mathematics

MDPI

Article

On Hilfer Generalized Proportional Nabla Fractional Difference Operators

Qiushuang Wang and Run Xu *

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China; qswang2020@163.com
* Correspondence: xurun2005@163.com

Abstract: In this paper, the Hilfer type generalized proportional nabla fractional differences are defined. A few important properties in the left case are derived and the properties in the right case are proved by Q-operator. The discrete Laplace transform in the sense of the left Hilfer generalized proportional fractional difference is explored. Furthermore, An initial value problem with the new operator and its generalized solution are considered.

Keywords: hilfer operator; generalized proportional fractional difference; Q-operator, discrete laplace transform; the initial value problem

MSC: 26A33

check for updates

Citation: Wang, Q.; Xu, R. On Hilfer Generalized Proportional Nabla Fractional Difference Operators. *Mathematics* **2022**, *10*, 2654. https://doi.org/10.3390/math10152654

Academic Editors: António Lopes, Alireza Alfi, Liping Chen and Sergio Adriani David

Received: 5 July 2022
Accepted: 26 July 2022
Published: 28 July 2022

1. Introduction

Fractional Calculus (FC), which can be traced back to the 17th century, is derived from integral calculus. A wide variety of concepts for fractional operators in the continuous setting have been defined in the literature so far, such as Riemann–Liouville, Hadamard, Caputo, proportional, Hilfer fractional operators, and so on; the reader can refer to [1–4] and the references therein. Fractional models are of great theoretical significance and practical value, compared to integer models, in real world problems. Therefore, FC has been widely used in mathematics, physics, engineering, etc. For more recent developments on fractional calculus, see the monographs [5–12].

It is generally known that Discrete Fractional Calculus (DFC) is the extension of FC. The models for DFC play an important role in modeling complex problems of discontinuous systems, which are far superior to their counterparts in continuous settings. Unlike FC of the continuous system, whose history is more than hundreds of years old, the idea of DFC is very recent. The theory of DFC has been investigated extensively since the 20th century, when Chapman [13] presented the definitions of the fractional delta sequential differences, in 1911. Similarly to the case of FC, there are many forms of definitions, such as Riemann–Liouville, Caputo, Hilfer, proportional discrete fractional operators, and so on (see [14–17]).

In addition to the study of fractional operators in FC or DFC, there have also been many directions to develop, for instance, fractional inequalities, fractional equations, etc. In particular, initial value problems with fractional differential or difference operators have been extensively studied. In 2020, Jonnalagadda and Gopal [18] defined the nabla αth-order and βth-type Hilfer fractional difference of f

$$\nabla_a^{\alpha,\beta} f(t) = \nabla_{a+n}^{-\beta(n-\alpha)} \nabla^n \nabla_a^{-(1-\beta)(n-\alpha)} f(t),\ t \in \mathbb{N}_{a+n},$$

where $0 \leq \beta \leq 1$, $n-1 < \alpha \leq n$ with $n \in \mathbb{N}^+$, and $\nabla_a^{-\alpha} f(t) = \sum_{k=a}^{t} \frac{(t-k+1)^{\overline{\alpha}}}{\Gamma(\alpha)} f(k)$ is the nabla Riemann–Liouville fractional sum defined in [19]. Furthermore, they explored the solution of the following initial value problem

$$\begin{cases} \nabla_a^{\alpha,\beta} y(t) = f(t, y(t)), & t \in \mathbb{N}_{a+1}, \\ \nabla_a^{-(1-\gamma)} y(t)|_{t=a} = y(a), \end{cases}$$

where $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. Recently, motivated by the generalized proportional and Hilfer fractional continuous operators, which are defined in [20,21], respectively, Ahmed et al. [22] introduced the Hilfer generalized proportional fractional derivative of order α and type β of a function f

$$\mathcal{D}_a^{\alpha,\beta,\rho} f(x) = \mathcal{I}_a^{\beta(n-\alpha),\rho}\left[\mathcal{D}^\rho\left(\mathcal{I}_a^{(1-\beta)(n-\alpha),\rho} f\right)\right](x),$$

where $n - 1 < \alpha < n$, $\rho \in (0, 1]$, $0 \leq \beta \leq 1$ with $n \in \mathbb{N}_1$, $\mathcal{D}^\rho f(x) = (1-\rho)f(x) + \rho f'(x)$, and $\mathcal{I}$ is the generalized proportional fractional integral operator defined in [21]. Furthermore, they discussed the existence and uniqueness of the solution for the following nonlinear differential equation with a nonlocal initial condition

$$\begin{cases} \mathcal{D}_{a^+}^{\alpha,\beta,\rho} y(t) = f(t, y(t)), & t \in [a, T],\ T > a \geq 0, \\ \mathcal{I}_{a+}^{1-\gamma,\rho} y(t)|_{t=a} = \sum_{i=1}^m c_i x(\tau_i), & \gamma = \alpha + \beta - \alpha\beta,\ \tau_i \in (a, T), \end{cases}$$

where $0 < \alpha < 1$, $c_i \in \mathbb{R}$, $f : [a, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\tau_i \in (a, T)$ satisfying $a < \tau_i < \cdots < \tau_m < T$ for $i = 1, \ldots, m$. For more studies that investigate and extend the fractional differential or fractional difference equation, we refer the reader to [16,17,23,24].

The goal of this paper is to introduce the Hilfer-type generalized proportional fractional difference, which is a discrete counterpart of the fractional derivative defined in [22]. Moreover, we shall study the following initial value problem

$$\begin{cases} {}_a\nabla_h^{\alpha,\beta,\rho} y(t) = f(t, y(t)), & t \in \mathbb{N}_{a+h,h}, \\ {}_a\nabla_h^{-(1-\gamma),\rho} y(t)|_{t=a+h} = \frac{h^{1-\gamma}}{(\rho-(\rho-1)h)^{1-\gamma}} y(a+h), \end{cases} \tag{1}$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $0 < \rho \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$, ${}_a\nabla_h^{\alpha,\beta,\rho}(\cdot)$ is the new difference operator of order α and type β (see Definition 7), and ${}_a\nabla_h^{-(1-\gamma),\rho}(\cdot)$ is the proportional fractional sum operator of order $(1-\gamma)$ (see Definition 4). The new operator can reduce to some known operators. Additionally, our results can provide a powerful tool for studying the qualitative properties for the solution of (1), such as existence, uniqueness, oscillation, and so on.

The structure of this article is as follows: In Section 2, we review some basic definitions and results of discrete calculus. In Section 3, two new fractional difference operators are introduced, and some corresponding properties for the left case are proved based on the definitions. We also prove the properties of the right case by Q-operator. Moreover, the h-Laplace transform for the left Hilfer generalized proportional fractional difference operator is developed. Additionally, the general solution of an initial value problem (1) with the new operator is discussed. Finally, the conclusion of the paper is given in Section 4.

2. Preliminaries

In this section, some definitions and results are given for later use in the following sections. The sets considered in this paper are $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$, ${}_b\mathbb{N} = \{\ldots b-2, b-1, b\}$, $\mathbb{N}_{a,h} = \{a, a+h, a+2h, \ldots\}$ and ${}_{b,h}\mathbb{N} = \{\ldots b-2h, b-h, b\}$ with the step $h > 0$.

For convenience, we give some of the notations to be used here. The h-backward operator is given by $\rho_h(t) = t - h$ for $t \in \mathbb{N}_{a,h}$. The nabla and delta h-difference operators are given as

$$\nabla_h f(t) = \frac{f(t) - f(t-h)}{h}, \quad t \in \mathbb{N}_{a+h,h},$$

$$\Delta_h f(t) = \frac{f(t+h) - f(t)}{h}, \quad t \in {}_{b-h,h}\mathbb{N}.$$

For $h = 1$, we get the following nabla and delta difference operators

$$\nabla f(t) = f(t) - f(t-1), \quad \Delta f(t) = f(t+1) - f(t).$$

They are also called the backward and the forward difference operator, respectively. The nabla and delta h-sums are given as

$$\left(\nabla_h^{-1} f\right)(t) = \int_a^t f(s)\nabla_h s = \sum_{k=\frac{a}{h}+1}^{\frac{t}{h}} f(kh)h, \quad t \in \mathbb{N}_{a+h,h},$$

$$\left(\Delta_h^{-1} f\right)(t) = \int_t^b f(s)\Delta_h s = \sum_{k=\frac{t}{h}}^{\frac{b}{h}-1} f(kh)h, \quad t \in {}_{b-h,h}\mathbb{N},$$

where ∇_h and Δ_h are derivative operators on the time scales $\{a, a+h, \ldots, t\}$ and $\{t, \ldots, b-h, b\}$, respectively.

