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Modular Geometric Properties in Variable Exponent Spaces

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Abstract: Much has been written on variable exponent spaces in recent years. Most of the literature deals with the normed space structure of such spaces. However, because of the variability of the exponent, the underlying modular structure of these spaces is radically different from that induced by the norm. In this article, we focus our attention on the progress made toward the study of the modular structure of the sequence Lebesgue spaces of variable exponents. In particular, we present a survey of the state of the art regarding modular geometric properties in variable exponent spaces.

Keywords: electrorheological fluid; fixed point; modular vector space; Nakano modular; strictly convex; uniformly convex modular

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1. Introduction

It was recognized as early as during the 1930s by many prominent mathematicians that the L^p spaces and the mathematical methods inherent in their study, though an essential mathematical tool, created many complications and were insufficient to treat non-power type integral equations; see [1]. In particular, Orlicz and Birnbaum considered spaces of functions with growth properties different from those of the power type growth control provided by the L^p norms. More precisely, they realized that by replacing the power function $\varphi(t) = t^p, 1 \leq p < \infty$, with a more general function φ with similar properties (see below) and by defining the function space

$$L^\varphi = \{f : \mathbb{R} \to \mathbb{R} : \text{ there exists } \lambda > 0 \text{ such that } \int_{\mathbb{R}} \varphi(\lambda|f(x)|)\, dm(x) < \infty\},$$

one is provided with a fruitful generalization of the L^p spaces, which is not only non-trivial from the mathematical point of view, but also much more flexible in the realm of applications.

Throughout the present work, we assume that the function $\varphi : [0,\infty) \to [0,\infty)$ is convex and increasing to infinity; thus, it behaves similarly to the power functions $\varphi(t) = t^p$. A typical example of such a function is $\varphi(t) = e^{t^2} - 1$.

These L^φ spaces, now known as Orlicz spaces, can be endowed with a linear metric structure that induces a vast richness of mathematical properties; in particular, Orlicz spaces are of the utmost importance in the study of differential and integral equations with non-power type kernels. For these reasons, the theory of Orlicz spaces, as well as their applications and generalizations, experienced a vigorous development during the second half of the twentieth century [2].

In the seminal work by Orlicz, [3], Lebesgue spaces of variable exponents, $L^{p(\cdot)}$, were introduced as an example; later in the twentieth century, this generalization of the classical Lebesgue spaces was bound to transcend the field of pure mathematics after their importance in the field of partial differential equations with non-standard growth was realized.

The variability of the exponent amounts to a substantial, highly non-trivial deviation from the classical L^p setting. From the standpoint of the present survey, it suffices to say that the modular structure of $L^{p(\cdot)}$ differs dramatically from the Banach space structure. This is to be contrasted with the case of constant exponent, in which both, the modular and the norm topology coincide. Another level of difficulty emerges from the point of view of harmonic analysis. Specifically, optimal conditions on the variable exponent $p(x)$ under which the Hardy–Littlewood maximal function is bounded on $L^{p(\cdot)}$ are still unknown. Variable exponent spaces were first presented in a systematic way in [4]. Some questions related to electromagnetism studied by Zhikov [5] required the consideration of integrals such as

$$\int_{\Omega} |\nabla f(x)|^{p(x)} dx,$$

the minimization of which is related to the following Lagrange–Euler equation:

$$\Delta_{p(\cdot)} u := \operatorname{div}\left(|\nabla u|^{p(\cdot)-2} \nabla u\right) = 0. \tag{1}$$

Due to the dependence of $p(x)$ on x, (1) is said to possess non-standard growth. The solution spaces of differential equations of this type must necessarily account for the variability of $p(x)$. This is the reason why the classical L^p theory, which presupposes a constant value for the exponent p, is inadequate in this context. This can be remedied by requiring the solution to satisfy the following condition:

$$\int_{\Omega} |\nabla u(x)|^{p(x)} dx < \infty.$$

A similar discussion is necessary in the study of the hydrodynamic equations that describe the behavior of non-Newtonian fluids [6,7]. Electrorheological fluids, whose viscosity changes dramatically and abruptly when exposed to an electric or magnetic field, are examples of such fluids. The study of electrorheological fluids is a field of vigorous mathematical research; their importance in applications to civil engineering, military science and medicine cannot be overemphasized [8–12]. The necessity of a clear understanding of spaces with variable integrability is reinforced by their potential applications.

The material outlined in the sequel requires tools from the fields of fixed point theory and modular vector spaces, for which the reader is referred to the books [13,14].

2. Basic Definitions and Results

In 1931 Orlicz gave an interesting example which we denote by $\ell_{p(\cdot)}$.

Definition 1 ([3])**.** *For $p : \mathbb{N} \to [1, \infty)$, the linear space $\ell_{p(\cdot)}$ is defined by*

$$\ell_{p(\cdot)} = \left\{ (y_n) \subset \mathbb{R}^{\mathbb{N}};\ \exists \beta > 0 \text{ for which } \sum_{n=0}^{\infty} \frac{1}{p(n)} \left| \frac{y_n}{\beta} \right|^{p(n)} < +\infty, \right\}.$$

These spaces inspired Nakano, who developed a more general theory of modular vector spaces [15–17].

Definition 2 ([15,18])**.** *Let X be a linear vector space over the field $\mathbb{R}$. A modular on X is a function $\varrho : X \to [0, \infty]$ satisfying the following conditions:*

(1) $\varrho(x) = 0$ *if and only if* $x = 0$,
(2) $\varrho(\alpha x) = \varrho(x)$, *if* $|\alpha| = 1$,

(3) $\varrho(\alpha x + (1-\alpha)y) \leq \varrho(x) + \varrho(y)$, *for any* $\alpha \in [0,1]$ *and any* $x, y \in X$.

If (3) is replaced by

$$\varrho(\alpha x + (1-\alpha)y) \leq \alpha\varrho(x) + (1-\alpha)\varrho(y)$$

for any $\alpha \in [0,1]$ *and* $x, y \in X$, *then* ϱ *is called a convex modular. In addition,* ϱ *is said to be left-continuous if* $\lim\limits_{r\to 1-} \varrho(rx) = \varrho(x)$ *for any* $x \in X$.

A modular function on a vector space X engenders a modular space in a natural fashion.

Definition 3 ([18]). *Given a convex modular* ϱ *defined on the vector space* X, *the modular space generated by* ϱ *is the set*

$$X_\varrho = \{x \in X;\ \lim_{\alpha\to 0} \varrho(\alpha x) = 0\}.$$

The Luxemburg norm on X, $\|.\|_\varrho : X_\varrho \to [0,\infty)$, *is defined by*

$$\|x\|_\varrho := \inf\Big\{\alpha > 0 :\ \varrho\Big(\frac{x}{\alpha}\Big) \leq 1\Big\}.$$

In what follows it is assumed that ϱ is left-continuous

Example 1 ([15,19,20]). *Consider the vector space* $\ell_{p(\cdot)}$ *introduced in Definition 1. The functional* $\varrho : \ell_{p(\cdot)} \to [0,\infty]$ *defined by*

$$\varrho(u) = \varrho((u_n)) = \sum_{n=0}^{\infty} \frac{1}{p(n)}\, |u_n|^{p(n)}$$

is a convex modular functional. Note that ϱ *is left-continuous. The spaces defined above have a rich mathematical structure and have been widely studied, in particular,* $\ell_{p(\cdot)}$ *is reflexive if and only if* $1 < \inf\limits_n p(n) \leq \sup\limits_n p(n) < \infty$ *[21].*

The $\ell_{p(\cdot)}$ spaces have a continuous counterpart, as the next example shows.

Example 2. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain. The notation* $\mathcal{M}(\Omega)$ *will be used for the vector space of all real-valued Borel-measurable functions defined on* Ω. *Let* $\mathcal{P}(\Omega)$ *be the subset of* $\mathcal{M}$ *consisting of functions* $p : \Omega \longrightarrow [1,\infty]$. *For each such* p, *define the set* $\Omega_\infty := \{x \in \Omega : p(x) = \infty\}$. *The function* $\varrho_p : \mathcal{M}(\Omega) \longrightarrow [0,\infty]$, *defined by*

$$\varrho_p(u) = \int\limits_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)} d\mu + \sup_{x\in\Omega_\infty} |u(x)|,$$

is a convex and continuous modular on $\mathcal{M}(\Omega)$. *The associated modular vector space is denoted by* $L^{p(\cdot)}$.

