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Pseudo Steady-State Period in Non-Stationary Infinite-Server Queue with State Dependent Arrival Intensity

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Abstract: An infinite-server queueing model with state-dependent arrival process and exponential distribution of service time is analyzed. It is assumed that the difference between the value of the arrival rate and total service rate becomes positive starting from a certain value of the number of customers in the system. In this paper, time until reaching this value by the number of customers in the system is called the pseudo steady-state period ($PSSP$). Distribution of duration of $PSSP$, its raw moments and its simple approximation under a certain scaling of the number of customers in the system are analyzed. Novelty of the considered problem consists of an arbitrary dependence of the rate of customer arrival on the current number of customers in the system and analysis of time until reaching from below a certain level by the number of customers in the system. The relevant existing papers focus on the analysis of time interval since exceeding a certain level until the number of customers goes down to this level (congestion period). Our main contribution consists of the derivation of a simple approximation of the considered time distribution by the exponential distribution. Numerical examples are presented, which confirm good quality of the proposed approximation.

Keywords: infinite-server queue; non-stationary regime; steady-state period; asymptotic analysis

MSC: 60K25

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1. Introduction

Queueing models can be helpful for solving various problems related to optimization of customer access to a certain restricted resource. One of the popular schemes for organizing such an access is to split the resource into several (more often equal) parts called servers and assign a dedicated server to provide the service (use of the resource) to each newly arriving request for service (customer). One of the most exhaustively investigated kind of such models assumes that the number of available servers is infinite [1–7]. The assumption that the number of servers in the system is infinite is not always realistic, but it greatly simplifies mathematical analysis of the corresponding queueing systems. For example, while the constructive results for the system providing service to the stationary Poisson arrival flow of customers and having a finite number, say, N, $N \geq 2$, of servers are known only for the cases when the system has no buffer, see [8], or service time distribution function has some partial forms, e.g., the phase-type (PH) distribution, see, e.g., [9–11], the distribution of the number of customers present in the system with the infinite number of servers is well known.

Some known results for infinite-server queues with the stationary Poisson arrival process (with rate λ) are as follows.

Let us first consider the $M/M/\infty$ system. Service time has an exponential distribution with rate μ and let $\rho = \frac{\lambda}{\mu}$ be the traffic intensity.

Let $n(t)$ be the number of customers in the system at moment t and $n(0) = k$ where $k \geq 0$. Denote by $\pi_k(n,t)$ the probability that $n(t) = n,\ n \geq 0$, and by $\Pi_k(z,t)$ the generating function of probabilities $\pi_k(n,t)\ n \geq 0:$

$$\Pi_k(z,t) = \sum_{n=0}^{\infty} \pi_k(n,t) z^n,\ |z| \leq 1.$$

Generating function $\Pi_k(z,t)$ has the following form:

$$\Pi_k(z,t) = (1 + (z-1)e^{-\mu t})^k e^{\rho(z-1)(1-e^{-\mu t})}.$$

In the case when $k = 0$ (the system is empty at the moment $t = 0$), probabilities $\pi_0(n,t)$ are defined by formula:

$$\pi_0(n,t) = \frac{(\rho(1-e^{-\mu t}))^n}{n!} e^{-\rho(1-e^{-\mu t})}, n \geq 0,$$

i.e., they define the Poisson distribution with parameter $\rho(1 - e^{-\mu t})$.

Let also $j(t)$ be the number of customers that received service in the system during the interval $(0,t)$. Denote by $q_k(n,j,t)$ the probability that $n(t) = n,\ n \geq 0,\ j(t) = j, j \geq 0$ and by $Q_k(z,y,t)$ the generating function of probabilities $q_k(n,j,t)\ n \geq 0:$

$$Q_k(z,y,t) = \sum_{j=0}^{\infty} \sum_{n=\max\{0,k-j\}}^{\infty} q_k(n,j,t) z^n y^j,\ |z| \leq 1,\ |y| < 1.$$

Generating function $Q_k(z,y,t)$ has the following form:

$$Q_k(z,y,t) = (y + (z-y)e^{-\mu t})^k e^{\lambda t(y-1) + \rho(z-y)(1-e^{-\mu t})},$$

see [12].

