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Differential Game for an Infinite System of Two-Block Differential Equations

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Abstract: We present a pursuit differential game for an infinite system of two-block differential equations in Hilbert space l_2. The pursuer and evader control functions are subject to integral constraints. The differential game is said to be completed if the state of the system falls into the origin of l_2 at some finite time. The purpose of the pursuer is to bring the state of the controlled system to the origin of the space l_2, whereas the evader's aim is to prevent this. For the optimal pursuit time, we obtain an equation and construct the optimal strategies for the players.

Keywords: differential game; pursuit; control; strategy; infinite system of differential equations; integral constraint

MSC: 91A23; 49N75

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1. Introduction

The concept of differential games was introduced by R. Isaacs [1], although some differential games were considered before Isaacs. The majority of the past literature on this subject is devoted to finite-space differential games (see, for example, [2–5]).

Control problems described by partial differential equations can be studied in some Banach spaces (see, for example, [6–9]).

In many cases, control problems governed by partial differential equations can be reduced to those written by an infinite system of differential equations, such as those in [6,8,10–14].

In the paper of Azamov et al. [15], for an infinite system of linear ordinary differential equations, the principle diagonal of the coefficient matrix is λ and the upper diagonal entries of the matrix are 1 s. The stability of $\lambda \le -1$ and the controllability of the system were studied. The exact controllability for the trajectories the Korteweg–de Vries–Burgers equation was studied through a linearized system in [13]. Furthermore, the work in [16] relates to a control problem for an infinite system of differential equations.

Differential games problems were studied for the first time in [8,17] in systems with distributed parameters.

The studies presented in [18–26] were devoted to differential games for infinite-dimensional spaces.

The study presented in [22] was devoted to the differential games described by the partial differential equation of the parabolic type given on some interval, $[0, T]$. The differential games were reduced to those described by the following infinite system of differential equations:

$$\dot{z}_i = \lambda_i z_i - u_i + v_i, \;\; z_i(0) = z_{i0}, \;\; i = 1, 2, \ldots, \;\; 0 > \lambda_1 \ge \lambda_2 \ge \ldots \to -\infty,$$

where u_i, v_i, $i = 1, 2, \dots$ are the control parameters of pursuer and evader, respectively. In that paper, four differential games corresponding to the following constraints on controls were considered and sufficient conditions of evasion and completion pursuit were obtained:

$$
\begin{array}{llll}
(i) & \int_0^T \sum\limits_{i=1}^{\infty} u_i^2(t)dt \le \rho^2,\ \int_0^T \sum\limits_{i=1}^{\infty} v_i^2(t)dt \le \sigma^2 & (ii) & \sum\limits_{i=1}^{\infty} u_i^2(t) \le \rho^2,\ \sum\limits_{i=1}^{\infty} v_i^2(t) \le \sigma^2 \\
(iii) & \int_0^T \sum\limits_{i=1}^{\infty} u_i^2(t)dt \le \rho^2,\ \sum\limits_{i=1}^{\infty} v_i^2(t) \le \sigma^2 & (iv) & \sum\limits_{i=1}^{\infty} u_i^2(t) \le \rho^2, \int_0^T \sum\limits_{i=1}^{\infty} v_i^2(t)dt \le \sigma^2
\end{array}
$$

A simple motion-evasion differential game of infinitely numerous evaders and pursuers was studied in [18] in the Hilbert space l_2. The fact that l_2 is an infinite-dimensional space played a key role in the construction of the evasion strategy.

The soft-landing differential game problem of one pursuer and one evader described by the following infinite system of equations was studied in [19]:

$$
\begin{array}{lll}
\ddot{x}_k = -\alpha_k x_k + \beta_k y_k - u_{k1} + v_{k1}, & x_k(0) = x_{k0}, & \dot{x}_k(0) = x_{k1} \\
\ddot{y}_k = \quad \beta_k x_k + \alpha_k y_k - u_{k2} + v_{k2}, & y_k(0) = y_{k0}, & \dot{y}_k(0) = y_{k1}
\end{array}, \quad k = 1, 2, \dots,
$$

In the case of integral constraints on the controls of players, a theorem on the completion of the game was proved.

The construction of optimal strategies of players is a difficult and important task in differential games (see, for example, [20,27]). The study presented in [27] is devoted to the construction of such strategies for the differential game governed by the following infinite system of differential equations:

$$
\begin{array}{ll}
\dot{x}_j = \gamma_j x_j - \delta_j y_j + u_{j1} - v_{j1}, & x_j(0) = x_{j0}, \\
\dot{y}_j = \delta_j x_j + \gamma_j y_j + u_{j2} - v_{j2}, & y_j(0) = y_{j0},
\end{array} \tag{1}
$$

$j = 1, 2, \dots$ in Hilbert space l_2, where γ_j, δ_j are given real numbers; $\gamma_j \le 0$, $\xi_0 = (\xi_{10}, \xi_{20}, \dots) = (x_{10}, y_{10}, x_{20}, y_{20}, \dots)$ is the initial state with $\xi_{j0} = (x_{j0}, y_{j0})$; $u = (u_{11}, u_{12}, u_{21}, u_{22}, \dots)$ is the pursuer's control parameter; and $v = (v_{11}, v_{12}, v_{21}, v_{22}, \dots)$ is the evader's control parameter. It is assumed that $\xi_0 \in l_2$ and $\xi_0 \neq 0$.