We next introduce the Δ_2 condition, the deep implications of which have an essential significance in the theory of modular vector spaces. Specifically:

Definition 4 ([18]). *A modular* ϱ *defined on a vector space* X *is said to satisfy the* Δ_2*-condition if there exists* $K \geq 0$ *such that, for any* $x \in X_\varrho$, *we have*

$$\varrho(2x) \leq K\, \varrho(x). \tag{2}$$

We set $\omega(2)$ *to be the infimum of all constants* K *for which the preceding condition holds [22].*

Further discussions regarding the Δ_2-condition, its importance and its variants may be found in [14,18,23].

A central idea in the geometry of Banach spaces is that of uniform convexity. More precisely, the norm $\|\cdot\|$ on a vector space X with unit sphere S_X is said to be uniformly convex if for each $0 < \epsilon \leq 2$, there exists $\delta(\epsilon) > 0$ such that

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \ x, y \in S_X, \|x - y\| \geq \epsilon\right\} > 0.$$

A natural question arises in the realm of modular vector spaces, namely whether the normed vector space $(X_\varrho, \|.\|_\varrho)$ is uniformly convex. It is not surprising that the answer to this question depends on the behavior of the modular ϱ. This problem has been fully investigated in Orlicz function spaces in [18,24].

On the other hand, the idea of uniform convexity can be studied directly as a property of the modular, with no reference whatsoever to the norm. Modular uniform convexity was introduced and studied by Nakano [16]. In more precise terms:

Definition 5 ([23])**.** *Let ϱ be a modular on a vector space X.*

(a) *Let $r > 0$ and $\varepsilon > 0$ be given. Define*

$$D_1(r, \varepsilon) := \big\{(x, y); \ x, y \in X_\varrho, \varrho(x) \leq r, \varrho(y) \leq r, \varrho(x - y) \geq \varepsilon r\big\}.$$

If $D_1(r, \varepsilon) \neq \emptyset$, let

$$\delta_1(r, \varepsilon) := \inf\left\{1 - \frac{1}{r}\,\varrho\left(\frac{x+y}{2}\right); \ (x, y) \in D_1(r, \varepsilon)\right\}.$$

If $D_1(r, \varepsilon) = \emptyset$, we set $\delta_1(r, \varepsilon) = 1$. The modular ϱ is said to satisfy $(UC1)$ if for every $r > 0$ and $\varepsilon > 0$, we have $\delta_1(r, \varepsilon) > 0$ [16]. Observe that by selecting $\varepsilon > 0$ small enough, we have $D_1(r, \varepsilon) \neq \emptyset$ for any $r > 0$.

(b) *ϱ is said to satisfy $(UUC1)$ [14] if for every $s \geq 0$ and $\varepsilon > 0$, there exists $\eta_1(s, \varepsilon) > 0$ depending on s and ε such that*

$$\delta_1(r, \varepsilon) > \eta_1(s, \varepsilon) > 0 \text{ for } r > s.$$

(c) *ϱ is said to be uniformly convex in every direction (in short, (UCED)) [25,26] if for any $z_1 \neq z_2$ in X_ϱ and $R > 0$, there exists $\delta = \delta(z_1, z_2, R) > 0$ such that*

$$\left\{\begin{array}{lcl} \varrho(x - z_1) & \leq & R \\ \varrho(x - z_2) & \leq & R \end{array}\right. \implies \varrho\left(x - \frac{z_1 + z_2}{2}\right) \leq R(1 - \delta)$$

for any $x \in X_\varrho$. X_ϱ is said to be (UUCED) if $\delta(z_1, z_2, R) \geq \delta(z_1, z_2, R^*)$ whenever $R \leq R^*$.

(d) *ϱ is said to be strictly convex, (SC), if for every $x, y \in X_\varrho$ such that*

$$\varrho\left(\frac{x+y}{2}\right) = \varrho(x) = \varrho(y),$$

we have $x = y$.

The above modular geometric properties were introduced by mimicking the geometric properties of the norm in Banach spaces. After carefully studying the proof of the uniform convexity of the classical Banach spaces L^p, for $p \in (1, +\infty)$, the authors in [27] introduced a new geometric property which became central in understanding the modular geometric nature of some spaces, the geometry of which was previously out of reach.

Definition 6 ([23])**.** *Given a modular ϱ on a vector space X, we introduce the following uniform convexity type properties of ϱ:*

(a) *Let $r > 0$ and $\varepsilon > 0$ be given. Define*

$$D_2(r,\varepsilon) := \left\{(x,y);\ x,y \in X_\varrho, \varrho(x) \le r, \varrho(y) \le r, \varrho\left(\frac{x-y}{2}\right) \ge \varepsilon r\right\}.$$

If $D_2(r,\varepsilon) \neq \emptyset$, let

$$\delta_2(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\,\varrho\left(\frac{x+y}{2}\right);\ (x,y) \in D_2(r,\varepsilon)\right\}.$$

If $D_2(r,\varepsilon) = \emptyset$, we set $\delta_2(r,\varepsilon) = 1$. ϱ is said to satisfy $(UC2)$ if for every $r > 0$ and $\varepsilon > 0$, one has $\delta_2(r,\varepsilon) > 0$. Observe that given $r > 0$, $\varepsilon > 0$ can be chosen small enough so that $D_2(r,\varepsilon) \neq \emptyset$.

(b) *ϱ is said to satisfy $(UUC2)$ if for every $s \ge 0$ and $\varepsilon > 0$, there exists $\eta_2(s,\varepsilon) > 0$ depending on s and ε such that*

$$\delta_2(r,\varepsilon) > \eta_2(s,\varepsilon) > 0 \text{ for } r > s.$$

We underline the observations that for $i = 1,2$, we have $\delta_i(r,0) = 0$ and that for fixed $r > 0$,

$$\varepsilon \longrightarrow \delta_i(r,\varepsilon)$$

is an increasing function. The following properties follow easily from the above definition [14].

Proposition 1. *The following statements hold for the geometric concepts introduced in the preceding paragraph:*

(a) *$(UUCi)$ implies (UCi) for $i = 1,2.$;*
(b) *$\delta_1(r,\varepsilon) \le \delta_2(r,\varepsilon)$;*
(c) *$(UC1)$ implies $(UC2)$ and $(UUC1)$ implies $(UUC2)$;*
(d) *$(UUC2)$ implies $(UUCED)$, which in turn implies (SC).*

We emphasize the fact that $(UC1)$ and $(UC2)$ are equivalent as long as ϱ satisfies the Δ_2-condition. In the next section, these modular geometric properties are discussed in both the $\ell_{p(\cdot)}$ and the $L^{p(\cdot)}$ spaces.

3. The Case of the $\ell_{p(\cdot)}$ and $L^{P(\cdot)}$ Spaces

Contrary to what might seem intuitive, modular uniform convexity results for the $L^{p(\cdot)}$ spaces cannot be derived by slightly modifying the arguments used for the case of the $\ell^{p(\cdot)}$ spaces. Profound differences emerge between the two cases. These differences are of interest even in the classical, constant exponent ℓ^p and L^p spaces.

The following example, introduced by Orlicz [3], is of central importance in the sequel. It helps one appreciate the novelty behind the modular uniform convexity property $(UUC2)$.