For arbitrary $t, \alpha \in \mathbb{R}$, the generalized rising and falling h-factorial functions are defined by

$$t_h^{\overline{\alpha}} = h^{\alpha} \frac{\Gamma\left(\frac{t}{h} + \alpha\right)}{\Gamma\left(\frac{t}{h}\right)}, \quad \frac{t}{h}, \frac{t}{h} + \alpha \notin \{\cdots, -2, -1, 0\},$$

$$t_h^{(\alpha)} = h^{\alpha} \frac{\Gamma\left(\frac{t}{h} + 1\right)}{\Gamma\left(\frac{t}{h} + 1 - \alpha\right)}, \quad \frac{t}{h} + 1, \frac{t}{h} + 1 - \alpha \notin \{\cdots, -2, -1, 0\},$$

where $\Gamma(\cdot)$ is the Gamma function given as $\Gamma(x) = \int_0^\infty \xi^{x-1} e^{-\xi} d\xi$. When $h = 1$, we obtain the rising and falling factorial function: $t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}$, $t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}$. It is clear that

$$\nabla_h t_h^{\overline{\alpha}} = \alpha\, t_h^{\overline{\alpha-1}}.$$

For $\rho \in (0,1] \setminus \frac{h}{1-h}$, we introduce the h-proportional differences of order ρ defined in [16]

$$(\nabla_h^{\rho} f)(t) = (1-\rho) f(t) + \rho(\nabla_h f)(t), \quad t \in \mathbb{N}_{a+h,h},$$

$$({}_{\ominus}\Delta_h^{\rho} f)(t) = (1-\rho) f(t) - \rho(\Delta_h f)(t), \quad t \in {}_{b-h,h}\mathbb{N},$$

and

$$(\nabla_h^{n,\rho} f)(t) = \underbrace{(\nabla_h^{\rho}\, \nabla_h^{\rho} \cdots\, \nabla_h^{\rho} f)(t)}_{n \text{ times}}, \quad ({}_{\ominus}\Delta_h^{n,\rho} f)(t) = \underbrace{({}_{\ominus}\Delta_h^{\rho}\, {}_{\ominus}\Delta_h^{\rho} \cdots\, {}_{\ominus}\Delta_h^{\rho} f)(t)}_{n \text{ times}}.$$

When $h = 1$, we denote $(\nabla_1^{\rho} f)(t) = f(t) - \rho f(t-1)$ and $({}_{\ominus}\Delta_1^{\rho} f)(t) = f(t) - \rho f(t+1)$. Next, we recall some definitions and properties of discrete fractional operators as follows.

Definition 1 ([25]). *For $\alpha > 0$, the nabla left and right h-Riemann–Liouville fractional sums of f are given by*

$$({}_a\nabla_h^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \rho_h(s))_h^{\overline{\alpha-1}} f(s)\nabla_h s, \quad t \in \mathbb{N}_{a+h,h}, \tag{2}$$

$$({}_h\nabla_b^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s - \rho_h(t))_h^{\overline{\alpha-1}} f(s)\Delta_h s, \quad t \in {}_{b-h,h}\mathbb{N}. \tag{3}$$

Definition 2 ([25]). *For $\alpha > 0$, the nabla left and right h-Riemann–Liouville fractional differences of f are given by*

$$({}_a\nabla_h^{\alpha} f)(t) = \nabla_h^n \Big({}_a\nabla_h^{-(n-\alpha)} f\Big)(t), \quad t \in \mathbb{N}_{a+h,h}, \tag{4}$$

$$({}_h\nabla_b^{\alpha} f)(t) = (-1)^n \Delta_h^n \Big({}_h\nabla_b^{-(n-\alpha)} f\Big)(t), \quad t \in {}_{b-h,h}\mathbb{N}, \tag{5}$$

where $n-1 < \alpha < n$, $n := [\alpha] + 1$, and $[\alpha]$ is the greatest integer that is less than or equal to α.

Definition 3 ([26]). *For $\alpha > 0$, the nabla left and right h-Caputo fractional differences are defined by*

$$({}_a^C\nabla_h^{\alpha} f)(t) = {}_{a_h(\alpha)}\nabla_h^{-(n-\alpha)} (\nabla_h^n f)(t), \quad t \in \mathbb{N}_{a+nh,h}, \tag{6}$$

$$({}_h^C\nabla_b^{\alpha} f)(t) = (-1)^n \, {}_h\nabla_{b_h(\alpha)}^{-(n-\alpha)} (\Delta_h^n f)(t), \quad t \in {}_{b-nh,h}\mathbb{N}, \tag{7}$$

where $n = [\alpha] + 1$, and $a_h(\alpha) = a + (n-1)h$, $b_h(\alpha) = b - (n-1)h$.

Definition 4 ([16]). *For $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, the left and right generalized proportional fractional sums are defined by*

$$\begin{aligned}({}_a\nabla_h^{-\alpha,\rho} f)(t) =& \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t {}_h\hat{e}_p(t-\tau+\alpha h, 0)(t-\rho_h(\tau))_h^{\overline{\alpha-1}} f(\tau) \nabla_h \tau \\ =& \frac{h}{\rho^\alpha \Gamma(\alpha)} \sum_{k=\frac{a}{h}+1}^{\frac{t}{h}} {}_h\hat{e}_p(t-kh+\alpha h, 0)(t-\rho_h(kh))_h^{\overline{\alpha-1}} f(kh), \quad t \in \mathbb{N}_{a+h,h},\end{aligned} \tag{8}$$

$$\begin{aligned}({}_h\nabla_b^{-\alpha,\rho} f)(t) =& \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_t^b {}_h\hat{e}_p(\tau-t+\alpha h, 0)(\tau-\rho_h(t))_h^{\overline{\alpha-1}} f(\tau) \Delta_h \tau \\ =& \frac{h}{\rho^\alpha \Gamma(\alpha)} \sum_{k=\frac{t}{h}}^{\frac{b}{h}-1} {}_h\hat{e}_p(kh-t+\alpha h, 0)(kh-\rho_h(t))_h^{\overline{\alpha-1}} f(kh), \quad t \in {}_{b-h,h}\mathbb{N},\end{aligned} \tag{9}$$

where the proportionality index $\rho \in (0,1]$, and the exponential function is given as

$${}_h\hat{e}_p(t,a) = \left(\frac{1}{1-ph}\right)^{\frac{t-a}{h}} = \left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\frac{t-a}{h}}, \quad \text{for } p = \frac{\rho-1}{\rho}.$$

Some properties of the exponential function that will be important in the development of this article are described in the following remark.

Remark 1 ([16]). *For $t \in \mathbb{N}_{a,h}$, $\alpha > 0$, $\beta > 0$ and $0 < \rho \leq 1$, the following identities hold,*

(i) ${}_h\hat{e}_p(t,a) = {}_h\hat{e}_p(t-a,0) = {}_h\hat{e}_p(0,a-t)$.

(ii) $\nabla_h^\rho \left(c \cdot {}_h\hat{e}_p(t,a)\right) = 0$, for c is a constant.

(iii) $\nabla_h^\rho (g(t) \cdot {}_h\hat{e}_p(t,0)) = \rho(\nabla_h g)(t) \cdot {}_h\hat{e}_p(t-h,0)$.

(iv) ${}_a\nabla_h^{-\alpha,\rho}\left({}_h\hat{e}_p(t,0)(t-a)_h^{\overline{\beta-1}}\right) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)\rho^\alpha} \, {}_h\hat{e}_p(t+\alpha h,0)(t-a)_h^{\overline{\alpha+\beta-1}}$.

Definition 5 ([16]). *For $\rho \in (0,1]$ and $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, the left and right generalized proportional fractional differences are defined by*

$$({}_a\nabla_h^{\alpha,\rho} f)(t) = \nabla_h^{n,\rho} \Big({}_a\nabla_h^{-(n-\alpha),\rho} f\Big)(t), \quad t \in \mathbb{N}_{a+h,h}, \tag{10}$$

$$({}_h\nabla_b^{\alpha,\rho} f)(t) = {}_\ominus\Delta_h^{n,\rho} \Big({}_h\nabla_b^{-(n-\alpha),\rho} f\Big)(t), \quad t \in {}_{b-h,h}\mathbb{N}, \tag{11}$$

where $n = [Re(\alpha)] + 1$.

Remark 2 ([16]). *Clearly,* $\lim_{\alpha\to 0}({}_a\nabla_h^{\alpha,\rho} f)(t) = f(t), \quad \lim_{\alpha\to 1}({}_a\nabla_h^{\alpha,\rho} f)(t) = (\nabla_h^{\rho} f)(t).$

Definition 6 ([16]). *For* $\rho \in (0,1]$ *and* $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, *the left and right Caputo generalized proportional fractional differences are defined by*

$$\left({}_a^C\nabla_h^{\alpha,\rho} f\right)(t) = {}_{a_h(\alpha)}\nabla_h^{-(n-\alpha),\rho}(\nabla_h^{n,\rho} f)(t), \tag{12}$$

$$\left({}_h^C\nabla_b^{\alpha,\rho} f\right)(t) = {}_h\nabla_{b_h(\alpha)}^{-(n-\alpha),\rho}({}_\ominus\Delta_h^{n,\rho} f)(t). \tag{13}$$

where $n = [Re(\alpha)] + 1$.

