finite-dimensional normed space, then*

$$\|T\|_{\mathcal{N}_E^0} = \|T\|_{\mathcal{N}_E}, \|T\|_{\mathcal{N}_0^E} = \|T\|_{\mathcal{N}^E}, \|T\|_{{}_u\mathcal{N}_E^0} = \|T\|_{{}_u\mathcal{N}_E}, \|T\|_{{}_u\mathcal{N}_0^E} = \|T\|_{{}_u\mathcal{N}^E}$$

for every operartor T from X to Y.

Proof. We only consider $\mathcal{N}_0^E$. Let $T : X \to Y$ be an operator, and let $\delta > 0$ be given. Let

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n$$

be a dual E-nuclear representation such that

$$\|(x_n^*)_n\|_{E_*^w(X^*)}\|(y_n)_n\|_{E(Y)} \le (1+\delta)\|T\|_{\mathcal{N}^E}.$$

If X is finite-dimensional, then there exists an $l \in \mathbb{N}$ such that

$$\begin{aligned}\Big\| \sum_{n \geq l+1} x_n^* \underline{\otimes} y_n \Big\| &\leq \sup_{x \in B_X} \sum_{n \geq l+1} |x_n^*(x)| \|y_n\| \\ &= \sup_{x \in B_X} \Big(\sum_{n \geq l+1} |x_n^*(x)| e_n^* \Big) \Big(\sum_{n \geq l+1} \|y_n\| e_n \Big) \\ &\leq \|(x_n^*)_n\|_{E_*^w(X^*)} \Big\| \sum_{n \geq l+1} \|y_n\| e_n \Big\|_E \\ &\leq \delta \|T\|_{\mathcal{N}^E} / \|id_X\|_{\mathcal{N}_0^E},\end{aligned}$$

where id_X is the identity map on X. We have

$$\begin{aligned}\|T\|_{\mathcal{N}_0^E} &\leq \Big\| \sum_{n=1}^{l} x_n^* \underline{\otimes} y_n \Big\|_{\mathcal{N}_0^E} + \Big\| \sum_{n \geq l+1} x_n^* \underline{\otimes} y_n \Big\|_{\mathcal{N}_0^E} \\ &\leq \|(x_n^*)_n\|_{E_*^w(X^*)} \|(y_n)_n\|_{E^w(Y)} + \Big\| \sum_{n \geq l+1} x_n^* \underline{\otimes} y_n \Big\| \|id_X\|_{\mathcal{N}_0^E} \\ &\leq (1+2\delta) \|T\|_{\mathcal{N}^E}.\end{aligned}$$

If Y is finite-dimensional, then id_X can be replaced by id_Y in the above proof. □

From Proposition 2, we have:

Corollary 1. *The tensor norms g_E, g_E^t, w_E and w_E^t, respectively, are associated with $[\mathcal{N}_E, \|\cdot\|_{\mathcal{N}_E}]$, $[\mathcal{N}^E, \|\cdot\|_{\mathcal{N}^E}]$, $[_u\mathcal{N}_E, \|\cdot\|_{_u\mathcal{N}_E}]$ and $[_u\mathcal{N}^E, \|\cdot\|_{_u\mathcal{N}^E}]$.*

3. Some Results of the g_E- and w_E-Tensor Norms

A tensor norm α is called *left-projective* if, for every quotient operator $q : Z \to X$, the operator

$$q \otimes id_Y : Z \otimes_\alpha Y \to X \otimes_\alpha Y$$

is a quotient operator for all Banach spaces X, Y and Z. If the transposed α^t of α is left-projective, then α is called *right-projective*.

Proposition 3. *The tensor norm g_E is left-projective.*

Proof. Let $q : Z \to X$ be a quotient operator. To show that the map

$$q \otimes id_Y : Z \otimes_{g_E} Y \to X \otimes_{g_E} Y$$

is a quotient operator, let $u = \sum_{n \in F} x_n \otimes y_n \in X \otimes_{g_E} Y$. We should show that

$$g_E(u; X, Y) \geq \inf\{g_E(v; Z, Y) : v \in Z \otimes_{g_E} Y, q \otimes id_Y(v) = u\}.$$

