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Some Geometrical Results Associated with Secant Hyperbolic Functions

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Abstract: In this paper, we examine the differential subordination implication related with the Janowski and secant hyperbolic functions. Furthermore, we explore a few results, for example, the necessary and sufficient condition in light of the convolution concept, growth and distortion bounds, radii of starlikeness and partial sums related to the class $\mathcal{S}^*_{\text{sech}}$.

Keywords: univalent functions; subordination; analytic functions; secant hyperbolic function; Janowski function

MSC: 30C45; 30C50

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1. Introduction and Preliminaries

Let $\mathcal{H}(c,n)$ be the class of analytic functions $f(\zeta)$ having the series form

$$f(\zeta) = c + c_n\zeta^n + c_{n+1}\zeta^{n+1} + c_{n+2}\zeta^{n+2} + c_{n+3}\zeta^{n+3} + \dots, \quad \zeta \in \mho := \{\zeta \in \mathbb{C} : |\zeta| < 1\}. \tag{1}$$

We denote $\mathcal{H}(0,1)$ with $c_1 = 1$ by $\mathcal{H}$ and $\mathcal{H}(1,1)$ by $\mathcal{P}$. Let $\mathcal{S}$ denote the subclasses of $\mathcal{H}$ consisting of functions that are univalent in $\mho$. We say $f(\zeta) \in \mathcal{H}$ is subordinate to $g(\zeta) \in \mathcal{H}$ (written as $f \prec g$ or $f(\zeta) \prec g(\zeta)$) if there exists a Schwarz function $w(\zeta)$ such that $f(\zeta) = g(w(\zeta))$ for all $\zeta \in \mho$. For $f, g \in \mathcal{H}$ with $f(\zeta) = \zeta + \sum\limits_{n=2}^{\infty} a_n\zeta^n$ and $g(\zeta) = \zeta + \sum\limits_{n=2}^{\infty} b_n\zeta^n$, the convolution of f and g depicted by $f(\zeta) * g(\zeta)$ is defined in [1] as:

$$f(\zeta) * g(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n b_n \zeta^n, \quad \zeta \in \mho$$

Let $\mathcal{P}(\mathcal{A},\mathcal{B})$ denotes the class of all functions $p(\zeta)$ such that $p(\zeta) \prec \frac{1+\mathcal{A}\zeta}{1+\mathcal{B}\zeta}$, $\zeta \in \mho$, $-1 \le \mathcal{B} < \mathcal{A} \le 1$. Equivalently, $p \in \mathcal{P}(A,B)$ if and only if $p(\zeta)$ satisfies the following inequality:

$$\left| \frac{p(\zeta) - 1}{\mathcal{A} - \mathcal{B}p(\zeta)} \right| \le 1, \quad \zeta \in \mho. \tag{2}$$

For $f \in \mathcal{H}$, if we choose $p(\zeta) = \zeta f'(\zeta)/f(\zeta)$ in (2), then $f \in \mathcal{S}^*(A,B)$. In particular, if $\mathcal{A} = 1 - 2\sigma$, $\mathcal{B} = -1$, the class $\mathcal{S}^*(A,B)$ reduces to the class $\mathcal{S}^*(\sigma)$ of starlike function of order σ.

The class $\mathcal{S}$ is one of the most vital categories of Geometric functions theory due to its wide applications in sciences and engineering, such as in the study of ODEs and

PDEs, operators' theory and image processing techniques. Most subclasses of S emanated in an attempt to solve the great Bieberbach conjecture, and these were classified based on the geometries of their image domains. For example, the subclass $\mathcal{S}^*$ of $\mathcal{S}$ consists of those functions $f \in \mathcal{H}$ that map $\mho$ onto a starlike domain, whereas those that map $\mho$ onto a convex domain are denoted by $\mathcal{C}$. These functions are known as starlike and convex functions, respectively.

In 1992, Ma and Minda [2] gave a unified characterization of the subclasses of $\mathcal{S}^*$. For this reason, they considered analytic functions $\varphi(\zeta)$ with $\mathrm{Re}\varphi(\zeta) > 0$ in $\mho$ and normalized by $\varphi(0) = 1$ and $\varphi'(0) > 0$. Thus, the Ma and Minda class of starlike functions denoted by $\mathcal{S}^*(\varphi(\zeta))$ was defined by the subordination

$$\frac{\zeta f'(\zeta)}{f(\zeta)} \prec \varphi(\zeta), f \in A,\ \zeta \in \mho.$$

Many known and new subclasses of $\mathcal{S}$ whose image domains have nice geometries can be obtained by specializing the superordinate function $\varphi(\zeta)$. For example, if

(a) $\varphi(\zeta) = \frac{1+\zeta}{1-\zeta}$, we are led to the well-konwn class of starlike functions.
(b) $\varphi(\zeta) = \frac{1+\mathcal{A}\zeta}{1-\mathcal{B}\zeta}$, $\mathcal{S}^*(\varphi(\zeta))$ reduces to the class of Janowski starlike function introduced and studied by Janowski [3].
(c) $\varphi(\zeta) = \sqrt{1+\zeta}$, we have the class $\mathcal{S}^*_{\mathcal{L}}$, which illustrates the starlike functions mapping $\mho$ onto a region bounded by lemniscate of Bernoulli in right half plan, and was introduced by Sokół and Stankiewic [4].
(d) $\varphi(\zeta) = \zeta + \sqrt{1+\zeta^2}$, we have the class $\mathcal{S}^*_{\mathcal{CR}}$ of functions mapping $\mho$ onto a region bounded by crescent domains, and was introduced by Raina and Sokół [5].
(e) $\varphi(\zeta) = (1+s\zeta)^2$, $0 < s \leq \frac{\sqrt{2}}{2}$, then the class $\mathcal{S}^*(\varphi)$ reduces to the class of starlike limaçon functions, which was developed and examined by Masih and Kanas [6].