Theorem 1 (Composition Rule [16]). *Assume* $\alpha > 0$, $n = [\alpha] + 1$ *and* $\beta > 0$. *Then for any* $0 < \rho \le 1$, *we have*

(i) $\quad {}_a\nabla_h^{\alpha,\rho}\left({}_a\nabla_h^{-\alpha,\rho} f\right)(t) = f(t).$

(ii) $\quad {}_a\nabla_h^{-\alpha,\rho}\left(\nabla_h^{\rho} f\right)(t) = \nabla_h^{\rho}\left({}_a\nabla_h^{-\alpha,\rho} f\right)(t) - \frac{(t-a)_h^{\overline{\alpha-1}}\ {}_h\hat{e}_p(t,a)}{\rho^{\alpha-1}\Gamma(\alpha)}\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\alpha-1} f(a).$

(iii) $\quad {}_a\nabla_h^{-\alpha,\rho}({}_a\nabla_h^{-\beta,\rho} f)(t) = {}_a\nabla_h^{-\beta,\rho}({}_a\nabla_h^{-\alpha,\rho} f)(t) = ({}_a\nabla_h^{-(\alpha+\beta),\rho} f)(t).$

(iv) $\quad \left({}_{a_h(\alpha)}\nabla_h^{-\alpha,\rho}\ {}_{a_h(\alpha)}\nabla_h^{\alpha,\rho} f\right)(t) = f(t) - {}_h\hat{e}_p(t-(n-1)h, a)$

$\quad \cdot \sum_{j=1}^{n}\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\alpha-1} \frac{(t-a_h(\alpha))_h^{\overline{\alpha-j}}}{\rho^{\alpha-j}\Gamma(\alpha+1-j)}\left({}_{a_h(\alpha)}\nabla_h^{-(j-\alpha),\rho} f\right)(a+(n-1)h).$

3. Main Results

In this section, we define the left and right generalized proportional fractional difference operators in the Hilfer sense and discuss some of their properties. In addition, we demonstrate a general solution of problem (1).

3.1. The Hilfer Generalized Proportional Fractional Difference and Some Related Operators

(1) First, like the nabla Hilfer-type fractional difference that is defined by the composition of the nabla Riemann–Liouville fractional sum and nabla integral difference ([18]), the Hilfer generalized proportional fractional difference operators are introduced as follows, based on the generalized proportional fractional sum and h-proportional difference.

Definition 7. *Let* $n-1 < \alpha < n$ *with* $n \in \mathbb{N}_1$, $\rho \in (0,1]$ *and* $0 \le \beta \le 1$. *Then the left and right Hilfer generalized proportional fractional difference of order* α *and type* β *of a function* f *are defined by*

$$\left({}_a\nabla_h^{\alpha,\beta,\rho} f\right)(t) = {}_a\nabla_h^{-\beta(n-\alpha),\rho} \cdot \nabla_h^{\rho} \cdot {}_a\nabla_h^{-(n-\alpha)(1-\beta),\rho} f(t),\ t \in \mathbb{N}_{a+h,h}, \tag{14}$$

$$\left({}_h\nabla_b^{\alpha,\beta,\rho} f\right)(t) = {}_h\nabla_b^{-\beta(n-\alpha),\rho} \cdot {}_\ominus\Delta_h^{\rho} \cdot {}_h\nabla_b^{-(n-\alpha)(1-\beta),\rho} f(t),\ t \in {}_{b-h,h}\mathbb{N}, \tag{15}$$

where ${}_a\nabla_h^{-\beta(n-\alpha),\rho}(\cdot)$, ${}_h\nabla_b^{-\beta(n-\alpha),\rho}(\cdot)$ *are generalized proportional fractional sum operators defined in* (8) *and* (9), *respectively.*

In particular, when $n = 1$, *Definition* 7 *is equivalent with*

$$\left({}_a\nabla_h^{\alpha,\beta,\rho} f\right)(t) = {}_a\nabla_h^{-\beta(1-\alpha),\rho} \cdot \nabla_h^{\rho} \cdot {}_a\nabla_h^{-(1-\alpha)(1-\beta),\rho} f(t), \tag{16}$$

$$\left({}_h\nabla_b^{\alpha,\beta,\rho} f\right)(t) = {}_h\nabla_b^{-\beta(1-\alpha),\rho} \cdot {}_\ominus\Delta_h^{\rho} \cdot {}_h\nabla_b^{-(1-\alpha)(1-\beta),\rho} f(t). \tag{17}$$

When $n = 1$ and $h = 1$, Definition 7 is equivalent with

$$\left({}_{a}\nabla_{1}^{\alpha,\beta,\rho} f\right)(t) = {}_{a}\nabla_{1}^{-\beta(1-\alpha),\rho} \cdot \nabla_{1}^{\rho} \cdot {}_{a}\nabla_{1}^{-(1-\alpha)(1-\beta),\rho} f(t), \tag{18}$$

$$\left({}_{1}\nabla_{b}^{\alpha,\beta,\rho} f\right)(t) = {}_{1}\nabla_{b}^{-\beta(1-\alpha),\rho} \cdot {}_{\ominus}\Delta_{1}^{\rho} \cdot {}_{1}\nabla_{b}^{-(1-\alpha)(1-\beta),\rho} f(t). \tag{19}$$

Remark 3. *It is worth noting that:*

(i) *For the special value of β, (14) coincides with the generalized Riemann–Liouville and Caputo type proportional fractional difference, respectively (see Definitions 5 and 6 with $n = 1$)*

$$\begin{cases} \left({}_{a}\nabla_{h}^{\alpha,\beta,\rho} f\right)(t) = \nabla_{h}^{\rho} \cdot {}_{a}\nabla_{h}^{-(1-\alpha),\rho} f(t) = \left({}_{a}\nabla_{h}^{\alpha,\rho} f\right)(t), & \beta = 0, \\ {}_{a}\nabla_{h}^{-(1-\alpha),\rho} \cdot \nabla_{h}^{\rho} f(t) = \left({}_{a}^{C}\nabla_{h}^{\alpha,\rho} f\right)(t), & \beta = 1. \end{cases}$$

In addition, when $\beta = 0,\ \rho = 1$, we recover the h-Riemann–Liouville fractional difference (see Definition 2), and when $\beta = 1,\ \rho = 1$, we get the h-Caputo fractional difference (see Definition 3)

$$\begin{cases} \left({}_{a}\nabla_{h}^{\alpha,\beta,\rho} f\right)(t) = \nabla_{h}^{\rho} \cdot {}_{a}\nabla_{h}^{-(1-\alpha),\rho} f(t) = \left({}_{a}\nabla_{h}^{\alpha} f\right)(t), & \beta = 0,\ \rho = 1, \\ {}_{a}\nabla_{h}^{-(1-\alpha),\rho} \cdot \nabla_{h}^{\rho} f(t) = \left({}_{a}^{C}\nabla_{h}^{\alpha} f\right)(t), & \beta = 1,\ \rho = 1. \end{cases}$$

The corresponding results for the right case ${}_{h}\nabla_{b}^{\alpha,\beta,\rho}$ are similar.

(ii) *Clearly,*

$$\begin{aligned} &\lim_{\alpha\to 0}\left({}_{a}\nabla_{h}^{\alpha,\beta,\rho} f\right)(t) = f(t),\ \lim_{\alpha\to 1}\left({}_{a}\nabla_{h}^{\alpha,\beta,\rho} f\right)(t) = \left(\nabla_{h}^{\rho} f\right)(t), \\ &\lim_{\alpha\to 0}\left({}_{h}\nabla_{b}^{\alpha,\beta,\rho} f\right)(t) = f(t),\ \lim_{\alpha\to 1}\left({}_{h}\nabla_{b}^{\alpha,\beta,\rho} f\right)(t) = \left({}_{\ominus}\Delta_{h}^{\rho} f\right)(t). \end{aligned}$$

Here are some properties for the left Hilfer generalized proportional fractional difference operator.

Theorem 2 (Composition Rule)**.** *Assume $0 < \alpha < 1,\ 0 \le \beta \le 1,\ \rho \in (0,1]$ and f is defined on $\mathbb{N}_{a+h,h}$. Let $\gamma = \alpha + \beta - \alpha\beta$. Then we obtain*

(i) $\left({}_{a}\nabla_{h}^{\alpha,\beta,\rho} f\right)(t) = {}_{a}\nabla_{h}^{-\beta(1-\alpha),\rho}\left({}_{a}\nabla_{h}^{\gamma,\rho} f\right)(t).$

(ii) ${}_{a}\nabla_{h}^{-\alpha,\rho}\left({}_{a}\nabla_{h}^{\alpha,\beta,\rho} f\right)(t) = {}_{a}\nabla_{h}^{-\gamma,\rho}\left({}_{a}\nabla_{h}^{\gamma,\rho} f\right)(t).$

(iii) ${}_{a}\nabla_{h}^{\alpha,\beta,\rho}\left({}_{a}\nabla_{h}^{-\alpha,\rho} f\right)(t) = {}_{a}\nabla_{h}^{-\beta(1-\alpha),\rho}\left({}_{a}\nabla_{h}^{\beta(1-\alpha),\rho} f\right)(t).$

(iv) ${}_{a}\nabla_{h}^{\alpha,\beta,\rho}\left({}_{a}\nabla_{h}^{-\alpha,\rho} f\right)(t) = f(t) - {}_{h}\hat{e}_{p}(t,a)\left(\frac{1}{\rho-(\rho-1)h}\right)^{\beta-\alpha\beta-1}\frac{(t-a)_{h}^{\overline{\beta-\alpha\beta-1}}}{\Gamma(\beta-\alpha\beta)} \left({}_{a}\nabla_{h}^{-(1-\beta+\alpha\beta),\rho} f\right)(a).$

Proof. According to (16), we have

$$\begin{aligned} \left({}_{a}\nabla_{h}^{\alpha,\beta,\rho} f\right)(t) &= {}_{a}\nabla_{h}^{-\beta(1-\alpha),\rho} \cdot \nabla_{h}^{\rho} \cdot {}_{a}\nabla_{h}^{-(1-\alpha)(1-\beta),\rho} f(t) \\ &= {}_{a}\nabla_{h}^{-\beta(1-\alpha),\rho}\left({}_{a}\nabla_{h}^{\gamma,\rho} f\right)(t). \end{aligned}$$

The proof of (i) is completed.