Convergence results for an $M/M/\infty$ system are given in [13] for transient characteristics, such as the period of time the occupation process remains above a given state and the number of customers arriving during this period. In [14], the authors consider the $M/M/\infty$ queue with traffic intensity ρ. They analyze the first passage distribution of the time the number of customers $n(t)$ reaches the certain level c, starting from $n(0) = m > c$. If $m = c + 1$, they refer to this time period as the congestion period above the level c. They give asymptotic expansions for the distribution of the first passage time for $\rho \to \infty$, various ranges of m and c, and several different time scales.

In [15], the authors derive the queue size distribution for the case times when the parameters of both exponential distributions of inter-arrival and service times are not constant. They are allowed to vary with time. Both continuous and discrete variation are examined. A similar model was examined in [16].

Quite a lot of attention is paid to the infinite-server systems where the parameters of exponential distributions of inter-arrival and service times depend on time via their dependence on the state of some external stochastic process (random environment). In [17], the steady-state behavior of the $M/M/\infty$ queue operating in a Markovian random environment, which modulates the arrival and service rates, is studied. Explicit results are obtained for the factorial moments, the impossibility of a 'matrix-Poisson' steady-state distribution is demonstrated. Similar results were obtained in [18]. In [19], the similar system operating in a semi-Markovian random environment is considered. The mean number of customers in the system in steady state is found. In a particular case when the random environment has only two states, the distribution of the number of customers in the system is found. In papers [20,21], similar to [19] results are obtained. In [22], the authors consider an $M/M/\infty$ queueing system subject to random interruptions of exponentially distributed durations. System breakdowns, where none of the servers work, as well as partial failures, where all servers work with lower efficiency, are investigated. In both cases, it is shown that the number of customers present in the system in equilibrium is the sum of two independent random variables. One of these is the number of customers present in an ordinary $M/M/\infty$ queue without interruptions. Some results on infinite-server queues with state dependent arrivals may be found also in [23].

Let us now briefly touch the $M/G/\infty$ system. Let service time be defined by the distribution function $B(t)$. Mean service time is denoted by $b_1 : \ b_1 = \int_0^\infty (1 - B(u))du$. For this system, the generating function $\Pi_k(z,t)$ has the following form:

$$\Pi_k(z,t) = (1 + (z-1)(1 - \tilde{B}(t)))^k e^{\rho(z-1)\tilde{B}(t)},$$

see [24]. Here, $\tilde{B}(t)$ is a distribution function of elapsed to or residual from an arbitrary moment service time. It is defined by formula $\tilde{B}(t) = b_1^{-1} \int_0^t (1 - B(u))du$.

The $M/G/\infty$ system operating in a Markovian random environment is analyzed in [25]. A similar model was considered in [26]. The model in a random fastly oscillating environment was considered in [27,28].

It is worth noting that the distribution given by the generating function $\Pi_k(z,t)$ presented above has a nice analytical form and is obtained not only for the so-called stationary regime of a system operation, but also for the time-dependent scenario. On the other hand, there are approaches for how to apply results obtained for infinite-server queues for real world systems with a limited number of servers, see [29–33].

Computation of the time-dependent distribution [34,35] of the states of any queueing system is very important for practical purposes. This computation is, as a rule, a very difficult task and researchers are compelled to restrict themselves to the analysis of a time-independent (or so called stationary) distribution of the states of the queueing system. The stationary distribution does not always exist and often a solid amount of work should be spent to establish limitations on the system parameters for which the stationary distribution exists. These limitations usually constitute the so called ergodicity condition for the corresponding queueing system. Roughly speaking, such a condition requires that in a situation when the number of customers presenting in a queueing system is very large, the rate of new customer arrivals is less than the rate of customer departures from the system.

