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A Seasonally Competitive M-Prey and N-Predator Impulsive System Modeled by General Functional Response for Integrated Pest Management

Juan Liu ¹, Jie Hu ², Peter Yuen ^{3,*}  and Fuzhong Li ^{2,*}

¹ Department of Basic Sciences, Shanxi Agricultural University, Jinzhong 030810, China; liujuannk@163.com

² School of Software, Shanxi Agricultural University, Jinzhong 030810, China; 17835425155@163.com

³ Centre for Electronics Warfare, Information and Cyber (CEWIC), Cranfield University, Shrivenham SN6 8LA, UK

* Correspondence: p.yuen@cranfield.ac.uk (P.Y.); sxaulfz@126.com (F.L.)

Abstract: Considering the harvesting of prey and stocking of predator impulsively at different fixed moments of time, this paper studies the dynamics of a seasonally competitive m-prey and n-predator impulsive system, which is focused more specifically in four areas as follows: (i) we emphasize the dynamics of m-prey and n-predator in the ecosystem with a view to understanding how the present work may be able to apply to real environment applications; (ii) this work uses the general functional response instead of using specific impulse responses; (iii) considering the intra- and inter-competitions between species and (iv) the system is subjected to the influences of seasonal factors which imposes direct impacts to the delicate balance of biological systems. By using the comparison techniques and the Floquet theorems, the sufficient conditions for the ecosystem permanence and the asymptotic stabilities of the global and local prey-free periodic solutions have been subsequently obtained. This work is concluded with an in-depth discussion of the biological significance of the results obtained in this research. The obtained results can provide theoretical support for protecting endangered species and to help maintain the ecological balance, especially when it is applied to practical pest management, such as rodent controls in the farmland.

Keywords: intra- and inter-competition; m-prey and n-predator impulsive system; general functional response; seasonally perturbed

MSC: 34E05; 37M05; 37M10



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1. Introduction

Pests in the natural environment not only cause impacts to the well-being of human life in a negative way, they also disrupt food production and spread of diseases, and in the worst scenario, they even bring death and serious disasters to plants and other living things on Earth. Pests such as locusts, mosquitoes, weeds, fleas, cockroaches, pathogens, mice, and so on, are common in our daily life. In China, the loss of grains caused by pest damage amounts up to 6 million tons every year, which is equivalent to the worth of more than CNY 2 billion of economic loss annually. This alarming issue has become an urgent matter for controlling the negative impacts of pests. Chemical control by spraying pesticides and simultaneous deployment of biological control by using natural enemies have been commonly applied in the integrated pest management (IPM) system. In practice, the timing for spraying the pesticides and when natural enemies should be release, as well as the quantity and frequency of the pesticides spray and the population of natural enemies to be released, are necessary to know in advance. Due to the discontinuous nature of the pesticides spray and the release of natural enemy events, the predator–prey system such

as this kind can be modeled by using impulsive differential equations which have been widely adopted by mathematical ecologists [1–20]. Since the pesticide is harmful to the pest, as well as to the natural enemies, they are ideal to deploy in two different time instances rather than for both to be applied at the same time. Furthermore, much of the previous works consider two species, namely one prey and one predator, for modeling the dynamics of the predator–prey ecosystem, which is far from realistic [1–20]. Although recent work has reported the study of three species system which considers two-prey and one-predator, as well as the one-prey and two-predator systems [2], the consideration of the effects due to the intra- and inter-species competitions in these studies are lacking [1–5,7–20]. The co-existence of three or more species competing with each other often occurs in the real ecosystem, even when they are active in close range of geographical surroundings. We will add competitive factors to the model to make it more realistic, even when they are confined in local close range of geographical areas. One good example to understand this intra-competition can be illustrated through the food chain of prey and predator in the domestic farmland environment: the eagles prey on mice and snakes, at the same time rabbits are preyed on by the eagles and foxes, insects are also preyed by the spiders and frogs, meanwhile the snakes prey on frogs, and the frogs prey on spiders, and so on. Therefore, there are more complex food chain networks that can be commonly found even for a domestic farmland scenario in the real world: there are more than 200 kinds of pests which feed on corn, 650 kinds of pests that feed on elm trees, 1400 species of pests that feed on oak, and so on. Thus, in this paper, we have modeled the balance of m -preys and n -predators in the ecosystem by including the intra- and inter-species prey–predator competitions for the very first time. Furthermore, this paper considers the application of the pesticide and the natural enemies in two different instances of time, to maximize the effectiveness of the pest control without harming the natural enemies of the pest.

The relationship between the predator and prey in the predator–prey model is classically defined through a predator’s functional response, which represents the predator’s rate of feeding on the prey. Research in the ecological management discipline conventionally models the behavior of such dynamic systems through specific responsible functions [1,3,5–11,13,14,16–18,20]; however, the results of these studies are applicable only if the environments conform exactly to the conditions as defined by the specific responsible functions which have been implemented for specific scenarios. Thus, there may be limitations regarding the application of previous work for solving real-world problems. In this paper, we will use a general functional response $u_i(t)\varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t))$ ($1 \leq i \leq m, 1 \leq j \leq n$), where $\mathbf{U}(t) = (u_1(t), u_2(t), \dots, u_m(t))$ and $\mathbf{V}(t) = (v_1(t), v_2(t), \dots, v_n(t))$ represent the population of the prey and predator, respectively. The results of this paper, which utilized the general functional response for the modeling, can be validated under specialized environmental conditions. For example, the solution of our models can be validated by using (i) the prey-dependent functional responses, such as the Holling-type, Ivlev-type, Monod–Haldance-type, and so on [13,14,16]; and (ii) the predator-dependent functional response, such as Leslie–Gower-type, Beddington–DeAngelies-type, ratio-dependent-type, Hassell–Varley-type, square-root-type, Crowley–Martin-type, Watt-type, etc. [8,17,18]. In this paper we will conclude the generalized solutions obtained from this work and how to validate them by using a couple of specific functional responses in the light of reported results in the literature.

The change of environment resulting from the effects due to seasonal variations, climate, and natural disasters imposes impacts on the delicate balance of biological ecosystems. The survival and breeding of biological species are affected by the environmental change, which also influences their relationships such as their competitions, mutual co-operations, and predator–prey habitats [4,10]. Due to the diverse configurations in the biological systems, biological species’ evolution in the natural environment has routinely been considered as an example of dynamic ecosystems, which can be investigated mathematically [1–20]. For example, there is enough food in the warm seasons, such as in the spring and summer, that favors the animals and insects; thus, they enjoy high intrinsic growth rates. One

objective of the present work is to extend previous research [1–3,5–12,14–20] by including environmental factors which modulate the prey's intrinsic growth rate periodically.