Using (iii) of Theorem 1 and Definition 5, we have

$$\begin{aligned}
{}_a\nabla_h^{-\alpha,\rho}\left({}_a\nabla_h^{\alpha,\beta,\rho} f\right)(t) &= {}_a\nabla_h^{-\alpha,\rho}\left({}_a\nabla_h^{-\beta(1-\alpha),\rho}\cdot\nabla_h^{\rho}\cdot\, {}_a\nabla_h^{-(1-\alpha)(1-\beta),\rho}\right) f(t) \\
&= {}_a\nabla_h^{-\alpha-\beta+\alpha\beta,\rho}\cdot\nabla_h^{\rho}\left({}_a\nabla_h^{-(1-\alpha)(1-\beta),\rho} f\right)(t) \\
&= {}_a\nabla_h^{-\gamma,\rho}\cdot\nabla_h^{\rho}\left({}_a\nabla_h^{-1+\gamma,\rho} f\right)(t) \\
&= {}_a\nabla_h^{-\gamma,\rho}\left({}_a\nabla_h^{\gamma,\rho} f\right)(t).
\end{aligned}$$

The proof of (ii) is completed.
We use Theorem 1 (iii) and Definition 5 to prove (iii). Consider

$$\begin{aligned}
{}_a\nabla_h^{\alpha,\beta,\rho}\left({}_a\nabla_h^{-\alpha,\rho} f\right)(t) &= {}_a\nabla_h^{-\beta(1-\alpha),\rho}\cdot\nabla_h^{\rho}\cdot\, {}_a\nabla_h^{-(1-\beta)(1-\alpha),\rho}\left({}_a\nabla_h^{-\alpha,\rho} f\right)(t) \\
&= {}_a\nabla_h^{-\beta(1-\alpha),\rho}\cdot\nabla_h^{\rho}\left({}_a\nabla_h^{-[1-\beta(1-\alpha)],\rho} f\right)(t) \\
&= {}_a\nabla_h^{-\beta(1-\alpha),\rho}\left({}_a\nabla_h^{\beta(1-\alpha),\rho} f\right)(t).
\end{aligned}$$

The proof of (iii) is completed.
Consider the left-hand side of (iv). Using (iii) and Theorem 1 (iv) with $n = 1$, we have

$$\begin{aligned}
&{}_a\nabla_h^{\alpha,\beta,\rho}\left({}_a\nabla_h^{-\alpha,\rho} f\right)(t) \\
&= {}_a\nabla_h^{-\beta(1-\alpha),\rho}\left({}_a\nabla_h^{\beta(1-\alpha),\rho} f\right)(t) \\
&= f(t) - {}_h\hat{e}_p(t,a)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\beta-\alpha\beta-1}\frac{(t-a)_h^{\overline{\beta-\alpha\beta-1}}}{\rho^{\beta-\alpha\beta-1}\Gamma(\beta-\alpha\beta)}\left({}_a\nabla_h^{-(1-\beta+\alpha\beta),\rho} f\right)(a) \\
&= f(t) - {}_h\hat{e}_p(t,a)\left(\frac{1}{\rho-(\rho-1)h}\right)^{\beta-\alpha\beta-1}\frac{(t-a)_h^{\overline{\beta-\alpha\beta-1}}}{\Gamma(\beta-\alpha\beta)}\left({}_a\nabla_h^{-(1-\beta+\alpha\beta),\rho} f\right)(a).
\end{aligned}$$

The proof of (iv) is completed. □

(2) Now, we will consider the Q-operator, which is used to demonstrate the results corresponding to Theorem 2 (i)–(iii) for the right case.

The Q-operator is defined as follows: Suppose $a \equiv b \bmod 1$ and $f(t)$ is defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$, then

$$(Qf)(t) = f(a+b-t),$$

which is used to connect the left and right fractional discrete operators.

Lemma 1 ([16]). *Assume $n-1 < \alpha < n$ with $n \in \mathbb{N}_1$, $a \equiv b \bmod h$ and function f is defined on $\mathbb{N}_{a+h,h} \cap {}_{b-h,h}\mathbb{N}$. Then we have*

(i) $Q(\nabla_h^{\rho} f)(t) = {}_{\ominus}\Delta_h^{\rho}(Qf)(t)$.
(ii) $Q({}_a\nabla_h^{-\alpha,\rho} f)(t) = {}_h\nabla_b^{-\alpha,\rho}(Qf)(t)$.
(iii) $Q({}_a\nabla_h^{\alpha,\rho} f)(t) = {}_h\nabla_b^{\alpha,\rho}(Qf)(t)$.

Theorem 3. *Let $n-1 < \alpha < n$ with $n \in \mathbb{N}_1$, $0 \le \beta \le 1$, $\rho \in (0,1]$ and $a \equiv b \bmod h$. Suppose f is defined on $\mathbb{N}_{a+h,h} \cap {}_{b-h,h}\mathbb{N}$. Then,*

$$Q({}_a\nabla_h^{\alpha,\beta,\rho} f)(t) = {}_h\nabla_b^{\alpha,\beta,\rho}(Qf)(t). \tag{20}$$

Proof. With the help of Lemma 1, we arrive at

$$\begin{aligned}
Q({}_a\nabla_h^{\alpha,\beta,\rho} f)(t) &= Q({}_a\nabla_h^{-\beta(n-\alpha),\rho} \cdot \nabla_h^{\rho} \cdot {}_a\nabla_h^{-(n-\alpha)(1-\beta),\rho} f)(t) \\
&= {}_h\nabla_b^{-\beta(n-\alpha),\rho} Q(\nabla_h^{\rho} \cdot {}_a\nabla_h^{-(n-\alpha)(1-\beta),\rho} f)(t) \\
&= \left({}_h\nabla_b^{-\beta(n-\alpha),\rho} \cdot {}_{\ominus}\Delta_h^{\rho}\right) Q({}_a\nabla_h^{-(n-\alpha)(1-\beta),\rho} f)(t) \\
&= \left({}_h\nabla_b^{-\beta(n-\alpha),\rho} \cdot {}_{\ominus}\Delta_h^{\rho} \cdot {}_h\nabla_b^{-(n-\alpha)(1-\beta),\rho}\right)(Qf)(t) = {}_h\nabla_b^{\alpha,\beta,\rho}(Qf)(t).
\end{aligned}$$

The proof is completed. □

Theorem 4. *Assume* $0 < \alpha < 1$, $0 \le \beta \le 1$, $\rho \in (0,1]$ *and* $a \equiv b \bmod h$. *Let* f *be defined on* $\mathbb{N}_{a+h,h} \cap {}_{b-h,h}\mathbb{N}$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *Then we obtain*

(i) $\left({}_h\nabla_b^{\alpha,\beta,\rho} f\right)(t) = {}_h\nabla_b^{-\beta(1-\alpha),\rho}\left({}_h\nabla_b^{\gamma,\rho} f\right)(t).$

(ii) ${}_h\nabla_b^{\alpha,\beta,\rho}\left({}_h\nabla_b^{-\alpha,\rho} f\right)(t) = {}_h\nabla_b^{-\beta(1-\alpha),\rho}\left({}_h\nabla_b^{\beta(1-\alpha),\rho} f\right)(t).$

(iii) ${}_h\nabla_b^{-\alpha,\rho}\left({}_h\nabla_b^{\alpha,\beta,\rho} f\right)(t) = {}_h\nabla_b^{-\gamma,\rho}\left({}_h\nabla_b^{\gamma,\rho} f\right)(t).$

Proof. Let $t \in \mathbb{N}_{a+h,h} \cap {}_{b-h,h}$. Then $a + b - t \in \mathbb{N}_{a+h,h} \cap {}_{b-h,h}$. If we apply Q-operator to equations of Theorem 2 (i)–(iii), then we can get the following identities

$$\begin{aligned}
&{}_h\nabla_b^{\alpha,\beta,\rho}(Qf)(t) = {}_h\nabla_b^{-\beta(1-\alpha),\rho} \cdot {}_h\nabla_b^{\gamma,\rho}(Qf)(t), \\
&{}_h\nabla_b^{\alpha,\beta,\rho} \cdot {}_h\nabla_b^{-\alpha,\rho}(Qf)(t) = {}_h\nabla_b^{-\beta(1-\alpha),\rho} \cdot {}_h\nabla_b^{\beta(1-\alpha),\rho}(Qf)(t), \\
&{}_h\nabla_b^{-\alpha,\rho} \cdot {}_h\nabla_b^{\alpha,\beta,\rho}(Qf)(t) = {}_h\nabla_b^{-\gamma,\rho} \cdot {}_h\nabla_b^{\gamma,\rho}(Qf)(t),
\end{aligned}$$

which are equal to the desired equations. Thus we complete the proof. □

(3) We review two types of the discrete Laplace transform to obtain the h-Laplace transform for ${}_a\nabla_h^{\alpha,\beta,\rho}$.