For more information for other choices of $\varphi(\zeta)$, we refer to [7] (p. 6), and [8] (p. 2). Furthermore, in recent times, the choice of $\varphi(\zeta)$ has been extended to trigonometry and hyperbolic functions. In this direction, for the choice of $\varphi(\zeta) = \mathrm{e}^{\zeta}, \cos(\zeta), 1+\sin(\zeta)$, $1+\sinh^{-1}(\zeta)$ and $\cosh(\zeta)$, Mendiratta et al. [9], Bano and Mohsa [10], Cho et al. [11], Kumar and Arora [12] and Alotaibi et al. [13] developed and examined the respective subclasses $\mathcal{S}^*_{\mathrm{e}}, \mathcal{S}^*_{\cos}, \mathcal{S}^*_{\sin}, \mathcal{S}^*_{\sinh^{-1}}$ and $\mathcal{S}^*_{\cosh}$ of starlike functions. In a more recent article by Bano and Mohsan [14], the choice of secant hyperbolic function was unvailed and the geometric properties such as the structural formula, inclusion results, and some sharp radii of convexity and Janowski starlikeness associated with the class

$$\mathcal{S}^*_{\mathrm{sech}} = \left\{ f \in \mathcal{H} \colon \frac{\zeta f'(\zeta)}{f(\zeta)} \prec \mathrm{sech}(\zeta),\ \zeta \in \mho \right\} \tag{3}$$

were discussed.

In the light of these studies by Bano and Mohsan [14] and Alotaibi et al. [13], we study the differential subordination implication related to the Janowski and secant hyperbolic functions. Moreover, we examine a few geometric characterization of this function, such as necessary and sufficient conditions with the concept of convolution, growth and distortion bounds, radii of starlikeness and partial sums.

The following Jack's Lemma is significant to establish our findings.

Lemma 1 ([15])**.** *Let $w(\zeta)$ be analytic in $\mho$ with $w(0) = 0$. If $|w(\zeta)|$ attains its maximum value on the circle $|\zeta| = r$ at a point $\zeta_0 \in \mho$, then we have $\zeta_0 w'(\zeta_0) = kw(\zeta_0)$, for some $k \geq 1$.*

In the subsequent sections, we assume the analytic function $p \in \mathcal{P}$, state and prove the main results of this current work.

2. Sufficient Conditions Related with sech(ζ)

Theorem 1. *Let $-1 \leq \mathcal{B} < \frac{\mathrm{sech}(1)\tanh(1)}{\sec(1)\tan(1)} \leq \mathcal{A} \leq 1$ and suppose*

$$1+\beta\zeta p'(\zeta) \prec \frac{1+\mathcal{A}\zeta}{1+\mathcal{B}\zeta}, \quad \zeta \in \mho. \tag{4}$$

If

$$|\beta| > \frac{\mathcal{A}-\mathcal{B}}{\operatorname{sech}(1)\tanh(1) - |\mathcal{B}|\sec(1)\tan(1)}, \tag{5}$$

then

$$p(\zeta) \prec \operatorname{sech}(\zeta), \quad \zeta \in \mho.$$

Proof. Let $p(\zeta) = \operatorname{sech}(w(\zeta))$, where $w(\zeta)$ is analytic in $\mho$ with $w(0)=0$. Let

$$\mathcal{R}_1(\zeta) = 1+\beta\zeta p'(\zeta) = 1 - \beta\zeta w'(\zeta)\operatorname{sech}(w(\zeta))\tanh(w(\zeta)).$$

Then

$$\left|\frac{\mathcal{R}_1(\zeta)-1}{\mathcal{A}-\mathcal{B}\mathcal{R}_1(\zeta)}\right| = \left|\frac{\beta\zeta w'(\zeta)\operatorname{sech}(w(\zeta))\tanh(w(\zeta))}{\mathcal{A}-\mathcal{B}+\mathcal{B}\beta\zeta w'(\zeta)\operatorname{sech}(w(\zeta))\tanh(w(\zeta))}\right|.$$

To achieve our goal, we have to prove that $|w(\zeta)|<1$ in $\mho$. On the contrary, assume $\zeta_0 \in \mho$ such that $\max\limits_{|\zeta|\le|\zeta_0|} |w(\zeta)| = |w(\zeta_0)| = 1$. By Lemma 1, there exists $k \ge 1$ such that $\zeta_0 w'(\zeta_0) = k w(\zeta_0)$. Let $w(\zeta_0) = e^{i\theta}$ for $\theta \in [0,\pi]$. Then

$$\begin{aligned}\left|\frac{\mathcal{R}_1(\zeta_0)-1}{\mathcal{A}-\mathcal{B}\mathcal{R}_1(\zeta_0)}\right| =& \left|\frac{\beta\, e^{i\theta}\, w'(e^{i\theta})\operatorname{sech}(e^{i\theta})\tanh(e^{i\theta})}{\mathcal{A}-\mathcal{B}+\mathcal{B}\beta\, e^{i\theta}\, w'(e^{i\theta})\operatorname{sech}(e^{i\theta})\tanh(e^{i\theta})}\right| \\ \ge& \frac{|\beta|k|\operatorname{sech}(e^{i\theta})|\,|\tanh(e^{i\theta})|}{\mathcal{A}-\mathcal{B}+k|\beta|\,|\mathcal{B}|\,|\operatorname{sech}(e^{i\theta})|\,|\tanh(e^{i\theta})|}.\end{aligned} \tag{6}$$

A direct computation gives that

$$\begin{aligned}|\operatorname{sech}(e^{i\theta})|^2 =& \left|\frac{\cos(\sin\theta)\cosh(\cos\theta)}{\sinh^2(\cos\theta)+\cos^2(\sin\theta)} - i\frac{\sin(\sin\theta)\sinh(\cos\theta)}{\sinh^2(\cos\theta)+\cos^2(\sin\theta)}\right|^2 \\ =& \frac{1}{\cosh^2(\cos\theta)+\cos^2(\sin\theta)-1} \\ :=& \phi_1(\theta)\end{aligned}$$

and

$$\begin{aligned}|\tanh(e^{i\theta})|^2 =& \left|\frac{\sinh(\cos\theta)\cosh(\cos\theta)}{\sinh^2(\cos\theta)+\cos^2(\sin\theta)} - i\frac{\sin(\sin\theta)\cos(\sin\theta)}{\sinh^2(\cos\theta)+\cos^2(\sin\theta)}\right|^2 \\ =& \frac{\cosh^2(\cos\theta)-\cos^2(\sin\theta)}{\cosh^2(\cos\theta)+\cos^2(\sin\theta)-1} \\ :=& \phi_2(\theta).\end{aligned}$$