Let $\delta > 0$ be given. Since q is a quotient operator, there exists $\{z_n\}_{n \in F} \subset Z$ such that

$$qz_n = x_n, \|z_n\| \leq (1+\delta)\|x_n\|$$

for every $n \in F$. Then we have

$$\begin{aligned}
&\inf\{g_E(v; Z, Y) : v \in Z \otimes_{g_E} Y, q \otimes id_Y(v) = u\} \\
&\leq g_E\Big(\sum_{n\in F} z_n \otimes y_n; Z, Y\Big) \\
&\leq \|(z_n)_{n\in F}\|_{E(Z)} \|(y_n)_{n\in F}\|_{(E_*)^w(Y)} \\
&= \Big\|\sum_{n\in F} \|z_n\| e_n\Big\|_E \|(y_n)_{n\in F}\|_{(E_*)^w(Y)} \\
&\leq (1+\delta)\Big\|\sum_{n\in F} \|x_n\| e_n\Big\|_E \|(y_n)_{n\in F}\|_{(E_*)^w(Y)}.
\end{aligned}$$

Since $u = \sum_{n\in F} x_n \otimes y_n$ was an arbitrary representation,

$$\inf\{g_E(v; Z, Y) : v \in Z \otimes_{g_E} Y, q \otimes id_Y(v) = u\} \leq (1+\delta) g_E(u; X, Y).$$

Since $\delta > 0$ was also arbitrary, we complete the proof. □

For a tensor norm α, we will denote by $X \hat{\otimes}_\alpha Y$ the completion of the normed space $X \otimes_\alpha Y$.

Lemma 1 ([2], Proposition 21.7(1))**.** *For a finitely generated tensor norm α, if a Banach space X has the approximation property, then for every Banach space Y, the natural map*

$$I_\alpha : Y \hat{\otimes}_\alpha X \longrightarrow Y \hat{\otimes}_\varepsilon X$$

is injective.

Theorem 2. *For every Banach space X,*

$$X \otimes_\varepsilon E = X \otimes_{w_E} E$$

holds isometrically, and if $(e_n)_n$ is shrinking, then

$$E^* \otimes_\varepsilon X = E^* \otimes_{w_E} X$$

holds isometrically.

Proof. In order to prove the first statement, let $u \in X \otimes E$, and let $U : X^* \to E$ be the corresponding finite rank operator for u. Then, $U^*(E^*)$ can be viewed with a subset of X. Thus, for every $x^* \in X^*$,

$$Ux^* = \sum_{i=1}^{\infty} (e_i^* U x^*) e_i = \sum_{i=1}^{\infty} x^*(U^* e_i^*) e_i.$$

Since $U(B_{X^*})$ is a relatively compact subset of E,

$$\begin{aligned}
\lim_{l\to\infty} \varepsilon\Big(\sum_{i=1}^{l} U^* e_i^* \otimes e_i - u; X, E\Big) &= \lim_{l\to\infty} \Big\|\sum_{i=1}^{l} U^* e_i^* \underline{\otimes} e_i - U\Big\| \\
&= \lim_{l\to\infty} \sup_{x^*\in B_{X^*}} \Big\|\sum_{i=1}^{l} (e_i^* U x^*) e_i - U x^*\Big\|_E = 0.
\end{aligned}$$

Consequently,

$$u = \sum_{i=1}^{\infty} U^* e_i^* \otimes e_i$$

converges in $X\hat{\otimes}_\varepsilon E$.

To show that the above series unconditionally converges in $X\hat{\otimes}_{w_E}E$, let $\delta > 0$ be given. Let $\{Ux_k^*\}_{k=1}^m$ be a $\delta/2$-net for $U(B_{X^*})$. Choose an $l_\delta \in \mathbb{N}$ so that

$$\Big\| \sum_{i \geq l_\delta} (e_i^* U x_k^*) e_i \Big\|_E \leq \frac{\delta}{2}$$

for every $k = 1, ..., m$. Now, let G be an arbitrary finite subset of $\mathbb{N}$ with $\min G > l_\delta$. Let $x^* \in B_{X^*}$ and $e^* \in B_{E^*}$. Let $k_0 \in \{1, ..., m\}$ be such that