In this paper, we analyze the infinite-server queueing system in which the condition that the rate of new customer arrivals is less than the rate of customer departures from the system is not fulfilled starting from some, probably large, number, say, i_2 of customers in the system. Because the rate of new customer arrivals exceeds the rate of customers departure after the reaching the number i_2 of customers in the system, it is clear that the stationary regime of operation of this system does not exist. After reaching this level i_2, the number of customers in the system will increase to infinity. However, during a certain, probably pretty long, period of time, until the number of customers in the system reaches value i_2, the system has the period of relatively stable operation that we call here the pseudo steady-state period. Duration of such a period may be of a practical interest in many real world systems, e.g., in the description of various chemical or biological reactions, see, e.g., [36]. As another example of potential applications of the considered queueing system, we can consider the modelling of the spread of some disease, such as COVID-19. If the rate of new cases of disease occurrence, depending on the current number of infected patients, can be evaluated from the statistical data as well as the individual rate of the patients recovering are known, the results presented in our paper can be used for exact evaluation of the probabilistic distribution of time until the number of ill patients reaches some threshold value and a certain external intervention into the process of the disease spreading will be required. Note, that, in contrast to the existing papers, see, e.g., [13,14], where the subject of interest is the period of exceeding some level by the number of customers in the system, arrival rate is constant and the existence of the stationary state probabilities is suggested, we analyze the period until reaching some level by the number of customers in the system, arrival rate is state dependent and the existence of the stationary state probabilities is not supposed.

In the example with the COVID-19 (or other infectious disease) potential application, we consider the evaluation of the speed of a disease spreading (indirectly characterized by the distribution of the length period of time since disease occurrence moment until reaching some epidemic announcement threshold), but not the duration of the epidemy after it is announced, conditional that it will end. From the practical point of view, the former period's evaluation looks more important than the latter ones.

The rest of the paper is organized as follows. In Section 2, the mathematical model under study is formulated and the notion of the pseudo steady-state period ($PSSP$) is introduced. Conditional moment-generating functions of the $PSSP$ are introduced in Section 3 and the system of the linear algebraic equations for these functions is derived. In Section 4, a similar system for the conditional characteristic functions is written down. The system is rewritten in the matrix form and the expression for characteristic function $h(u)$ of $PSSP$ length is obtained. The problem of sequential computation of raw moments of the $PSSP$ length is solved in Section 5. Numerical examples are presented in Section 6. Results of the computation of raw moments of orders 1, 2, 3 are presented. A good quality of approximation of distribution of $PSSP$ length by exponential distribution is demonstrated. Section 7 concludes the paper.

2. Mathematical Model

Consider an infinite-server queueing system. Denote the number of customers in the system at instant t by $i = i(t),\ t \geq 0$. Let the arrival process be non-stationary Poisson and its intensity λ_i depends on the number of customers in the system $i = i(t)$. Let us write the intensity in the form $\lambda_i = \lambda(i/N)$, where $\lambda(x)$ is a positive differentiable function with continuous argument x and parameter $N > 0$ has a meaning of a scale or sensitivity in relation to the number of customers.

Let service time of a customer have the exponential distribution with rate μ_1, which we choose in the form $\mu_1 = \mu/N$ for the convenience of the further derivations (here, μ is some fixed value).

We define function $\lambda(x)$ in the following way. Let us denote:

$$a(i) = \lambda_i - i\frac{\mu}{N} = \lambda\left(\frac{i}{N}\right) - i\frac{\mu}{N}. \tag{1}$$

Function $a(i)$ of discrete argument i has the meaning of the excess of the flow rate over the departure rate when the number of customers in the system is equal to $i,\ i \geq 0$. We assume that function $a(i)$ of argument i satisfies the following condition: there exist such values of argument $i = i_1$ and $i = i_2$ that provide:

$$a(i) = \begin{cases} > 0 & \text{for} \quad 0 \leq i < i_1, \\ < 0 & \text{for} \quad i_1 \leq i \leq i_2, \\ > 0 & \text{for} \quad i_2 < i < \infty. \end{cases} \tag{2}$$

The practical meaning of conditions (2) are in the existence of some interval between values i_1 and i_2 of the number of customers in the system when 'classical' stationarity condition $\rho < 1$ is satisfied, but if the number of customers i grows over i_2, the overflow regime begins. On the other hand, $a(i)$ plays a role of the drift coefficient of a certain diffusion process that determines the distribution of a scaled number of customers in the system, similar to the same process studied in [37].