Since $\phi_i(-\theta) = \phi_i(\theta)$ for $i=1,2$, we consider $\theta \in [0,\pi]$. Then

$$\begin{aligned}\max\{\phi_1(\theta)\} =& \phi_1\left(\frac{\pi}{2}\right) = \sec^2(1) \\ \min\{\phi_1(\theta)\} =& \phi_1(0) = \phi_1(\pi) = \operatorname{sech}^2(1) \\ \max\{\phi_2(\theta)\} =& \phi_2\left(\frac{\pi}{2}\right) = \tan^2(1) \\ \min\{\phi_2(\theta)\} =& \phi_2(0) = \phi_2(\pi) = \tanh^2(1)\end{aligned}$$

Therefore,

$$\operatorname{sech}(1) \le |\operatorname{sech}(e^{i\theta})| \le \sec(1) \tag{7}$$

and

$$\tanh(1) \leq |\tanh(e^{i\theta})| \leq \tan(1). \tag{8}$$

On the account of (7) and (8) in (6), we have

$$\left|\frac{\mathcal{R}_1(\zeta_0)-1}{\mathcal{A}-\mathcal{B}\mathcal{R}_1(\zeta_0)}\right| \geq \frac{|\beta|k\ \mathrm{sech}(1)\tanh(1)}{\mathcal{A}-\mathcal{B}+k|B||\beta|k\ \sec(1)\tan(1)} :=\phi(k).$$

Then

$$\phi'(k) = \frac{(\mathcal{A}-\mathcal{B})|\beta|\,\mathrm{sech}(1)\tanh(1)}{(\mathcal{A}-\mathcal{B}+|\beta||B|k\ \sec(1)\tan(1))^2}.$$

This shows that $\phi(k)$ is an increasing function of $k \in [0,\infty)$. Thus, $\phi(k) \geq \phi(1)$. Therefore,

$$\left|\frac{\mathcal{R}_1(\zeta_0)-1}{\mathcal{A}-\mathcal{B}\mathcal{R}_1(\zeta_0)}\right| \geq \frac{|\beta|\,\mathrm{sech}(1)\tanh(1)}{\mathcal{A}-\mathcal{B}+|B||\beta|k\ \sec(1)\tan(1)} > 1,$$

where we have used (5). This contradicts the hypothesis of the Theorem. Hence, there is no $\zeta_0 \in \mho$ such that $|w(\zeta_0)| = 1$. So, $|w(\zeta)| < 1$ for all $\zeta \in \mho$. This proves the Theorem. □

Theorem 2. *Let* $-1 \leq \mathcal{B} < \frac{\tanh(1)}{\tan(1)} \leq \mathcal{A} \leq 1$ *and suppose*

$$1+\beta\frac{\zeta p'(\zeta)}{p(\zeta)} \prec \frac{1+\mathcal{A}\zeta}{1+\mathcal{B}\zeta}, \quad \zeta \in \mho. \tag{9}$$

If

$$|\beta| > \frac{\mathcal{A}-\mathcal{B}}{\tanh(1)-|\mathcal{B}|\tan(1)}, \tag{10}$$

then

$$p(\zeta) \prec \mathrm{sech}(\zeta), \quad \zeta \in \mho.$$

Proof. Let $p(\zeta) = \mathrm{sech}(w(\zeta))$, where $w(\zeta)$ is analytic in $\mho$ with $w(0) = 0$. Let

$$\mathcal{R}_2(\zeta) = 1+\beta\frac{\zeta p'(\zeta)}{p(\zeta)} = 1-\beta\zeta w'(\zeta)\tanh(w(\zeta)).$$

Then

$$\left|\frac{\mathcal{R}_2(\zeta)-1}{\mathcal{A}-\mathcal{B}\mathcal{R}_2(\zeta)}\right| = \left|\frac{\beta\zeta w'(\zeta)\tanh(w(\zeta))}{\mathcal{A}-\mathcal{B}+\mathcal{B}\beta\zeta w'(\zeta)\tanh(w(\zeta))}\right|.$$

Let $\zeta_0 \in \mho$ such that $\max_{|\zeta|\leq|\zeta_0|}|w(\zeta)| = |w(\zeta_0)| = 1$. Then in view of Lemma 1, there exists $k \geq 1$ such that $\zeta_0 w'(\zeta_0) = kw(\zeta_0)$. Let $w(\zeta_0) = e^{i\theta}$ for $\theta \in [0,\pi]$. Then

$$\begin{aligned}\left|\frac{\mathcal{R}_2(\zeta_0)-1}{\mathcal{A}-\mathcal{B}\mathcal{R}_2(\zeta_0)}\right| &= \left|\frac{\beta\, e^{i\theta}\, w'(e^{i\theta})\tanh(e^{i\theta})}{\mathcal{A}-\mathcal{B}+\mathcal{B}\beta\, e^{i\theta}\, w'(e^{i\theta})\tanh(e^{i\theta})}\right| \\ &\geq \frac{|\beta|k|\tanh(e^{i\theta})|}{\mathcal{A}-\mathcal{B}+k|\beta|\,|\mathcal{B}|\,|\tanh(e^{i\theta})|} \\ &\geq \frac{|\beta|k\ \tanh(1)}{\mathcal{A}-\mathcal{B}+|\beta|\,|\mathcal{B}|\,k\tan(1)},\end{aligned} \tag{11}$$

where we have used (7) and (8). It is easy to see that the right side of the inequality (11) is an increasing function of $k \in [1,\infty)$. So,

$$\left|\frac{\mathcal{R}_2(\zeta_0)-1}{\mathcal{A}-\mathcal{B}\mathcal{R}_2(\zeta_0)}\right| \geq \frac{|\beta|\tanh(1)}{\mathcal{A}-\mathcal{B}+|\beta|\,|\mathcal{B}|\ \tan(1)} \geq 1,$$

provided (10) holds. This contradicts the assumption of the Theorem. Hence, we obtain our result. □

Theorem 3. *Let* $-1 \leq \mathcal{B} < \frac{\tanh(1)}{\tan(1)} \leq \mathcal{A} \leq 1$ *and assume*

$$1 + \beta \frac{\zeta p'(\zeta)}{p^2(\zeta)} \prec \frac{1 + \mathcal{A}\zeta}{1 + \mathcal{B}\zeta}, \quad \zeta \in \mho. \tag{12}$$