Example 3 ([3,17,20]). *Consider the function ϱ defined on $X = \mathbb{R}^{\mathbb{N}}$ by*

$$\varrho(x) := \varrho((x_n)) = \sum_{n=1}^{\infty} |x_n|^{n+1}.$$

It can easily be verified that ϱ is a convex modular as specified in Definition 2. In this case, ϱ does not satisfy the Δ_2-condition: to see this, it suffices to observe that for $x = (x_n)$ with $x_n = 1/2$ for $n \ge 1$, we have $\varrho(x) < \infty$ whereas $\varrho(2x) = \infty$. For $p \ge 2$, the inequality

$$|a+b|^p + |a-b|^p \le 2^{p-1}\left(|a|^p + |b|^p\right)$$

holds, from which it follows that

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leq \frac{1}{2}\left(|a|^p + |b|^p\right)$$

for any $a, b \in \mathbb{R}$. *This easily implies that*

$$\varrho\left(\frac{x+y}{2}\right) + \varrho\left(\frac{x-y}{2}\right) \leq \frac{1}{2}\left(\varrho(x) + \varrho(x)\right)$$

for any $x, y \in X_\varrho$. *Thus,* ϱ *is* $(UC2)$ *with* $\delta_2(r, \varepsilon) \geq \varepsilon$ *for each* $r > 0$ *and* $\varepsilon > 0$. *In fact,* ϱ *is* $(UUC2)$*, but it can easily be verified that* ϱ *is* **not** $(UC1)$. *This is because setting* $e_m = (x_n)$*, with* $x_n = 0$ *if* $n \neq m$ *and* $x_m = 1$ *for any* $m \geq 1$ *and considering the sequences*

$$x_m = \left(1 + \frac{1}{m+1}\right)e_m + b\, e_{m+1} \ \text{ and } \ y_m = \left(1 + \frac{1}{m+1}\right)e_m - b\, e_{m+1}$$

for $m \geq 1$ *and* $1/2 < b < 1$*, we see that*

$$\varrho(x_m) = \varrho(y_m) = \left(1 + \frac{1}{m+1}\right)^{m+1} + b^{m+2},\ \varrho\left(\frac{x_m + y_m}{2}\right) = \left(1 + \frac{1}{m+1}\right)^{m+1}$$

and $\varrho(x_m - y_m) = \varrho(2b\, e_{m+1}) = (2b)^{m+2}$. *This yields*

$$\lim_{m\to\infty} \varrho(x_m) = \lim_{m\to\infty} \varrho(y_m) = \lim_{m\to\infty} \varrho\left(\frac{x_m + y_m}{2}\right) = e$$

and $\lim_{m\to\infty} \varrho(x_m - y_m) = \infty$*, which would be impossible if* ϱ *were* $(UC1)$.

The preceding example shows the difference between the two modular uniform convexity properties. Most of the published research that deals with uniform convexity in modular spaces focuses mainly on $(UC1)$. As will be seen later, this is an important observation: in fact, a number of important modular geometric properties that are not known to hold in the absence of $(UC1)$, can be dealt with using $(UC2)$.

The following lemma, of a rather technical nature, is crucial when dealing with variable exponent spaces.

Lemma 1. *The following inequalities hold:*

(i) *[28] If* $p \geq 2$*, then*

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leq \frac{1}{2}\left(|a|^p + |b|^p\right)$$

for any $a, b \in \mathbb{R}$.

(ii) *[20] If* $1 < p \leq 2$*, then*

$$\left|\frac{a+b}{2}\right|^p + \frac{p(p-1)}{2}\left|\frac{a-b}{|a|+|b|}\right|^{2-p}\left|\frac{a-b}{2}\right|^p \leq \frac{1}{2}\left(|a|^p + |b|^p\right)$$

for any $a, b \in \mathbb{R}$ *such that* $|a| + |b| \neq 0$.

One of the first results regarding modular uniform convexity was obtained in [29]. In the interest of completeness and with the intention of providing the reader with a glimpse of the theory of variable exponent spaces, we include the proof of this result.

Theorem 1. *For* $p : \mathbb{N} \to [1, \infty]$ *such that* $p^- = \inf_{n\in\mathbb{N}} p(n) > 1$*, the modular* ϱ *on* $\ell_{p(\cdot)}$ *introduced in Example 1 is* $(UUC2)$.

Proof. Assume that $p^- = \inf_{n\in\mathbb{N}} p(n) > 1$, $r > 0$ and $\varepsilon > 0$. Pick $x, y \in \ell_{p(\cdot)}$ in such a way that

$$\varrho(x) \le r, \;\; \varrho(y) \le r \;\; and \;\; \varrho\left(\frac{x-y}{2}\right) \ge r\,\varepsilon.$$

Since ϱ is convex, we have

$$r\,\varepsilon \le \varrho\left(\frac{x-y}{2}\right) \le \frac{\varrho(x)+\varrho(y)}{2} \le r.$$

It follows that $\varepsilon \le 1$. Now let $I = \{n \in \mathbb{N}; p(n) \ge 2\}$ and $J = \{n \in \mathbb{N}; p(n) < 2\} = \mathbb{N} \setminus I$. For any subset K of $\mathbb{N}$, we set

$$\varrho_K(x) = \varrho_K((x_n)) = \sum_{n\in K} \frac{1}{p(n)}\, |x_n|^{p(n)}.$$

If $K = \emptyset$, we set $\varrho_K(x) = 0$. Note that we have $\varrho(z) = \varrho_I(z) + \varrho_J(z)$ for any $z \in \ell_{p(\cdot)}$. It is clear from the assumptions that either $\varrho_I((x-y)/2) \ge r\,\varepsilon/2$ or $\varrho_J((x-y)/2) \ge r\,\varepsilon/2$.

Suppose that $\varrho_I((x-y)/2) \ge r\,\varepsilon/2$. Then, Lemma 1 yields

$$\varrho_I\left(\frac{x+y}{2}\right) + \varrho_I\left(\frac{x-y}{2}\right) \le \frac{\varrho_I(x)+\varrho_I(y)}{2},$$

which implies that

$$\varrho_I\left(\frac{x+y}{2}\right) \le \frac{\varrho_I(x)+\varrho_I(y)}{2} - \frac{r\,\varepsilon}{2}.$$

Since

$$\varrho_J\left(\frac{x+y}{2}\right) \le \frac{\varrho_J(x)+\varrho_J(y)}{2},$$

we obtain

$$\varrho\left(\frac{x+y}{2}\right) \le \frac{\varrho(x)+\varrho(y)}{2} - \frac{r\,\varepsilon}{2} \le r\Big(1 - \frac{\varepsilon}{2}\Big).$$

On the other hand, if one assumes $\varrho_J((x-y)/2) \ge r\,\varepsilon/2$, then setting $C = \varepsilon/4$,

$$J_1 = \Big\{n \in J;\; |x_n - y_n| \le C(|x_n| + |y_n|)\Big\} \;\; and \;\; J_2 = J \setminus J_1,$$

we see that

$$\varrho_{J_1}\left(\frac{x-y}{2}\right) \le \sum_{n\in J_1} \frac{C^{p(n)}}{p(n)} \left|\frac{|x_n|+|y_n|}{2}\right|^{p(n)} \le \frac{C}{2} \sum_{n\in J_1} \frac{|x_n|^{p(n)} + |y_n|^{p(n)}}{p(n)},$$

because the power function is convex and $C \le 1$. Hence

$$\varrho_{J_1}\left(\frac{x-y}{2}\right) \le \frac{C}{2}\Big(\varrho_{J_1}(x) + \varrho_{J_1}(y)\Big) \le \frac{C}{2}\Big(\varrho(x)+\varrho(y)\Big) \le C\,r.$$

Since $\varrho_J((x-y)/2) \ge r\,\varepsilon/2$, we obtain

$$\varrho_{J_2}\left(\frac{x-y}{2}\right) = \varrho_J\left(\frac{x-y}{2}\right) - \varrho_{J_1}\left(\frac{x-y}{2}\right) \ge \frac{r\,\varepsilon}{2} - C\,r.$$

For any $n \in J_2$, we have

$$p^- - 1 \le p(n)(p(n)-1) \;\; and \;\; C \le C^{2-p(n)} \le \left|\frac{x_n - y_n}{|x_n|+|y_n|}\right|^{2-p(n)},$$

which implies by Lemma 1 that

$$\left|\frac{x_n+y_n}{2}\right|^{p(n)}+\frac{(p^--1)C}{2}\left|\frac{x_n-y_n}{2}\right|^{p(n)}\leq\frac{1}{2}\Big(|x_n|^{p(n)}+|y_n|^{p(n)}\Big).$$

Hence

$$\varrho_{J_2}\left(\frac{x+y}{2}\right)+\frac{(p^--1)C}{2}\,\varrho_{J_2}\left(\frac{x-y}{2}\right)\leq\frac{\varrho_{J_2}(x)+\varrho_{J_2}(y)}{2}$$

and this yields

$$\varrho_{J_2}\left(\frac{x+y}{2}\right)\leq\frac{\varrho_{J_2}(x)+\varrho_{J_2}(y)}{2}-r\,\frac{(p^--1)\varepsilon^2}{8}$$

because $C=\varepsilon/4$. Thus,

$$\varrho\left(\frac{x+y}{2}\right)\leq r-r\,\frac{(p^--1)\varepsilon^2}{8}=r\left(1-\frac{(p^--1)\varepsilon^2}{8}\right).$$

Using the definition of $\delta_2(r,\varepsilon)$, we conclude that

$$\delta_2(r,\varepsilon)\geq\min\left(\frac{\varepsilon}{2},(p^--1)\frac{\varepsilon^2}{8}\right)>0.$$

Therefore, ϱ is $(UC2)$ and setting $\eta_2(r,\varepsilon)=\min\big(\varepsilon/2,(p^--1)\varepsilon^2/8\big)$, we see that ϱ is, in fact, $(UUC2)$. $\square$

It is easy to realize that the function $\eta_2(r,\varepsilon)$ introduced in the preceding proof is, in fact, a function of ε only. It will be noticed later that this observation is of the utmost importance in the derivation of some uniform convexity type modular properties.