Definition 8 ([19])**.** *Assume* $f : \mathbb{N}_a \to \mathbb{R}$ *and* $s \in \mathbb{C} \setminus \{1\}$, *then the Laplace transform of* f *is defined by*

$$F(s) = \mathcal{N}_a\{f(t)\}(s) = \sum_{t=1}^{\infty}(1-s)^{t-1} f(t+a) = \sum_{t=a+1}^{\infty}(1-s)^{t-a-1} f(t). \tag{21}$$

Definition 9 ([16])**.** *Assume* $f : \mathbb{N}_{a,h} \to \mathbb{R}$, *then the h-Laplace transform of* f *is defined by*

$$F(s) = \mathcal{N}_{a,h}\{f(t)\}(s) = h \sum_{t=\frac{a}{h}+1}^{\infty}(1-hs)^{t-\frac{a}{h}-1} f(ht). \tag{22}$$

Note that (22) *is consistent with* (21) *when* $h = 1$, *and when* $a = 0$, (22) *is reduced to*

$$\mathcal{N}_{0,h}\{f(t)\}(s) = h\sum_{t=1}^{\infty}(1-hs)^{t-1} f(ht).$$

Lemma 2 ([16])**.** *Let* $\rho \in (0,1]$, $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, *and* $n = [Re(\alpha)] + 1$. *Then the h-discrete Laplace transforms for fractional proportional difference and sum are given by*

$$\mathcal{N}_{a,h}\left\{\left({}_a\nabla_h^{\alpha,\rho} f\right)(t)\right\}(s) = \left(\frac{\rho}{\rho-(\rho-1)h}\right)^{n-\alpha-1} \frac{\mathcal{N}_{a,h}\{f(t)\}(s)}{(\rho s+1-\rho)^{-\alpha}},$$

and

$$\mathcal{N}_{a,h}\left\{\left({}_a\nabla_h^{-\alpha,\rho}f\right)(t)\right\}(s)=\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\alpha-1}\frac{\mathcal{N}_{a,h}\{f(t)\}(s)}{(\rho s+1-\rho)^{\alpha}}.$$

After carefully checking, it is worth noting that there is a typing mistake in [16], Remark 3.2: $\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{h(\alpha-1)}$*, which should be* $\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\alpha-1}$*. By calculating, we find that the same problem occurs in [16], Lemma 3.1, Theorems 4.1 and 4.3. We have revised it in Theorem 1 (ii) and Lemma 2.*

Theorem 5 (The h-Laplace transform for ${}_a\nabla_h^{\alpha,\beta,\rho}$)**.** *Assume* $0<\alpha<1,\ 0\le\beta\le1,\ \rho\in(0,1]$*, and let* $f:\mathbb{N}_{a+h,h}\to\mathbb{R}$*. Then, we have the h-Laplace transform for the Hilfer generalized proportional fractional difference operator given as*

$$\mathcal{N}_{a,h}\left\{\left({}_a\nabla_h^{\alpha,\beta,\rho}f\right)(t)\right\}(s)=\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{-\alpha-1}(\rho s+1-\rho)^{\alpha}\mathcal{N}_{a,h}\{f(t)\}(s). \tag{23}$$

Proof. Set $\gamma=\alpha+\beta-\alpha\beta$, then $0<\gamma<1$. Using Lemma 2, we obtain

$$\begin{aligned}\mathcal{N}_{a,h}\left\{\left({}_a\nabla_h^{\alpha,\beta,\rho}f\right)(t)\right\}(s)=&\mathcal{N}_{a,h}\left\{{}_a\nabla_h^{-\beta(1-\alpha),\rho}\left({}_a\nabla_h^{\gamma,\rho}f\right)(t)\right\}(s)\\ =&\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\beta(1-\alpha)-1}\frac{\mathcal{N}_{a,h}\left\{\left({}_a\nabla_h^{\gamma,\rho}f\right)(t)\right\}(s)}{(\rho s+1-\rho)^{\beta(1-\alpha)}}\\ =&\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\beta(1-\alpha)-1-\gamma}\cdot(\rho s+1-\rho)^{\gamma-\beta(1-\alpha)}\mathcal{N}_{a,h}\{f(t)\}(s)\\ =&\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{-\alpha-1}(\rho s+1-\rho)^{\alpha}\mathcal{N}_{a,h}\{f(t)\}(s).\end{aligned}$$

Thus, we complete the proof. □

3.2. The Initial Value Problem for the New Fractional Difference

Here, we give a general solution of an initial value problem for the new fractional difference.

According to the generalized proportional fractional sum given in Definition 4, we have the following identity

$${}_a\nabla_h^{-(1-\gamma),\rho}y(t)|_{t=a+h}=\frac{h^{1-\gamma}}{(\rho-(\rho-1)h)^{1-\gamma}}y(a+h).$$

Hence, consider the following initial value problem for a nonlinear fractional difference equation,

$$\begin{cases}{}_a\nabla_h^{\alpha,\beta,\rho}y(t)=f(t,y(t)), & t\in\mathbb{N}_{a+2h,h}, \quad (24)\\ {}_a\nabla_h^{-(1-\gamma),\rho}y(t)|_{t=a+h}=\dfrac{h^{1-\gamma}}{(\rho-(\rho-1)h)^{1-\gamma}}y(a+h)\triangleq m, & \quad (25)\end{cases}$$

where $0<\alpha<1,\ 0\le\beta\le1,\ 0<\rho\le1,\ \gamma=\alpha+\beta-\alpha\beta$, and m is a constant.

Theorem 6. *Let* $f:\mathbb{N}_{a+h,h}\to\mathbb{R}$ *be given and* $\alpha\in(0,1)$*,* $\beta\in[0,1]$*,* $\rho\in(0,1]$*,* $\gamma=\alpha+\beta-\alpha\beta$*. Then the initial value problem (24) and (25) has a general solution*

$$\begin{aligned}y(t)=&{}_{a+h}\nabla_h^{-\gamma,\rho}\cdot{}_a\nabla_h^{\beta(1-\alpha),\rho}f(t,y(t))+\frac{(t-a-h)_h^{\overline{\gamma-1}}}{\rho^{\gamma-1}\Gamma(\gamma)}{}_h\hat{e}_p(t,a+h)\eta(h,\rho,\gamma)m\\ &+\frac{h^{2-\gamma}(\gamma-1)(t-a)_h^{\overline{\gamma-2}}}{\Gamma(\gamma)}{}_h\hat{e}_p(t,a+h)y(a+h),\end{aligned} \tag{26}$$

where $\eta(h,\rho,\gamma) = \left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\gamma-1}$.

Proof. Applying the operator ${}_a\nabla_h^{\beta(1-\alpha),\rho}$ to the both sides of (24), we have for $t \in \mathbb{N}_{a+2h,h}$,

$$ {}_a\nabla_h^{\beta(1-\alpha),\rho} \cdot {}_a\nabla_h^{\alpha,\beta,\rho} y(t) = {}_a\nabla_h^{\beta(1-\alpha),\rho} f(t,y(t)). \tag{27}$$

Let $F(t,y(t)) = {}_a\nabla_h^{\beta(1-\alpha)\rho} f(t,y(t))$. Then using (16), we get

$$ {}_a\nabla_h^{\beta(1-\alpha),\rho} \cdot {}_a\nabla_h^{-\beta(1-\alpha),\rho} \cdot \nabla_h^{\rho} \cdot {}_a\nabla_h^{-(1-\alpha)(1-\beta),\rho} y(t) = F(t,y(t)). \tag{28}$$

Besides, with the help of Theorem 1 (i), we have

$$ \nabla_h^{\rho} \cdot {}_a\nabla_h^{-(1-\gamma),\rho} y(t) = F(t,y(t)). \tag{29}$$

where $0 < 1-\gamma < 1$.