If

$$|\beta| > \frac{(\mathcal{A} - \mathcal{B}) \operatorname{sec}(1)}{\tanh(1) - |\mathcal{B}| \tan(1)}, \tag{13}$$

then

$$p(\zeta) \prec \operatorname{sech}(\zeta), \quad \zeta \in \mho.$$

Proof. Let $p(\zeta) = \operatorname{sech}(w(\zeta))$, where $w(\zeta)$ is analytic in $\mho$ with $w(0) = 0$. Consider

$$\mathcal{R}_3(\zeta) = 1 + \beta \frac{\zeta p'(\zeta)}{p(\zeta)} = 1 - \frac{\beta \zeta w'(\zeta) \tanh(w(\zeta))}{\operatorname{sech}(w(\zeta))}.$$

Then

$$\left| \frac{\mathcal{R}_3(\zeta) - 1}{\mathcal{A} - \mathcal{B}\mathcal{R}_3(\zeta)} \right| = \left| \frac{\beta \zeta w'(\zeta) \tanh(w(\zeta))}{(\mathcal{A} - \mathcal{B}) \operatorname{sech}(w(\zeta)) + \mathcal{B}\beta \zeta w'(\zeta) \tanh(w(\zeta))} \right|.$$

Let $\zeta_0 \in \mho$ such that $\max_{|\zeta| \leq |\zeta_0|} |w(\zeta)| = |w(\zeta_0)| = 1$. Then by Lemma 1, there exists $k \geq 1$ such that $\zeta_0 w'(\zeta_0) = k w(\zeta_0)$. Let $w(\zeta_0) = \mathrm{e}^{i\theta}$ for $\theta \in [0, \pi]$. Then using (7) and (8), we have

$$\begin{aligned}
\left| \frac{\mathcal{R}_3(\zeta_0) - 1}{\mathcal{A} - \mathcal{B}\mathcal{R}_3(\zeta_0)} \right| &= \left| \frac{\beta \, \mathrm{e}^{i\theta} \, w'(\mathrm{e}^{i\theta}) \tanh(\mathrm{e}^{i\theta})}{(\mathcal{A} - \mathcal{B}) \operatorname{sech}(\mathrm{e}^{i\theta}) + \mathcal{B}\beta \, \mathrm{e}^{i\theta} \, w'(\mathrm{e}^{i\theta}) \tanh(\mathrm{e}^{i\theta})} \right| \\
&\geq \frac{|\beta| k |\tanh(\mathrm{e}^{i\theta})|}{(\mathcal{A} - \mathcal{B}) |\operatorname{sech}(\mathrm{e}^{i\theta})| + k |\beta| \, |\mathcal{B}| \, |\tanh(\mathrm{e}^{i\theta})|} \\
&\geq \frac{|\beta| k \tanh(1)}{(\mathcal{A} - \mathcal{B}) \operatorname{sec}(1) + |\beta| \, |\mathcal{B}| \, k \tan(1)}
\end{aligned} \tag{14}$$

It is not difficult to see that the right side of the inequality (14) is an increasing function of $k \in [1, \infty)$. So,

$$\left| \frac{\mathcal{R}_3(\zeta_0) - 1}{\mathcal{A} - \mathcal{B}\mathcal{R}_3(\zeta_0)} \right| \geq \frac{|\beta| \tanh(1)}{(\mathcal{A} - \mathcal{B}) \operatorname{sec}(1) + |\beta| \, |\mathcal{B}| \, \tan(1)} \geq 1,$$

where we have used (13). This contradicts the assumption of the Theorem. Hence, the result is proved. □

Consider the Alexander integral operator

$$F(\zeta) = \int_0^z \frac{h(s)}{s} ds, \quad h \in \mathcal{H}. \tag{15}$$

This operator was the first integral operator known in the field of Geometric function theory. It is very important and resourceful in studying many geometric properties of several subclasses of univalent functions. Therefore, in the next Theorem, we examine the differential subordination of the sectant hyperbolic function under this integral transformation.

Theorem 4. *Let* $-1 \leq \mathcal{B} < \frac{\tanh(1) - \operatorname{sech}(1)}{\tan(1) + \sec(1)} \leq \mathcal{A} \leq 1$ *and suppose*

$$1+\beta\frac{zh'(\zeta)}{h(\zeta)}\prec\frac{1+\mathcal{A}\zeta}{1+\mathcal{B}\zeta},\quad \zeta\in\mho. \tag{16}$$

If

$$|\beta|>\frac{\mathcal{A}-\mathcal{B}}{\tanh(1)-\operatorname{sech}(1)-|\mathcal{B}|(\tan(1)+\sec(1))}, \tag{17}$$

then

$$\frac{\zeta F'(\zeta)}{F(\zeta)}\prec\operatorname{sech}(\zeta),\quad \zeta\in\mho.$$

Proof. Let $\frac{\zeta F'(\zeta)}{F(\zeta)}=\operatorname{sech}(w(\zeta))$ with $w(\zeta)$ analytic in $\mho$ such that $w(0)=0$. To achieve the aim of this Theorem, we need to establish that $|w(\zeta)|<1$ in $\mho$. From the integral transformation (15), we have

$$h(\zeta)=\zeta F'(\zeta),$$

and logarithmic differentiation gives

$$\begin{aligned}\frac{zh'(\zeta)}{h(\zeta)}&=\frac{\zeta F'(\zeta)}{F(\zeta)}+\frac{z\left(\frac{\zeta F'(\zeta)}{F(\zeta)}\right)'}{\frac{\zeta F'(\zeta)}{F(\zeta)}}\\&=-\left(\zeta w'(\zeta)\tanh(w(\zeta))-\operatorname{sech}(w(\zeta))\right).\end{aligned}$$

Let

$$\mathcal{R}_4(\zeta)=1+\beta\frac{zh'(\zeta)}{h(\zeta)}=1-\beta\left(\zeta w'(\zeta)\tanh(w(\zeta))-\operatorname{sech}(w(\zeta))\right).$$

Then

$$\left|\frac{\mathcal{R}_4(\zeta)-1}{\mathcal{A}-\mathcal{B}\mathcal{R}_4(\zeta)}\right|=\left|\frac{\beta(\zeta w'(\zeta)\tanh(w(\zeta))-\operatorname{sech}(w(\zeta)))}{\mathcal{A}-\mathcal{B}+\mathcal{B}\beta\left(\zeta w'(\zeta)\tanh(w(\zeta))-\operatorname{sech}(w(\zeta))\right)}\right|.$$

