mathematics

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An Application of Miller–Ross-Type Poisson Distribution on Certain Subclasses of Bi-Univalent Functions Subordinate to Gegenbauer Polynomials

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Abstract: The Miller–Ross-type Poisson distribution is an important model for plenty of real-world applications. In the present analysis, we study and introduce a new class of bi-univalent functions defined by means of Gegenbauer polynomials with a Miller–Ross-type Poisson distribution series. For functions in each of these bi-univalent function classes, we have derived and explored estimates of the Taylor coefficients $|a_2|$ and $|a_3|$ and Fekete-Szegö functional problems for functions belonging to these new subclasses.

Keywords: Poisson distribution series; Gegenbauer polynomials; bi-univalent functions; analytic functions; Fekete-Szegö problem

MSC: 30C45

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Citation: Amourah, A.; Frasin, B.A.; Seoudy, T.M. An Application of Miller–Ross-Type Poisson Distribution on Certain Subclasses of Bi-Univalent Functions Subordinate to Gegenbauer Polynomials. *Mathematics* **2022**, *10*, 2462. https://doi.org/10.3390/math10142462

Academic Editor: Jay Jahangiri

Received: 14 June 2022
Accepted: 12 July 2022
Published: 15 July 2022

1. Definitions and Preliminaries

In recent years, the distributions of random variables have generated a great deal of interest. Their probability density functions have played an important role in statistics and probability theory. Because of this, the study of distributions has been considerable. Many forms of distributions are regarded from real-life situations, such as binomial distribution, Poisson distribution and hyper geometric distribution.

A distribution is a Poisson distribution if its probability density function for a random variable x is given by:

$$f(x) = \frac{e^{-m}}{x!} m^x, \ x = 0, 1, 2, \cdots . \tag{1}$$

and m is the parameter of the distribution.

Let $\mathcal{A}$ denote the class of all normalized analytic functions f of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{2}$$

In addition, the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

Let the functions f and g be analytic in $\mathbb{U}$. We say that the function f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \qquad (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)).$$

In addition, if the function g is univalent in $\mathbb{U}$, then the following equivalence holds:

$$f(z) \prec g(z) \quad \text{if and only if} \quad f(0) = g(0)$$

and

$$f(\mathbb{U}) \subset g(\mathbb{U}).$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4})$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{3}$$

A function is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$.

Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (2). For interesting subclasses of functions in the class Σ, see [1–21].

Orthogonal polynomials have been extensively studied in recent years from various perspectives due to their importance in mathematical statistics, mathematical physics, probability theory and engineering. From a mathematical point of view, orthogonal polynomials often arise from solutions of ordinary differential equations under certain conditions imposed by a certain model. Orthogonal polynomials that appear most commonly in applications are the classical orthogonal polynomials (Legendre polynomials, Chebyshev polynomials, Horadam polynomials, Fibonacci polynomials and Jacobi polynomials). For a recent connection between the geometric function theory and orthogonal polynomials, see [7,22–24].

In 2020, Amourah et al. [1] considered the following generating function of Gegenbauer polynomials:

$$H_\alpha(x,z) = \frac{1}{(1 - 2xz + z^2)^\alpha}. \tag{4}$$

For a fixed x, the function H_α is analytic in $\mathbb{U}$, so it can be expanded in a Taylor series as:

$$H_\alpha(x,z) = \sum_{n=0}^{\infty} C_n^\alpha(x) z^n, \tag{5}$$

where $-1 \leq x \leq 1$, $z \in \mathbb{U}$ and $C_n^\alpha(x)$ is a Gegenbauer polynomial of degree n.

Clearly, H_α generates nothing when $\alpha = 0$. Therefore, the generating function of the Gegenbauer polynomial is set to be:

$$H_0(x,z) = \sum_{n=0}^{\infty} C_n^0(x) z^n \tag{6}$$

for $\alpha = 0$. Moreover, it is worth mentioning that a normalization of α to be greater than $-1/2$ is desirable [25]. Gegenbauer polynomials can also be defined by the following recurrence relations:

$$C_n^\alpha(x) = \frac{1}{n}\left[2x(n + \alpha - 1)C_{n-1}^\alpha(x) - (n + 2\alpha - 2)C_{n-2}^\alpha(x)\right], \tag{7}$$

with the initial values:

$$C_0^\alpha(x) = 1,\ C_1^\alpha(x) = 2\alpha x \text{ and } C_2^\alpha(x) = \alpha\Big[(2 + 2\alpha)x^2 - 1\Big]. \tag{8}$$

Special cases:

i When $\alpha = 1$, we obtain the Chebyshev Polynomials.
ii When $\alpha = \frac{1}{2}$, we obtain the Legendre Polynomials.