This paper attempts to model the balance of m -preys and n -predators ecosystem as motivated by previous work in [1–20], which is subjected to a periodic variation of intrinsic growth rate of the prey due to the seasonal effect, and under the deployment of impulsive control strategies, by using a more general impulsive functional response of $u_i(t)\varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t))$, and to compound it with a periodic variation of intrinsic growth rate of the prey:

$$\left\{ \begin{array}{l} \frac{du_i(t)}{dt} = u_i(t)(a_i - \sum_{l=1}^m b_{il}u_l(t) + \lambda_i \sin(\omega_i t)) - \sum_{j=1}^n u_i(t)v_j(t)\varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t)) \\ \frac{dv_j(t)}{dt} = v_j(t)(-D_j + \sum_{i=1}^m k_{ij}u_i(t)\varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t))) \\ t \neq (m+q-1)T, t \neq mT, 1 \leq i \leq m, 1 \leq j \leq n \\ \left. \begin{array}{l} \Delta u_i(t) = -\delta_i u_i(t) \\ \Delta v_j(t) = -l_j v_j(t) \end{array} \right\} t = (m+q-1)T \\ \left. \begin{array}{l} \Delta u_i(t) = 0 \\ \Delta v_j(t) = \mu_j \end{array} \right\} t = mT \end{array} \right\} \quad (1)$$

in which $\mathbf{U}(t) = (u_1(t), u_2(t), \dots, u_m(t))$ is the vector of all prey densities and $\mathbf{V}(t) = (v_1(t), v_2(t), \dots, v_n(t))$ represents the vector of all predator densities. $u_i(t)$ is the density of the prey species i , $1 \leq i \leq m$, $v_j(t)$ is the density of the predator species j , $1 \leq j \leq n$. $a_i > 0$ denotes the intrinsic birth rate of the prey in the absence of predation and competition. T is the period of the impulsive stocking and harvesting. b_{il} is the effects of intraspecific competition on species i when $i = l$, which represents the effects of inter-species competition on species i and species j when $i \neq l$. ω_i is the i -th pest species' angular frequency of the fluctuation caused by seasonal periodicity, λ_i is the i -th pest species' perturbation magnitude, $D_j > 0$ is the death rate of the predator species j . k_{ij} denotes the efficiency for the species j of the prey that is converted into newborn species i of predators, $u_i(t)\varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t))$ is the general functional response. The following four conditions have been assumed to satisfy: (i) the $\varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t))$ is assumed to be monotonous decreasing with respect to prey species $u_i(t)$ and predator species $v_j(t)$, respectively; (ii) $u_i(t)\varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t))$ is the monotonous increasing density of prey species $u_i(t)$, respectively; (iii) $v_j(t)\varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t))$ is the monotonous increasing density of predator species $v_j(t)$, respectively; (iv) $u_i(t)v_j(t)\varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t))$ is locally Lipschitz. These assumptions conform to the natural events that have been observed in the actual ecosystem. $\Delta u_i(t) = u_i(t^+) - u_i(t)$, $\Delta v_j(t) = v_j(t^+) - v_j(t)$, $u_i(t^+) = \lim_{t \rightarrow t^+} u_i(t)$, $v_j(t^+) = \lim_{t \rightarrow t^+} v_j(t)$. $\mu_j > 0$ is the predator species' density which is released at time $t = mT$, and m is a positive integer. Because of the spraying of pesticides, when $(m+q-1)T$, $0 \leq \delta_i, l_j < 1$ is the prey population ($u_i(t)$)'s fixed death rate and the predator population ($v_j(t)$)'s fixed death rate, respectively, where $0 \leq q < 1$, $1 \leq i \leq m$ and $1 \leq j \leq n$.

The organization of this paper is outlined as follows: Section 2 summarizes the Lemmas which will be used in the later sections. In Section 3, the stability of the prey-eradication periodic solution, including the local asymptotically stability and global asymptotically stability, are derived. The sufficient conditions for the permanence of system in (1) are investigated. Subsequently, the theoretical results are discussed from a biological viewpoint in Section 4, and the paper is concluded in Section 5.

2. Preliminaries

Definition 1. Suppose $(\mathbf{U}(t), \mathbf{V}(t))$ is the positive solution of the ecosystem (1), then for all large enough t and provided that there exist positive constants M and m , when the conditions $m \leq u_i(t) \leq M$, $m \leq v_j(t) \leq M$, $1 \leq i \leq m$, $1 \leq j \leq n$ are satisfied, then the system (1) is said to be permanent.

Definition 2. Suppose $(\mathbf{U}(t), \mathbf{V}(t))$ is the positive solution of the ecosystem (1) and when the conditions $\lim_{t \rightarrow \infty} u_i(t) = 0$ or $\lim_{t \rightarrow \infty} v_j(t) = 0, 1 \leq i \leq m, 1 \leq j \leq n$ are satisfied, then the prey or the predator is said to become extinct.

Lemma 1. Suppose $(\mathbf{U}(t), \mathbf{V}(t))$ is the positive solution of the ecosystem (1), then it is a piecewise continuous (PC) function, which has the property of uniqueness and the global existence when the ecosystem (1) has that the first $m + n$ term of the equations is smooth, that means that $f = (f_1, \dots, f_m, g_1, \dots, g_n)$ satisfies the Lipschitz condition.

Lemma 2. Suppose $u(t) \in PC^1(R_+, R)$ satisfies the following inequalities:

$$\begin{cases} \frac{du}{dt} \leq \omega(t)u(t) + z(t), & t \neq t_k, t > 0, \\ u(t_k^+) \leq c_k u(t_k) + a_k & t = t_k > 0, \\ u(0^+) \leq u_0, \end{cases}$$

in which $\omega(t), z(t) \in C(R_+, R), a_k$ and $c_k \geq 0$ are constants ($k = 1, 2, \dots$). The following result can be established when $t > 0$:

$$\begin{aligned} u(t) \leq & u_0 \left(\prod_{t_0 < t_k < t} c_k \right) \exp\left(\int_{t_0}^t \omega(s) ds\right) + \int_{t_0}^t \left(\prod_{s < t_k < t} c_k \right) \exp\left(\int_s^t \omega(\gamma) d\gamma\right) z(s) ds \\ & + \sum_{t_0 < t_k < t} \left(\prod_{t_k < t_j < t} c_j \right) \exp\left(\int_{t_k}^t \omega(s) ds\right) a_k. \end{aligned}$$

Lemma 3. [21] Suppose $\mathbf{Z}(t) = (\mathbf{U}(t), \mathbf{V}(t))$ is the positive solution of the ecosystem (1), then the following conditions are valid: (1) when $\mathbf{Z}(0^+) \geq \mathbf{0}$, then for all $t \geq 0$, we have $\mathbf{Z}(t) \geq \mathbf{0}$, namely, $u_i(t) \geq 0, v_j(t) \geq 0$; (2) when $\mathbf{Z}(0^+) > \mathbf{0}$, then for all $t \geq 0$, we also have $\mathbf{Z}(t) > \mathbf{0}$, namely, $u_i(t) > 0, v_j(t) > 0$.