Let $\zeta_0\in\mho$ such that $\max\limits_{|\zeta|\le|\zeta_0|}|w(\zeta)|=|w(\zeta_0)|=1$. Then by Lemma 1, there exists $k\ge1$ such that $\zeta_0 w'(\zeta_0)=kw(\zeta_0)$. Let $w(\zeta_0)=e^{i\theta}$ for $\theta\in[0,\pi]$. Then using (7) and (8), we have

$$\begin{aligned}\left|\frac{\mathcal{R}_4(\zeta_0)-1}{\mathcal{A}-\mathcal{B}\mathcal{R}_4(\zeta_0)}\right|&\ge\left|\frac{\beta\left(k\,e^{i\theta}\tanh(e^{i\theta})-\operatorname{sech}(e^{i\theta})\right)}{\mathcal{A}-\mathcal{B}+\mathcal{B}\beta\left(k\,e^{i\theta}\tanh(e^{i\theta})-\operatorname{sech}(e^{i\theta})\right)}\right|\\&\ge\frac{|\beta|\left(k|\tanh(e^{i\theta})|-|\operatorname{sech}(e^{i\theta})|\right)}{\mathcal{A}-\mathcal{B}+|\beta||\mathcal{B}|\left(k|\tanh(e^{i\theta})|-|\operatorname{sech}(e^{i\theta})|\right)}\\&\ge\frac{|\beta|(k\tanh(1)-\sec(1))}{\mathcal{A}-\mathcal{B}+|\beta||\mathcal{B}|(k\tan(1)-\sec(1))}\\&:=\phi(k),\end{aligned}$$

where $\phi(k)$ is an increasing function of $k\in[1,\infty)$. Therefore, $\phi(k)\ge\phi(1)$, and so,

$$\left|\frac{\mathcal{R}_4(\zeta_0)-1}{\mathcal{A}-\mathcal{B}\mathcal{R}_4(\zeta_0)}\right|\ge\frac{|\beta|(\tanh(1)-\sec(1))}{\mathcal{A}-\mathcal{B}+|\beta||\mathcal{B}|(\tan(1)-\sec(1))}>1$$

when we used (17). This is a contradiction, and hence, the proof is completed. □

Setting $p(\zeta)=\zeta f'(\zeta)/f(\zeta)$ in Theorems 1–3, we arrive at the following results.

Corollary 1. *Let $f\in\mathcal{H}$. Then each of the following is sufficient for $f\in\mathbb{S}^*_{sech}$:*

(a)

$$1+\beta\frac{\zeta f'(\zeta)}{f(\zeta)}\left(\frac{(\zeta f'(\zeta))'}{f'(\zeta)}-\frac{\zeta f'(\zeta)}{f(\zeta)}\right)\prec\frac{1+\mathcal{A}\zeta}{1+\mathcal{B}\zeta},\zeta\in\mho,\ -1\leq\mathcal{B}<\frac{\operatorname{sech}(1)\tanh(1)}{\sec(1)\tan(1)}\leq\mathcal{A}\leq1, \tag{18}$$

for

$$|\beta|>\frac{\mathcal{A}-\mathcal{B}}{\operatorname{sech}(1)\tanh(1)-|\mathcal{B}|\sec(1)\tan(1)}.$$

(b)

$$1+\beta\left(\frac{(\zeta f'(\zeta))'}{f'(\zeta)}-\frac{\zeta f'(\zeta)}{f(\zeta)}\right)\prec\frac{1+\mathcal{A}\zeta}{1+\mathcal{B}\zeta},\ \zeta\in\mho,\ -1\leq\mathcal{B}<\frac{\tanh(1)}{\tan(1)}\leq\mathcal{A}\leq1 \tag{19}$$

for

$$|\beta|>\frac{\mathcal{A}-\mathcal{B}}{\tanh(1)-|\mathcal{B}|\tan(1)}.$$

(c)

$$1+\beta\frac{f(\zeta)}{\zeta f'(\zeta)}\left(\frac{(\zeta f'(\zeta))'}{f'(\zeta)}-\frac{\zeta f'(\zeta)}{f(\zeta)}\right)\prec\frac{1+\mathcal{A}\zeta}{1+\mathcal{B}\zeta},\ \zeta\in\mho,\ -1\leq\mathcal{B}<\frac{\tanh(1)}{\tan(1)}\leq\mathcal{A}\leq1 \tag{20}$$

for

$$|\beta|>\frac{(\mathcal{A}-\mathcal{B})\sec(1)}{\tanh(1)-|\mathcal{B}|\tan(1)}.$$

As we set $\mathcal{A}=1$ and $\mathcal{B}=0$ in Corollary 1, we are led to the following results, respectively:

Corollary 2. *Let $f\in\mathcal{H}$. Then, each of the following is sufficient for $f\in\mathbb{S}^*_{\text{sech}}$:*

(a)

$$1+\beta\frac{\zeta f'(\zeta)}{f(\zeta)}\left(\frac{(\zeta f'(\zeta))'}{f'(\zeta)}-\frac{\zeta f'(\zeta)}{f(\zeta)}\right)\prec1+\zeta,\zeta\in\mho,$$

for

$$|\beta|>\frac{1}{\operatorname{sech}(1)\tanh(1)}.$$

(b)

$$1+\beta\left(\frac{(\zeta f'(\zeta))'}{f'(\zeta)}-\frac{\zeta f'(\zeta)}{f(\zeta)}\right)\prec1+\zeta,\ \zeta\in\mho,$$

for

$$|\beta|>\frac{1}{\tanh(1)}.$$

(c)

$$1+\beta\frac{f(\zeta)}{\zeta f'(\zeta)}\left(\frac{(\zeta f'(\zeta))'}{f'(\zeta)}-\frac{\zeta f'(\zeta)}{f(\zeta)}\right)\prec1+\zeta,\ \zeta\in\mho,$$

for

$$|\beta|>\frac{\operatorname{sech}(1)}{\tanh(1)}.$$

Corollary 3. *The following is sufficient for the transformation (15) to be a member of the secant hyperbolic class when $\mathcal{A}=0$ and $\mathcal{B}=-1$*

$$1+\beta\frac{zh'(\zeta)}{h(\zeta)}\prec\frac{1}{1-\zeta},\ \zeta\in\mho,$$

for

$$|\beta|>\frac{1}{2\sec(1)+\tan(1)-\tanh(1)}.$$

In view of Theorems 1–3 and Corollary 1, we conclude this section by the following remark:

Remark 1. *If for a real* β *such that* $\beta > \frac{2}{\operatorname{sech}(1)\tanh(1)-\sec(1)\tan(1)} \approx -0.83724$, *a function* $p \in \mathcal{P}$ *satisfies*

$$1+\beta\frac{\zeta p'(\zeta)}{(p(\zeta))^n} \prec \frac{1+\zeta}{1-\zeta}, \ n=0,1,2,$$

then $p(\zeta) \prec \operatorname{sech}(\zeta)$, *and hence for the same* β, *if* $f \in \mathcal{H}$ *satisfies any of the conditions given by* (18), (19) *and* (20), *then* $f \in \mathcal{S}^*_{\mathrm{sech}}$.