Remark 1. *Even if $p^-=\inf_{n\in\mathbb{N}}p(n)>1$, it may happen that $p^+=+\infty$. In this case ϱ fails to satisfy the Δ_2-condition. The modular geometry of $\ell_{p(\cdot)}$ in the absence of the Δ_2-condition, that is, in the case where $p^+=+\infty$, remained an unsolved challenge up to the publication of the preceding result.*

Once Theorem 1 was established, the interest in the extreme cases $p^-=1$ and $p^+=+\infty$ intensified. The first result that delved into these cases was discovered in [30].

Theorem 2. *For $p:\mathbb{N}\to[1,\infty]$, the following statements are equivalent:*

(i) *The cardinality of the set $\{n\in\mathbb{N};p(n)=1\}$ is at most one (that is, $p(\cdot)$ satisfies condition (AO));*

(ii) *The modular ϱ is (UUCED) on the vector space; $\ell_{p(\cdot)}$;*

(iii) *The modular ϱ is (SC) on the vector space $\ell_{p(\cdot)}$.*

This conclusion represents a major breakthrough because uniform convexity in every direction, initially introduced in Banach spaces, is stronger than strict convexity.

These two results have been extended to the spaces $L^{p(\cdot)}$. This endeavor is far from a straightforward consequence of the discrete case since it necessitates a profound control of the underlying functional nature of these spaces and the measure involved. Theorem 3 is the extension of Theorem 1 to the $L^{p(\cdot)}$ spaces.

Theorem 3 ([31])**.** *Consider an open set $\Omega\subseteq\mathbb{R}^n$ and let $p\in\mathcal{P}(\Omega)$. If $|\Omega_\infty|=0$ and $p^->1$, then for fixed $r>0$, $0<\varepsilon\leq1$ and for arbitrary $x\in L^{p(\cdot)}(\Omega)$, $y\in L^{p(\cdot)}(\Omega)$ such that $\varrho(x)\leq r$, $\varrho(y)\leq r$ and $\varrho((x-y)/2)\geq\varepsilon r$, one has the inequality*

$$\varrho\left(\frac{x+y}{2}\right)\leq r\left(1-\min\left\{\frac{\varepsilon}{2},(p^--1)\frac{\varepsilon^2}{2}\right\}\right),$$

which implies that the modular ϱ satisfies the $(UUC2)$ property.

In the notation of Example 2, it is indispensable that $|\Omega_\infty| = 0$, since it is easy to prove that if $|\Omega| > 0$, then $L^\infty(\Omega)$ does not have the $(UUC2)$ property.

The following technical lemma is needed for the next result.

Lemma 2 ([4,24,32])**.** *Consider a domain $\Omega \subseteq \mathbb{R}^n$ and assume that $p \in \mathcal{P}(\Omega)$ satisfies $p^+ < \infty$. Then*

$$\|u\|_{p(\cdot)} \leq \max\left\{ \left(\int_\Omega |u|^p \, dt\right)^{\frac{1}{p^-}}, \left(\int_\Omega |u|^p \, dt\right)^{\frac{1}{p^+}} \right\}.$$

The next result is an extension of Theorem 2 to the $L^{p(\cdot)}$ spaces. In order to convey a feeling of the degree of difficulty involved in the passage from $\ell_{p(\cdot)}$ to $L^{p(\cdot)}$, the proof of this result is given below.

Theorem 4 ([33])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and let $p \in \mathcal{P}(\Omega)$. Then, the following properties are equivalent:*

(a) $|\Omega_1| = |\Omega_\infty| = 0$, *where*

$$\Omega_1 = \{t \in \Omega : \; p(t) = 1\} \;\; \text{and} \;\; \Omega_\infty = \{t \in \Omega : \; p(t) = \infty\}.$$

(b) *The modular ϱ is strictly convex.*

(c) *The modular ϱ is $(UUCED)$.*

Proof. It can readily be seen that (c) implies (b). Since neither $L^1(\Omega_1)$ nor $L^\infty(\Omega_\infty)$ are strictly convex, (b) implies (a). The proof will be completed by showing that (a) implies (c). By virtue of assumption (a) it follows that $1 < p(t) < \infty$ a.e.. Pick z_1 and z_2 in $L^{p(\cdot)}(\Omega)$ such that $z_1 \neq z_2$. Then the set

$$\widetilde{\Omega} = \{t \in \Omega; \, z_1(t) \neq z_2(t)\}$$

has positive measure, that is, $|\widetilde{\Omega}| > 0$. Fix $a \in (1, 2)$. Setting

$$\Omega_{1a} = \{t \in \widetilde{\Omega} : 1 < p(t) < a\} \quad \text{and} \quad \Omega_{a\infty} = \{t \in \widetilde{\Omega} : \; a \leq p(t)\},$$

we see that $\widetilde{\Omega} = \widetilde{\Omega}_{1a} \cup \widetilde{\Omega}_{a\infty}$. Fix $R > 0$ and let $u \in L^{p(\cdot)}(\Omega)$ be chosen in such a way that

$$\varrho(u - z_1) \leq R \quad \text{and} \quad \varrho(u - z_2) \leq R.$$

In what follows, we will find $\delta(z_1, z_2, R) > 0$ such that

$$\varrho\left(u - \frac{z_1 + z_2}{2}\right) \leq R\,(1 - \delta(z_1, z_2, R)).$$

For notational convenience, we set, for $v \in L^{p(\cdot)}(\Omega)$,

$$\varrho_{1a}(v) = \int\limits_{\widetilde{\Omega}_{1a}} |v(t)|^{p(t)} dt \;\; \text{and} \;\; \varrho_{1a}^c(v) = \int\limits_{(\widetilde{\Omega}_{1a})^c} |v(t)|^{p(t)} dt.$$

In the proof of Theorem 4, we consider two separate cases: $|\widetilde{\Omega}_{1a}| > 0$ and $|\widetilde{\Omega}_{1a}| = 0$.