The present paper studies a pursuit–evasion differential game of one pursuer and a single evader governed by an infinite system of two-block differential Equation (1) under the condition of $\gamma_j \ge 0$. The control functions of both the pursuer and evader are subjected to integral constraints. The pursuer's purpose is to bring the state of system (1) to the origin of the space l_2, the evader's purpose is to avoid this. We find the optimal pursuit time, and construct optimal strategies of pursuer and evader. First, for an auxiliary control problem, the optimal time is found and the optimal control is constructed. Then, an equation for the optimal pursuit time is given and the optimal strategies of the players are constructed.

2. Statement of the Problem

Let ρ_0 be a given positive number.

Definition 1. *We call the function $w(t) = (w_{11}(t), w_{12}(t), w_{21}(t), w_{22}(t), \dots)$, $0 \le t \le T$, with the values in l_2 an admissible control if its coordinates, $w_{j1}(t)$, $w_{j2}(t)$, $0 \le t \le T$, $j = 1, 2, \dots$, are measurable and satisfy the following condition:*

$$
\int_0^T ||w(s)||^2 ds \le \rho_0^2, \ \ ||w(s)|| = \left(\sum_{j=1}^{\infty} \left(w_{j1}^2(s) + w_{j2}^2(s) \right) \right)^{1/2},
$$

where T is a sufficiently large fixed positive number.

We use $S(\rho_0)$ to denote the set of all admissible controls. Let

$$C_j(t) = e^{\gamma_j t} \begin{bmatrix} \cos \delta_j t & -\sin \delta_j t \\ \sin \delta_j t & \cos \delta_j t \end{bmatrix}, \; j = 1, 2, \ldots. \tag{2}$$

It can be easily seen that, for the matrices $C_j(t)$,

$$C_j(t+h) = C_j(t)C_j(h) = C_j(h)C_j(t), \;\; |C_j(t)\xi_j| = |C_j^*(t)\xi_j| = e^{\gamma_j t}|\xi_j|, \tag{3}$$

where C^* denotes the transpose of the matrix C. Let

$$\xi_j(t) = (x_j(t), y_j(t)), \;\; \xi(t) = (\xi_1(t), \xi_2(t), \ldots) = (x_1(t), y_1(t), x_2(t), y_2(t), \ldots),$$
$$|\xi_j| = \sqrt{x_j^2 + y_j^2}, \;\; \|\xi\| = \left(\sum_{j=1}^{\infty} (x_j^2 + y_j^2) \right)^{1/2}, \;\; \|\xi_0\| = \left(\sum_{j=1}^{\infty} (x_{j0}^2 + y_{j0}^2) \right)^{1/2}.$$

We will use the symbol $C(0, T; l_2)$ to denote the space of continuous functions $\xi(t) \in l_2$, $0 \le t \le T$. The following statement will be needed.

Proposition 1 ([28]). *If $w(\cdot) \in S(\rho_0)$ and $0 \le \gamma_j \le a$ for some $a > 0$, then, for any given $T > 0$, the following infinite system of two-block differential equations*

$$\begin{matrix} \dot{x}_j = \gamma_j x_j - \delta_j y_j + w_{j1}, & x_j(0) = x_{j0} \\ \dot{y}_j = \delta_j x_j + \gamma_j y_j + w_{j2}, & y_j(0) = y_{j0} \end{matrix}, \;\; j = 1, 2, \ldots, \tag{4}$$

has a unique solution $\xi(t) = (\xi_1(t), \xi_2(t), \ldots)$, $0 \le t \le T$, in the space $C(0, T; l_2)$, where

$$\xi_j(t) = C_j(t)\xi_{j0} + \int_0^t C_j(t-s)w(s)ds, \;\; j = 1, 2, \ldots.$$

From now on, we consider the solution of system (4) on the finite time interval $[0, T]$ since Proposition 1 was proved [28] for any fixed $T > 0$. Therefore, T is assumed to be sufficiently large number.

Definition 2. *A function $u(\cdot) \in S(\rho)$ and $v(\cdot) \in S(\sigma)$, where ρ and σ are given positive numbers, are referred to as the admissible controls of pursuer and evader, respectively.*

Let $w_j = (w_{j1}, w_{j2})$, $u_j = (u_{j1}, u_{j2})$, $v_j = (v_{j1}, v_{j2})$, $j = 1, 2, \ldots$. We give the definition for the strategy of pursuer.