Because $a(i) > 0$ for all $i > i_2$, the considered queueing system is non-stationary and the true steady-state regime of its operation does not exist. However, there is a period of the system evolution inside which the queue behavior may look like a steady-state one. We call this period a pseudo steady-state period ($PSSP$). It corresponds to values $\{i : 0 \leq i < i_2\}$ of the number of customers in the system. When the number of customers $i(t)$ reaches values in interval $[0; i_2)$, it may lay inside this interval during a long enough time, so it seems like the queue is in the steady-state regime during this period.

Denote the length of the interval starting from time t, when $i(t) = i \leq i_2$, until the time when process $i(u),\ u \geq t$, reaches value $i_2 + 1$, by $T(i,t)$. We mean that if process $i(t)$ reaches value $i_2 + 1$, then the system leaves the $PSSP$ and, therefore, this period ends. We assume that the $PSSP$ starts exactly when $i(t) = i_1$ and continues until the process $i(t)$ transits to the state $i(t) = i_2 + 1$. Therefore, the length of $PSSP$ is equal to the value $T(i_1, t)$. The goal of the paper is to obtain probabilistic characteristics of the length of $PSSP$.

3. Moment-Generating Function of the *PSSP*

Denote the conditional moment-generating functions of the $PSSP$ length by:

$$G(\alpha, i, t) = \mathbb{E}\left\{e^{-\alpha T(i(t),t)} | i(t) = i\right\},\ i = 0, 1, \dots, i_2. \tag{3}$$

Let us derive the Kolmogorov equation for these functions. It is easy to obtain the following system of the difference equations:

$$G(\alpha, i, t - \Delta t) = \mathbb{E}\left\{e^{-\alpha T(i(t-\Delta t), i-\Delta t)} | i(t - \Delta t) = i\right\} =$$

$$(1 - \lambda_i \Delta t)\left(1 - i\frac{\mu}{N}\Delta t\right)\mathbb{E}\left\{e^{-\alpha(T(i(t),t)+\Delta t} | i(t) = i\right\} +$$

$$\lambda_i \Delta t G(\alpha, i+1, t) + i\frac{\mu}{N}\Delta t G(\alpha, i-1, t) + o(\Delta t) =$$

$$(1 - \lambda_i \Delta t)\left(1 - i\frac{\mu}{N}\Delta t\right)e^{-\alpha\Delta t} G(\alpha, i, t) + \lambda_i \Delta t G(\alpha, i+1, t) + i\frac{\mu}{N}\Delta t G(\alpha, i-1, t) + o(\Delta t)$$

and, then, the system of the differential equations:

$$-\frac{\partial G(\alpha, i, t)}{\partial t} = -\left(\lambda_i + i\frac{\mu}{N} + \alpha\right)G(\alpha, i, t) + \lambda_i G(\alpha, i+1, t) + i\frac{\mu}{N}G(\alpha, i-1, t) \tag{4}$$

for $i = 0, 1, \dots, i_2$.

Due to the time homogeneity of process $i(t)$, i.e., due to independence of parameters N, μ, λ_i of t, we can use the relation:

$$G(\alpha, i, t) \equiv G(\alpha, i)$$

and rewrite the system of differential Equation (4) as follows:

$$-(\lambda_0 + \alpha)G(\alpha, 0) + \lambda_0 G(\alpha, 1) = 0,$$

$$-\left(\lambda_i + i\frac{\mu}{N} + \alpha\right)G(\alpha, i) + \lambda_i G(\alpha, i+1) + i\frac{\mu}{N}G(\alpha, i-1) = 0,\ i = 1, \dots, i_2. \tag{5}$$

Here, $G(\alpha, i_2 + 1) \equiv 1$ because $T(i,t) = 0$ for $i > i_2$.

The solution to (5) for $i = i_1$ is a moment-generating function of the $PSSP$ length.