Let $\Phi_{\nu,c}(z)$ be the Miller–Ross function [26] (see also, [10,27,28]) defined by

$$\Phi_{\nu,d}(z) = z^{\nu} \sum_{n=0}^{\infty} \frac{(dz)^n}{\Gamma(n+\nu+1)}, \quad (\nu, d, z \in \mathbb{C}\ ). \tag{9}$$

In addition, let $E_{\varsigma,\mu}(z)$ be the two parameters of the Mittag–Leffler function [18] defined by:

$$E_{\varsigma,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\varsigma n + \beta)}, \quad (z, \varsigma, \mu \in \mathbb{C}, \mathrm{Re}(\varsigma) > 0,\ \mathrm{Re}(\mu) > 0\ ). \tag{10}$$

If $\mu = 1$, from (10), we obtain the one-parameter Mittag–Leffler function [29]:

$$E_{\varsigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\varsigma n + 1)}, \quad (z, \varsigma \in \mathbb{C}, \mathrm{Re}(\varsigma) > 0). \tag{11}$$

Several properties of the Mittag–Leffler function and the generalized Mittag–Leffler function can be found in [3,4,6,8].

From (9) and (10), the Miller–Ross function may be written as:

$$\Phi_{\nu,d}(z) = z^{\nu} E_{1,1+\nu}(dz).$$

Very recently, Şeker et al. [30] introduced a power series whose coefficients are Miller–Ross-type Poisson distributions as follows:

$$\mathrm{Y}^m_{\nu,d}(z) = z + \sum_{n=2}^{\infty} \frac{m^{\nu}(dm)^{n-1}}{\Gamma(n+\nu)\Phi_{\nu,d}(m)} z^n,\ z \in \mathbb{U},$$

where $\nu > -1$, $d > 0$.

In addition, they define the series

$$\mathbb{K}^m_{\nu,d}(z) = 2z - \mathrm{Y}^m_{\nu,d}(z) = z - \sum_{n=2}^{\infty} \frac{m^{\nu}(dm)^{n-1}}{\Gamma(n+\nu)\Phi_{\nu,d}(m)} z^n,\ z \in \mathbb{U}. \tag{12}$$

Now, we consider the linear operator $\mathbb{I}^m_{\nu,c} : \mathcal{A} \to \mathcal{A}$ defined by the convolution or Hadamard product

$$\mathbb{I}^m_{\nu,d} f(z) = \mathrm{Y}^m_{\nu,d}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{\nu}(dm)^{n-1}}{\Gamma(n+\nu)\Phi_{\nu,d}(m)} a_n z^n,\ z \in \mathbb{U}, \tag{13}$$

where $\nu > -1$ and $d > 0$.

Motivated essentially by the work of Amourah et al. [20], we introduce a new subclass of Σ involving the Pascal distribution associated with Gegenbauer polynomial and obtain bounds for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ and Fekete-Szegö functional problems [31] for functions in this new class.

2. Coefficient Bounds of the Class $\mathfrak{G}^{\alpha}_{\Sigma}(x, \gamma, \beta)$

We begin this section by defining the new subclass $\mathfrak{G}^{\alpha}_{\Sigma}(x, \gamma, \beta)$ associated with the Miller–Ross-type Poisson distribution