$$\|Ux^* - Ux_{k_0}^*\|_E \leq \frac{\delta}{2}.$$

Then we have

$$\begin{aligned}
\Big\| \sum_{i\in G} x^*(U^*e_i^*)e_i \Big\|_E \Big\| \sum_{i\in G} (e^*e_i)e_i^* \Big\|_{E^*} &= \Big\| \sum_{i\in G} (e_i^*Ux^*)e_i \Big\|_E \sup_{\sum_k \alpha_k e_k \in B_E} \Big| \sum_{i\in G} e^*(\alpha_i e_i) \Big| \\
&\leq \Big\| \sum_{i\in G} (e_i^*Ux^*)e_i \Big\|_E \\
&\leq \Big\| \sum_{i\in G} (e_i^*U(x^* - x_{k_0}^*))e_i \Big\|_E + \Big\| \sum_{i\in G} (e_i^*Ux_{k_0}^*)e_i \Big\|_E \\
&\leq \Big\| \sum_{i=1}^{\infty} (e_i^*U(x^* - x_{k_0}^*))e_i \Big\|_E + \Big\| \sum_{i\geq l_\delta} (e_i^*Ux_{k_0}^*)e_i \Big\|_E \\
&\leq \|Ux^* - Ux_{k_0}^*\|_E + \frac{\delta}{2} \leq \delta.
\end{aligned}$$

Consequently,

$$w_E\Big(\sum_{i\in G} U^*e_i^* \otimes e_i; X, E \Big) \leq \|(U^*e_i^*)_{i\in G}\|_{E^w(X)} \|(e_i)_{i\in G}\|_{E_*^w(E)} \leq \delta$$

and so

$$v := \sum_{i=1}^{\infty} U^*e_i^* \otimes e_i$$

unconditionally converges in $X\hat{\otimes}_{w_E}E$. Since a Banach space with a basis has the approximation property, by Lemma 1, $u = v$ in $X\hat{\otimes}_{w_E}E$. Then, since for every $l \in \mathbb{N}$,

$$\begin{aligned}
w_E\Big(\sum_{i=1}^{l} U^*e_i^* \otimes e_i; X, E \Big) &\leq \|(U^*e_i^*)_{i=1}^l\|_{E^w(X)} \|(e_i)_{i=1}^l\|_{E_*^w(E)} \\
&\leq \sup_{x^* \in B_{X^*}} \Big\| \sum_{i=1}^{l} (e_i^*Ux^*)e_i \Big\|_E,
\end{aligned}$$

$$w_E(u; X, E) \leq \|U\| = \varepsilon(u; X, E).$$

In order to prove the second statement, let $v \in E^* \otimes X$ and let $V : E \to X$ be the corresponding finite rank operator for v. For every $e = \sum_i \alpha_i e_i \in E$,

$$Ve = \sum_{i=1}^{\infty} \alpha_i Ve_i = \sum_{i=1}^{\infty} (e_i^*e)Ve_i.$$

Since $(e_i^*)_i$ is a basis for E^*, and $V^*(B_{X^*})$ is a relatively compact subset of E^*,

$$\begin{aligned}\lim_{l\to\infty}\varepsilon\Big(\sum_{i=1}^{l}e_i^*\otimes Ve_i-v;E^*,X\Big)&=\lim_{l\to\infty}\Big\|\sum_{i=1}^{l}e_i^*\underline{\otimes}Ve_i-V\Big\|\\&=\lim_{l\to\infty}\Big\|\sum_{i=1}^{l}Ve_i\underline{\otimes}e_i^*-V^*\Big\|\\&=\lim_{l\to\infty}\sup_{x^*\in B_{X^*}}\Big\|\sum_{i=1}^{l}(V^*x^*)(e_i)e_i^*-V^*x^*\Big\|_{E^*}=0.\end{aligned}$$

Consequently,

$$v=\sum_{i=1}^{\infty}e_i^*\otimes Ve_i$$

converges in $E^*\hat{\otimes}_\varepsilon X$.