Case 1 : $|\widetilde{\Omega}_{1a}| > 0$

By definition, using the convention agreed to above, one has

$$\varrho\left(u - \frac{z_1 + z_2}{2}\right) = \varrho_{1a}\left(u - \frac{z_1 + z_2}{2}\right) + \varrho_{1a}^c\left(u - \frac{z_1 + z_2}{2}\right).$$

For $t \in \widetilde{\Omega}_{1a}$, Lemma 1 implies that

$$\left|u(t) - \frac{z_1(t) + z_2(t)}{2}\right|^{p(t)} + Z(t) \le \frac{1}{2}\Big(|u(t) - z_1(t)|^{p(t)} + |u(t) - z_2(t)|^{p(t)}\Big), \tag{3}$$

where

$$\begin{aligned} Z(t) &= \frac{p(t)(p(t)-1)}{2}\left|\frac{z_1(t) - z_2(t)}{|u(t) - z_1(t)| + |u(t) - z_2(t)|}\right|^{2-p(t)}\left|\frac{z_1(t) - z_2(t)}{2}\right|^{p(t)} \\ &= \frac{p(t)(p(t)-1)}{2^{p(t)+1}} \frac{|z_1(t) - z_2(t)|^2}{\Big||u(t) - z_1(t)| + |u(t) - z_2(t)|\Big|^{2-p(t)}}. \end{aligned}$$

To facilitate the computations, for $t \in \widetilde{\Omega}_{1a}$, set

$$\begin{cases} \gamma(t) &= \dfrac{p(t)(p(t)-1)}{2^{p(t)+1}} < \dfrac{a(a-1)}{4} < \dfrac{1}{2}, \\ f(t) &= \gamma(t)|z_1(t) - z_2(t)|^2, \\ g(t) &= \dfrac{1}{\Big(|u(t) - z_1(t)| + |u(t) - z_2(t)|\Big)^{2-p(t)}}. \end{cases}$$

By assumption, one has

$$\begin{aligned} \int_{\widetilde{\Omega}_{1a}} \left(\frac{1}{g(t)^{p(t)/2}}\right)^{2/(2-p(t))} dt &= \int_{\widetilde{\Omega}_{1a}} \Big(|u(t) + z_1(t)| + |u(t) - z_2(t)|\Big)^{p(t)} dt \\ &\le \int_{\widetilde{\Omega}_{1a}} 2^{p(t)-1}\Big(|u(t) - z_1(t)|^{p(t)} + |u(t) - z_2(t)|^{p(t)}\Big) dt \\ &\le 2\int_{\widetilde{\Omega}_{1a}} \Big(|u(t) - z_1(t)|^{p(t)} + |u(t) - z_2(t)|^{p(t)}\Big) dt \\ &\le 2(2R) = 4R. \end{aligned}$$

A straightforward application of Hölder's inequality [[4], Theorem 2.1] yields

$$\begin{aligned} \int_{\widetilde{\Omega}_{1a}} f(t)^{p(t)/2} dt &= \int_{\widetilde{\Omega}_{1a}} (f(t)g(t))^{p(t)/2} \frac{1}{g(t)^{p(t)/2}} dt \\ &\le C_p \left\|(f(t)g(t))^{p(t)/2}\right\|_{2/p} \left\|\frac{1}{g(t)^{p(t)/2}}\right\|_{2/2-p}. \end{aligned}$$

The constant $C_p > 0$ only depends on the exponent function $p(\cdot)$; moreover, it is possible to select C_p in such a way that it only depends on the constant a. Next, observe that for $t \in \widetilde{\Omega}_{1a}$, one has $2/p(t) \le 2$ and $2/(2 - p(t)) \le 2/(2 - a)$. Next, in view of Lemma 2, we have

$$\begin{aligned} \int_{\widetilde{\Omega}_{1a}} f(t)^{p(t)/2} dt &\le C_p \left(\int_{\widetilde{\Omega}_{1a}} f(t)g(t) dt\right)^{\alpha} \left(\int_{\widetilde{\Omega}_{1a}} \frac{1}{g(t)^{p(t)/(2-p(t))}} dt\right)^{\beta} \\ &\le C_p\, (4\,R)^{\beta} \left(\int_{\widetilde{\Omega}_{1a}} f(t)g(t) dt\right)^{\alpha}. \end{aligned}$$

In the last statement, we have set $\alpha \in A = \{(2/p)^+, (2/p)^-\}$ and $\beta \in B = \{(2/(2-p))^+, (2/(2-p))^-\}$.

Now put

$$\Delta(R, z_1, z_2, \alpha, \beta) = \frac{1}{\left(C_p\ (4\,R)^\beta\right)^{1/\alpha}} \left(\int_{\widetilde{\Omega}_{1a}} \gamma(t)^{p(t)/2} |z_1(t) - z_2(t)|^{p(t)} dt \right)^{1/\alpha}$$

and define

$$\Delta(R, z_1, z_2) = \min\{\Delta(R, z_1, z_2, \alpha, \beta), \alpha \in A, \beta \in B\}.$$

It is clear that the condition $1 < p(t) < \infty$ a.e. implies that $\Delta(R, z_1, z_2) > 0$, from which it follows that

$$\int_{\widetilde{\Omega}_{1a}} f(t) g(t) dt = \int_{\widetilde{\Omega}_{1a}} \frac{\gamma(t) |z_1(t) - z_2(t)|^2}{\left(|u(t) - z_1(t)| + |u(t) - z_2(t)|\right)^{2-p(t)}} dt \geq \Delta(R, z_1, z_2).$$

In conclusion, on account of inequality (3), we have

$$\varrho_{1a}\left(u - \frac{z_1 + z_2}{2}\right) + \Delta(R, z_1, z_2) \leq \frac{\varrho_{1a}(u - z_1) + \varrho_{1a}(u - z_2)}{2}. \tag{4}$$

When combined with the convexity of ϱ^c_{1a}, which in particular implies that

$$\varrho^c_{1a}\left(u - \frac{z_1 + z_2}{2}\right) \leq \frac{\varrho^c_{1a}(u - z_1) + \varrho^c_{1a}(u - z_2)}{2},$$

inequality (4) yields the following inequality:

$$\varrho\left(u - \frac{z_1 + z_2}{2}\right) + \Delta(R, z_1, z_2) \leq \frac{\varrho(u - z_1) + \varrho(u - z_2)}{2}.$$

Set $\delta_1(z_1, z_2, R) = \dfrac{1}{R}\Delta(R, z_1, z_2)$. Then, $\delta_1(z_1, z_2, R) > 0$ and

$$\varrho\left(u - \frac{z_1 + z_2}{2}\right) \leq R(1 - \delta_1(z_1, z_2, R)).$$

Case 2: $|\widetilde{\Omega}_{1a}| = 0$, that is, $\widetilde{\Omega} = \widetilde{\Omega}_{a\infty}$

Under this condition, the restriction of $p(\cdot)$ to Ω_1 satisfies $p_- \geq a > 1$. Let u_1, z_{11} and z_{12} be the restrictions to Ω_1 of u, z_1 and z_2, respectively. For $v \in L^{p(\cdot)}(\Omega)$, write

$$\varrho_{1a}(v) = \int_{\widetilde{\Omega}} |v(t)|^{p(t)} dt \quad and \quad \varrho^c_{1a}(v) = \int_{(\widetilde{\Omega})^c} |v(t)|^{p(t)} dt.$$

It follows from the fact that $z_1 = z_2$ on $(\widetilde{\Omega})^c$ that

$$\varrho^c_{1a}(u - z_1) = \varrho^c_{1a}(u - z_2) = \varrho^c_{1a}\left(u - \frac{z_1 + z_2}{2}\right) = R_u \leq R.$$

For the same reason, we have

$$\varrho_{1a}\left(\frac{z_1 - z_2}{2}\right) = \varrho\left(\frac{z_1 - z_2}{2}\right) > 0.$$

Set $\varepsilon = \varrho((z_1 - z_2)/2)/R$. Then,

$$R\,\varepsilon = \varrho_{1a}\left(\frac{z_1 - z_2}{2}\right) \leq \frac{\varrho_{1a}(u - z_1) + \varrho_{1a}(u - z_2)}{2} \leq R - R_u \leq R.$$

Thus,

$$\begin{cases} \varrho_{1a}\left(\frac{z_1 - z_2}{2}\right) & \geq & (R - R_u)\,\varepsilon, \\ \varrho_{1a}(u - z_1) & \leq & R - R_u, \\ \varrho_{1a}(u - z_2) & \leq & R - R_u. \end{cases}$$

Since $|\Omega_\infty| = 0$, it follows from an application of Theorem 3 to the modular ϱ_{1a} on $L^{p(\cdot)}(\Omega_1)$ with $r = R - R_u$ that

$$\varrho_{1a}\left(u - \frac{z_1 + z_2}{2}\right) \leq (R - R_u)\,\Big(1 - \delta_2(\varepsilon)\Big).$$

Here we wrote

$$\delta_2(\varepsilon) = \min\left\{\frac{\varepsilon}{2}, (p^- - 1)\frac{\varepsilon^2}{2}\right\}.$$