Definition 1. *A function $f \in \Sigma$ given by (2) is said to be in the class $\mathfrak{G}_{\Sigma}^{\alpha}(x,\gamma,\beta)$ if the following subordinations are satisfied:*

$$(1-\gamma+2\beta)\frac{\mathbb{I}_{\nu,d}^{m}f(z)}{z}+(\gamma-2\beta)\left(\mathbb{I}_{\nu,d}^{m}f(z)\right)'+\beta z\left(\mathbb{I}_{\nu,d}^{m}f(z)\right)''\prec H_{\alpha}(x,z) \tag{14}$$

and

$$(1-\gamma+2\beta)\frac{\mathbb{I}_{\nu,d}^{m}f(w)}{w}+(\gamma-2\beta)\left(\mathbb{I}_{\nu,d}^{m}f(w)\right)'+\beta w\left(\mathbb{I}_{\nu,d}^{m}f(w)\right)''\prec H_{\alpha}(x,w), \tag{15}$$

where $\gamma,\beta \geq 0$, $x \in (\frac{1}{2},1]$ and the function $g = f^{-1}$ are given by (3), and H_{α} is the generating function of the Gegenbauer polynomial given by (4).

Upon specializing the parameters γ and β, one can obtain the various new subclasses of Σ, as illustrated in the following examples.

Example 1. *For $\beta = 0$, we have, $\mathfrak{G}_{\Sigma}^{\alpha}(x,\gamma,0) = \mathfrak{G}_{\Sigma}^{\alpha}(x,\gamma)$, in which $\mathfrak{G}_{\Sigma}^{\alpha}(x,\gamma)$ denotes the class of functions $f \in \Sigma$ given by (2) and satisfying the following conditions:*

$$(1-\gamma)\frac{\mathbb{I}_{\nu,d}^{m}f(z)}{z}+\gamma\left(\mathbb{I}_{\nu,d}^{m}f(z)\right)'\prec H_{\alpha}(x,z) \tag{16}$$

and

$$(1-\gamma)\frac{\mathbb{I}_{\nu,d}^{m}f(z)}{w}+\gamma\left(\mathbb{I}_{\nu,d}^{m}f(z)\right)'\prec H_{\alpha}(x,w), \tag{17}$$

where $\alpha > 0$, $\gamma \geq 0$, $x \in (\frac{1}{2},1]$ and the function $g = f^{-1}$ are given by (3), and H_{α} is the generating function of the Gegenbauer polynomial given by (4).

Example 2. *For $\beta = 0$ and $\gamma = 1$, we have, $\mathfrak{G}_{\Sigma}^{\alpha}(x,1,0) = \mathfrak{G}_{\Sigma}^{\alpha}(x)$, in which $\mathfrak{G}_{\Sigma}^{\alpha}(x)$ denotes the class of functions $f \in \Sigma$ given by (2) and satisfying the following conditions:*

$$\left(\mathbb{I}_{\nu,d}^{m}f(z)\right)'\prec H_{\alpha}(x,z) \tag{18}$$

and

$$\left(\mathbb{I}_{\nu,d}^{m}f(z)\right)'\prec H_{\alpha}(x,w), \tag{19}$$

where $\alpha > 0$, $x \in (\frac{1}{2},1]$ and the function $g = f^{-1}$ are given by (3), and H_{α} is the generating function of the Gegenbauer polynomial given by (4).

Example 3. *For $\beta = 1/2$, we have, $\mathfrak{G}_{\Sigma}^{\alpha}(x,\gamma,1/2) = \widetilde{\mathfrak{G}}_{\Sigma}^{\alpha}(x,\gamma)$, in which $\widetilde{\mathfrak{G}}_{\Sigma}^{\alpha}(x,\gamma)$ denotes the class of functions $f \in \Sigma$ given by (2) and satisfying the following conditions:*

$$(2-\gamma)\frac{\mathbb{I}_{\nu,d}^{m}f(z)}{z}+(\gamma-1)\left(\mathbb{I}_{\nu,d}^{m}f(z)\right)'+\frac{1}{2}z\left(\mathbb{I}_{\nu,d}^{m}f(z)\right)''\prec H_{\alpha}(x,z) \tag{20}$$

and

$$(2-\gamma)\frac{\mathbb{I}_{\nu,d}^{m}f(w)}{w}+(\gamma-1)\left(\mathbb{I}_{\nu,d}^{m}f(w)\right)'+\frac{1}{2}w\left(\mathbb{I}_{\nu,d}^{m}f(w)\right)''\prec H_{\alpha}(x,w), \tag{21}$$

where $\alpha > 0$, $x \in (\frac{1}{2},1]$ and the function $g = f^{-1}$ are given by (3), and H_{α} is the generating function of the Gegenbauer polynomial given by (4).