Definition 3. *A function*

$$u(t, v) = (u_1(t, v), u_2(t, v), \ldots), \;\; u : [0, T] \times l_2 \to l_2,$$

of the form

$$u_j(t, v) = v_j(t) + w_j(t), \;\; w(\cdot) = (w_1(\cdot), w_2(\cdot), \ldots) \in S(\rho - \sigma),$$

subject to the integral constraint

$$\sum_{j=1}^{\infty} \int_0^T |u_j(t, v(t))|^2 dt \le \rho^2 \;\; for \;\; any \;\; v(\cdot) \in S(\sigma),$$

is called a strategy of the pursuer.

Definition 4. *If, for any admissible control of evader, the equality $\xi(\tau) = 0$ holds at some time $\tau \in [0, \theta]$ depending on the admissible control of evader, we state that the strategy $u(\cdot)$ guarantees the completion of pursuit for the time θ. Here, the number θ is called a guaranteed pursuit time.*

Let

$$\mu(t) = \left(\rho^2 - \int_0^t \|u(s)\|^2 ds\right)^{1/2}, \;\; \nu(t) = \left(\sigma^2 - \int_0^t \|v(s)\|^2 ds\right)^{1/2}.$$

Definition 5. *A function*

$$v(t, \xi, \mu, \nu, u(t-\varepsilon)), \;\; v : [0, T] \times l_2 \times [0, \rho_0] \times [0, \sigma_0] \times l_2 \to l_2,$$

of the form

$$v(t, \xi, \mu, \nu, u(t-\varepsilon)) = \begin{cases} v_0(t), & \mu(t) > \nu(t), \\ 0, & \tau \le t \le \tau + \varepsilon, \\ u(t-\varepsilon), & \tau + \varepsilon < t \le T, \end{cases} \tag{5}$$

that satisfies the constraint

$$\sum_{j=1}^{\infty} \int_0^T |v_j(t, \xi, \mu, \nu, u(t-\varepsilon))|^2 dt \le \sigma^2,$$

is called a strategy of the evader, where ε is a positive number which will be chosen during the game, and τ is the first time when $\mu(\tau) = \nu(\tau)$.

Next, we give a definition for the optimal pursuit time and optimal strategies of players.

Definition 6. *A number θ is called the optimal pursuit time if the following conditions are met:*

(i) There exists a strategy $u_0(\cdot)$ of the pursuer of pursuer such that, for any admissible control of the evader, the differential game can be completed on the time interval $[0, \theta]$;

(ii) There exists a strategy $v_0(\cdot)$ of the evader, for which $\xi(t) \ne 0$, $0 \le t < \theta$, for an arbitrary admissible control of the pursuer.

In this case, we say that evasion is possible on $[0, \theta)$. Furthermore, the strategies $u_0(\cdot)$ and $v_0(\cdot)$ are called optimal strategies of the pursuer and evader, respectively.

The main problem of the paper is to find the optimal pursuit time and to construct the optimal strategies of pursuer and evader in the differential game (1).

3. Time-Optimal Control Problem

First, we consider the following time-optimal control problem for the system (4).

Problem 1. *Find the optimal time to transfer the state of the system (4) from the given initial state $\xi(0) = \xi_0$ to the origin of l_2.*

To solve this problem, we let

$$\phi_j(t) = \int_0^t e^{-2\gamma_j t} dt = \begin{cases} \frac{1-e^{-2\gamma_j t}}{2\gamma_j}, & \gamma_j \ne 0, \\ t, & \gamma_j = 0. \end{cases}$$

Observe $\frac{1}{\phi_j(t)} \to +\infty$ as $t \to 0^+$ for each $j = 1, 2, \ldots$. Since $\xi_0 \neq 0$, at least one term of the sum in the left-hand side of the equation

$$\sum_{j=1}^{\infty} \frac{|\xi_{j0}|^2}{\phi_j(t)} = \rho_0^2, \tag{6}$$

which corresponds to the non-zero ξ_{j0}, approaches $+\infty$ as $t \to 0^+$. Consequently, the left-hand side of Equation (6) approaches $+\infty$ as $t \to 0^+$. Moreover, the left part of (6) is decreasing function of t, $t > 0$, because the functions $\phi_j(t)$ are increasing. Since, for $\gamma_j > 0$,

$$\begin{aligned} 0 < \frac{1}{\phi_j(t)} - 2\gamma_j &= \frac{2\gamma_j}{1 - e^{-2\gamma_j t}} - 2\gamma_j = \frac{2\gamma_j e^{-2\gamma_j t}}{1 - e^{-2\gamma_j t}} \\ &= \frac{2\gamma_j}{e^{2\gamma_j t} - 1} \leq \frac{2\gamma_j}{2\gamma_j t + 1 - 1} = \frac{1}{t}, \end{aligned}$$

then

$$\begin{aligned} \sum_{j=1}^{\infty} 2\gamma_j |\xi_{j0}|^2 &\leq \sum_{j=1}^{\infty} \frac{|\xi_{j0}|^2}{\phi_j(t)} = \sum_{\gamma_j = 0} \frac{|\xi_{j0}|^2}{t} + \sum_{\gamma_j > 0} \frac{|\xi_{j0}|^2}{\phi_j(t)} \\ &\leq \sum_{\gamma_j = 0} \frac{|\xi_{j0}|^2}{t} + \sum_{\gamma_j > 0} \left(2\gamma_j + \frac{1}{t}\right) |\xi_{j0}|^2 \\ &= \sum_{j=1}^{\infty} 2\gamma_j |\xi_{j0}|^2 + \frac{||\xi_0||^2}{t}. \end{aligned}$$