Hence

$$\begin{aligned} \varrho\left(u - \frac{z_1 + z_2}{2}\right) &= \varrho_{1a}\left(u - \frac{z_1 + z_2}{2}\right) + \varrho_{1a}^{c}\left(u - \frac{z_1 + z_2}{2}\right) \\ &= \varrho_{1a}\left(u - \frac{z_1 + z_2}{2}\right) + R_u \\ &\leq (R - R_u)\,\Big(1 - \delta_2(\varepsilon)\Big) + R_u \\ &= R\left(1 - \frac{R - R_u}{R}\delta_2(\varepsilon)\right) \\ &\leq R(1 - \varepsilon\,\delta_2(\varepsilon)) \end{aligned}$$

because $R - R_u \geq R\varepsilon$. Set

$$\delta(z_1, z_2, R) = \min\left\{\delta_1(z_1, z_2, R), \frac{\varrho((z_1 - z_2)/2)}{R}\,\delta_2\left(\frac{\varrho((z_1 - z_2)/2)}{R}\right)\right\}.$$

Then $\delta(z_1, z_2, R) > 0$ and we have

$$\varrho\left(u - \frac{z_1 + z_2}{2}\right) \leq R\,(1 - \delta(z_1, z_2, R)),$$

which completes the proof of our assertion. $\square$

4. Applications

As described in the previous section, even when $p^- = 1$ or $p^+ = +\infty$, the spaces $\ell_{p(\cdot)}$ and $L^{p(\cdot)}$ may enjoy modular convexity properties, hitherto unknown to hold in these extreme cases. Our next order of business is to explore the functional analytic significance of these properties. We focus, in particular, on the implications of the modular uniform convexity properties in fixed point theory. With this objective in mind, we first recall some standard notation and terminology before proceeding.

Firstly, we point out that any modular on a vector space X induces a topology (referred to as the modular topology) which is reminiscent of the one generated by a metric. Specific details are given in the following definition.

Definition 7 ([34]). *Let ϱ be a modular defined on a vector space X.*

(a) *We say that a sequence $\{x_n\} \subset X_\varrho$ is ϱ-convergent to $x \in X_\varrho$ if $\varrho(x_n - x) \to 0$. If the ϱ limit exists, its uniqueness follows easily.*

(b) *If a sequence $\{x_n\} \subset X_\varrho$ satisfies $\varrho(x_n - x_m) \to 0$ as $n, m \to \infty$, then it is said to be ϱ-Cauchy.*

(c) *X_ϱ is said to be ϱ-complete if and only if any ϱ-Cauchy sequence in X_ϱ is ϱ-convergent.*

(d) *A set $C \subset X_\varrho$ is said to be ϱ-closed if for any sequence of $\{x_n\} \subset C$ which ϱ-converges to x, we have $x \in C$.*

(e) *A set $C \subset X_\varrho$ is said to be ϱ-bounded if $\delta_\varrho(C) = \sup\{\varrho(x - y); x, y \in C\} < \infty$.*

(f) *If $K \subset X_\varrho$ has the property that any sequence $\{x_n\}$ in K has a subsequence that ϱ-converges to a point in K, then K is said to be ϱ-compact.*

(g) *If whenever $\{y_n\}$ ϱ-converges to $y \in X_\varrho$, we have $\varrho(x - y) \le \liminf\limits_{n\to\infty} \varrho(x - y_n)$ for any $x \in X_\varrho$, then ϱ is said to have the Fatou property.*

The Fatou property is closely tied to the modular geometry. As a particular instance we emphasize the important fact that the validity of the Fatou property implies that ϱ-balls are ϱ-closed. More precisely, any subset of the form

$$B_\varrho(x, r) = \{y \in X_\varrho;\ \varrho(x - y) \le r\},$$

for any $x \in X_\varrho$ and $r \ge 0$ (a ϱ-ball), is ϱ-closed, provided ϱ has the Fatou property. The following technical lemma is a powerful tool in many applications.

Lemma 3. *Let ϱ be a convex modular on a vector space X. Assume that ϱ satisfies the Fatou property, is $(UUC2)$ and that X_ϱ is complete. Then the following properties hold true.*

(i) *Let $\emptyset \ne C \subset X_\varrho$ be ϱ-closed and convex, and let $x \in X_\varrho$ satisfy*

$$d_\varrho(x, C) = \inf\{\varrho(x - y);\ y \in C\} < \infty.$$

Then there exists a unique $c \in C$, for which $d_\varrho(x, C) = \varrho(x - c)$.

(ii) *X_ϱ has property (R). Specifically, for any decreasing sequence $\{C_n\}_{n\ge1}$, $\emptyset \ne C_n \subset X_\varrho$ of ϱ-closed and convex sets such that, for some $x \in X_\varrho$, $\sup\limits_{n\ge1} d_\varrho(x, C_n) < \infty$, we have $\bigcap\limits_{n\ge1} C_n \ne \emptyset$.*

Proof. Since C is ϱ-closed, it can be assumed without any loss of generality that $x \notin C$ It follows that $d_\varrho(x, C) > 0$. Write $R = d_\varrho(x, C)$, so that for each $n \ge 1$, there exists $y_n \in C$ such that $\varrho(x - y_n) < R(1 + 1/n)$. We contend that the sequence $\{y_n/2\}$ is ϱ-Cauchy. For if it were not, there would exist a subsequence $\{y_{\varphi(n)}\}$ and $\varepsilon_0 > 0$ satisfying $\varrho\Big((y_{\varphi(n)} - y_{\varphi(m)})/2\Big) \ge \varepsilon_0$ for all $n > m \ge 1$. Since for each $n \ge 1$, $R(1 + 1/n) > R/2 = s$, it would follow that

$$\delta_2(R(1 + 1/n), 2\varepsilon_0/R) \ge \eta_2(R/2, 2\varepsilon_0/R) > 0$$

for each $n \ge 1$. Using the inequalities $\max\Big(\varrho(x - y_{\varphi(n)}), \varrho(x - y_{\varphi(m)})\Big) \le R(1 + 1/\varphi(m))$ and

$$\varrho\left(\frac{y_{\varphi(n)} - y_{\varphi(m)}}{2}\right) \ge \varepsilon_0 \ge R\left(1 + \frac{1}{\varphi(m)}\right)\frac{\varepsilon_0}{2R},$$

valid for all $n > m \ge 1$, we can easily conclude that

$$\varrho\left(x - \frac{y_{\varphi(n)} + y_{\varphi(m)}}{2}\right) \le R\left(1 + \frac{1}{\varphi(m)}\right)(1 - \eta_2(R/2, 2\varepsilon_0/R)).$$

Thus, for any $m \geq 1$, one has

$$R = d_\varrho(x, C) \leq R\left(1 + \frac{1}{\varphi(m)}\right)(1 - \eta_2(R/2, 2\varepsilon_0/R)).$$

Letting $m \to \infty$, we see that $R \leq R(1 - \eta_2(R/2, 2\varepsilon_0/R))$. However, the latter inequality contradicts the inequalities $R > 0$ and $\eta(R/2, 2\varepsilon_0/R) > 0$. Thus, $\{y_n/2\}$ is ϱ-Cauchy and by virtue of the ϱ-completeness of X_ϱ, $\{y_n/2\}$ ϱ-converges to some y. Next, we show that $2y \in C$. To see this, observe that for any $m \geq 1$, the sequence $\{(y_n + y_m)/2\}$ ϱ-converges to $y + y_m/2$ and that C is ϱ-closed and convex, which implies $y + y_m/2 \in C$. Moreover, the sequence $\{y + y_m/2\}$ ϱ-converges to $2y$, from which it follows that $2y \in C$. Let $c = 2y$. Using the Fatou property, which ϱ is assumed to have, it follows that

$$\begin{aligned} d_\varrho(x, C) &\leq \varrho(x - c) \\ &\leq \liminf_{m \to \infty} \varrho\Big(x - (y + y_m/2)\Big) \\ &\leq \liminf_{m \to \infty} \liminf_{n \to \infty} \varrho\Big(x - (y_n + y_m/2)\Big) \\ &\leq \liminf_{m \to \infty} \liminf_{n \to \infty} \Big(\varrho(x - y_n) + \varrho(x - y_m)\Big)/2 \\ &= R = d_\varrho(x, C). \end{aligned}$$

Hence, $\varrho(x - c) = d_\varrho(x, C)$. Since ϱ is (SC), it also follows that c is unique.