3. Convolution Properties

In this section, we prove the convolution conditions for the analytic functions $f \in \mathcal{S}^*_{\mathrm{sech}}$.

Theorem 5. *Let* $f \in \mathcal{H}$. *Then*

$$f \in \mathcal{S}^*_{\mathrm{sech}} \Longleftrightarrow \frac{1}{\zeta}\left(f(\zeta) * \frac{\zeta-\mu\zeta^2}{(1-\zeta)^2}\right) \neq 0, \tag{21}$$

for $\mu = \frac{\operatorname{sech}(e^{i\theta})}{\operatorname{sech}(e^{i\theta})-1}$, *as well as* $\mu = 1$.

Proof. Let $f \in \mathcal{S}^*_{\mathrm{sech}}$. Then $f(\zeta)$ is analytic in $\mho$, and so $f(\zeta)/\zeta \neq 0$ in $\mho$. This proves the case $\mu = 1$. On the other hand, there exists $w(\zeta)$ analytic in $\mho$ with $w(0) = 0$ and $|w(\zeta)| < 1$ in $\mho$ such that $\frac{\zeta f'(\zeta)}{f(\zeta)} = \operatorname{sech}(w(\zeta))$. This is equivalent to $\frac{\zeta f'(\zeta)}{f(\zeta)} \neq \operatorname{sech}(e^{i\theta})$, $0 \leq \theta \leq 2\pi$. That is

$$\begin{aligned}
0 \neq& \frac{1}{\zeta}\left(\zeta f'(\zeta) - f(\zeta)\operatorname{sech}(e^{i\theta})\right)\\
=&\frac{1}{\zeta}\left(f(\zeta) * \frac{\zeta}{(1-\zeta)^2} - f(\zeta) * \frac{\zeta}{1-\zeta}\operatorname{sech}(e^{i\theta})\right)\\
=&\frac{1}{\zeta}\left[f(\zeta) * \left(\frac{\zeta}{(1-\zeta)^2} - \frac{\zeta}{1-\zeta}\operatorname{sech}(e^{i\theta})\right)\right]\\
=&\frac{1-\operatorname{sech}(e^{i\theta})}{\zeta}\left(f(\zeta) * \frac{\zeta - \frac{\operatorname{sech}(e^{i\theta})}{\operatorname{sech}(e^{i\theta})-1}\zeta^2}{(1-\zeta)^2}\right),
\end{aligned}$$

where we have used the fact that

$$\zeta f'(\zeta) = f(\zeta) * \frac{\zeta}{(1-\zeta)^2} \quad \text{and} \quad f(\zeta) = f(\zeta) * \frac{\zeta}{1-\zeta}.$$

This completes the proof in the forward direction.

For the backward proof, let $\mu = 1$. Then $f(\zeta)/\zeta \neq 0$ in $\mho$. Therefore, the function $\mathcal{V}(\zeta) = \zeta f'(\zeta)/f(\zeta)$ is holomorphic in $\mho$ along with $\mathcal{V}(0) = 1$. In the first part of the proof, we observe that

$$\frac{\zeta f'(\zeta)}{f(\zeta)} \neq \operatorname{sech}(e^{i\theta})$$

and

$$\frac{1}{\zeta}\left(f(\zeta) * \frac{\zeta-\mu\zeta^2}{(1-\zeta)^2}\right) \neq 0$$

are identical. Let $\mathcal{X}(\zeta) = \operatorname{sech}(e^{i\theta})$ for $\zeta \in \mho$. Then $\mathcal{V} \cap \mathcal{X} = \emptyset$. Hence, a connected part of $\mathbb{C} - \mathcal{X}(\partial\mho)$ contains the simply connected domain $\mathcal{V}(\mho)$. The univalence of the function $\mathcal{X}$, along with the fact $\mathcal{X}(0) = \mathcal{V}(0) = 1$, shows that $\mathcal{V}(\zeta) \prec \mathcal{X}(\zeta)$ and it means that $f \in \mathcal{S}^*_{\mathrm{sech}}$. $\square$

Corollary 4. *Let $f \in \mathcal{H}$. Then*

$$f \in \mathcal{S}^*_{\text{sech}} \Longleftrightarrow \sum_{n=2}^{\infty} \frac{n - \text{sech}(e^{i\theta})}{\text{sech}(e^{i\theta}) - 1} a_n \zeta^{n-1} \neq 1, \tag{22}$$

Proof. We have alrealdy established that $f \in \mathcal{S}^*_{\text{sech}}$ if and only if (21) is satisfied. Thus, rewrite the right side of (21) as

$$\begin{aligned}
0 \neq &\frac{1}{\zeta}\left(f(\zeta) * \frac{\zeta - \mu\zeta^2}{(1-\zeta)^2}\right) \\
= &\frac{1}{\zeta}\left(f(\zeta) * \frac{\zeta}{(1-\zeta)^2} - f(\zeta) * \frac{\zeta^2}{(1-\zeta)^2}\mu\right) \\
= &\frac{1}{\zeta}\left[f(\zeta) * \frac{\zeta}{(1-\zeta)^2} - \mu\left(f(\zeta) * \frac{\zeta}{(1-\zeta)^2} - f(\zeta) * \frac{\zeta}{1-\zeta}\right)\right] \\
= &(1-\mu)f'(\zeta) + \mu\frac{f(\zeta)}{z} \\
= &1 - \sum_{n=2}^{\infty}(n(\mu - 1) - \mu)a_n\zeta^{n-1} \\
= &1 - \sum_{n=2}^{\infty} \frac{n - \text{sech}(e^{i\theta})}{\text{sech}(e^{i\theta}) - 1} a_n \zeta^{n-1}.
\end{aligned}$$

□

Corollary 5. *Let $f \in \mathcal{H}$. If*

$$\sum_{n=2}^{\infty} \left| \frac{n - \text{sech}(e^{i\theta})}{\text{sech}(e^{i\theta}) - 1} \right| |a_n| < 1, \tag{23}$$

then $f \in \mathbb{S}^*_{\text{sech}}$.