From the definition of the generalized proportional fraction sum given as (8), we get

$$\begin{aligned}
\left({}_a\nabla_h^{-(1-\gamma),\rho} y\right)(t) &= \frac{h}{\rho^{1-\gamma}\Gamma(1-\gamma)} \sum_{k=\frac{a}{h}+1}^{\frac{t}{h}} {}_h\hat{e}_p(t-kh+(1-\gamma)h,0)(t-\rho_h(kh))_h^{\overline{-\gamma}} y(kh) \\
&= \frac{h}{\rho^{1-\gamma}\Gamma(1-\gamma)} \sum_{k=\frac{a+h}{h}+1}^{\frac{t}{h}} {}_h\hat{e}_p(t-kh+(1-\gamma)h,0)(t-\rho_h(kh))_h^{\overline{-\gamma}} y(kh) \\
&\quad + \frac{h\cdot(t-\rho_h(a+h))_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)} {}_h\hat{e}_p(t-a-h+(1-\gamma)h,0)y(a+h) \\
&= \left({}_{a+h}\nabla_h^{-(1-\gamma),\rho} y\right)(t) \\
&\quad + \frac{h\cdot(t-a)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)} {}_h\hat{e}_p(t-a-h+(1-\gamma)h,0)y(a+h),
\end{aligned} \tag{30}$$

where the properties for ${}_h\hat{e}_p(\cdot,\cdot)$ are in Remark 1. Then, applying both sides of (29) by the operator ${}_{a+h}\nabla_h^{-\gamma,\rho}$, we obtain

$$\begin{aligned}
&{}_{a+h}\nabla_h^{-\gamma,\rho} \cdot \nabla_h^{\rho}\left\{ {}_{a+h}\nabla_h^{-(1-\gamma),\rho} y(t) + \frac{h\cdot(t-a)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)} {}_h\hat{e}_p(t-a-h+(1-\gamma)h,0)y(a+h)\right\} \\
&\quad = G(t,y(t)),
\end{aligned} \tag{31}$$

with $G(t,y(t)) = {}_{a+h}\nabla_h^{-\gamma,\rho} F(t,y(t))$. That is

$$\begin{aligned}
&{}_{a+h}\nabla_h^{-\gamma,\rho} \cdot \nabla_h^{\rho} \cdot {}_{a+h}\nabla_h^{-(1-\gamma),\rho} y(t) \\
&+ {}_{a+h}\nabla_h^{-\gamma,\rho} \cdot \nabla_h^{\rho}\left\{ \frac{h\cdot(t-a)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)} {}_h\hat{e}_p(t-a-h+(1-\gamma)h,0)y(a+h)\right\} \\
&= G(t,y(t)).
\end{aligned} \tag{32}$$

For the convenience of calculations, we rewrite the above equation as

$$ I + J = G(t,y(t)), $$

where

$$I = {}_{a+h}\nabla_h^{-\gamma,\rho} \cdot \nabla_h^{\rho} \cdot {}_{a+h}\nabla_h^{-(1-\gamma),\rho} y(t),$$
$$J = {}_{a+h}\nabla_h^{-\gamma,\rho} \cdot \nabla_h^{\rho} \left\{ \frac{h \cdot (t-a)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)} \, {}_h\hat{e}_p(t-a-h+(1-\gamma)h, 0) y(a+h) \right\}.$$

In the following, we come to deal with the above two terms one by one.

First, we consider I. It follows from (30) and the fact $(h)_h^{\overline{-\gamma}} = h^{-\gamma}\Gamma(1-\gamma)$ that

$$\begin{aligned} &\left({}_{a+h}\nabla_h^{-(1-\gamma),\rho} y\right)(t)|_{t=a+h} \\ &= \left({}_a\nabla_h^{-(1-\gamma),\rho} y(t) - \frac{h \cdot (t-a)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)} \, {}_h\hat{e}_p(t-a-h+(1-\gamma)h, 0) y(a+h) \right)_{t=a+h} \\ &= m - \frac{h^{1-\gamma}}{\rho^{1-\gamma}} \, {}_h\hat{e}_p((1-\gamma)h, 0) y(a+h). \end{aligned} \tag{33}$$

In addition, from $\lim_{\alpha \to 0} ({}_a\nabla_h^{\alpha,\rho} f)(t) = f(t)$ (see Remark 2) and Definition 5, we have

$$\nabla_h^{\rho} \, {}_{a+h}\nabla_h^{-1,\rho} y(t) = \lim_{\alpha \to 1^-} \nabla_h^{\rho} \, {}_{a+h}\nabla_h^{-\alpha,\rho} y(t) = \lim_{\alpha \to 1^-} \, {}_{a+h}\nabla_h^{1-\alpha,\rho} y(t) = y(t).$$

Therefore, with the help of Theorem 1 (ii)–(iii), we get

$$\begin{aligned} I = & \; {}_{a+h}\nabla_h^{-\gamma,\rho} \cdot \nabla_h^{\rho} \left({}_{a+h}\nabla_h^{-(1-\gamma),\rho} y\right)(t) \\ = & \; \nabla_h^{\rho} \cdot {}_{a+h}\nabla_h^{-\gamma,\rho} \left({}_{a+h}\nabla_h^{-(1-\gamma),\rho} y\right)(t) \\ & - \tfrac{(t-a-h)_h^{\overline{\gamma-1}}}{\rho^{\gamma-1}\Gamma(\gamma)} \, {}_h\hat{e}_p(t, a+h) \left(\tfrac{\rho}{\rho-(\rho-1)h}\right)^{\gamma-1} \left\{ \left({}_{a+h}\nabla_h^{-(1-\gamma),\rho} y\right)(t)|_{t=a+h} \right\} \\ = & \; \nabla_h^{\rho} \cdot {}_{a+h}\nabla_h^{-1,\rho} y(t) - \tfrac{(t-a-h)_h^{\overline{\gamma-1}}}{\rho^{\gamma-1}\Gamma(\gamma)} \, {}_h\hat{e}_p(t-a-h, 0) \eta(h,\rho,\gamma) \\ & \cdot \left(m - \tfrac{h^{1-\gamma}}{\rho^{1-\gamma}} \, {}_h\hat{e}_p((1-\gamma)h, 0) y(a+h) \right) \\ = & \; y(t) - \tfrac{(t-a-h)_h^{\overline{\gamma-1}}}{\rho^{\gamma-1}\Gamma(\gamma)} \, {}_h\hat{e}_p(t-a-h, 0) \eta(h,\rho,\gamma) \left(m - \tfrac{h^{1-\gamma}}{\rho^{1-\gamma}} \, {}_h\hat{e}_p((1-\gamma)h, 0) y(a+h) \right). \end{aligned} \tag{34}$$

Using the fact that

$${}_h\hat{e}_p(t-a-h, 0) \cdot {}_h\hat{e}_p((1-\gamma)h, 0) = {}_h\hat{e}_p(t-a-\gamma h, 0),$$

where we use the definition of the exponential function, then

$$\begin{aligned} I = y(t) &- \frac{(t-a-h)_h^{\overline{\gamma-1}}}{\rho^{\gamma-1}\Gamma(\gamma)} \, {}_h\hat{e}_p(t-a-h, 0) \eta(h,\rho,\gamma) m \\ &+ \frac{h^{1-\gamma}(t-a-h)_h^{\overline{\gamma-1}}}{\Gamma(\gamma)} \, {}_h\hat{e}_p(t-a-\gamma h, 0) \eta(h,\rho,\gamma) y(a+h). \end{aligned} \tag{35}$$

Define the last term of the above equation as

$$\Psi(t,h,\rho,\gamma) = \frac{h^{1-\gamma}(t-a-h)_h^{\overline{\gamma-1}}}{\Gamma(\gamma)} \, {}_h\hat{e}_p(t-a-\gamma h, 0) \eta(h,\rho,\gamma) y(a+h),$$

hence, (35) becomes

$$I = y(t) - \frac{(t-a-h)_h^{\overline{\gamma-1}}}{\rho^{\gamma-1}\Gamma(\gamma)} \, {}_h\hat{e}_p(t-a-h, 0) \eta(h,\rho,\gamma) m + \Psi(t,h,\rho,\gamma). \tag{36}$$

Now, consider the second term J in (32). Define

$$\Phi(t,h,\rho,\gamma) = \nabla_h^\rho \cdot {}_{a+h}\nabla_h^{-\gamma,\rho}\left\{\frac{h\cdot(t-a)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)}\, {}_h\hat{e}_p(t-a-h+(1-\gamma)h,0)y(a+h)\right\}.$$

Then, using Theorem 1 (ii) and Remark 1, we get

$$\begin{aligned}
J &= {}_{a+h}\nabla_h^{-\gamma,\rho}\cdot\nabla_h^\rho\left\{\frac{h\cdot(t-a)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)}\, {}_h\hat{e}_p(t-a-h+(1-\gamma)h,0)y(a+h)\right\}\\
&= \nabla_h^\rho\cdot {}_{a+h}\nabla_h^{-\gamma,\rho}\left\{\frac{h\cdot(t-a)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)}\, {}_h\hat{e}_p(t-a-h+(1-\gamma)h,0)y(a+h)\right\}\\
&\quad -\frac{(t-a-h)_h^{\overline{\gamma-1}}}{\rho^{\gamma-1}\Gamma(\gamma)}\, {}_h\hat{e}_p(t-a-h,0)\eta(h,\rho,\gamma)\left(\frac{h\cdot(h)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)}\, {}_h\hat{e}_p((1-\gamma)h,0)y(a+h)\right)\\
&= \Phi(t,h,\rho,\gamma)-\frac{h^{1-\gamma}(t-a-h)_h^{\overline{\gamma-1}}}{\Gamma(\gamma)}\, {}_h\hat{e}_p(t-a-\gamma h,0)\eta(h,\rho,\gamma)y(a+h)\\
&= \Phi(t,h,\rho,\gamma)-\Psi(t,h,\rho,\gamma).
\end{aligned} \tag{37}$$