Let $u_i(t) = 0$ for all $1 \leq i \leq m$, then for all $1 \leq j \leq n$, the system (1) becomes:

$$\begin{cases} \frac{dv_j(t)}{dt} = -D_j v_j(t) & t \neq (m+q-1)T, t \neq mT \\ v_j(t^+) = (1-l_j)v_j(t) & t = (m+q-1)T \\ v_j(t^+) = v_j(t) + \mu_j & t = mT \\ v_{j0} = v_j(0^+) \end{cases} \quad (2)$$

By calculation, we can obtain the following positive periodic solution:

$$\tilde{v}_j(t) = \begin{cases} \frac{\mu_j \exp\{-D_j[t-(m-1)T]\}}{1-(1-l_j)\exp(-D_jT)} & (m-1)T < t \leq (m+q-1)T \\ \frac{\mu_j(1-l_j)\exp\{-D_j[t-(m-1)T]\}}{1-(1-l_j)\exp(-D_jT)} & (m+q-1)T < t \leq mT \end{cases} \quad (3)$$

and $\tilde{v}_j(0^+) = \tilde{v}_j(mT^+) = \frac{\mu_j}{1-(1-l_j)\exp(-D_jT)}, \tilde{v}_j(qT^+) = \tilde{v}_j((m+q-1)T^+) = \frac{\mu_j(1-l_j)\exp(-D_jqT)}{1-(1-l_j)\exp(-D_jT)}$. When the initial value is $v_{j0} \geq 0$, the solution of ecosystem (2) can be obtained through the expansion of the above:

$$v_j(t) = \begin{cases} (1-l_j)^{m-1} \left(v_j(0^+) - \frac{\mu_j}{1-(1-l_j)\exp(-D_jT)} \right) \exp(-D_j t) + \tilde{v}_j(t), & (m-1)T < t \leq (m+q-1)T, \\ (1-l_j)^m \left(v_j(0^+) - \frac{\mu_j}{1-(1-l_j)\exp(-D_jT)} \right) \exp(-D_j t) + \tilde{v}_j(t), & (m+q-1)T < t \leq mT. \end{cases} \quad (4)$$

The following result can be readily obtained according to the equalities (3) and (4):

Lemma 4. Suppose $v_j(t)$ is the positive solution of ecosystem (2), and the initial value is $v_{j0} \geq 0$, when $t \rightarrow \infty$, then $|v_j(t) - \tilde{v}_j(t)| \rightarrow 0$ is established.

Lemma 5. [21] Suppose $(\mathbf{U}(t), \mathbf{V}(t))$ is the positive solutions of the ecosystem (1), then for all large enough t , there exists a positive constant M such that $u_i(t) \leq M, v_j(t) \leq M, 1 \leq i \leq m, 1 \leq j \leq n$.

Thus we can obtain the positive periodic solution of the prey-free $(\mathbf{0}, \tilde{\mathbf{V}}(t))$ in the ecosystem (1), where $\tilde{\mathbf{V}}(t) = (\tilde{v}_1(t), \tilde{v}_2(t), \dots, \tilde{v}_n(t)), 1 \leq j \leq n$ and $\mathbf{0} = (0, 0, \dots, 0)$ is m -dimensional zero vector.

3. Locally and Globally Asymptotically Stable and Permanent

Theorem 1. When the parameters in the ecosystem (1) satisfy the following inequalities for $1 \leq i \leq m$,

$$\ln(1 - \delta_i) + (a_i + \lambda_i)T - \sum_{j=1}^n \int_0^T \tilde{v}_j(t) \varphi_{ij}(\mathbf{0}, \tilde{\mathbf{V}}(t)) dt < 0. \quad (5)$$

then the locally asymptotically stable of prey-free's periodic solution $(\mathbf{0}, \tilde{\mathbf{V}}(t))$ is obtained.

Proof. Firstly, we change the system (1) as:

$$\left\{ \begin{array}{l} \frac{du_{i1}(t)}{dt} = u_{i1}(t) \left(a_i - \sum_{l=1}^m b_{il} u_{l1}(t) + \lambda_i \right) - \sum_{j=1}^n u_{i1}(t) v_{j1}(t) \varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t)) \\ \frac{dv_{j1}(t)}{dt} = v_{j1}(t) \left(-D_j + \sum_{i=1}^m k_{ij} u_{i1}(t) \varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t)) \right) \\ t \neq (m+q-1)T, t \neq mT, 1 \leq i \leq m, 1 \leq j \leq n \\ \Delta u_{i1}(t) = -\delta_i u_{i1}(t) \\ \Delta v_{j1}(t) = -l_j v_{j1}(t) \end{array} \right\}_{t=(m+q-1)T} \quad (6)$$

$$\left\{ \begin{array}{l} \Delta u_{i1}(t) = 0 \\ \Delta v_{j1}(t) = \mu_j \end{array} \right\}_{t=mT}$$

Since the prey-free periodic solution $(\mathbf{0}, \tilde{\mathbf{V}}(t))$ of the system (6) and system (1) is the same, and $u_{i1}(t) \geq u_i(t), v_{j1}(t) \geq v_j(t)$, thus it is necessary to prove that the prey-free periodic solution of system (6) is locally asymptotically stable.

Let us denote $z_i(t) = u_{i1}(t), w_j(t) = v_{j1}(t) - \tilde{v}_j(t)$. We rewrite the form in system (6) by taking the linear part of the Taylor expansion as:

$$\left\{ \begin{array}{l} \frac{dz_i(t)}{dt} = z_i(t) \left(a_i + \lambda_i - \sum_{j=1}^n \tilde{v}_j(t) \varphi_{ij}(\mathbf{0}, \tilde{\mathbf{V}}(t)) \right) \\ \frac{dw_j(t)}{dt} = -D_j w_j(t) + \tilde{v}_j(t) \sum_{i=1}^m k_{ij} z_i(t) \varphi_{ij}(\mathbf{0}, \tilde{\mathbf{V}}(t)) \\ t \neq (m+q-1)T, t \neq mT, 1 \leq i \leq m, 1 \leq j \leq n \\ z_i((n+l-1)T^+) = (1 - \delta_i) z_i((n+l-1)T) \\ w_j((n+l-1)T^+) = (1 - l_j) w_j((n+l-1)T) \end{array} \right\}_{t=(m+q-1)T} \quad (7)$$

$$\left\{ \begin{array}{l} z_i(nT^+) = z_i(nT) \\ w_j(nT^+) = w_j(nT) \end{array} \right\}_{t=mT}$$