4. Characteristic Function of the *PSSP*

Denoting,

$$j = \sqrt{-1}, \quad \alpha = -ju, \quad G(\alpha, i) = H(u, i)$$

and taking into account that $H(u, i_2 + 1) \equiv 1$, we can rewrite system (5) as the following system of equations for conditional characteristic functions $H(u,i)$ of the $PSSP$ length:

$$-(\lambda_0 - ju)H(u, 0) + \lambda_0 H(u, 1) = 0,$$

$$i\frac{\mu}{N}H(u, i-1) - \left(\lambda_i + i\frac{\mu}{N} - ju\right)H(u, i) + \lambda_i H(u, i+1) = 0,\ i = 1, \dots, i_2 - 1, \tag{6}$$

$$i_2\frac{\mu}{N}H(u, i_2 - 1) - \left(\lambda_{i_2} + i_2\frac{\mu}{N} - ju\right)H(u, i_2) = -\lambda_{i_2}.$$

Let us denote the following column vectors and matrix with dimension i_2+1 (we numerate their entries from 0 to i_2):

$$\mathbf{H}(u) = \begin{bmatrix} H(u,0) & H(u,1) & \dots & H(u,i_2-1) & H(u,i_2) \end{bmatrix}^T,$$

$$\mathbf{v} = \begin{bmatrix} 0 & 0 & \dots & 0 & -\lambda_{i_2} \end{bmatrix}^T,$$

$$\mathbf{A} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & \dots & 0 & 0 \\ \frac{\mu}{N} & -(\lambda_1+\frac{\mu}{N}) & \lambda_1 & \dots & 0 & 0 \\ 0 & 2\frac{\mu}{N} & -(\lambda_2+2\frac{\mu}{N}) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -(\lambda_{i_2-1}+(i_2-1)\frac{\mu}{N}) & \lambda_{i_2-1} \\ 0 & 0 & 0 & \dots & i_2\frac{\mu}{N} & -(\lambda_{i_2}+i_2\frac{\mu}{N}) \end{bmatrix}.$$

Now we can write system (6) in the form:

$$(\mathbf{A}+ju\mathbf{I})\mathbf{H}(u) = \mathbf{v},$$

where $\mathbf{I}$ is an identity matrix. Therefore, we express the vector $\mathbf{H}(u)$ as:

$$\mathbf{H}(u) = (\mathbf{A}+ju\mathbf{I})^{-1}\mathbf{v}. \tag{7}$$

Existence of the inverse matrix on the right hand side of (7) follows from the well-known O. Taussky theorem, see, e.g., [38], and strong dominance of the diagonal entry in the last row of the matrix.

Only one entry $H(u,i_1)$ of vector $\mathbf{H}(u)$ given by (7) is under our interest. Taking into account a structure of vector $\mathbf{v}$ whose entries are equal to 0 except the last one (with number i_2), which is equal to $-\lambda_{i_2}$, we can write the following expression for characteristic function $h(u)$ of $PSSP$ length:

$$h(u) = H(u,i_1) = -\lambda_{i_2}\left[(\mathbf{A}+ju\mathbf{I})^{-1}\right]_{i_1,i_2} \tag{8}$$

where $[\cdot]_{p,q}$ means an entry of the matrix with indices p,q.

5. Raw Moments of the Distribution of the *PSSP* Length

Expression (8) is simple enough but requires evaluation of an inverse matrix. If it is not necessary to obtain the entire probability distribution, we can reduce calculation cost by applying some direct evaluations. For example, in this section, we consider the evaluation of raw moments of the $PSSP$ length.

Denote the conditional raw moment of n-th order of the $PSSP$ length by: $T_n(i)$:

$$T_n(i) = \mathbb{E}\{T^n(i(t),t)|i(t)=i\}.$$

Here, $T_n(i)$ depends only on value i and does not depend on t due to the homogeneity of process $i(t)$. Because:

$$G(\alpha,i) = \sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{n!}T_n(i)$$

and $G(\alpha,i+1) \equiv 1$, we can substitute this expression into equations of system (5) and derive relations:

$$-\lambda_0\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{n!}T_n(0) + \alpha\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{n!}T_n(0) + \lambda_0\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{n!}T_n(1) = 0,$$

$$-\left(\lambda_i+i\frac{\mu}{N}\right)\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{n!}T_n(i) + \alpha\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{n!}T_n(i) + \lambda_i\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{n!}T_n(i+1)$$

$$+i\frac{\mu}{N}\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{n!}T_n(i-1)=0,\ i=1,\ldots,i_2-1,$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{n!}T_n(i_2+1)\equiv 1.$$

Now, we can write equalities for each value n of power α^n:

- for $n=1$:

$$-\lambda_0 T_1(0)+\lambda_0 T_1(1)+1=0,$$
$$-\left(\lambda_i+i\frac{\mu}{N}\right)T_1(i)+\lambda_i T_1(i+1)+i\frac{\mu}{N}T_1(i-1)+1=0, \tag{9}$$
$$T_1(i_2+1)=0.$$

- for $n\geq 2$:

$$-\lambda_0 T_n(0)+\lambda_0 T_n(1)+nT_{n-1}(0)=0,$$
$$-\left(\lambda_i+i\frac{\mu}{N}\right)T_n(i)+\lambda_i T_n(i+1)+i\frac{\mu}{N}T_n(i-1)+nT_{n-1}(i)=0, \tag{10}$$
$$T_n(i_2+1)=0.$$

So, we obtain the sequence of boundary value problems for finite difference equations of the second order with variable coefficients. Let us find their solutions.

5.1. Solution for $n=1$

Let us denote:

$$T_1(i)-T_1(i+1)=V_1(i). \tag{11}$$

Summing up these equalities, we obtain:

$$\sum_{m=0}^{i}T_1(m)-\sum_{m=0}^{i}T_1(m+1)=\sum_{m=0}^{i}V_1(m),$$

and it follows that:

$$T_1(0)-T_1(i+1)=\sum_{m=0}^{i}V_1(m). \tag{12}$$

Substituting $i=i_2$ and taking into account condition $T_1(i_2+1)=0$, we obtain:

$$T_1(0)=\sum_{m=0}^{i_2}V_1(m).$$

From (12), we have:

$$T_1(i+1)=T_1(0)-\sum_{m=0}^{i}V_1(m)=\sum_{m=0}^{i_2}V_1(m)-\sum_{m=0}^{i}V_1(m)=\sum_{m=i+1}^{i_2}V_1(m),$$

therefore,

$$T_1(i)=\sum_{m=i}^{i_2}V_1(m). \tag{13}$$

Using (9) and (11), we can write equalities:

$$\lambda_0 V_1(0)=1,\qquad \lambda_i V_1(i)-i\frac{\mu}{N}V_1(i-1)=1,$$

from which we derive the following recurrent expressions that determine $V_1(i)$ for all $0\leq i\leq i_2$:

$$V_1(0)=\frac{1}{\lambda_0},\qquad V_1(i)=\frac{1}{\lambda_i}\left(1+i\frac{\mu}{N}V_1(i-1)\right). \tag{14}$$

So, we can evaluate numerically all values $V_1(i)$ and, using (13), obtain values of $T_1(i)$ for all $0 \le i \le i_2$.

Substituting $i = i_1$, we can calculate the value of the first raw moment T_1 of the $PSSP$ length:

$$T_1 = T_1(i_1). \tag{15}$$

5.2. Solution for $n \ge 2$

We can build a recurrent procedure for evaluation $T_n(i)$ from (10) similar to the previous subsection.

Using (13) and (14), we can evaluate $T_1(i)$ for all $0 \ge i \ge i_2$. Let us suppose that we have found values $T_{n-1}(i)$, $0 \ge i \ge i_2$. Denote:

$$T_n(i) - T_n(i+1) = V_n(i). \tag{16}$$

Summing up these equalities, we obtain:

$$T_n(0) - T_n(i+1) = \sum_{m=0}^{i} V_n(m). \tag{17}$$

Substituting here $i = i_2$ and taking into account condition $T_n(i_2 + 1) = 0$, we obtain:

$$T_n(0) = \sum_{m=0}^{i_2} V_n(m).$$

From (17), we have:

$$T_n(i+1) = T_n(0) - \sum_{m=0}^{i} V_n(m) = \sum_{m=0}^{i_2} V_n(m) - \sum_{m=0}^{i} V_n(m) = \sum_{m=i+1}^{i_2} V_n(m),$$

therefore,

$$T_n(i) = \sum_{m=i}^{i_2} V_n(m). \tag{18}$$

Using (10) and (16), we can write equalities:

$$\lambda_0 V_n(0) = nT_{n-1}(0), \qquad \lambda_i V_n(i) - i\frac{\mu}{N} V_n(i-1) = nT_{n-1}(i),$$

from which we derive the following recurrent expressions that determine $V_n(i)$ for all $0 \le i \le i_2$:

$$V_n(0) = \frac{1}{\lambda_0} nT_{n-1}(0), \qquad V_n(i) = \frac{1}{\lambda_i}\Big(nT_{n-1}(i) + i\frac{\mu}{N} V_n(i-1)\Big). \tag{19}$$

Thus, we can evaluate numerically values $V_n(i)$ and, using (18), obtain values of $T_n(i)$ for all $0 \le i \le i_2$.