Similar to (30), we have

$$\begin{aligned}
&{}_{a+h}\nabla_h^{-\gamma,\rho}\left\{\frac{h\cdot(t-a)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)}\, {}_h\hat{e}_p(t-a-h+(1-\gamma)h,0)y(a+h)\right\}\\
&= {}_{a}\nabla_h^{-\gamma,\rho}\left\{\frac{h\cdot(t-a)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)}\, {}_h\hat{e}_p(t-a-h+(1-\gamma)h,0)y(a+h)\right\}\\
&\quad -\frac{h\cdot(t-a)_h^{\overline{\gamma-1}}}{\rho^{\gamma}\Gamma(\gamma)}\, {}_h\hat{e}_p(t-a-h+\gamma h,0)\left(\frac{h\cdot(h)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)}\, {}_h\hat{e}_p((1-\gamma)h,0)y(a+h)\right)\\
&= {}_{a}\nabla_h^{-\gamma,\rho}\left\{\frac{h\cdot(t-a)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)}\, {}_h\hat{e}_p(t-a-h+(1-\gamma)h,0)y(a+h)\right\}\\
&\quad -\frac{h^{2-\gamma}}{\rho\Gamma(\gamma)}(t-a)_h^{\overline{\gamma-1}}\, {}_h\hat{e}_p(t-a,0)y(a+h).
\end{aligned} \tag{38}$$

Using Remark 1 and $\nabla_h(t-a)_h^{\overline{\gamma-1}} = (\gamma-1)(t-a)_h^{\overline{\gamma-2}}N$, it follows that

$$\begin{aligned}
\Phi(t,h,\rho,\gamma) &= \nabla_h^\rho\cdot {}_{a}\nabla_h^{-\gamma,\rho}\left\{\frac{h\cdot(t-a)_h^{\overline{-\gamma}}}{\rho^{1-\gamma}\Gamma(1-\gamma)}\, {}_h\hat{e}_p(t-a-h+(1-\gamma)h,0)y(a+h)\right\}\\
&\quad -\nabla_h^\rho\left(\frac{h^{2-\gamma}}{\rho\Gamma(\gamma)}(t-a)_h^{\overline{\gamma-1}}\, {}_h\hat{e}_p(t-a,0)y(a+h)\right)\\
&= \frac{h}{\rho^{1-\gamma}\Gamma(1-\gamma)}\, {}_h\hat{e}_p(-a-h+(1-\gamma)h,0)y(a+h)\\
&\quad \cdot\nabla_h^\rho\cdot {}_{a}\nabla_h^{-\gamma,\rho}\left\{{}_h\hat{e}_p(t,0)(t-a)_h^{\overline{-\gamma}}\right\}\\
&\quad -\frac{h^{2-\gamma}}{\rho\Gamma(\gamma)}\, {}_h\hat{e}_p(-a,0)y(a+h)\cdot\nabla_h^\rho\left\{{}_h\hat{e}_p(t,0)(t-a)_h^{\overline{\gamma-1}}\right\}\\
&= \frac{h}{\rho}y(a+h)\cdot\nabla_h^\rho\, {}_h\hat{e}_p(t-a,0)-\frac{h^{2-\gamma}}{\Gamma(\gamma)}\, {}_h\hat{e}_p(t-a-h,0)y(a+h)\cdot\nabla_h(t-a)_h^{\overline{\gamma-1}}\\
&= -\frac{h^{2-\gamma}(\gamma-1)(t-a)_h^{\overline{\gamma-2}}}{\Gamma(\gamma)}\, {}_h\hat{e}_p(t-a-h,0)y(a+h).
\end{aligned} \tag{39}$$

Thus,

$$J = -\frac{h^{2-\gamma}(\gamma-1)(t-a)_h^{\overline{\gamma-2}}}{\Gamma(\gamma)}\, {}_h\hat{e}_p(t-a-h,0)y(a+h) - \Psi(t,h,\rho,\gamma). \tag{40}$$

Finally, substituting (36) and (40) back in (32) and arranging, we can obtain the general solution representation

$$\begin{aligned} y(t) = G(t,y(t)) &+ \frac{(t-a-h)_h^{\overline{\gamma-1}}}{\rho^{\gamma-1}\Gamma(\gamma)}\, {}_h\hat{e}_p(t-a-h,0)\eta(h,\rho,\gamma)m - \Psi(t,h,\rho,\gamma) \\ &+ \frac{h^{2-\gamma}(\gamma-1)(t-a)_h^{\overline{\gamma-2}}}{\Gamma(\gamma)}\, {}_h\hat{e}_p(t-a-h,0)y(a+h) + \Psi(t,h,\rho,\gamma). \end{aligned} \tag{41}$$

That is,

$$\begin{aligned} y(t) = {}_{a+h}\nabla_h^{-\gamma,\rho} \cdot {}_a\nabla_h^{\beta(1-\alpha),\rho} f(t,y(t)) &+ \frac{(t-a-h)_h^{\overline{\gamma-1}}}{\rho^{\gamma-1}\Gamma(\gamma)}\, {}_h\hat{e}_p(t,a+h)\eta(h,\rho,\gamma)m \\ &+ \frac{h^{2-\gamma}(\gamma-1)(t-a)_h^{\overline{\gamma-2}}}{\Gamma(\gamma)}\, {}_h\hat{e}_p(t,a+h)y(a+h). \end{aligned} \tag{42}$$

The proof of Theorem 6 is complete. □

Example 1. *For a given function $g(t) : \mathbb{N}_{a+2h,h} \to \mathbb{R}$ and a constant $\lambda \neq 0$, we give two examples to illustrate Theorem 6.*

(i) *Consider the initial value problem*

$$\begin{cases} {}_a\nabla_h^{\alpha,\beta,\rho} y(t) = g(t), & t \in \mathbb{N}_{a+2h,h}, \\ {}_a\nabla_h^{-(1-\gamma),\rho} y(t)|_{t=a+h} \triangleq m. \end{cases} \tag{43}$$

Then we deduce from Theorem 6 that the general solution of the above initial value problem is given by

$$\begin{aligned} y(t) = {}_{a+h}\nabla_h^{-\gamma,\rho} \cdot {}_a\nabla_h^{\beta(1-\alpha),\rho} g(t) &+ \frac{(t-a-h)_h^{\overline{\gamma-1}}}{\rho^{\gamma-1}\Gamma(\gamma)}\, {}_h\hat{e}_p(t,a+h)\eta(h,\rho,\gamma)m \\ &+ \frac{h^{2-\gamma}(\gamma-1)(t-a)_h^{\overline{\gamma-2}}}{\Gamma(\gamma)}\, {}_h\hat{e}_p(t,a+h)y(a+h). \end{aligned} \tag{44}$$

(ii) *Consider the initial value problem*

$$\begin{cases} {}_a\nabla_h^{\alpha,\beta,\rho} y(t) = \lambda y(t), & t \in \mathbb{N}_{a+2h,h}, \\ {}_a\nabla_h^{-(1-\gamma),\rho} y(t)|_{t=a+h} \triangleq m. \end{cases} \tag{45}$$

From Theorem 6, the general solution is given by

$$\begin{aligned} y(t) =\lambda\, {}_{a+h}\nabla_h^{-\gamma,\rho} \cdot {}_a\nabla_h^{\beta(1-\alpha),\rho} y(t) &+ \frac{(t-a-h)_h^{\overline{\gamma-1}}}{\rho^{\gamma-1}\Gamma(\gamma)}\, {}_h\hat{e}_p(t,a+h)\eta(h,\rho,\gamma)m \\ &+ \frac{h^{2-\gamma}(\gamma-1)(t-a)_h^{\overline{\gamma-2}}}{\Gamma(\gamma)}\, {}_h\hat{e}_p(t,a+h)y(a+h). \end{aligned} \tag{46}$$

With a similar proof to Theorem 6, we obtain the following corollary.

Corollary 1. *Consider the initial value problem*

$$\begin{cases} {}_{a-h}\nabla_h^{\alpha,\beta,\rho} y(t) = f(t, y(t)), & t \in \mathbb{N}_{a+h,h}, \\ {}_{a-h}\nabla_h^{-(1-\gamma),\rho} y(t)|_{t=a} = \frac{h^{1-\gamma}}{(\rho-(\rho-1)h)^{1-\gamma}} y(a) \triangleq c, \end{cases} \tag{47}$$

where $0 < \alpha < 1$, $0 \le \beta \le 1$, $0 < \rho \le 1, \gamma = \alpha + \beta - \alpha\beta$ *and* c *is a constant. We can get the general solution representation*

$$\begin{aligned} y(t) = {}_a\nabla_h^{-\gamma,\rho} \cdot {}_{a-h}\nabla_h^{\beta(1-\alpha),\rho} f(t, y(t)) + \frac{(t-a)_h^{\overline{\gamma-1}}}{\rho^{\gamma-1}\Gamma(\gamma)} {}_h\hat{e}_p(t, a)\eta(h, \rho, \gamma)c \\ + \frac{h^{2-\gamma}(\gamma-1)(t-a+h)_h^{\overline{\gamma-2}}}{\Gamma(\gamma)} {}_h\hat{e}_p(t, a)y(a). \end{aligned} \tag{48}$$

where $\eta(h, \rho, \gamma) = \left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\gamma-1}$.