To show that the above series unconditionally converges in $E^*\hat{\otimes}_{w_E}X$, let $\delta>0$ be given. Let $\{V^*x_k^*\}_{k=1}^m$ be a $\delta/2$-net for $V^*(B_{X^*})$. Choose an $l_\delta\in\mathbb{N}$ so that

$$\Big\|\sum_{i\geq l_\delta}V^*x_k^*(e_i)e_i^*\Big\|_{E^*}\leq\frac{\delta}{2}$$

for every $k=1,...,m$. Now, let G be an arbitrary finite subset of $\mathbb{N}$ with $\min G>l_\delta$. Let $x^*\in B_{X^*}$ and $e^{**}\in B_{E^{**}}$. Let $k_0\in\{1,...,m\}$ be such that

$$\|V^*x^*-V^*x_{k_0}^*\|_{E^*}\leq\frac{\delta}{2}.$$

Then, we have

$$\begin{aligned}\Big\|\sum_{i\in G}e^{**}(e_i^*)e_i\Big\|_E\Big\|\sum_{i\in G}(x^*Ve_i)e_i^*\Big\|_{E^*}&=\sup_{\sum_k\alpha_ke_k^*\in B_{E^*}}\Big|\sum_{i\in G}e^{**}(\alpha_ie_i^*)\Big|\Big\|\sum_{i\in G}V^*x^*(e_i)e_i^*\Big\|_{E^*}\\&\leq\Big\|\sum_{i\in G}V^*x^*(e_i)e_i^*\Big\|_{E^*}\\&\leq\Big\|\sum_{i\in G}V^*(x^*-x_{k_0}^*)(e_i)e_i^*\Big\|_{E^*}+\Big\|\sum_{i\in G}V^*x_{k_0}^*(e_i)e_i^*\Big\|_{E^*}\\&\leq\Big\|\sum_{i=1}^{\infty}V^*(x^*-x_{k_0}^*)(e_i)e_i^*\Big\|_{E^*}+\Big\|\sum_{i\geq l_\delta}V^*x_{k_0}^*(e_i)e_i^*\Big\|_{E^*}\\&\leq\|V^*(x^*-x_{k_0}^*)\|_{E^*}+\frac{\delta}{2}\leq\delta.\end{aligned}$$

Consequently,

$$w_E\Big(\sum_{i\in G}e_i^*\otimes Ve_i;E^*,X\Big)\leq\|(e_i^*)_{i\in G}\|_{E^w(E^*)}\|(Ve_i)_{i\in G}\|_{E_*^w(X)}\leq\delta$$

and so

$$u:=\sum_{i=1}^{\infty}e_i^*\otimes Ve_i$$

unconditionally converges in $E^*\hat{\otimes}_{w_E}X$. By Lemma 1, since $v^t = u^t$ in $X\hat{\otimes}_\varepsilon E^*$, $v^t = u^t$ in $X\hat{\otimes}_{w_E^t}E^*$, and so $v = u$ in $E^*\hat{\otimes}_{w_E}X$. Since for every $l \in \mathbb{N}$,

$$w_E\Big(\sum_{i=1}^{l} e_i^* \otimes Ve_i; E^*, X\Big) \le \|(e_i^*)_{i=1}^l\|_{E^w(E^*)}\|(Ve_i)_{i=1}^l\|_{E_*^w(X)}$$
$$\le \sup_{x^*\in B_{X^*}} \Big\|\sum_{i=1}^{l} V^*x^*(e_i)e_i^*\Big\|_{E^*},$$
$$w_E(v; E^*, X) \le \|V\| = \varepsilon(v; E^*, X).$$

□

Now, we consider the completions of our tensor norms. The following lemma is well known.

Lemma 2. *Let $(Z, \|\cdot\|)$ be a normed space, and let $(\hat{Z}, \|\cdot\|)$ be its completion. If $z \in \hat{Z}$, then for every $\delta > 0$, there exists a sequence $(z_n)_n$ in Z such that*

$$\sum_{n=1}^{\infty} \|z_n\| \le (1+\delta)\|z\|$$

and $z = \sum_{n=1}^{\infty} z_n$ converges in $\hat{Z}$.