To prove (ii), suppose that for some $n_0 \geq 1$, we have $x \notin C_{n_0}$. The sequence $\{d_\varrho(x, C_n)\}$ is increasing and bounded; write $\lim_{n \to \infty} d_\varrho(x, C_n) = R$. No generality is lost by assuming that $R > 0$, for if this were not so, we would have $x \in C_n$ for any $n \geq 1$. It can be observed that due to (i), there exists a unique $y_n \in C_n$ for which $d_\varrho(x, C_n) = \varrho(x - y_n)$ for each $n \geq 1$. It can be shown analogously that $\{y_n/2\}$ ϱ-converges to some $y \in X_\varrho$. Next, we observe that $\{C_n\}$ is decreasing, and that each C_n is convex and ϱ-closed. This implies that $2y \in \bigcap_{n \geq 1} C_n$. $\square$

At this point, a natural question arises, namely, whether property (R) extends to arbitrary decreasing families of subsets. In this connection, we have the following proposition.

Proposition 2. *Let ϱ be a convex modular on the space X. Suppose X_ϱ is complete and ϱ is $(UUC2)$. Let $\emptyset \neq C \subset X_\varrho$ be ϱ-closed, convex and ϱ-bounded. Consider a family of nonempty, ϱ-closed and convex subsets of C, say $\{C_i\}_{i \in I}$, and suppose that for any finite subset $F \subset I$, one has $\bigcap_{i \in F} C_i \neq \emptyset$. Then $\bigcap_{i \in I} C_i \neq \emptyset$.*

Property (R) was first introduced in metric spaces in [35]. It was inspired by the fact that a Banach space has property (R) if and only if it is reflexive. It is still not known how to extract from given sequences, subsequences which converge in some sense, when property (R) holds.

Following the work of Garkavi [25,26], similar properties can be derived when the modular is $(UCED)$.

Proposition 3. *Let ϱ be a convex modular defined on X. Assume X_ϱ is complete. Let $\emptyset \neq C \subset X_\varrho$ be ϱ-closed, convex and ϱ-bounded.*

(i) If C has property (R) and $\emptyset \neq K \subset C$ is ϱ-closed and convex, then K is ϱ-proximinal in C. This is to say that for any $x \in C$, the set $P_{\varrho,K}(x) = \{y \in C;\ \varrho(x - y) = \inf_{z \in K} \varrho(x - z)\}$ is not empty. Moreover, if ϱ is (SC), then K is a Čebyšev subset, that is, $P_{\varrho,K}(x)$ is a singleton for any $x \in C$.

(ii) *If C has property (R), ϱ is (UUCED) and $\emptyset \neq K \subset C$ is ϱ-closed and convex, then K has a unique ϱ-Čebyšev center $x \in K$. In other words,*

$$\sup\{\varrho(x-y);\ y \in K\} = \inf_{z\in K} \Big(\sup\{\varrho(z-y);\ y \in K\}\Big).$$

This implies, in particular, that X_ϱ has the ϱ-normal structure property. More precisely, this means that for any $\emptyset \neq C \subset X_\varrho$, which is ϱ-closed, convex, ϱ-bounded and not a singleton, there exists a point $x \in C$ such that $\sup_{y\in C} \varrho(x-y) < \delta_\varrho(C)$.

The concept of ϱ-type functions plays a major role in many applications [14,36].

Definition 8 ([14,36])**.** *Consider a sequence $\{x_n\}$ X_ϱ and let $\emptyset \neq C \subset X_\varrho$. The function $\tau : C \to [0,\infty]$ defined by*

$$\tau(x) := \limsup_{n\to\infty} \varrho(x-x_n)$$

is referred to as a ϱ-type function. A minimizing sequence of τ in C is a sequence $\{c_n\}$ such that $\lim_{n\to\infty} \tau(c_n) = \inf_{x\in C} \tau(x)$.

We now recall that ϱ-type functions possess a number of powerful properties which are worth mentioning.

Proposition 4 ([14,36])**.** *Let X_ϱ be ϱ-complete and suppose that ϱ has the Fatou property. Let $\emptyset \neq C \subset X_\varrho$ be convex and ϱ-closed. Let $\tau : C \to [0,\infty]$ be the ϱ-type function generated by a sequence $\{x_n\}$ in X_ϱ and suppose that $\tau_0 = \inf_{x\in C} \tau(x) < \infty$. Then*

(i) *If ϱ is (SC), then τ has at most one minimum point.*
(ii) *If ϱ is $(UUC1)$, then any two minimizing sequences of τ ϱ-converge to the same limit.*
(iii) *If ϱ is $(UUC2)$ and $\{c_n\}$ is a minimizing sequence of τ, then $\{c_n/2\}$ ϱ-converges and its limit is independent of $\{c_n\}$.*

In general, it is very difficult to prove the existence of the minimum of modular types. In Banach spaces, the type functions are lower semi-continuous for the weak topology and continuous for the strong topology. Therefore, if some form of compactness is assumed, then the existence of the minimum point is guaranteed. The situation is considerably more involved in the case of modular vector spaces. We begin our discussion by recalling the definition of a uniformly continuous modular.

Definition 9 ([14])**.** *Consider a bounded domain $\Omega \subseteq \mathbb{R}^n$ and $p \in \mathcal{P}(\Omega)$. The modular ϱ on $L^{p(\cdot)}(\Omega)$ is said to be uniformly continuous if the following condition holds: for every $\varepsilon > 0$ and $L > 0$, there exists $\delta > 0$ such that for any $x, y \in L^{p(\cdot)}(\Omega)$ with $\varrho(y) \leq \delta$ and $\varrho(x) \leq L$, we have*

$$|\varrho(x+y) - \varrho(x)| \leq \varepsilon.$$

It was proved by Chen [37] and Kaminska [38] that the uniform continuity of the modular ϱ defined on $L^{p(\cdot)}(\Omega)$ is equivalent to the boundedness condition $p^+ < \infty$. The following result follows from Lemma 5.1 in [14].

Lemma 4. *Consider a bounded domain $\Omega \subseteq \mathbb{R}^n$; let $p \in \mathcal{P}(\Omega)$. If ϱ is uniformly continuous, then any ϱ-type function τ is ϱ-lower semicontinuous.*

Thus, we have the following interesting result, which has major applications in modular fixed point theory.

Lemma 5. *Consider a bounded domain $\Omega \subseteq \mathbb{R}^n$ such that $|\Omega_1| = 0$, let $p \in \mathcal{P}(\Omega)$ with $p^+ < \infty$ and assume that ϱ has property (R). Let $\emptyset \neq C \subset L^{p(\cdot)}(\Omega)$ be ϱ-bounded, ϱ-closed and convex. Then, any ϱ-type function $\tau : C \to [0, \infty]$ such that $\inf_{w \in C} \tau(w) < \infty$ has a minimum point in C.*

We finish this section by stating a fixed point result for modular nonexpansive mappings [14].

Definition 10 ([14]). *Let ϱ be a convex modular defined on X. Let C be a nonempty subset of X_ϱ and $T : C \to C$ be a mapping. If there exists a number $K \geq 0$ such that*

$$\varrho(T(x) - T(y)) \leq K\, \varrho(x - y), \quad \text{for any } x, y \in C,$$

then T is said to be ϱ-Lipschitzian.

(i) T is said to be a ϱ-contraction if $K < 1$.
(ii) T is said to be ϱ-nonexpansive if $K = 1$.
(iii) A point $x \in C$ that satisfies $T(x) = x$ is said to be a fixed point of T.

We finally arrive at the high point of this section, namely, the modular version of Kirk's celebrated fixed point theorem[39].