Through the simple calculation, we can obtain that the fundamental solution matrix of (7) satisfies:

$$\frac{d\phi(t)}{dt} = A(t)\phi(t),$$

and $A(t)$ is a $m+n$ order matrix,

$$A(t) = \begin{pmatrix} A_{11} & O \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$A_{11} = \begin{pmatrix} a_1 + \lambda_1 - \sum_{j=1}^n \tilde{v}_j(t) \varphi_{1j}(\mathbf{0}, \tilde{\mathbf{V}}(t)) & & & \\ & a_2 + \lambda_2 - \sum_{j=1}^n \tilde{v}_j(t) \varphi_{2j}(\mathbf{0}, \tilde{\mathbf{V}}(t)) & & \\ & & \ddots & \\ & & & a_m + \lambda_m - \sum_{j=1}^n \tilde{v}_j(t) \varphi_{mj}(\mathbf{0}, \tilde{\mathbf{V}}(t)) \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} \tilde{v}_1(t) k_{11} \varphi_{11}(\mathbf{0}, \tilde{\mathbf{V}}(t)) & \cdots & \tilde{v}_1(t) k_{m1} \varphi_{m1}(\mathbf{0}, \tilde{\mathbf{V}}(t)) \\ \tilde{v}_2(t) k_{12} \varphi_{12}(\mathbf{0}, \tilde{\mathbf{V}}(t)) & \cdots & \tilde{v}_2(t) k_{m2} \varphi_{m2}(\mathbf{0}, \tilde{\mathbf{V}}(t)) \\ \vdots & & \vdots \\ \tilde{v}_n(t) k_{1n} \varphi_{1n}(\mathbf{0}, \tilde{\mathbf{V}}(t)) & \cdots & \tilde{v}_n(t) k_{mn} \varphi_{mn}(\mathbf{0}, \tilde{\mathbf{V}}(t)) \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} -D_1 & & \\ & \ddots & \\ & & -D_n \end{pmatrix}.$$

and $\phi(t)$ is also a $m+n$ order matrix,

$$\phi(t) = \begin{pmatrix} B_{11} & O \\ B_{21} & B_{22} \end{pmatrix},$$

$$B_{11} = \begin{pmatrix} \exp\left(\int_0^T a_1 + \lambda_1 - \sum_{j=1}^n \tilde{v}_j(t) \varphi_{1j}(\mathbf{0}, \tilde{\mathbf{V}}(t))\right) & & \\ & \ddots & \\ & & \exp\left(\int_0^T a_m + \lambda_m - \sum_{j=1}^n \tilde{v}_j(t) \varphi_{mj}(\mathbf{0}, \tilde{\mathbf{V}}(t))\right) \end{pmatrix},$$

$$B_{22} = \begin{pmatrix} \exp(-D_1 T) & & \\ & \ddots & \\ & & \exp(-D_n T) \end{pmatrix}.$$

As the B_{21} is not required for the following analysis, its exact form is not necessary to obtain. The resetting impulsive conditions of ecosystem (7) can then be written as:

$$\begin{pmatrix} z_1((n+l-1)T^+) \\ \vdots \\ z_m((n+l-1)T^+) \\ w_1((n+l-1)T^+) \\ \vdots \\ w_n((n+l-1)T^+) \end{pmatrix} = B \begin{pmatrix} z_1((n+l-1)T) \\ \vdots \\ z_m((n+l-1)T) \\ w_1((n+l-1)T) \\ \vdots \\ w_n((n+l-1)T) \end{pmatrix},$$

$$B = \begin{pmatrix} 1 - \delta_1 & & & & \\ & \ddots & & & \\ & & 1 - \delta_m & & \\ & & & 1 - l_1 & \\ & & & & \ddots \\ & & & & & 1 - l_n \end{pmatrix},$$

$$\begin{pmatrix} z_1(nT^+) \\ \vdots \\ z_m(nT^+) \\ w_1(nT^+) \\ \vdots \\ w_n(nT^+) \end{pmatrix} = E_{m+n} \begin{pmatrix} z_1(nT) \\ \vdots \\ z_m(nT) \\ w_1(nT) \\ \vdots \\ w_n(nT) \end{pmatrix}.$$

Let $\lambda_1, \dots, \lambda_{m+n}$ be the monodromy matrix's eigenvalues

$$M = E_{m+n} \phi(T) \begin{pmatrix} 1 - \delta_1 & & & & & \\ & \ddots & & & & \\ & & 1 - \delta_m & & & \\ & & & 1 - l_1 & & \\ & & & & \ddots & \\ & & & & & 1 - l_n \end{pmatrix}_{(m+n) \times (m+n)},$$

where

$$\begin{aligned} \lambda_i &= (1 - \delta_i) \exp \left(\int_0^T a_i + \lambda_i - \sum_{j=1}^n \tilde{v}_j(t) \varphi_{ij}(\mathbf{0}, \tilde{\mathbf{V}}(t)) dt \right), 1 \leq i \leq m, \\ \lambda_{m+j} &= (1 - l_j) \exp(-D_j T) < 1, 1 \leq j \leq n. \end{aligned}$$

As $|\lambda_{m+j}| < 1$ ($1 \leq j \leq n$), and to apply the Floquet theory on the impulsive different equation, it can be seen that the following inequality satisfies

$$|\lambda_i| < 1 \quad (1 \leq i \leq m)$$

$$\ln(1 - \delta_i) + (a_i + \lambda_i)T - \sum_{j=1}^n \int_0^T \tilde{v}_j(t) \varphi_{ij}(\mathbf{0}, \tilde{\mathbf{V}}(t)) dt < 0 \quad (1 \leq i \leq m).$$

Then, the inequality (5) can be established. \square

Theorem 2. *Provided the following inequalities are established at the same time for $1 \leq i \leq m$,*

$$\ln(1 - \delta_i) + (a_i + \lambda_i)T - \sum_{j=1}^n \int_0^T \tilde{v}_j(t) \varphi_{ij}(\mathbf{K}, \tilde{\mathbf{V}}(t)) dt < 0. \quad (8)$$

then we can obtain the globally asymptotically stable periodic solution for the prey-free situation $(\mathbf{0}, \tilde{\mathbf{V}}(t))$, where $\mathbf{K} = (\frac{a_1 + \lambda_1}{b_{11}}, \frac{a_2 + \lambda_2}{b_{22}}, \dots, \frac{a_m + \lambda_m}{b_{mm}})$.

Proof. From the system (6) we can find that

$$\begin{cases} \frac{du_{i1}(t)}{dt} \leq u_{i1}(t)(a_i + \lambda_i - b_{ii}u_{i1}(t)), & t \neq (m+q-1)T, t \neq mT, \\ u_{i1}(t^+) = (1 - \delta_i)u_{i1}(t) \leq u_{i1}(t), & t = (m+q-1)T, t = mT. \end{cases}$$

By employing the comparison theorem of the impulsive different equation, $u_{i1}(t) \leq \tilde{u}_i(t)$ ($i = 1, 2$) can be obtained. Then

$$\tilde{u}_i(t) = \frac{\frac{a_i + \lambda_i}{b_{ii}} x_0}{\left(\frac{a_i + \lambda_i}{b_{ii}} - x_0\right) e^{-(a_i + \lambda_i)t} + x_0} \rightarrow \frac{a_i + \lambda_i}{b_{ii}} \quad (t \rightarrow \infty), \quad (9)$$

where $\tilde{u}_i(t)$ satisfies the following equation:

$$\begin{cases} \frac{d\tilde{u}_i(t)}{dt} = \tilde{u}_i(t)(a_i + \lambda_i - b_{ii}\tilde{u}_i(t)), \\ \tilde{u}_i(0^+) = x_0. \end{cases}$$

Therefore, for all sufficiently large t and for any $\varepsilon_i > 0$, it can be seen that $u_{i1}(t) \leq \frac{a_i + \lambda_i}{b_{ii}} + \varepsilon_i$ ($1 \leq i \leq m$).