Proposition 4. *Suppose that $E = (\sum \ell_q)_p$ $(1 \le p, q < \infty)$, $E = (\sum c_0)_p$ $(1 \le p < \infty)$ or $E = (\sum \ell_q)_{c_0}$ $(1 \le q < \infty)$. If $u \in X\hat{\otimes}_{w_E}Y$, then there exist $(x_n)_n \in E^u(X)$ and $(y_n)_n \in E_*^u(Y)$ such that*

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n$$

unconditionally converges in $X\hat{\otimes}_{w_E}Y$ and

$$w_E(u; X, Y) = \inf\Big\{\|(x_n)_n\|_{E^w(X)}\|(y_n)_n\|_{E_*^w(Y)} : u = \sum_{n=1}^{\infty} x_n \otimes y_n\Big\}.$$

Proof. Let $u \in X\hat{\otimes}_{w_E}Y$, and let $\delta > 0$ be given. Then, by Lemma 2, there exists a sequence $(u_n)_n$ in $X \otimes Y$ such that

$$\sum_{n=1}^{\infty} w_E(u_n; X, Y) \le (1+\delta)w_E(u; X, Y)$$

and $u = \sum_{n=1}^{\infty} u_n$ converges in $X\hat{\otimes}_{w_E}Y$.

We only consider the case $E = (\sum \ell_q)_p$ $(1 < p, q < \infty)$. The proofs of the other cases are similar. For every $n \in \mathbb{N}$, let

$$u_n = \sum_{i=1}^{m_n}\sum_{j=1}^{m_n} x_{ij}^n \otimes y_{ij}^n$$

be such that

$$\|((x_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n}\|_{E^w(X)}\|((y_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n}\|_{E_*^w(Y)} \le (1+\delta)w_E(u_n; X, Y).$$

We may assume that

$$\|((x_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n}\|_{E^w(X)} \le ((1+\delta)w_E(u_n; X, Y))^{1/p},$$

$$\|((y_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n}\|_{E_*^w(Y)} \le ((1+\delta)w_E(u_n;X,Y))^{1/p^*}.$$

In order to show that $u = \sum_{n=1}^{\infty}\sum_{i=1}^{m_n}\sum_{j=1}^{m_n} x_{ij}^n \otimes y_{ij}^n$ unconditionally converges in $X\hat{\otimes}_{w_E}Y$ and $(((x_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n})_n \in E^u(X)$ and $(((y_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n})_n \in E_*^u(Y)$, let $\gamma > 0$ be given. Choose an $N_\gamma \in \mathbb{N}$ so that for all $l \ge N_\gamma$,

$$w_E\Big(u - \sum_{n=1}^{l} u_n; X, Y\Big) \le \gamma \text{ and } \sum_{n \ge l} w_E(u_n;X,Y) \le \gamma.$$

Then, for all $l \ge N_\gamma$ and $1 \le a, b \le m_{l+1}$,

$$\begin{aligned}
&w_E\Big(u - \Big(\sum_{n=1}^{l} u_n + \sum_{i=1}^{a}\sum_{j=1}^{m_{l+1}} x_{ij}^{l+1}\otimes y_{ij}^{l+1} + \sum_{j=1}^{b} x_{(a+1)j}^{l+1}\otimes y_{(a+1)j}^{l+1}\Big); X, Y\Big)\\
&\le \gamma + w_E\Big(\sum_{i=1}^{a}\sum_{j=1}^{m_{l+1}} x_{ij}^{l+1}\otimes y_{ij}^{l+1} + \sum_{j=1}^{b} x_{(a+1)j}^{l+1}\otimes y_{(a+1)j}^{l+1}; X, Y\Big)\\
&\le \gamma + \|((x_{ij}^{l+1})_{j=1}^{m_{l+1}})_{i=1}^{m_{l+1}}\|_{E^w(X)}\|((y_{ij}^{l+1})_{j=1}^{m_{l+1}})_{i=1}^{m_{l+1}}\|_{E_*^w(Y)}\\
&\le \gamma + (1+\delta)w_E(u_{l+1};X,Y)\\
&\le \gamma + (1+\delta)\gamma.
\end{aligned}$$