Consequently,

$$\lim_{t \to \infty} \sum_{j=1}^{\infty} \frac{|\xi_{j0}|^2}{\phi_j(t)} = \sum_{j=1}^{\infty} 2\gamma_j |\xi_{j0}|^2,$$

where the series on the right-hand side is convergent since $\gamma_j \leq a$ and $\xi_0 \in l_2$; therefore,

$$\sum_{j=1}^{\infty} 2\gamma_j |\xi_{j0}|^2 \leq \sum_{j=1}^{\infty} 2a |\xi_{j0}|^2 = 2a ||\xi_0||^2.$$

Thus, if

$$\sum_{j=1}^{\infty} 2\gamma_j |\xi_{j0}|^2 < \rho_0^2, \tag{7}$$

then Equation (6) has a unique root $t = \theta$.

We use the following statement to prove the main results of the present paper.

Lemma 1 ([29]). *Let $C(t)$, $0 \leq t \leq \tau$, be a continuous-matrix function of the order n, and its determinant be not identical to 0 on $[0, \tau]$. Then, among the measurable functions, $\omega(\cdot)$, $\omega : [0, \tau] \to R^n$, satisfying the condition*

$$\int_0^{\tau} C(s)\omega(s)ds = \xi_0$$

the control defined at almost everywhere on $[0, \tau]$ by the formula

$$\omega(s) = C^*(s)W^{-1}(\tau)\xi_0, \; W(\tau) = \int_0^{\tau} C(s)C^*(s)ds, \tag{8}$$

gives the minimum to the functional $\int_0^{\tau} |\omega(s)|^2 ds$.

Theorem 1. *If ξ_0 satisfies Equation (7), then the number θ, the unique root of Equation (6), is the optimal time for the time-optimal control problem.*

Proof. Since

$$\xi_j(\theta) = C_j(\theta)\eta_j(\theta), \;\; \eta_j(\theta) = \xi_{j0} + \int_0^\theta C_j(-s)w_j(s)ds \;\; j = 1, 2, \ldots,$$

to show that $\xi(\theta) = 0$, we need only to show that $\eta_j(\theta) = 0$ for all $j = 1, 2, \ldots$, that is,

$$\int_0^\theta C_j(-s)w_j(s)ds = -\xi_{j0}, \;\; j = 1, 2, \ldots,$$

To apply Lemma 1 to this equation, we let

$$W_j(\theta) = \int_0^\theta C_j(-s)C_j^*(-s)ds,$$

where E_2 is the 2×2 identity matrix. It can be easily shown that $W_j(\theta) = \phi_j(\theta)E_2$ and $W_j^{-1}(\theta) = \frac{1}{\phi_j(\theta)}E_2$. Using Formula (8), we define the control $w(t) = (w_1(t), w_2(t), \ldots)$, $0 \le t \le \theta$, by the equation

$$w_j(t) = C_j^*(-t)W_j^{-1}(\theta)(-\xi_0) = -\frac{1}{\phi_j(\theta)}C_j^*(-t)\xi_{j0}, \;\; j = 1, 2, ..., \;\; 0 \le t \le \theta. \tag{9}$$

Then, by (3)

$$\sum_{j=1}^{\infty} \int_0^\theta |w_j(t)|^2 dt = \sum_{j=1}^{\infty} \frac{1}{\phi_j^2(\theta)} \int_0^\theta |C_j^*(-t)\xi_{j0}|^2 dt$$
$$= \sum_{j=1}^{\infty} \frac{|\xi_{j0}|^2}{\phi_j^2(\theta)} \int_0^\theta e^{-2\gamma_j t} dt = \sum_{j=1}^{\infty} \frac{|\xi_{j0}|^2}{\phi_j(\theta)} = \rho_0^2.$$

Hence, the control (9) is admissible.

Next, we show that $\xi_j(\theta) = 0$, $j = 1, 2, \ldots$. Indeed,

$$\eta_j(\theta) = \xi_{j0} + \int_0^\theta C_j(-s)w_j(s)ds = \xi_{j0} - \frac{1}{\phi_j(\theta)} \int_0^\theta C_j(-s)C_j^*(-s)\xi_{j0}ds$$
$$= \xi_{j0} - \frac{1}{\phi_j(\theta)} \int_0^\theta e^{-2\gamma_j s}\xi_{j0}ds = \xi_{j0} - \xi_{j0} = 0, \;\; j = 1, 2, \ldots.$$

Hence, $\xi_j(\theta) = 0$, $j = 1, 2, \ldots$.