Substituting $i = i_1$, we can calculate value of the n-th raw moment T_n of the $PSSP$ length:

$$T_n = T_n(i_1).$$

6. Numerical Examples and Proposed Approximation of Distribution of the *PSSP* Length

Consider a numerical example. Let function $\lambda(x)$ in definition (1) where $x = \frac{i}{n}$ have the following form:

$$\lambda(x) = a(x - x_0)^2 + b,$$

where we set the values of parameters as follows:

$$a = 1, \quad x_0 = 2, \quad b = 1.$$

Further, let the parameter μ be equal to 1.

Then, the points i_1 and i_2 of the changing sign of the value of function $a(i)$, which determine the $PSSP$ interval, for a given value N can be found by formulas:

$$i_1 = \lceil x_1 N \rceil, \quad i_2 = \lfloor x_2 N \rfloor$$

where x_1 and x_1 are the roots of equation:

$$\lambda(x) - x\mu = 0. \tag{20}$$

For the chosen values of parameters $a,\ x_0,\ b = 1$, these roots are given by:

$$x_1 = 1.382, \quad x_2 = 3.618.$$

Let us evaluate three raw moments T_1, T_2, T_3 of the $PSSP$ length for various values of parameter N. The results are presented in Table 1.

Table 1. Values of raw moments T_n of the n-th order for the $PSSP$ length ($n = 1, 2, 3$) for various values of parameter N.

N	5	10	15	20	25	30
T_1	2.324×10^3	6.974×10^5	1.643×10^8	3.490×10^{10}	6.999×10^{12}	1.353×10^{15}
T_2	1.078×10^7	9.736×10^{11}	5.398×10^{16}	2.437×10^{21}	9.798×10^{25}	3.659×10^{30}
T_3	7.495×10^{10}	2.035×10^{18}	2.661×10^{25}	2.552×10^{32}	2.057×10^{39}	1.485×10^{46}

We can notice from the table that:

$$T_2 \approx 2T_1^2, \quad T_3 \approx 3!T_1^3. \tag{21}$$

Therefore, expressions (21) may be used for the approximate evaluation of the high-order moments in a simpler way than using procedure (18) and (19). Let us find relative errors of expression (21):

$$\delta_2 = \frac{|T_2 - 2T_1^2|}{T_2}, \quad \delta_3 = \frac{|T_3 - 3!T_1^3|}{T_3}. \tag{22}$$

The results are presented in Table 2. Thus, we see that the errors of approximations (21) are small enough for $N = 5$ and greatly decrease with growing values of parameter N.

Table 2. Relative errors of expressions (21).

N	5	10	15	20	25	30
δ_2	2.4×10^{-3}	2.2×10^{-5}	1.6×10^{-7}	1.1×10^{-9}	7.0×10^{-12}	4.4×10^{-14}
δ_3	4.9×10^{-3}	4.3×10^{-5}	3.1×10^{-7}	2.1×10^{-9}	1.4×10^{-11}	8.8×10^{-14}

Consider another example. Let function $\lambda(x)$ have the following form:

$$\lambda(x) = a(x - x_0)^3 + b,$$

where we set the values of parameters as follows:

$$a = 2, \quad x_0 = 0.5, \quad b = 0.5.$$

Furthermore, let the parameter μ be equal to 2.

Then, the positive roots of Equation (20) are equal to:

$$x_1 = 0.230, \quad x_2 = 1.607.$$

The results of the evaluation of raw moments of the $PSSP$ for this example and errors (22) of their estimation (21) are presented in Tables 3 and 4, respectively.