This shows that

$$u = \sum_{n=1}^{\infty}\sum_{i=1}^{m_n}\sum_{j=1}^{m_n} x_{ij}^n \otimes y_{ij}^n$$

converges in $X\hat{\otimes}_{w_E}Y$. To show that the above series converges unconditionally, let F be an arbitrary finite subset of $\mathbb{N}$ with $\min F > \sum_{n=1}^{N_\gamma} m_n^2$, and let $\{s_k \otimes t_k\}_{k\in F}$ be the set of corresponding tensors. Then, there exists $l_1, l_2 > N_\gamma$ such that $\{s_k \otimes t_k\}_{k\in F} \subset \{\{x_{ij}^n \otimes y_{ij}^n\}_{i,j=1}^{m_n}\}_{n=l_1}^{l_2}$. We have

$$\begin{aligned}
w_E\Big(\sum_{k\in F} s_k \otimes t_k; X, Y\Big) &\le \sum_{n=l_1}^{l_2} \|((x_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n}\|_{E^w(X)}\|((y_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n}\|_{E_*^w(Y)}\\
&\le \sum_{n=l_1}^{l_2}(1+\delta)w_E(u_n;X,Y)\\
&\le (1+\delta)\gamma.
\end{aligned}$$

Since for all $l \ge N_\gamma$ and $1 \le a, b \le m_l$,

$$\begin{aligned}
&\sup_{x^*\in B_{X^*}}\Big(\Big(\sum_{j=b}^{m_l}|x^*(x_{aj}^l)|^q\Big)^{p/q} + \sum_{i=a+1}^{m_l}\Big(\sum_{j=1}^{m_l}|x^*(x_{ij}^l)|^q\Big)^{p/q} + \sum_{n\ge l+1}\sum_{i=1}^{m_n}\Big(\sum_{j=1}^{m_n}|x^*(x_{ij}^n)|^q\Big)^{p/q}\Big)^{1/p}\\
&\le \sup_{x^*\in B_{X^*}}\Big(\sum_{n\ge l}\sum_{i=1}^{m_n}\Big(\sum_{j=1}^{m_n}|x^*(x_{ij}^n)|^q\Big)^{p/q}\Big)^{1/p}\\
&\le \Big(\sum_{n\ge l}\sup_{x^*\in B_{X^*}}\sum_{i=1}^{m_n}\Big(\sum_{j=1}^{m_n}|x^*(x_{ij}^n)|^q\Big)^{p/q}\Big)^{1/p}\\
&\le \Big(\sum_{n\ge l}(1+\delta)w_E(u_n;X,Y)\Big)^{1/p} \le ((1+\delta)\gamma)^{1/p},
\end{aligned}$$

$(((x_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n})_n \in E^u(X)$ and we see that

$$\|(((x_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n})_n\|_{E^w(X)} \le \Big((1+\delta)\sum_{n=1}^{\infty} w_E(u_n;X,Y)\Big)^{1/p}.$$

Similarly,

$$(((y_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n})_n \in E_*^u(Y) \text{ and } \|(((y_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n})_n\|_{E_*^w(Y)} \le \Big((1+\delta)\sum_{n=1}^{\infty} w_E(u_n; X, Y)\Big)^{1/p^*}.$$

Consequently, the infimum

$$\inf\{\cdot\} \le \|(((x_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n})_n\|_{E^w(X)}\|(((y_{ij}^n)_{j=1}^{m_n})_{i=1}^{m_n})_n\|_{E_*^w(Y)} \le (1+\delta)^2 w_E(u; X, Y).$$

Since $\delta > 0$ was arbitrary, $\inf\{\cdot\} \le w_E(u; X, Y)$.
For every such representation

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n$$

unconditionally converging in $X\hat{\otimes}_{w_E} Y$,

$$\begin{aligned} w_E(u; X, Y) &= \lim_{l\to\infty} w_E\Big(\sum_{n=1}^{l} x_n \otimes y_n\Big) \\ &\le \lim_{l\to\infty} \|(x_n)_{n=1}^{l}\|_{E^w(X)}\|(y_n)_{n=1}^{l}\|_{E_*^w(Y)} \\ &= \|(x_n)_{n=1}^{\infty}\|_{E^w(X)}\|(y_n)_{n=1}^{\infty}\|_{E_*^w(Y)}. \end{aligned}$$

Thus, $w_E(u; X, Y) \le \inf\{\cdot\}$. □

As in the proof of Proposition 4, we have:

Proposition 5. *Suppose that $E = (\sum \ell_q)_p$ $(1 \le p, q < \infty)$, $E = (\sum c_0)_p$ $(1 \le p < \infty)$ or $E = (\sum \ell_q)_{c_0}$ $(1 \le q < \infty)$. If $u \in X\hat{\otimes}_{g_E} Y$, then there exist $(x_n)_n \in E(X)$ and $(y_n)_n \in E_*^u(Y)$ such that*