Theorem 5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and let $p \in \mathcal{P}(\Omega)$; assume that $|\Omega_1| = 0$, $p^+ < \infty$ and that ϱ has property (R). Let $\emptyset \neq C \subset L^{p(\cdot)}(\Omega)$ be ϱ-bounded, ϱ-closed and convex. If a map $T : C \to C$ is ϱ nonexpansive, then it has a fixed point.*

Proof. Fix $x_0 \in C$ and define the ϱ-type function $\tau : C \to [0, \infty]$ by

$$\tau(w) := \limsup_{n \to \infty} \varrho(T^n(x_0) - w).$$

By virtue of the ϱ-boundedness of C, it is clear that $\tau(w) \leq \sup_{w_1, w_2 \in C} \varrho(w_1 - w_2) < \infty$ for any $w \in C$, which implies that $\inf_{w \in C} \tau(w) < \infty$. Lemma 5 implies that τ possesses a unique minimum point $z \in C$. It now follows from

$$\begin{aligned} \tau(T(z)) &= \limsup_{n \to \infty} \varrho(T^n(x_0) - T(z)) \\ &\leq \limsup_{n \to \infty} \varrho(T^{n-1}(x_0) - z) \\ &= \tau(z), \end{aligned}$$

that $T(z)$ must also be a minimum point of τ. Since the minimum point must be unique, it follows that $T(z) = z$. Thus, the map T indeed has a fixed point, as asserted. □

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

References

1. Birnbaum, Z.; Orlicz, W. Uber die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen. *Studia Math.* **1931**, *3*, 1–67. [CrossRef]
2. Rao, M.; Ren, Z. *Theory of Orlicz Spaces*; Chapman and Hall Pure and Applied Mathematics; CRC Press: Boca Raton, FL, USA, 1991.
3. Orlicz, W. Über konjugierte Exponentenfolgen. *Studia Math.* **1931**, 3, 200–211. [CrossRef]
4. Kováčik, O.; Rákosník, J. On spaces $L^{p(x)}, W^{k,p(x)}$. *Czechoslov. Math. J.* **1991**, *41*, 592–618. [CrossRef]
5. Zhikov, V.V. On some variational problems. *Russ. J. Math. Phys.* **1997**, *5*, 105–116.
6. Rajagopal, K.; Ružička, M. On the modeling of electrorheological materials. *Mech. Research Comm.* **1996**, *23*, 401–407. [CrossRef]
7. Ružička, M. *Electrorheological Fluids: Modeling and Mathematical Theory*; Lecture Notes in Mathematics 1748; Springer: Berlin/Heidelberg, Germany, 2000.
8. Bansevicius, R.; Virbalis, J.A. Two-dimensional Braille readers based on electrorheological fluid valves controlled by electric field. *Mechatronics* **2007**, *17*, 570–577. [CrossRef]
9. Chen, J.; Liao, W. Design, testing and control of a magnetorheological actuator for assistive knee braces. *Smart Mater. Struct*. **2010**, *9*, 035029. [CrossRef]
10. Choi, S.H.; Kim, S.; Kim, P.; Park, J.; Choi, S.B. A new visual feedback-based magnetorheological haptic master for robot-assisted minimally invasive surgery. *Smart Mater. Struct*. **2015**, *24*, 065015. [CrossRef]
11. Spencer, B.; Yang, G.; Carlson, J.; Sain, M.M. Smart Dampers for Seismic Protection of Structures: A Full-Scale Study. In Proceedings of the Second World Conference on Structural Control, Kyoto, Japan, 28 June–1 July 1998.
12. Unyong, B.; Vadivel, R.; Govindaraju, M.; Anbuvithya, R.; Gunasekaran, N. Entropy analysis for ethylene glycol hybrid nanofluid flow with elastic deformation, radiation, non-uniform heat generation/absorption, and inclined Lorentz force effects. *Case Stud. Therm. Eng*. **2022**, *30*, 101639. [CrossRef]
13. Khamsi, M.A.; Kirk, W.A. *An Introduction to Metric Spaces and Fixed Point Theory*; John Wiley: New York, NY, USA, 2001.
14. Khamsi, M.A.; Kozlowski, W.M. *Fixed Point Theory in Modular Function Spaces*; Birkhäuser: New York, NY, USA, 2015.
15. Nakano, H. *Modulared Semi-Ordered Linear Spaces*; Maruzen Co.: Tokyo, Japan, 1950.
16. Nakano, H. Modulared sequence spaces. *Proc. Japan Acad.* **1951**, *27*, 508–512. [CrossRef]
17. Nakano, H. *Topology of Linear Topological Spaces*; Maruzen Co. Ltd.: Tokyo, Japan, 1951.
18. Musielak, J. *Orlicz Spaces and Modular Spaces*; Lecture Notes in Mathematics 1034; Springer: Berlin/Heidelberg, Germany; New York, NY, USA; Tokyo, Japan, 1983.
19. Klee, V. Summability in $\ell(p_{11}, p_{21}, \cdots)$ Spaces. *Studia Math.* **1965**, 25, 277–280. [CrossRef]
20. Sundaresan, K. Uniform convexity of Banach spaces $\ell(\{p_i\})$. *Studia Math.* **1971**, *39*, 227–231. [CrossRef]
21. Waterman, D.; Ito, T.; Barber, F.; Ratti, J. Reflexivity and Summability: The Nakano $\ell(p_i)$ spaces. *Studia Math.* **1969**, *331*, 141–146. [CrossRef]
22. Benavides, T.D.; Khamsi, M.A.; Samadi, S. Asymptotically Regular Mappings in Modular Function Spaces. *Sci. Math. Jpn.* **2001**, *53*, 295–304.
23. Kozlowski, W.M. *Modular Function Spaces*; Series of Monographs and Textbooks in Pure and Applied Mathematics; Dekker: New York, NY, USA; Basel, Switzerland, 1988; Volume 122.
24. Diening, L.; Harjulehto, P.; Hästö, P.; Ružiĉka, M. *Lebesgue and Sobolev Spaces with Variable Exponents*; Lecture Note in Mathematics 2017; Springer: Berlin/Heidelberg, Germany, 2011.
25. Garkavi, A. *On the Cebysev Center of a Set in a Normed Space*; Investigations of Contemporary Problems in the Constructive Theory of Functions: Moscow, Russia, 1961; pp. 328–331.
26. Garkavi, A. The best possible net and the best possible cross-section of a set in a normed space. *Izv. Akad. Nauk SSSR Ser. Mat.* **1962**, *26*, 87–106.
27. Khamsi, M.A.; Kozlowski, W.M. On Asymptotic Pointwise Nonexpansive Mappings in Modular Function Spaces. *J. Math. Anal. Appl*. **2011**, *380*, 697–708. [CrossRef]
28. Clarkson, J. Uniformly Convex Spaces. *Trans. Amer. Math. Soc.* **1936**, *40*, 396–414. [CrossRef]

29. Bachar, M.; Bounkhel, M.; Khamsi, M.A. Uniform Convexity in $\ell_{p(\cdot)}$. *J. Nonlinear Sci. Appl.* **2017**, *10*, 5292–5299. [CrossRef]
30. Bachar, M.; Khamsi, M.A.; Mendez, O.; Bounkhel, M. A geometric property in $\ell_{p(\cdot)}$ and its applications. *Math. Nachr.* **2019**, *292*, 1931–1940. [CrossRef]
31. Bachar, M.; Méndez, O.; Bounkhel, M. Modular Uniform Convexity of Lebesgue Spaces of Variable Integrability. *Symmetry* **2018**, *10*, 708. [CrossRef]
32. Méndez, O.; Lang, J. *Analysis on Function Spaces of Musielak-Orlicz Type*; Taylor and Francis: Abingdon, UK, 2018.
33. Bachar, M.; Méndez, O. Modular Uniform Convexity in Every Direction in $L^{p(\cdot)}$ and Its Applications. *Mathematics* **2020**, *8*, 870. [CrossRef]
34. Khamsi, M.A.; Kozlowski, W.K.; Reich, S. Fixed point theory in modular function spaces. *Nonlinear Anal.* **1990**, *14*, 935–953. [CrossRef]
35. Khamsi, M.A. On metric spaces with uniform normal structure. *Proc. Am. Math. Soc.* **1989**, *106*, 723–726. [CrossRef]
36. Abdou, A.A.; Khamsi, M.A. Fixed point theorems in modular vector spaces. *J. Nonlinear Sci. Appl.* **2017**, *10*, 4046–4057. [CrossRef]
37. Chen, S. Geometry of Orlicz spaces. *Dissertat. Math.* **1996**, *356*, 4–205.
38. Kaminska, A. On uniform convexity of Orlicz spaces. *Indag. Math.* **1982**, *44*, 27–36. [CrossRef]
39. Kirk, W.A. A fixed point theorem for mappings which do not increase distances. *Am. Math. Mon.* **1965**, *72*, 1004–1006. [CrossRef]