Proof. Let μ be given as in (21). Then, consider

$$\begin{aligned}
\left|1 - \sum_{n=2}^{\infty}(n(\mu-1) - \mu)a_n\zeta^{n-1}\right| \geq &1 - \sum_{n=2}^{\infty}\left|(n(\mu-1) - \mu)a_n\zeta^{n-1}\right| \\
> &1 - \sum_{n=2}^{\infty} |n(\mu - 1) - \mu| \, |a_n| \\
= &1 - \sum_{n=2}^{\infty} \left| \frac{n - \text{sech}(e^{i\theta})}{\text{sech}(e^{i\theta}) - 1} \right| |a_n| \\
> &0,
\end{aligned}$$

when we apply (24). Therefore, on the account of Corollary 4, we have the result. □

The next result is a direct consequence of Corollay 5.

Corollary 6. *Let $f \in \mathcal{H}$. If*

$$|a_n| < \frac{|\,\text{sech}(e^{i\theta}) - 1|}{|n - \text{sech}(e^{i\theta})|}, \quad n \geq 2 \tag{24}$$

then $f \in \mathbb{S}^*_{\text{sech}}$.

Corollary 7. *Let $f \in \mathcal{S}^*_{\text{sech}}$ and $|\zeta| = r$. Then*

$$r - \left(\frac{|\,\text{sech}(e^{i\theta}) - 1|}{|2 - \text{sech}(e^{i\theta})|}\right) r^2 \leq |f(\zeta)| \leq r + \left(\frac{|\,\text{sech}(e^{i\theta}) - 1|}{|2 - \text{sech}(e^{i\theta})|}\right) r^2.$$

Proof. Let $f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n$. Then

$$|f(\zeta)| \leq r + \sum_{n=2}^{\infty} |a_n| r^n \quad \text{and} \quad |f(\zeta)| \geq r - \sum_{n=2}^{\infty} |a_n| r^n. \tag{25}$$

Since $r^n \leq r^2$ for $n \geq 2$ and $r < 1$, we have

$$|f(\zeta)| \leq r + r^2 \sum_{n=2}^{\infty} |a_n| \tag{26}$$

and

$$|f(\zeta)| \geq r - r^2 \sum_{n=2}^{\infty} |a_n|. \tag{27}$$

It follows from Corollary 5 that

$$\left| \frac{2 - \operatorname{sech}(e^{i\theta})}{\operatorname{sech}(e^{i\theta}) - 1} \right| \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \left| \frac{n - \operatorname{sech}(e^{i\theta})}{\operatorname{sech}(e^{i\theta}) - 1} \right| |a_n| < 1,$$

and from this inequality, we obtain

$$\sum_{n=2}^{\infty} |a_n| < \left| \frac{\operatorname{sech}(e^{i\theta}) - 1}{2 - \operatorname{sech}(e^{i\theta})} \right|.$$

Using this last inequality in (26) and (27), we obtain the required result. □

Following the same line of proof as in Corollary 7, we obtain the following distortion result for the class $\mathcal{S}^*_{\text{sech}}$.

Corollary 8. *Let $f \in \mathcal{S}^*_{\text{sech}}$ and $|\zeta| = r$. Then*

$$1 - 2\left(\frac{|\operatorname{sech}(e^{i\theta}) - 1|}{|2 - \operatorname{sech}(e^{i\theta})|} \right) r \leq |f'(\zeta)| \leq 1 + 2\left(\frac{|\operatorname{sech}(e^{i\theta}) - 1|}{|2 - \operatorname{sech}(e^{i\theta})|} \right) r.$$

Corollary 9. *Let $f \in \mathcal{S}^*_{\text{sech}}$, then $f \in \mathcal{S}^*(\sigma)$, $0 \leq \sigma < 1$ in the disc $|\zeta| < r^*$, where*

$$r^* = \inf \left\{ \frac{(1-\sigma)|n - \operatorname{sech}(e^{i\theta})|}{(n-\sigma)|\operatorname{sech}(e^{i\theta}) - 1|} \right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

Proof. To prove that $f \in \mathcal{S}^*(\sigma)$, it is enough to show that

$$\left| \frac{\zeta f'(\zeta)}{f(\zeta)} - 1 \right| < 1 - \sigma,$$

which implies

$$\sum_{n=2}^{\infty} \frac{n - \sigma}{1 - \sigma} |a_n| \, r^{n-1} < 1,$$

where we deduce that $f \in \mathcal{S}^*(\sigma)$ if

$$|a_n| < \frac{1 - \sigma}{(n - \sigma) r^{n-1}}.$$

By the virtue of Corollary 6, $f \in \mathcal{S}^*(\sigma)$ if

$$\frac{|\operatorname{sech}(e^{i\theta}) - 1|}{|n - \operatorname{sech}(e^{i\theta})|} < \frac{1 - \sigma}{(n - \sigma) r^{n-1}}.$$

Thus, we have

$$r < \left(\frac{(1-\sigma)|n-\operatorname{sech}(e^{i\theta})|}{(n-\sigma)|\operatorname{sech}(e^{i\theta})-1|}\right)^{\frac{1}{n-1}},$$

which conclude the proof. □

4. Partial Sums of The Class $\mathcal{S}^*_{\text{sech}}$

Let $f_1(\zeta) = z$ and $f_n(\zeta) = \zeta + \sum_{k=2}^{n} a_k \zeta^k$ be the sequence of partial sum of the functions $f \in \mathcal{H}$, when the coefficients of $f(\zeta)$ are small enough to satisfy condition (24). In this section, we determine the sharp lower bounds for the geometric quantities

$$\operatorname{Re}\left(\frac{f(\zeta)}{f_n(\zeta)}\right), \operatorname{Re}\left(\frac{f_n(\zeta)}{f(\zeta)}\right), \operatorname{Re}\left(\frac{f'(\zeta)}{f'_n(\zeta)}\right) \text{ and } \operatorname{Re}\left(\frac{f'_n(\zeta)}{f'(\zeta)}\right).$$