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n$$

unconditionally converges in $X\hat{\otimes}_{g_E} Y$ and

$$g_E(u; X, Y) = \inf\Big\{\|(x_n)_n\|_{E(X)}\|(y_n)_n\|_{E_*^w(Y)} : u = \sum_{n=1}^{\infty} x_n \otimes y_n\Big\}.$$

Let α be a finitely generated tensor norm. Let $\mathcal{L}(X, Y)$ be the Banach space of all operators from X to Y. The operator $j_\alpha : X^* \otimes_\alpha Y \to \mathcal{L}(X, Y)$ is defined by $j_\alpha(\sum_{n=1}^{m} x_n^* \otimes y_n) = \sum_{n=1}^{m} x_n^* \underline{\otimes} y_n$, and let

$$J_\alpha : X^*\hat{\otimes}_\alpha Y \to \mathcal{L}(X, Y)$$

be the coninuous extension of j_α. We equip $J_\alpha(X^*\hat{\otimes}_\alpha Y)$ with the quotient norm of $X^*\hat{\otimes}_\alpha Y/\ker J_\alpha$, which will be denoted by $\|\cdot\|_{J_\alpha}$. According to a well-known result of Grothendieck [16] (cf. [10], Proposition 1.5.4), if X^* or Y has the approximation property (AP), then J_α is injective; hence, $X^*\hat{\otimes}_\alpha Y$ is isometric to $(J_\alpha(X^*\hat{\otimes}_\alpha Y), \|\cdot\|_{J_\alpha})$.

Lemma 3 ([21], Theorem 2.4)**.** *Assume that X^{***} or Y has the AP.*
If $T \in J_\alpha(X^{**}\hat{\otimes}_\alpha Y) \subset \mathcal{L}(X^*, Y)$ and $T^*(Y^*) \subset X$, then $T \in \overline{J_\alpha(X \otimes Y)}^{\|\cdot\|_{J_\alpha}}$.

The prototype of the following theorem is described in [21] (Theorem 3.1).

Theorem 3. *Suppose that $E = (\sum \ell_q)_p$ $(1 \le p, q < \infty)$, $E = (\sum c_0)_p$ $(1 \le p < \infty)$ or $E = (\sum \ell_q)_{c_0}$ $(1 \le q < \infty)$. Assume that X^{***} or Y has the AP. If $T \in \mathcal{N}_E(X^*, Y)$ (respectively,*

${}_u\mathcal{N}_E(X^*, Y)$) and $T^*(Y^*) \subset X$, then there exist $(x_n)_n \in E(X)$ (respectively, $E^u(X)$) and $(y_n)_n \in E^u_*(Y)$ such that

$$T = \sum_{n=1}^{\infty} x_n \underline{\otimes} y_n$$

unconditionally converges in $\mathcal{N}_E(X^*, Y)$ (respectively, ${}_u\mathcal{N}_E(X^*, Y)$).

Proof. We only consider $\mathcal{N}_E$. The proof of the case ${}_u\mathcal{N}_E$ is similar. First, we show that $(J_{g_E}(X^{**}\hat{\otimes}_{g_E} Y), \|\cdot\|_{J_{g_E}}) = (\mathcal{N}_E(X^*, Y), \|\cdot\|_{\mathcal{N}_E})$. Let $J_{g_E}(u) \in J_{g_E}(X^{**}\hat{\otimes}_{g_E} Y)$. Let $u = \sum_{n=1}^{\infty} x_n^{**} \otimes y_n$ be an arbitrary representation in Proposition 5. Then

$$J_{g_E}(u) = \sum_{n=1}^{\infty} x_n^{**} \underline{\otimes} y_n \in \mathcal{N}_E(X^*, Y)$$

and $\|J_{g_E}(u)\|_{\mathcal{N}_E} \le \|(x_n^{**})_n\|_{E(X)} \|(y_n)_n\|_{E^w_*(Y)}$. Since the representation of u was arbitrary, $\|J_{g_E}(u)\|_{\mathcal{N}_E} \le g_E(u; X^{**}, Y) = \|J_{g_E}(u)\|_{J_{g_E}}$.