Remark 4. *Corollary 1 is more general compared with corresponding results of the initial value problem with existing difference operators.*

(i) *Let* $\beta = 0$ *in the initial problem (47). Then the initial value problem*

$$\begin{cases} {}_{a-h}\nabla_h^{\alpha,\rho} y(t) = f(t, y(t)), & t \in \mathbb{N}_{a+h,h}, \\ {}_{a-h}\nabla_h^{-(1-\alpha),\rho} y(t)|_{t=a} = \frac{h^{1-\alpha}}{(\rho-(\rho-1)h)^{1-\alpha}} y(a) \triangleq c, \end{cases} \tag{49}$$

has the following general solution representation

$$\begin{aligned} y(t) = {}_a\nabla_h^{-\alpha,\rho} f(t, y(t)) + \frac{(t-a)_h^{\overline{\alpha-1}}}{\rho^{\alpha-1}\Gamma(\alpha)} {}_h\hat{e}_p(t, a)\eta(h, \rho, \alpha)c \\ + \frac{h^{2-\alpha}(\alpha-1)(t-a+h)_h^{\overline{\alpha-2}}}{\Gamma(\alpha)} {}_h\hat{e}_p(t, a)y(a). \end{aligned} \tag{50}$$

where $0 < \alpha < 1$, $\eta(h, \rho, \alpha) = \left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\alpha-1}$ *and* c *is a constant.*

(ii) *Let* $\beta = 0$, $\rho = 1$, *and* $h = 1$ *in (48). Then we obtain*

$$y(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} y(a) + {}_a\nabla^{-\alpha} f(t, y(t)), \tag{51}$$

which is the general solution representation of the following initial value problem [24]

$$\begin{cases} {}_{a-1}\nabla^{\alpha} y(t) = f(t, y(t)), & t \in \mathbb{N}_{a+1}, \\ {}_{a-1}\nabla^{-(1-\alpha)} y(t)|_{t=a} = y(a), \end{cases} \tag{52}$$

where $0 < \alpha < 1$. ${}_{a-1}\nabla^{-(1-\alpha)}(\cdot)$ *and* ${}_{a-1}\nabla^{\alpha}(\cdot)$ *are defined by Definition 1 and 2 for* $h = 1$, *respectively.*

Remark 5. *Here we only discuss the case of the left Hilfer generalized proportional fractional operator. The corresponding results for the right one can be obtained similarly.*

4. Conclusions

In this paper, we proposed the generalized proportional fractional difference in the sense of Hilfer, which is considered to be the analogy of the Hilfer generalized proportional fractional derivative. Also, our definition can reduce to some known operators, such as h-Riemann–Liouville, h-Caputo and generalized proportional fractional differences that are defined in [16,25,26] respectively. We derived some important properties of the left Hilfer proportional fractional difference. We also employed the Q-operator that enables us to prove properties for the right Hilfer proportional fractional difference based on the left one and considered the h-Laplace transform. Finally, following the newly left difference, we obtained a general solution of an initial value problem for $0 < \alpha < 1$. In the future, high-order case for $\alpha \geq 1$ can be considered. Furthermore, the general solution is one of most important ways to studying the qualitative properties of the solutions of difference equations, such as existence, uniqueness, oscillation, and so on.

Author Contributions: Conceptualization, Q.W. and R.X.; results and proofs, Q.W. and R.X.; formal analysis, Q.W. and R.X.; writing—original manuscript, Q.W. and R.X.; writing—review and editing, Q.W. and R.X. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Science Foundation of China grant number (11971015).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are very grateful to the anonymous referees for their valuable suggestions and comments, which helped to improve the quality of the paper.

Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of this paper.

References

1. Kilbas, A.A.; Marichev, O.I.; Samko, S.G. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: Yverdon, Switzerland, 1993.
2. Podlubny, I. Fractional differential equations. *Math. Sci. Eng.* **1999**, *198*, 41–119.
3. De Oliveira, E.C.; Machado, J.A.T. A review of definitions for fractional derivatives and integral. *Math. Probl. Eng.* **2014**, *2014*, 238459. [CrossRef]
4. Teodoro, G.S.; Machado, J.A.T.; De Oliveira, E.C. A review of definitions of fractional derivatives and other operators. *J. Comput. Phys.* **2019**, *388*, 195–208. [CrossRef]
5. Yang, F.; Wang, X. Dynamic characteristic of a new fractional-order chaotic system based on the Hopfield Neural Network and its digital circuit implementation. *Phys. Scr.* **2021**, *96*, 3. [CrossRef]
6. Jajarmi, A.; Baleanu, D.; Sajjadi, S.S.; Nieto, J.J. Analysis and some applications of a regularized Ψ-Hilfer fractional derivative. *J. Comput. Appl. Math.* **2022**, *415*, 114476. [CrossRef]
7. Yusuf, A.; Qureshi, S.; Mustapha, U.T.; Musa, S.S.; Sulaiman, T.A. Fractional modeling for improving scholastic performance of students with optimal control. *Int. J. Appl. Comput. Math.* **2022**, *8*, 1–20. [CrossRef]
8. Jajarmi, A.; Baleanu, D.; Zarghami Vahid, K.; Mobayen, S. A general fractional formulation and tracking control for immunogenic tumor dynamics. *Math. Methods Appl. Sci.* **2022**, *45*, 667–680. [CrossRef]
9. Qiu, L.; Zhang, M.; Qin, Q.H. Homogenization function method for time-fractional inverse heat conduction problem in 3D functionally graded materials. *Appl. Math. Lett.* **2021**, *122*, 107478. [CrossRef]
10. Abro, K.A.; Gomez-Aguilar, J.F. Fractional modeling of fin on non-Fourier heat conduction via modern fractional differential operators. *Arab. J. Sci. Eng.* **2021**, *46*, 2901–2910. [CrossRef]
11. Mozafarifard, M.; Toghraie, D.; Sobhani, H. Numerical study of fast transient non-diffusive heat conduction in a porous medium composed of solid-glass spheres and air using fractional Cattaneo subdiffusion model. *Int. Commun. Heat Mass Transf.* **2021**, *122*, 105192. [CrossRef]
12. Yavtushenko, I.O.; Makhmud-Akhunov, M.Y; Sibatov, R.T.; Kitsyuk, E.P.; Svetukhin, V.V. Temperature-Dependent Fractional Dynamics in Pseudo-Capacitors with Carbon Nanotube Array/Polyaniline Electrodes. *Nanomaterials* **2022**, *12*, 739. [CrossRef] [PubMed]
13. Chapman, S. On non-integral orders of summability of series and integrals. *Proc. Lond. Math. Soc.* **1911**, *2*, 369–409. [CrossRef]
14. Miller, K.S.; Ross, B. Fractional difference calculus. In Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and Their Applications, Koriyama, Japan, May 1988.
15. Bohner, M.; Peterson, A.C. *Advances in Dynamic Equations on Time Scales*; Springer Science and Business Media: Boston, MA, USA, 2002.
16. Abdeljawad, T.; Jarad, F.; Alzabut, J. Fractional proportional differences with memory. *Eur. Phys. J. Spec. Top.* **2017**, *226*, 3333–3354. [CrossRef]
17. Haider, S.S.; Abdeljawad, T. On Hilfer fractional difference operator. *Adv. Differ. Equ.* **2020**, *2020*, 1–20. [CrossRef]

18. Jonnalagadda, J.M.; Gopal, N.S. On Hilfer-type nabla fractional differences. *Int. J. Differ. Equ.* **2020**, *15*, 91–107.
19. Atıcı, F.M.; Eloe, P.W. Discrete fractional calculus with the nabla operator. *Electron. J. Qual. Theory Differ. Equ.* **2009**, *2009*, 1–12. [CrossRef]
20. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.
21. Jarad, F.; Abdeljawad, T.; Alzabut, J. Generalized fractional derivatives generated by a class of local proportional derivatives. *Eur. Phys. J. Spec. Top.* **2017**, *226*, 3457–3471. [CrossRef]
22. Ahmed, I.; Kumam, P.; Jarad, F.; Borisut, P.; Jirakitpuwapat, W. On Hilfer generalized proportional fractional derivative. *Adv. Differ. Equ.* **2020**, *2020*, 1–18. [CrossRef]
23. Atıcı, F.M.; Eloe, P.W. Gronwall's inequality on discrete fractional calculus. *Comput. Math. Appl.* **2012**, *64*, 3193–3200. [CrossRef]
24. Abdeljawad, T.; Atıcı, F.M. On the definitions of nabla fractional operators. *Abstr. Appl. Anal.* **2012**, *2012*, 1–13. [CrossRef]
25. Jia, B.G.; Du, F.F.; Erbe, L.; Peterson, A. Asymptotic behavior of nabla half order h-difference equations. *J. Appl. Anal. Comput.* **2018**, *8*, 1707–1726.
26. Abdeljawad, T. On delta and nabla Caputo fractional differences and dual identities. *Discret. Dyn. Nat. Soc.* **2013**, *2013*, 406910. [CrossRef]