What is left is to show is that $\xi(t) \neq 0$, $0 \le t < \theta$, where $\xi(t)$ is the state of the system (4). To obtain a contradiction, we assume that there exists time τ, $0 < \tau < \theta$, and an admissible control $\overline{w}(\cdot) = (\overline{w}_1(\cdot), \overline{w}_2(\cdot), \ldots)$, such that $\xi(\tau) = 0$. Consequently, $\eta(\tau) = (\eta_1(\tau), \eta_2(\tau), \ldots) = 0$, i.e.,

$$\int_0^\tau C_j(-s)\overline{w}_j(s)ds = -\xi_{j0}, \;\; \int_0^\tau ||\overline{w}(s)||^2 ds \le \rho_0^2. \tag{10}$$

Then, by Lemma 1 for the control

$$w_0(t) = (w_{10}(t), w_{20}(t), \ldots), \;\; w_{j0}(t) = -\frac{1}{\phi_j(\tau)}C_j^*(-t)\xi_{j0}, \;\; j = 1, 2, \ldots, \tag{11}$$

(10) is satisfied and the functional

$$I(w(\cdot)) = \sum_{j=1}^{\infty} \int_0^{\tau} |w_j(t)|^2 dt$$

takes its minimum. Substituting the control (11) into the functional I yields

$$I(\bar{w}(\cdot)) \geq I(w_0(\cdot)) = \sum_{j=1}^{\infty} \frac{|\xi_{j0}|^2}{\phi_j(\tau)} > \sum_{j=1}^{\infty} \frac{|\xi_{j0}|^2}{\phi_j(\theta)} = \rho_0^2$$

This means the control $\bar{w}(\cdot)$ is not admissible—this is a contradiction. This completes the Proof of Theorem 1. □

4. Differential Game Problem

Now, we examine the differential game (1). From the reasoning of the previous section, we conclude that the equation

$$\sum_{j=1}^{\infty} \frac{|\xi_{j0}|^2}{\phi_j(t)} = (\rho - \sigma)^2$$

has the only solution $t = \theta_1$. Since T is sufficiently large number, we assume that $\theta_1 < T$.

Theorem 2. *The number θ_1 is the optimal pursuit time in the game (1).*

Proof. First, we establish that the time θ_1 is a guaranteed pursuit time in a differential game (1). To establish this, we let the pursuer use the following strategy

$$u_j(t, v) = v_j(t) - \frac{1}{\phi_j(\theta_1)} C^*(-t)\xi_{j0},\ j = 1, 2, \ldots, \tag{12}$$

where $v(t)$, $0 \leq t \leq \theta_1$, is an arbitrary admissible control of the evader. To see that the strategy (12) is admissible, we use the definition of the time θ_1 and apply the Minkowski inequality

$$\left(\sum_{j=1}^{\infty} \int_a^b (f_j(t) + g_j(t))^2 dt \right)^{1/2} \leq \left(\sum_{j=1}^{\infty} \int_a^b f_j^2(t) dt \right)^{1/2} + \left(\sum_{j=1}^{\infty} \int_a^b g_j^2(t) dt \right)^{1/2}$$

where $a \leq b$, and $f_j(t), g_j(t)$, $i = 1, 2, \ldots$, $t \in [a, b]$, are scalar measurable functions, to obtain

$$\begin{aligned}
&\left(\sum_{j=1}^{\infty} \int_0^{\theta_1} |u_j(t, v(t))|^2 dt \right)^{1/2} \\
&= \left(\sum_{j=1}^{\infty} \int_0^{\theta_1} |v_j(t) - \frac{1}{\phi_j(\theta_1)} C^*(-t)\xi_{j0}|^2 dt \right)^{1/2} \\
&\leq \left(\sum_{j=1}^{\infty} \int_0^{\theta_1} |v_j(t)|^2 dt \right)^{1/2} + \left(\sum_{j=1}^{\infty} \int_0^{\theta_1} \frac{1}{\phi_j^2(\theta_1)} |C^*(-t)\xi_{j0}|^2 dt \right)^{1/2} \\
&\leq \sigma + \left(\sum_{j=1}^{\infty} \frac{|\xi_{j0}|^2}{\phi_j(\theta_1)} \right)^{1/2} \leq \sigma + \rho - \sigma = \rho.
\end{aligned}$$

It is straightforward to show that $\xi_j(\theta_1) = 0$, $j = 1, 2, \ldots$. Thus, θ_1 is a guaranteed pursuit time.

We next claim that evasion is possible on $[0, \theta_1)$. We divide the construction of evader's strategy into two parts. If $\mu(t) > \nu(t)$, we set

$$v_j(t) = -\frac{1}{\phi_j(\theta_1)} \cdot \frac{\sigma}{\rho - \sigma} C_j^*(-t)\xi_{j0}, \ \ j = 1, 2, \dots. \tag{13}$$

If $\mu(\tau) = \nu(\tau)$ at some $\tau \in [0, \theta_1)$, then starting from the time τ, the evader applies the second part of its strategy to be constructed later.

We let the evader use the control (13). We claim that $\xi(t) \neq 0$, $t \in [0, \theta_1)$, while $\mu(t) \geq \nu(t)$.