Unless otherwise mentioned, we shall assume in this paper that $\alpha > 0$, $\gamma,\beta \geq 0$ *and* $x \in (\frac{1}{2},1]$.

First, we give the coefficient estimates for the class $\mathfrak{G}_{\Sigma}^{\alpha}(x,\gamma,\beta)$ given in Definition 1.

Theorem 1. *Let $f \in \Sigma$ given by (2) belong to the class $\mathfrak{G}_{\Sigma}^{\alpha}(x,\gamma,\beta)$. Then,*

$$|a_2| \leq \frac{2|\alpha|x\sqrt{2|\alpha|x}\Gamma(2+\nu)\Phi_{\nu,d}(m)}{\sqrt{\left|\left[2x^2\Psi_{\nu,d}(m,\alpha,\gamma,\beta)+\alpha(1+\gamma)^2m^{\nu}\right]m^{\nu}(dm)^2\right|}},$$

and

$$|a_3| \leq \frac{4\alpha^2x^2(\Gamma(2+\nu)\Phi_{\nu,d}(m))^2}{m^{2\nu}(dm)^2(1+\gamma)^2}+\frac{2|\alpha|x\Gamma(3+\nu)\Phi_{\nu,d}(m)}{(1+2\gamma+2\beta)m^{\nu}(dm)^2},$$

where

$$\Psi_{\nu,d}(m,\alpha,\gamma,\beta)=\frac{2(1+2\gamma+2\beta)}{\Gamma(3+\nu)}(\Gamma(2+\nu))^2\Phi_{\nu,d}(m)\alpha^2-(1+\gamma)^2m^{\nu}\alpha(1+\alpha).$$

Proof. Let $f \in \mathfrak{G}_{\Sigma}^{\alpha}(x,\gamma,\beta)$. From Definition 1, for some analytic functions w, v such that $w(0)=v(0)=0$ and $|w(z)|<1$, $|v(w)|<1$ for all $z,w \in \mathbb{U}$, then we can write:

$$(1-\gamma+2\beta)\frac{\mathbb{I}_{\nu,d}^{m}f(z)}{z}+(\gamma-2\beta)\left(\mathbb{I}_{\nu,d}^{m}f(z)\right)'+\beta z\left(\mathbb{I}_{\nu,d}^{m}f(z)\right)''=H_{\alpha}(x,w(z)) \tag{22}$$

and

$$(1-\gamma+2\beta)\frac{\mathbb{I}_{\nu,d}^{m}f(w)}{w}+(\gamma-2\beta)\left(\mathbb{I}_{\nu,d}^{m}f(w)\right)'+\beta w\left(\mathbb{I}_{\nu,d}^{m}f(w)\right)''=H_{\alpha}(x,v(w)). \tag{23}$$

From the equalities (22) and (23), we obtain that

$$\begin{aligned}&(1-\gamma+2\beta)\frac{\mathbb{I}_{\nu,d}^{m}f(z)}{z}+(\gamma-2\beta)\left(\mathbb{I}_{\nu,d}^{m}f(z)\right)'+\beta z\left(\mathbb{I}_{\nu,d}^{m}f(z)\right)''\\&=1+C_1^{\alpha}(x)c_1z+\left[C_1^{\alpha}(x)c_2+C_2^{\alpha}(x)c_1^2\right]z^2+\cdots\end{aligned} \tag{24}$$

and

$$\begin{aligned}&(1-\gamma+2\beta)\frac{\mathbb{I}_{\nu,d}^{m}f(w)}{w}+(\gamma-2\beta)\left(\mathbb{I}_{\nu,d}^{m}f(w)\right)'+\beta w\left(\mathbb{I}_{\nu,d}^{m}f(w)\right)''\\&=1+C_1^{\alpha}(x)d_1w+\left[C_1^{\alpha}(x)d_2+C_2^{\alpha}(x)d_1^2\right])w^2+\cdots.\end{aligned} \tag{25}$$