Theorem 6. *Let $f \in \mathcal{H}$ satisfies condition (24), then*

$$\operatorname{Re}\left(\frac{f(\zeta)}{f_n(\zeta)}\right) \geq 1 - \frac{1}{\mu_{n+1}}, \ \zeta \in \mho \tag{28}$$

and

$$\operatorname{Re}\left(\frac{f_n(\zeta)}{f(\zeta)}\right) \geq \frac{\mu_{n+1}}{\mu_{n+1}+1}, \ \zeta \in \mho, \tag{29}$$

where

$$\mu_n = \left|\frac{n-\operatorname{sech}(e^{i\theta})}{\operatorname{sech}(e^{i\theta})-1}\right|. \tag{30}$$

The result is sharp for every n with the leading function

$$f(\zeta) = \zeta + \frac{\zeta^{n+1}}{\mu_{n+1}} \tag{31}$$

Proof. Consider

$$w(\zeta) = (1+\mu_{n+1})\left[\frac{f(\zeta)}{f_n(\zeta)} - \left(1 - \frac{\mu_{n+1}}{1+\mu_{n+1}}\right)\right] = 1 + \frac{\mu_{n+1}\sum_{k=n+1}^{\infty} a_k \zeta^{k-1}}{1+\sum_{k=2}^{n} a_k \zeta^{k-1}} := \frac{1+g(\zeta)}{1-g(\zeta)}.$$

Then

$$g(\zeta) = \frac{w(\zeta)-1}{w(\zeta)+1} = \frac{\mu_{n+1}\sum_{k=n+1}^{\infty} a_k \zeta^{k-1}}{2+2\sum_{k=2}^{n} a_k \zeta^{k-1} + \mu_{n+1}\sum_{k=n+1}^{\infty} a_k \zeta^{k-1}}.$$

To prove our result, we need to demonstrate that $\operatorname{Re} w(\zeta) \geq 0$ in $\mho$, which is equivalent to showing $|g(\zeta)| \leq 1$ in $\mho$. Therefore,

$$|g(\zeta)| \leq \frac{\mu_{n+1}\sum_{k=n+1}^{\infty} |a_k|}{2-2\sum_{k=2}^{n} |a_k| - \mu_{n+1}\sum_{k=n+1}^{\infty} |a_k|} \leq 1,$$

provided

$$\sum_{k=2}^{n} |a_k| + \mu_{n+1}\sum_{k=n+1}^{\infty} |a_k| \leq 1. \tag{32}$$

Since $f(\zeta)$ satisfies (25), then to prove (34), it is enough to demonstrate that the left side of the inequality (32) is bounded by $\sum_{k=2}^{\infty} \mu_k |a_k|$. This is equivalent to

$$\sum_{k=2}^{n}(1-\mu_k)|a_k| + \sum_{k=n+1}^{\infty} (\mu_k - \mu_{n+1})|a_k| \geq 0. \tag{33}$$

On the account of this inequality (33), the proof of the inequality (34) is completed. To see the sharpness of the result, we consider the function in (31) and observe that for $\zeta = r\, e^{i\frac{\pi}{n}}$, we have

$$\frac{f(\zeta)}{f_n(\zeta)} = f(\zeta) = 1 + \frac{\zeta^n}{\mu_n} \to 1 - \frac{1}{\mu_{n+1}}, \quad r \to 1^-.$$

Similarly, if we consider

$$w(\zeta) = \mu_{n+1}\left[\frac{f_n(\zeta)}{f(\zeta)} - \left(1 - \frac{1}{\mu_{n+1}}\right)\right] = 1 + \frac{\mu_{n+1} \sum\limits_{k=n+1}^{\infty} a_k \zeta^{k-1}}{1 + \sum\limits_{k=2}^{n} a_k \zeta^{k-1}} := \frac{1+g(\zeta)}{1-g(\zeta)}.$$

Then

$$g(\zeta) = \frac{w(\zeta)-1}{w(\zeta)+1} = \frac{(1+\mu_{n+1}) \sum\limits_{k=n+1}^{\infty} a_k \zeta^{k-1}}{2 + 2\sum\limits_{k=2}^{n} a_k \zeta^{k-1} + (\mu_{n+1}-1) \sum\limits_{k=n+1}^{\infty} a_k \zeta^{k-1}}.$$

Therefore,

$$|g(\zeta)| \leq \frac{(1+\mu_{n+1}) \sum\limits_{k=n+1}^{\infty} |a_k|}{2 - 2\sum\limits_{k=2}^{n} |a_k| - (\mu_{n+1}-1) \sum\limits_{k=n+1}^{\infty} |a_k|} \leq 1,$$

which leads us directly to the assertion (35) of Theorem 6. The bound of (35) cannot be improved since the function given by (31) assumes the equality. This complete the proof. □

The following results involving the ratio of derivative can be obtained *mutatis mutandis* as in Theorem 6, thus we omit the proofs.

Theorem 7. *Let $f \in \mathcal{H}$ satisfy condition* (24)*, then*

$$\mathrm{Re}\left(\frac{f'(\zeta)}{f_n'(\zeta)}\right) \geq 1 - \frac{n+1}{\mu_{n+1}},\ \zeta \in \mho \tag{34}$$

and

$$\mathrm{Re}\left(\frac{f_n'(\zeta)}{f'(\zeta)}\right) \geq \frac{\mu_{n+1}}{n+1+\mu_{n+1}},\ \zeta \in \mho, \tag{35}$$

where μ_n is given by (30)*. The result is sharp for the function defined by* (31)*.*

5. Conclusions

These current findings are motivated by the various families of Ma and Minda's class connected to trigonometric functions, which have surfaced in the existing literature in Geometric function theory. Here, in this article, we have successfully found the condition on $\beta \in \mathbb{C}$ (given by (5), (10), (13) and (17)) such that the following differential subordination implication holds:

$$1+\beta\frac{\zeta p'(\zeta)}{(p(\zeta))^n}\prec\frac{1+\mathcal{A}\zeta}{1+\mathcal{B}\zeta}\Longrightarrow p(\zeta)\prec\operatorname{sech}(\zeta)\ \zeta\in\mho,\ n=0,1,2.$$

Consequently, we found the sufficient conditions for $f\in\mathcal{H}$ to be in the class $\mathcal{S}^*_{\text{sech}}$. On this note, condition on β associated with certain differential subordination of the Janowski type for which Alexander integral transformation is preserved was estimated.

Moreover, we investigated the convolution property related to the class $\mathcal{S}^*_{\text{sech}}$ and presented many of its geometrical properties, such as coefficient estimate, growth and distortion results, radii of the starkness of order σ and partial sums results.

Other geometrical features related to secant hyperbolic functions such as the third and fourth Hankel and Toeplitz determinants could be examined as future work.

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