On the contrary, suppose that the game is completed at some $t = \theta_0 < \theta_1$, that is,

$$\xi_j(\theta_0) = C_j(\theta_0)\eta_j(\theta_0) = 0, \ \ j = 1, 2, \dots,$$

hence

$$\eta_j(\theta_0) = \xi_{j0} - \int_0^{\theta_0} C_j(-s)u_j(s)ds + \int_0^{\theta_0} C_j(-s)v_j(s)ds = 0, \ \ j = 1, 2, \dots,$$

when

$$\mu(\theta_0) \geq \nu(\theta_0). \tag{14}$$

We have from (13) that

$$\begin{aligned} \int_0^{\theta_0} C_j(-s)u_j(s)ds &= \xi_{j0} + \int_0^{\theta_0} C_j(-s)v_j(s)ds \\ &= \xi_{j0} + \frac{\sigma}{\rho - \sigma}\frac{\phi_j(\theta_0)}{\phi_j(\theta_1)}\xi_{j0}, \ \ j = 1, 2, \dots. \end{aligned}$$

For this equation, applying Lemma 1, we obtain the minimum of the functional $\int_0^{\theta_0} |u_j(s)|^2 ds$ at

$$u_j(s) = \frac{1}{\phi_j(\theta_0)} C_j^*(-s)\xi_{j0}\left(1 + \frac{\sigma}{\rho - \sigma}\frac{\phi_j(\theta_0)}{\phi_j(\theta_1)}\right)$$

almost everywhere on $[0, \theta_0]$. Therefore,

$$\int_0^{\theta_0} |u_j(s)|^2 ds = \frac{1}{\phi_j(\theta_0)}|\xi_{j0}|^2\left(1 + \frac{\sigma}{\rho - \sigma}\frac{\phi_j(\theta_0)}{\phi_j(\theta_1)}\right)^2, \ \ j = 1, 2, \dots.$$

By (13)

$$\int_0^{\theta_0} |v_j(s)|^2 ds = \frac{\sigma^2}{(\rho - \sigma)^2}\frac{\phi_j(\rho)}{\phi_j^2(\theta_1)}|\xi_{j0}|^2, \ j = 1, 2, \dots.$$

This gives, for $j = 1, 2, \dots$, that

$$\int_0^{\theta_0} |u_j(s)|^2 ds - \int_0^{\theta_0} |v_j(s)|^2 ds = \frac{1}{\phi_j(\theta_0)}|\xi_{j0}|^2 + \frac{2\sigma}{\rho - \sigma}\frac{1}{\phi_j(\theta_1)}|\xi_{j0}|^2.$$

Consequently,

$$\sum_{j=1}^{\infty}\left(\int_0^{\theta_0} |u_j(s)|^2 ds - \int_0^{\theta_0} |v_j(s)|^2 ds\right) = \sum_{j=1}^{\infty} \frac{1}{\phi_j(\theta_0)}|\xi_{j0}|^2 + \frac{2\sigma}{\rho - \sigma}\sum_{j=1}^{\infty}\frac{1}{\phi_j(\theta_1)}|\xi_{j0}|^2 \tag{15}$$

Since $\theta_0 < \theta_1$ and $\phi_j(t)$, $t > 0$, is an increasing function, therefore in by the definition of θ_1 we obtain

$$\sum_{j=1}^{\infty} \frac{1}{\phi_j(\theta_0)}|\xi_{j0}|^2 > \sum_{j=1}^{\infty} \frac{1}{\phi_j(\theta_1)}|\xi_{j0}|^2 = (\rho - \sigma)^2.$$

Hence, by (15)

$$\sum_{j=1}^{\infty}\left(\int_0^{\theta_0}|u_j(s)|^2ds-\int_0^{\theta_0}|v_j(s)|^2ds\right)>(\rho-\sigma)^2+\frac{2\sigma}{\rho-\sigma}\cdot(\rho-\sigma)^2=\rho^2-\sigma^2,$$

or, equivalently,

$$\sigma^2-\sum_{j=1}^{\infty}\int_0^{\theta_0}|v_j(s)|^2ds>\rho^2-\sum_{j=1}^{\infty}\int_0^{\theta_0}|u_j(s)|^2ds.$$

This gives $\nu^2(\theta_0)>\mu^2(\theta_0)$, contrary to our assumption $\mu(\theta_0)\geq\nu(\theta_0)$. Thus, when the evader uses control (13), we have $\xi(t)\neq 0$, $t\in[0,\theta_1)$ whenever $\mu(t)\geq\nu(t)$.

What if $\mu(t_1)<\nu(t_1)$ at some time $t_1\in[0,\theta_1)$? If so, using the continuity of the functions $\mu(t),\nu(t)$, $t\geq 0$, and the relation $\mu(0)>\nu(0)$ (that is, $\rho>\sigma$), we deduce that there exists τ, $0\leq\tau<t_1$, such that $\mu(\tau)=\nu(\tau)$. In view of the inequality $\xi(t)\neq 0$, $0\leq t\leq\tau$, proved above, we have $\xi(\tau)\neq 0$.