By selecting an $\varepsilon_i > 0$ and for $1 \leq i \leq m$, such that

$$\eta_i = (1 - \delta_i) \exp\left(\int_0^T a_i + \lambda_i - Q ds\right) \in (0, 1),$$

where $Q \triangleq \sum_{j=1}^n (\tilde{v}_j(t) - \varepsilon_j) \varphi_{ij}(\mathbf{K} + \underline{\mathcal{Q}}, M_1, \dots, M_{i-1}, \tilde{v}_j(t) - \varepsilon_j, M_{i+1}, \dots, M_n)$ and $\underline{\mathcal{Q}} = (\varepsilon_1, \dots, \varepsilon_m)$. Similarly we can see that $\frac{dv_{j1}(t)}{dt} \geq -D_j v_{j1}(t)$; therefore, $v_{j1}(t) \geq \tilde{v}_j(t)$. From Lemma 4, it can be obtained that $v_{j1}(t) \rightarrow \tilde{v}_j(t)$ when t is large enough, then

$$v_{j1}(t) \geq \tilde{v}_j(t) > \tilde{v}_j(t) - \varepsilon. \quad (10)$$

Note that:

$$\frac{du_{i1}}{dt} \leq u_{i1}(a_i + \lambda_i - Q),$$

and the following can be readily obtained:

$$\begin{aligned} u_{i1}((m+q)T) &\leq u_{i1}((m+q-1)T^+)A_i \\ &= u_{i1}((m+q-1)T)(1 - \delta_i)A_i \\ &= u_{i1}((m+q-1)T)\eta_i. \end{aligned}$$

where $A = \exp\left(\int_{(m+q-1)T}^{(m+q)T} a_i + \lambda_i - Q ds\right)$. Therefore, we have:

$$u_{i1}((m+q)T) \leq u_{i1}(qT)\eta_i^m \rightarrow 0 \quad (m \rightarrow \infty).$$

As

$$0 \leq u_{i1}(t) \leq u_{i1}((m+q-1)T)(1 - \delta_i)e^{rT}$$

holds for $t \in [(m+q-1)T, (m+q)T]$; thus, $u_{i1}(t) \rightarrow 0$ ($1 \leq i \leq m$) as $t \rightarrow \infty$.

The claim that when $u_{i1}(t) \rightarrow 0$ ($1 \leq i \leq m$), then $\lim_{t \rightarrow \infty} v_{j1}(t) = \tilde{v}_j(t)$ ($1 \leq j \leq n$) will be proved in the following. For $\varepsilon_i > 0$, we can obtain that:

$$-D_j v_j(t) \leq \frac{dv_j(t)}{dt} \leq v_j(t) \left(-D_j + \sum_{i=1}^m k_{ij} \varepsilon_i \varphi_{ij}(\mathbf{0}, \mathbf{0}) \right)$$

and

$$\tilde{v}_{j1}(t) \leq v_j(t) \leq \tilde{v}_{j2}(t), \quad (11)$$

in which $\tilde{v}_{j1}(t)$ is the system (2)'s positive periodic solutions and $\tilde{v}_{j2}(t)$ is the system (2)'s positive periodic solutions with $-D_j$ changes into $-D_j + \sum_{i=1}^m k_{ij} \varepsilon_i \varphi_{ij}(\mathbf{0}, \mathbf{0})$. By employing Lemma 4 and for all sufficiently large t , we can obtain

$$\tilde{v}_{j1}(t) \rightarrow \tilde{v}_j(t), \tilde{v}_{j2}(t) \rightarrow \tilde{v}_j(t).$$

From inequality (11) we can obtain $\lim_{t \rightarrow \infty} v_j(t) \rightarrow \tilde{v}_j(t)$ ($1 \leq j \leq n$). \square

According to Theorem 1, we select the parameters $m = 1$, $n = 1$, and allow the functional response $\varphi(u(t), v(t))$ to be in the following forms, respectively:

$$\varphi(u(t), v(t)) = \frac{a}{b + cu(t)}, \varphi(u(t), v(t)) = \frac{m}{a + u^2(t)},$$

$$\varphi(u(t), v(t)) = \frac{h(1 - e^{-cu(t)})}{u(t)}, \varphi(u(t), v(t)) = \frac{c}{mv(t) + u(t)}.$$

Then, the following corollary can be obtained:

Corollary 1. *Provided the following inequalities are established,*

$$\ln(1 - p_1) + rT - \frac{aB}{b} < 0. \quad (12)$$

$$\ln(1 - p_1) + rT - \frac{mB}{a} < 0. \quad (13)$$

$$\ln(1 - p_1) + rT - hcB < 0. \quad (14)$$

$$\ln(1 - p_1) + rT - \frac{cT}{m} < 0. \quad (15)$$

then we can obtain the locally asymptotically stable periodic solution about prey-free $(0, \tilde{v}(t))$, where $B \triangleq \int_0^T \tilde{y}(s) ds = \frac{\mu[1 - p_2 \exp(-DIT) - (1 - p_2) \exp(-DT)]}{d[1 - (1 - p_2) \exp(-DT)]}$.

Remark 1. The inequalities (12)–(15) of the Corollary 3.3 are identical to the Theorem 3.2 in [14], the Theorem 3.3 in [15], the Theorem 3.2 in [18], and also the Theorem 3.3 of the literature [20], respectively. This again shows that the above results in Theorems 3.1 and 3.2 are the generalized solutions for modeling the predator–prey system using the general response function $\varphi(u(t), v(t))$.

Theorem 3. When the parameters in the ecosystem (1) satisfy the following inequalities for $1 \leq i \leq m$,

$$\ln(1 - \delta_i) + (a_i - \lambda_i)T - \sum_{j=1}^n \int_0^T \tilde{v}_j(t) \varphi_{ij}(\mathbf{0}, \tilde{\mathbf{V}}(t)) dt > 0 \quad (1 \leq i \leq m), \quad (16)$$

then the system (1) is permanent.

Proof. According to Lemma 5, which implies that there exists a positive constant number M such that $u_i(t) \leq M, v_j(t) \leq M$ as $t \rightarrow \infty$, where $M = \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} (\frac{a_i + \lambda_i}{b_{ii}}, \tilde{v}_j(t) + \varepsilon_j)$. This

holds for all $t > 0$, according to Theorem 2. Let $\tilde{m}_j = \tilde{v}_j(t) - \varepsilon_j > 0$, and according to Lemma 4, such that $v_j(t) > \tilde{m}_j$ can be obtained. In the following, we only need to find $m_0 > 0$ such that $u_i(t) > m_0$ ($1 \leq i \leq m$) for all large enough t .