Table 3. Values of raw moments T_n of the n-th order for the $PSSP$ length ($n = 1, 2, 3$) for various values of parameter N (example 2).

N	5	10	15	20	25	30
T_1	2.173×10^{3}	3.256×10^{5}	4.054×10^{7}	4.636×10^{9}	5.045×10^{11}	5.313×10^{13}
T_2	9.460×10^{6}	2.120×10^{11}	3.287×10^{15}	4.298×10^{19}	5.090×10^{23}	5.646×10^{27}
T_3	6.176×10^{10}	2.071×10^{17}	3.998×10^{23}	5.978×10^{29}	7.702×10^{35}	8.999×10^{41}

Table 4. Relative errors of expressions (21) for example 2.

N	5	10	15	20	25	30
δ_2	1.3×10^{-3}	1.1×10^{-5}	2.9×10^{-7}	4.5×10^{-9}	6.1×10^{-11}	7.7×10^{-13}
δ_3	2.6×10^{-3}	2.2×10^{-5}	5.8×10^{-7}	8.9×10^{-9}	1.2×10^{-10}	1.5×10^{-12}

Based on the obtained results, we can formulate the following conjecture:

Hypothesis 1. *The probability distribution of the $PSSP$ length in the considered queueing system can be approximated with enough small error by the exponential distribution with parameter T_1^{-1} (with mean value T_1). The error of approximation quickly decreases with the increase of the scaling parameter N.*

To support this conjecture, let us compare the exact cumulative distribution function (c.d.f.) built based on (8):

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-jux}}{ju} h(u)\, du \tag{23}$$

and corresponding c.d.f. of the exponential distribution:

$$L(x) = 1 - e^{-\frac{x}{T_1}}. \tag{24}$$

We will compare these two probability distributions by using the Kolmogorov distance (see [39]):

$$\Delta = \max_{0 \le x < \infty} |F(x) - L(x)|.$$

The results of the comparison for the considered examples are presented in Table 5 (we use notation Δ_1 for the first example and Δ_2 for the second one). As one can see, these results support the conclusion that the exponential distribution can be used as an approximation for the $PSSP$ length distribution with a high accuracy (we suppose that $\Delta < 0.05$ is enough small error, thus, exponential approximation is acceptable even for small values of parameter N: $N \geq 1$ for example 1 and $N \geq 2$ for example 2).

Table 5. Kolmogorov distances Δ_1 and Δ_2 (examples 1 and 2) between distribution functions (23) and (24) for various values of parameter N.

N	1	2	3	4	5	6
Δ_1	6.1×10^{-3}	4.2×10^{-3}	1.0×10^{-3}	8.5×10^{-4}	4.4×10^{-4}	1.7×10^{-4}
Δ_2	0.127	0.017	3.0×10^{-3}	5.9×10^{-4}	5.9×10^{-4}	8.3×10^{-5}

Computation of the distribution of the $PSSP$ length via the use of the proposed approximation (24) requires only the knowledge of the first raw moment value (15) that can be obtained performing the procedure from Section 5.1.

This is a significant achievement because exact computation of the true c.d.f. defined by formula (23) requires computation of the inverse matrix in (23) in which the size can be quite large. High-order moments can also be computed much easier than via additional performing procedures from Section 5.2.

7. Discussion

In this paper, the notion of the pseudo steady-state period is introduced for the infinite-server queueing system with state dependent arrivals and arrival rate exceeding the customers departure rate when the number of customers in the system is large. The problem of computation of distribution of the length of this period and its raw moments is solved. The simple approximation of this distribution under the proper scaling of the number of customers in the system is offered and numerically illustrated.

Furthermore, the numerical examples demonstrate that this period can last a very long time (see values of the mean of its duration in Tables 1 and 3). Thus, such systems may be considered like models operating in the steady-state regime with the corresponding properties, at least for a long enough period.

The obtained results can be used for the evaluation of the length of the period of relatively stable operations of real world systems until the occurrence of congestion in the system with its overflow. This defines the importance of the results from the point of view of managerial insights.

Further studies with the described approach may be directed to considering more general models such as queues with non-exponential service time distribution and/or MAP arrivals [12,40] with state-dependent parameters. We suppose that the approach presented in the paper can be applied for the analysis of such models.

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