Let $T \in \mathcal{N}_E(X^*, Y)$ and let $\delta > 0$ be given. Let $T = \sum_{n=1}^{\infty} x_n^{**} \underline{\otimes} y_n$ be an arbitrary $\mathcal{N}_E$-representation. Since

$$\begin{aligned} g_E\Big(\sum_{n=m}^{l} x_n^{**} \otimes y_n; X^*, Y\Big) &\le \|(x_n^{**})_{n=m}^{l}\|_{E(X)} \|(y_n)_{n=m}^{l}\|_{E^w_*(Y)} \\ &\le \|(y_n)_n\|_{E^w_*(Y)} \Big\| \sum_{n=m}^{l} \|x_n^{**}\| e_n \Big\|_E, \end{aligned}$$

$\sum_{n=1}^{\infty} x_n^{**} \otimes y_n$ converges in $X^{**}\hat{\otimes}_{g_E} Y$. Thus,

$$T = J_{g_E}\Big(\sum_{n=1}^{\infty} x_n^{**} \otimes y_n\Big) \in J_{g_E}(X^{**}\hat{\otimes}_{g_E} Y).$$

Choose an $l \in \mathbb{N}$ so that $g_E(\sum_{n>l} x_n^{**} \otimes y_n; X^{**}, Y) \le \delta$. Then, we have

$$\begin{aligned} \|T\|_{J_{g_E}} &= g_E\Big(\sum_{n=1}^{\infty} x_n^{**} \otimes y_n; X^{**}, Y\Big) \\ &\le g_E\Big(\sum_{n=1}^{l} x_n^{**} \otimes y_n; X^{**}, Y\Big) + \delta \\ &\le \|(x_n^{**})_n\|_{E(X)} \|(y_n)_n\|_{E^w_*(Y)} + \delta. \end{aligned}$$

Since the representation of T was arbitrary, $\|T\|_{J_{g_E}} \le \|T\|_{\mathcal{N}_E}$.

Now, let $T \in \mathcal{N}_E(X^*, Y)$. Choose $u \in X^{**}\hat{\otimes}_{g_E} Y$ so that $T = J_{g_E}(u)$. By Lemma 3, $J_{g_E}(u) \in \overline{J_{g_E}(X \otimes Y)}^{\|\cdot\|_{J_{g_E}}}$. Since J_{g_E} is an isometry and $X\hat{\otimes}_{g_E} Y$ is isometrically embeded in $X^{**}\hat{\otimes}_{g_E} Y$ (cf. [3], Proposition 6.4), we see that $u \in X\hat{\otimes}_{g_E} Y$. By Proposition 5, there exist $(x_n)_n \in E(X)$ and $(y_n)_n \in E^u_*(Y)$ such that $u = \sum_{n=1}^{\infty} x_n \otimes y_n$ unconditionally converges in $X\hat{\otimes}_{g_E} Y$. Hence,

$$T = J_{g_E}(u) = \sum_{n=1}^{\infty} x_n \underline{\otimes} y_n$$

unconditionally converges in $\mathcal{N}_E(X^*, Y)$. □

4. Discussion

This work is the general and natural extension of some results about the tensor norms g_p and w_p. There have been many more investigations about g_p and w_p since their introduction. We expect that several more results on g_p and w_p, and the ideals of p-nuclear and p-compact operators, can be developed. For instance, for a finitely generated tensor norm α, a Banach space X is said to have the *α-approximation property* (α-AP) if for every Banach space Y, the natural map

$$J_\alpha : Y\hat{\otimes}_\alpha X \longrightarrow Y\hat{\otimes}_\varepsilon X$$

is injective (cf. [2]), Section 21.7. The g_p-AP and the w_p-AP were well studied, and the g_p-AP (respectively, w_p-AP) is closely related with an approximation property of the *ideal of p-summing operators* (respectively, *ideal of p-dominated operators*) (cf. [11]). We can consider the g_E-AP and the w_E-AP as the following subjects:

1. An investigation of the ideals of E-summing operators and E-dominated operators;
2. Some relationships of the ideals of E-summing operators and E-dominated operators, respectively, between the g_E-AP and the w_E-AP, respectively.

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