Similarly, let us also consider the following impulsive differential equations:

$$\left\{ \begin{array}{l} \frac{du_{i2}(t)}{dt} = u_{i2}(t)(a_i - \sum_{l=1}^m b_{il}u_{l2}(t) - \lambda_i) - \sum_{j=1}^n u_{i2}(t)v_{j2}(t)\varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t)) \\ \frac{dv_{j2}(t)}{dt} = v_{j2}(t)(-D_j + \sum_{i=1}^m k_{ij}u_{i2}(t)\varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t))) \\ t \neq (m+q-1)T, t \neq mT, 1 \leq i \leq m, 1 \leq j \leq n \\ \Delta u_{i2}(t) = -\delta_i u_{i2}(t) \\ \Delta v_{j2}(t) = -l_j v_{j2}(t) \end{array} \right\} t = (m+q-1)T$$

$$\left\{ \begin{array}{l} \Delta u_{i2}(t) = 0 \\ \Delta v_{j2}(t) = \mu_j \end{array} \right\} t = mT$$
(17)

Since $u_{i2}(t) \leq u_i(t)$ and $v_{j2}(t) \leq v_j(t)$ and both are valid for any $t > 0$, it is only necessary to prove that $u_{i2}(t) > m_0$ ($1 \leq i \leq m$) as $t \rightarrow \infty$. This can be achieved by the following two steps:

1. Now we can claim that there exist $t_1, t_2, \dots, t_m > 0$ such that $u_{i2}(t_i) > m_i$ for $1 \leq i \leq m$. Otherwise, without loss of generality, for any $t > 0$ we have $u_{12}(t) \leq m_1$. Let $m_1 > 0$ to be small enough such that

$$-D_j + k_{1j}m_1\varphi_{1j}(\mathbf{0}, \mathbf{0}) + \sum_{i=2}^m k_{ij}\frac{a_i + \lambda_i}{b_{ii}}\varphi_{ij}(\mathbf{0}, \mathbf{0}) < 0,$$

$$\sigma = \ln(1 - \delta_1) + (a_1 - \lambda_1 - b_{11}m_1)T - \int_0^T \left(\sum_{l=2}^m b_{1l}M - R \right) dt > 0,$$

where $R \triangleq \sum_{j=1}^n (\tilde{\omega}_j(t) + \varepsilon_1)\varphi_{ij}(\mathbf{0}, \mathbf{0}, \dots, \tilde{\omega}_j(t) + \varepsilon_1, \mathbf{0}, \dots, \mathbf{0})$. It can be seen from system (17) that

$$\frac{dv_{j2}(t)}{dt} \leq v_{j2}(t) \left(-D_j + k_{1j}m_1\varphi_{1j}(\mathbf{0}, \mathbf{0}) + \sum_{i=2}^m k_{ij}\frac{a_i + \lambda_i}{b_{ii}}\varphi_{ij}(\mathbf{0}, \mathbf{0}) \right).$$

Then, it can be seen that $v_{j2}(t) \leq \omega_j(t)$, $\omega_j(t) \rightarrow \tilde{\omega}_j(t)$ as $t \rightarrow \infty$, where $\omega_j(t)$ is the solution of the following cosystem:

$$\left\{ \begin{array}{l} \frac{d\omega_j(t)}{dt} = \left(-D_j + k_{1j}m_1\varphi_{1j}(\mathbf{0}, \mathbf{0}) + \sum_{i=2}^m k_{ij}\frac{a_i + \lambda_i}{b_{ii}}\varphi_{ij}(\mathbf{0}, \mathbf{0}) \right) \omega_j(t), \\ t \neq (m+q-1)T, t \neq mT, 1 \leq i \leq m, 1 \leq j \leq n \\ \omega_j(t^+) = (1 - l_j)\omega_j(t), \quad t = (m+q-1)T \\ \omega_j(t^+) = \omega_j(t) + \mu_j, \quad t = mT \\ \omega_{j0} = v_{j2}(0^+) \end{array} \right.$$
(18)

and we can also obtain the positive periodic solution:

$$\tilde{\omega}_j(t) = \begin{cases} \frac{\mu_j \exp \left\{ \left(-D_j + k_{1j}m_1\varphi_{1j}(\mathbf{0}, \mathbf{0}) + \sum_{i=2}^m k_{ij}\frac{a_i + \lambda_i}{b_{ii}}\varphi_{ij}(\mathbf{0}, \mathbf{0}) \right) [t - (m-1)T] \right\}}{1 - (1 - l_j) \exp \left[\left(-D_j + k_{1j}m_1\varphi_{1j}(\mathbf{0}, \mathbf{0}) + \sum_{i=2}^m k_{ij}\frac{a_i + \lambda_i}{b_{ii}}\varphi_{ij}(\mathbf{0}, \mathbf{0}) \right) T \right]}, & (m-1)T < t \leq (m+q-1)T \\ \frac{\mu_j (1 - l_j) \exp \left\{ \left(-D_j + k_{1j}m_1\varphi_{1j}(\mathbf{0}, \mathbf{0}) + \sum_{i=2}^m k_{ij}\frac{a_i + \lambda_i}{b_{ii}}\varphi_{ij}(\mathbf{0}, \mathbf{0}) \right) [t - (m-1)T] \right\}}{1 - (1 - l_j) \exp \left[\left(-D_j + k_{1j}m_1\varphi_{1j}(\mathbf{0}, \mathbf{0}) + \sum_{i=2}^m k_{ij}\frac{a_i + \lambda_i}{b_{ii}}\varphi_{ij}(\mathbf{0}, \mathbf{0}) \right) T \right]}, & (m+q-1)T < t \leq mT. \end{cases}$$

Thus, there exists $T_1 > 0$ such that $v_{j2}(t) \leq \omega_j(t) \leq \tilde{\omega}_j(t) + \varepsilon_1$, and

$$\frac{du_{12}(t)}{dt} \geq u_{12}(t) \left(a_1 - \lambda_1 - b_{1l}m_1 - \sum_{l=2}^m b_{il}M - R \right). \quad (19)$$

Through integrating the above inequality (19) over $[(m+q)T, (m+q+1)T]$, the following can be obtained:

$$\begin{aligned} u_{12}((m+q)T) &\geq u_{12}((m+q-1)T^+)C \\ &= u_{12}((m+q-1)T)(1-\delta_1)C \\ &= u_{12}((m+q-1)T) \exp(\sigma). \end{aligned}$$

where $C = \exp\left(\int_{(m+q-1)T}^{(m+q)T} (a_1 - \lambda_1 - b_{1l}m_1 - \sum_{l=2}^m b_{il}M - R)dt\right)$. Therefore,

$$u_{12}((m+q)T) \geq u_{12}(qT) \exp(m\sigma) \rightarrow \infty (m \rightarrow \infty),$$

which contradicts $u_{12}(t) \leq M$. Therefore, there must exist $t_1 > 0$ such that $u_{12}(t_1) > m_1$. Similarly, we can deduce the existence of $t_2, \dots, t_m > 0$ such that $u_{i2}(t_i) > m_i$ for $2 \leq i \leq m$.