Starting from the time τ the evader applies the second part of their strategy to be constructed below. Note that the inequality $\xi(\tau)=(\xi_1(\tau),\xi_2(\tau),\dots)\neq 0$, implies that $\xi_j(\tau)=C_j(\tau)\eta_j(\tau)\neq 0$ for some j. Hence, $\eta_j(\tau)\neq 0$. Consider the following equation:

$$\eta_j(t)=\eta_j(\tau)-\int_\tau^t C_j(-s)(u_j(s)-v_j(s))ds,\;\; t\geq\tau,$$

corresponding to the number j. We suggest the following strategy

$$v_j(s)=\begin{cases}0, & \tau\leq s\leq\tau+\varepsilon,\\ u_j(s-\varepsilon), & \tau+\varepsilon<s\leq T,\end{cases}$$

for the evader, where ε is a positive number which will be chosen below, and show that $\xi_j(t)\neq 0$, $\tau\leq t\leq T$. If $\tau\leq t\leq\tau+\varepsilon$, then

$$\eta_j(t)=\eta_j(\tau)-\int_\tau^t C_j(-s)u_j(s)ds$$

and

$$\begin{aligned}|\eta_j(t)|&\geq|\eta_j(\tau)|-\left|\int_\tau^t C_j(-s)u_j(s)ds\right|\geq|\eta_j(\tau)|-\int_\tau^t|C_j(-s)u_j(s)|ds\\&=|\eta_j(\tau)|-\int_\tau^t e^{-\gamma_j s}|u_j(s)|ds\geq|\eta_j(\tau)|-\left(\int_\tau^{\tau+\varepsilon}e^{-2\gamma_j s}ds\cdot\int_\tau^{\tau+\varepsilon}|u_j(s)|^2ds\right)^{1/2}.\end{aligned}$$

The right-hand side of the last inequality approaches $|\eta_j(\tau)|$ as $\varepsilon\to 0$ since

$$\int_\tau^{\tau+\varepsilon}|u_j(s)|^2ds\leq\int_0^T|u_j(s)|^2ds\leq\rho^2.$$

If $\tau+\varepsilon<t\leq T$, then we have

$$\begin{aligned}\eta_j(t)&=\eta_j(\tau)-\int_\tau^t C_j(-s)u_j(s)ds+\int_{\tau+\varepsilon}^t C_j(-s)u_j(s-\varepsilon)ds\\&=\eta_j(\tau)-\int_\tau^t C_j(-s)u_j(s)ds+\int_\tau^{t-\varepsilon}C_j(-s-\varepsilon)u_j(s)ds\\&=\eta_j(\tau)+\int_\tau^{t-\varepsilon}[C_j(-s-\varepsilon)-C_j(-s)]u_j(s)ds+\int_{t-\varepsilon}^t C_j(-s)u_j(s)ds.\end{aligned}$$

We use the Cauchy–Schwartz inequality to obtain

$$|\eta_j(t)| \geq |\eta_j(\tau)| - a_j \cdot \left(\int_\tau^{t-\varepsilon} |u_j(s)|^2 ds \right)^{1/2} - \left(\int_{t-\varepsilon}^{t} e^{-2\gamma_j s} ds \cdot \int_{t-\varepsilon}^{t} |u_j(s)|^2 ds \right)^{1/2},$$

where $a_j = \left(\int_\tau^{t-\varepsilon} \|C_j(-s-\varepsilon) - C_j(-s)\|^2 ds \right)^{1/2} \to 0$ as $\varepsilon \to 0$ since $\|C_j(-s-\varepsilon) - C_j(-s)\| \to 0$, $s \in [0, T]$, as $\varepsilon \to 0$. Clearly, $|\eta_j(\tau)| > 0$, and the second and the third summands on the right-hand side tends to 0 as $\varepsilon \to 0$.

Consequently, for some $\varepsilon_0 > 0$, we have $\eta_j(t) > |\eta_j(\tau)|/2$, $t \geq \tau$, whenever $0 < \varepsilon < \varepsilon_0$. This clearly ensures that $\eta_j(t) \neq 0$ for all $t \geq \tau$. Therefore, $\xi_j(t) \neq 0$, $\tau \leq t \leq T$. Hence, $\xi(t) \neq 0$, $\tau \leq t \leq T$. The proof of the theorem is complete. □

5. Conclusions

We have studied a pursuit differential game for an infinite system of two-block differential equations in the Hilbert space l_2. We found an equation for the optimal transfer time for an auxiliary optimal control problem, which is of independent importance. Furthermore, we have constructed the corresponding optimal control for this problem. For the differential game, we obtained a formula for the optimal pursuit time and constructed optimal strategies of players.