2. Next, we will prove the claim that when $t > t_1$, then $u_{12}(t_1) > m_1$ is obtained. Otherwise, for some $t > t_1$, it can be seen that $u_{12}(t) \leq m_1$. Placing $\tilde{t} = \inf_{t \geq t_1} \{u_{12}(t) < m_1\}$, it can be seen that $u_{12}(t_1) \geq m_1$ for $t \in [t_1, \tilde{t}]$ and $\tilde{t} \in [p_1T, (p_1+1)T]$, $p_1 \in N$. It is easy to deduce from the continuity of the $u_{12}(t)$ that $u_{12}(\tilde{t}) = m_1$. For $p_2, p_3 \in N$, such that

$$p_2T > \frac{1}{\left(-D_j + k_{1j}m_1\varphi_{1j}(\mathbf{0}, \mathbf{0}) + \sum_{i=2}^m k_{ij}\frac{a_i+\lambda_i}{b_{ii}}\varphi_{ij}(\mathbf{0}, \mathbf{0})\right)} \ln \frac{\varepsilon_1}{M + \mu_j},$$

$$\exp(\delta(p_2+1)T) \exp(p_3\sigma) > 1,$$

where $\delta \triangleq a_1 - \lambda_1 - b_{1l}m_1 - \sum_{l=2}^m b_{il}M - \sum_{j=1}^n M\varphi_{ij}(\mathbf{0}, \mathbf{0}, \dots, M, \mathbf{0}, \dots, \mathbf{0}) < 0$.

By setting $T' = (p_2 + p_3)T$, we can obtain the claim that there exists $t_2 \in [(p_1+1)T, (p_1+1)T + T']$ which implies that $u_{12}(t_1) \geq m_1$. Otherwise, $u_{12}(t_1) < m_1$, and from the system (17), we can obtain that when $\omega((p_1+1)T^+) = v_{j2}((p_1+1)T^+)$:

$$\omega_j(t) = \begin{cases} (1-l_j)^{m-1} \left(\omega_j((p_1+1)T^+) - \frac{\mu_j}{1-(1-l_j)\exp[C_jT]} \right) \times \\ \exp[C_j(t - (p_1+1)T)] + \tilde{\omega}_j(t) \\ (1-l_j)^m \left(\omega_j((p_1+1)T^+) - \frac{\mu_j}{1-(1-l_j)\exp[C_jT]} \right) \times \\ \exp[C_j(t - (p_1+1)T)] + \tilde{\omega}_j(t). \end{cases}$$

Therefore,

$$|\omega_j(t) - \tilde{\omega}_j(t)| < (M + \mu_j) \exp[C_j(t - (p_1+1)T)] < \varepsilon_1,$$

in which $C_j = -D_j + k_{1j}m_1\varphi_{1j}(\mathbf{0}, \mathbf{0}) + \sum_{i=2}^m k_{ij}\frac{a_i+\lambda_i}{b_{ii}}\varphi_{ij}(\mathbf{0}, \mathbf{0})$, and when $(p_1 + p_2 + 1)T \leq t \leq (p_1 + 1)T + T'$, it can be obtained that

$$v_{j2}(t) \leq \omega_j(t) \leq \tilde{\omega}_j(t) + \varepsilon_1.$$

Similarly, according to the inequality (19), it can be seen that

$$u_{12}((p_1 + p_2 + p_3 + 1)T) \geq u_{12}((p_1 + p_2 + 1)T) \exp(p_3\sigma). \quad (20)$$

It is found that when $t \in (\tilde{t}, (p_1 + 1)T]$, there are two possible cases.

The first case: If $u_{12}(t) \leq m_1$ for $t \in (\tilde{t}, (p_1 + 1)T]$, then $u_{12}(t) \leq m_1$ for all $t \in (\tilde{t}, (p_1 + p_2 + 1)T]$. From system (17) we can observe that

$$\frac{du_{12}(t)}{dt} \geq u_{12}(t)\delta, \quad (21)$$

where $\delta = a_1 - \lambda_1 - b_{11}m_1 - \sum_{l=2}^m b_{il}M - \sum_{j=1}^n M\varphi_{ij}(\mathbf{0}, 0, \dots, M, 0, \dots, 0)$. Then, by integrating the inequality (21) over $(\tilde{t}, (p_1 + p_2 + 1)T]$, it can be seen that

$$u_{12}((p_1 + p_2 + 1)T) \geq m_1 \exp(\delta(p_2 + 1)T). \quad (22)$$

By the simple calculation of inequalities (20) and (22), it can be seen that

$$u_{12}((p_1 + p_2 + p_3 + 1)T) \geq m_1 \exp(\delta(p_2 + 1)T) \exp(p_3\sigma) > m_1,$$

which is in contradiction to $u_{12}(t) \leq m_1$ for all t . Let us input $t^* = \inf_{t \geq \tilde{t}} \{u_{12}(t) \geq m_1\}$, then $u_{12}(t^*) = m_1$. The inequality (21) holds for $t \in [\tilde{t}, t^*)$ and the integration over $t \in [\tilde{t}, t^*)$ to obtain

$$u_{12}(t) \geq u_{12}(\tilde{t}) \exp(\delta(t - \tilde{t})) \geq m_1 \exp(\delta(p_2 + p_3 + 1)T) \triangleq m_0.$$

Due to $u_{12}(t^*) \geq m_1$, the same argument applies to $t > t^*$. Thus, $u_{12}(t) \geq m_0$ holds for all $t > t_1$.

The second case: There exists $t' \in (\tilde{t}, (n_1 + 1)T]$ which implies that $u_{12}(t') > m_1$. By setting $\hat{t} = \inf_{t \geq \tilde{t}} \{u_{12}(t) \geq m_1\}$, then $u_{12}(t) < m_1$ holds for $t \in [\tilde{t}, \hat{t})$ and $u_{12}(\hat{t}) = m_1$. Then, the inequality (21) holds for $t \in [\tilde{t}, \hat{t})$ and thus we can also observe that

$$u_{12}(t) \geq u_{12}(\hat{t}) \exp(\delta(t - \hat{t})) \geq m_1 \exp(\delta T) \geq m_0.$$

This process can continue as $u_{12}(t) \geq m_0$ and we can obtain $u_{12}(t) \geq m_0$ for all $t \geq t_1$.

Thus, in both cases, we can conclude that $u_1(t) \geq m_0$ as $u_{12}(t) \leq u_1(t)$ and it holds for all $t \geq t_1$. In the same way, we can also prove that other prey populations are permanent, namely, $u_i(t) \geq m_0$ ($2 \leq i \leq m$). \square

According to Theorem 3, we select the parameters $m = 1$, $n = 1$ and allow the functional response $\varphi(u(t), v(t))$ to be in the following forms, respectively:

$$\begin{aligned} \varphi(u(t), v(t)) &= \frac{a}{b + cu(t)}, \varphi(u(t), v(t)) = \frac{m}{a + u^2(t)}, \\ \varphi(u(t), v(t)) &= \frac{h(1 - e^{-cu(t)})}{u(t)}, \varphi(u(t), v(t)) = \frac{c}{mv(t) + u(t)}. \end{aligned}$$

Then, the following corollary can be obtained:

Corollary 2. *Provided the following inequalities are established*

$$\ln(1 - p_1) + rT - \frac{aB}{b} > 0. \quad (23)$$

$$\ln(1 - p_1) + rT - \frac{mB}{a} > 0. \quad (24)$$

$$\ln(1 - p_1) + rT - hcB > 0. \quad (25)$$

$$\ln(1 - p_1) + rT - \frac{cT}{m} > 0. \quad (26)$$

then the system (1) is permanent, where $B \triangleq \int_0^T \tilde{y}(s) ds = \frac{\mu[1-p_2 \exp(-DIT) - (1-p_2) \exp(-DT)]}{d[1 - (1-p_2) \exp(-DT)]}$.

Remark 2. The inequalities (23)–(26) of the Corollary 3.3 are identical to the Theorem 3.2 in [14], the Theorem 3.3 in [15], the Theorem 3.2 in [18], and also the Theorem 3.3 of the literature [20] respectively. This again shows that the above results in Theorems 3.1 and 3.2 are the generalized solutions for modeling the predator–prey system using the general response function $\varphi(u(t), v(t))$.

4. Biological Significance of the Model

This study extends previous work [1–20] by considering environmental factors, such as the seasonal effect, to model the stability of an m-prey and n-predator system which is subjected to the inter- and intra-species competition amongst the predators. In this work, an adaptive impulsive control strategy, which releases pesticides and natural enemy independently, was applied as a means to control pests in the model. One distinct feature of the present work is the use of a generalized functional response for modeling the evolution of m-prey species, rather than using specific functional responses such as those that have been commonly undertaken in the past. The generalized solutions of the theorems obtained from this work have been shown to agree well with previous studies reported in the literature [1–20], when various forms of the specific functional responses are substituted into our generalize solutions. The sufficient condition for the system (1)’s permanence and the prey-free periodic solutions asymptotic stability that were obtained in this work are shown in Section 3.

In this subsection, we would like to discuss the implication of the present work from the biological viewpoint. It is noted that the time derivative of the i -th pest species under model (1) can be expressed as

$$\frac{du_i(t)}{dt} = u_i(t) \left(a_i - \sum_{l=1}^m b_{il} u_l(t) + \lambda_i \sin(\omega_i t) - \sum_{j=1}^n v_j(t) \varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t)) \right).$$

Thus, when the i -th pest species is subjected to a loss in a period T due to the predation of the predator, and if the pest management is implemented for the biological control at the same time, then this situation can be represented by $\sum_{j=1}^n \int_0^T \tilde{v}_j(t) \varphi_{ij}(\mathbf{0}, \tilde{\mathbf{V}}(t)) dt$. If there is enough food, for example during the warm weather in the spring/summer seasons, the species growth rate may reach maximum as a result of the favorable living conditions. Since the populations of the species vary with the seasonal factors $\lambda_i \sin(\omega_i t)$, it can be adjusted to the maximum λ_i , and therefore the term $(a_i + \lambda_i)T$ will represent the i -th pest species’ maximum per-capita growth in the period T . Similarly, the term $(a_i - \lambda_i)T$ may also represent the i -th pest species’ minimum per-capita growth in a period T during the unfavorable cold season. The additional loss of the i -th pest species due to the chemical control through pesticide spraying can be given by $-\ln(1 - \delta_i)$.

Consequently, when the total loss exceeds the maximum total growth in a period T due to the chemical and the biological control:

$$(a_i + \lambda_i)T < \sum_{j=1}^n \int_0^T \tilde{v}_j(t) \varphi_{ij}(\mathbf{0}, \tilde{\mathbf{V}}(t)) dt - \ln(1 - \delta_i).$$

It is the natural conclusion that when $\lim_{t \rightarrow \infty} u_i(t) \rightarrow 0$ ($1 \leq i \leq m$), which satisfies the Theorem 1.

As the global stability condition is stronger than the condition of the local asymptotic stability, it requires that all prey population is near to extinction in any circumstances. Due to $\varphi_{ij}(\mathbf{U}(t), \mathbf{V}(t))$ is assumed to be monotonous decreasing with respect to $u_i(t)$ and $u_i(t) < \frac{a_i + \lambda_i}{b_{ii}}$, so $\sum_{j=1}^n \int_0^T \tilde{v}_j(t) \varphi_{ij}(\mathbf{K}, \tilde{\mathbf{V}}(t)) dt$ is the minimum loss due to predation by the predators, where $\mathbf{K} = (\frac{a_1 + \lambda_1}{b_{11}}, \frac{a_2 + \lambda_2}{b_{22}}, \dots, \frac{a_m + \lambda_m}{b_{mm}})$. Consequently, when the minimum loss surpasses the maximum total growth in a period T , it satisfies the Theorem 2, namely,

$$(a_i + \lambda_i)T < \sum_{j=1}^n \int_0^T \tilde{v}_j(t) \varphi_{ij}(\mathbf{K}, \tilde{\mathbf{V}}(t)) dt - \ln(1 - \delta_i).$$

Similarly, when the minimum total growth exceeds the maximum cumulative size loss due to the biological control and the chemical control in a period T , the ecosystem (1) is permanent, which satisfies the Theorem 3, namely,

$$(a_i - \lambda_i)T > \sum_{j=1}^n \int_0^T \tilde{v}_j(t) \varphi_{ij}(\mathbf{0}, \tilde{\mathbf{V}}(t)) dt - \ln(1 - \delta_i).$$

Consider the situations when the pesticide have a profound effect on the pests, i.e., when δ_i is larger than the inequality in (6) and it can be seen that the condition is more favorably to satisfy. Note also that the larger the magnitude of the perturbation λ_i , the more time needed for the pests to become extinct, which imposes high impacts to the system (1) and may induce various dynamic activity to occur in the system. These theories can be applied not only to the integrated pest and rodent control process, but also can be applied equally well for monitoring other biological resources in the ecosystem and to protect other species, such as those of endangered ones, to help maintain the ecological balance better.

5. Conclusions

A seasonally competitive m -prey and n -predator impulsive complex system was established by a more general functional response for integrated pest management in this paper. We obtain the prey-free periodic solution's locally and globally asymptotic stability according to Theorems 3.1 and 3.2, respectively, and the permanence of the ecosystem is obtained in Theorem 3.5. These theorems are, in fact, the extension of existing results previously established in recent papers, including two species or three species in [2,12].

The end result of the present work can serve as an effective and practical pest management guide for integrated pest management. According to the different situations of the practical farmland, we can select the moderate parameters to make the studied ecosystem permanent, dramatically reduce the environmental pollution, and effectively eliminate the pest.

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