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References

1. Isaacs, R. *Differential Games*; John Wiley & Sons: New York, NY, USA, 1965.
2. Berkovitz, L.D. A Survey of Differential Games. In *Mathematical Theory of Control*; Academic Press: New York, NY, USA, 1967; pp. 373–385.
3. Chikrii, A.A. *Conflict-Controlled Processes*; Kluwer: Dordrecht, The Netherlands, 1997.
4. Petrosyan, L.A. *Differential Games of Pursuit*; World Scientific: Singapore, 1993.
5. Pontryagin, L.S. *Selected Works*; Nauka: Moscow, Russia, 1988.
6. Avdonin, S.A.; Ivanov, S.A. *Families of Exponentials: The Method of Moments in Controllability Problems for Distributed Parameter Systems*; Cambridge University Press: Cambridge, UK, 1995.
7. Chaves-Silva, F.; Zhang, X.; Zuazua, E. Controllability of evolution equations with memory. *SIAM J. Control Optim.* **2017**, *55*, 2437–2459. [CrossRef]
8. Lions, J.L. *Contrôle Optimal de Systémes Gouvernées par des Equations aux Dérivées Partielles*; Dunod: Paris, France, 1968.
9. Martin, P.; Rosier, L.; Rouchon, P. Null controllability of the structurally damped wave equation with moving control. *SIAM J. Control Optim.* **2013**, *51*, 660–684. [CrossRef]
10. Albeverio, S.; Alimov, S.A. On a time-optimal control problem associated with the heat exchange process. *Appl. Math. Optim.* **2008**, *57*, 58–68. [CrossRef]
11. Butkovskiy, A.G. *Theory of Optimal Control of Distributed Parameter Systems*; Elsevier: New York, NY, USA, 1969.
12. Chernous'ko, F.L. Bounded controls in distributed-parameter systems. *J. Appl. Math. Mech.* **1992**, *56*, 707–723. [CrossRef]
13. Eduardo, C.; Montoya, C.; Zhang, B. Local exact controllability to the trajectories of the Korteweg-de Vries-Burgers equation on a bounded domain with mixed boundary conditions. *J. Differ. Equ.* **2020**, *268.9*, 4945–4972.

14. Fattorini, H.O. Time-Optimal control of solutions of operational differential equations. *SIAM J. Control* **1964**, *2*, 54–59.
15. Azamov, A.A.; Ibragimov, G.I.; Mamayusupov, K.; Ruzibaev, M. On the stability and null controllability of an infinite system of linear differential equations. *J. Dyn. Control Syst.* **2021**, 1–11. [CrossRef]
16. Azamov, A.A.; Ruziboev, M.B. The time-optimal problem for evolutionary partial differential equations. *J. Appl. Math. Mech.* **2013**, *77*, 220–224. [CrossRef]
17. Osipov, Y.S. The theory of differential games in systems with distributed parameters. *Dokl. Akad. Nauk SSSR* **1975**, *223*, 1314–1317.
18. Alias, I.A.; Ibragimov, G.I.; Rakhmanov, A.T. Evasion Differential Game of Infinitely Many Evaders from Infinitely Many Pursuers in Hilbert Space. *Dyn. Games Appl.* **2017**, *7*, 347–359. [CrossRef]
19. Ibragimov, G.I.; Allahabi, F.; Kuchkarov, A.S. A pursuit problem in an infinite system of second-order differential equations. *Ukr. Math. J.* **2014**, *65*, 1203–1216. [CrossRef]
20. Ibragimov, G.I. The optimal pursuit problem reduced to an infinite system of differential equation. *J. Appl. Math. Mech.* **2013**, *77*, 470–476. [CrossRef]
21. Satimov, N.Y.; Tukhtasinov, M. Game problems on a fixed interval in controlled first-order evolution equations. *Math. Notes* **2006**, *80*, 578–589. [CrossRef]
22. Satimov, N.Y.; Tukhtasinov, M. On Some Game Problems for First-Order Controlled Evolution Equations. *Differ. Equ.* **2005**, *41*, 1169–1177. [CrossRef]
23. Tukhtasinov, M.; Ibragimov, G.I.; Kuchkarova, S.A.; Hasim, R.M. Differential games for an infinite 2-systems of differential equations. *Mathematics* **2021**, *9*, 1467. [CrossRef]
24. Satimov, N.Y.; Tukhtasinov, M. On Game Problems for Second-Order Evolution Equations. *Russ. Math.* **2007**, *51*, 49–57. [CrossRef]
25. Tukhtasinov, M. Some problems in the theory of differential pursuit games in systems with distributed parameters. *J. Appl. Math. Mech.* **1995**, *59*, 979–984. [CrossRef]
26. Tukhtasinov, M.; Mamatov, M.S. On Pursuit Problems in Controlled Distributed Parameters Systems. *Math. Notes* **2008**, *84*, 256–262. [CrossRef]
27. Ibragimov, G.I. Optimal pursuit time for a differential game in the Hilbert space l_2. *ScienceAsia* **2013**, *39S*, 25–30. [CrossRef]
28. Ibragimov, G.I.; Kuchkarova, S. Existence and uniqueness of solutions of an infinite system of 2-block of differential equations. *Bull. Inst. Math.* **2021**, *4*, 70–75.
29. Lee, E.B.; Markus, L. *Foundations of Optimal Control Theory*; Wiley: New York, NY, USA, 1967.