It is fairly well known that if

$$|w(z)|=\left|c_1z+c_2z^2+c_3z^3+\cdots\right|<1,\ \ (z\in\mathbb{U})$$

and

$$|v(w)|=\left|d_1w+d_2w^2+d_3w^3+\cdots\right|<1,\ \ (w\in\mathbb{U}),$$

then

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{26}$$

Thus, upon comparing the corresponding coefficients in (24) and (25), we have:

$$(1+\gamma)\frac{m^{\nu}(dm)}{\Gamma(2+\nu)\Phi_{\nu,d}(m)}a_2=C_1^{\alpha}(x)c_1, \tag{27}$$

$$(1+2\gamma+2\beta)\frac{m^{\nu}(dm)^2}{\Gamma(3+\nu)\Phi_{\nu,d}(m)}a_3=C_1^{\alpha}(x)c_2+C_2^{\alpha}(x)c_1^2, \tag{28}$$

$$-(1+\gamma)\frac{m^{\nu}(dm)}{\Gamma(2+\nu)\Phi_{\nu,d}(m)}a_2 = C_1^{\alpha}(x)d_1, \tag{29}$$

and

$$(1+2\gamma+2\beta)\frac{m^{\nu}(dm)^2}{\Gamma(3+\nu)\Phi_{\nu,d}(m)}\left[2a_2^2 - a_3\right] = C_1^{\alpha}(x)d_2 + C_2^{\alpha}(x)d_1^2. \tag{30}$$

It follows from (27) and (29) that

$$c_1 = -d_1 \tag{31}$$

and

$$2(1+\gamma)^2\frac{m^{2\nu}(dm)^2}{(\Gamma(2+\nu)\Phi_{\nu,d}(m))^2}a_2^2 = [C_1^{\alpha}(x)]^2\left(c_1^2+d_1^2\right). \tag{32}$$

If we add (28) and (30), we obtain

$$2(1+2\gamma+2\beta)\frac{m^{\nu}(dm)^2}{\Gamma(3+\nu)\Phi_{\nu,d}(m)}a_2^2 = C_1^{\alpha}(x)(c_2+d_2) + C_2^{\alpha}(x)\left(c_1^2+d_1^2\right). \tag{33}$$

Substituting the value of $\left(c_1^2+d_1^2\right)$ from (32) the right-hand side of (33), we deduce that

$$\begin{aligned} &2\left[(1+2\gamma+2\beta)\frac{1}{\Gamma(3+\nu)} - (1+\gamma)^2\frac{m^{\nu}}{(\Gamma(2+\nu))^2\Phi_{\nu,d}(m)}\frac{C_2^{\alpha}(x)}{[C_1^{\alpha}(x)]^2}\right]\frac{m^{\nu}(dm)^2}{\Phi_{\nu,d}(m)}a_2^2 \\ &= C_1^{\alpha}(x)(c_2+d_2). \end{aligned} \tag{34}$$

Moreover, using computations (25), (26) and (34), we find that

$$\begin{aligned} &|a_2| \leq \\ &\frac{2|\alpha|x\sqrt{2|\alpha|x}\Gamma(2+\nu)\Phi_{\nu,d}(m)}{\sqrt{\left|\left[2x^2\Psi_{\nu,d}(m,\alpha,\gamma,\beta) + \alpha(1+\gamma)^2 m^{\nu}\right]m^{\nu}(dm)^2\right|}}. \end{aligned}$$

Moreover, if we subtract (30) from (28), we obtain

$$2(1+2\gamma+2\beta)\frac{m^{\nu}(dm)^2}{\Gamma(3+\nu)\Phi_{\nu,d}(m)}\left(a_3 - a_2^2\right) = C_1^{\alpha}(x)(c_2-d_2) + C_2^{\alpha}(x)\left(c_1^2-d_1^2\right). \tag{35}$$

Then, in view of (8) and (32), Equation (35) becomes:

$$\begin{aligned} a_3 = &\frac{(\Gamma(2+\nu)\Phi_{\nu,d}(m))^2[C_1^{\alpha}(x)]^2}{2m^{2\nu}(dm)^2(1+\gamma)^2}\left(c_1^2+d_1^2\right) \\ &+ \frac{C_1^{\alpha}(x)\Gamma(3+\nu)\Phi_{\nu,d}(m)}{2(1+2\gamma+2\beta)m^{\nu}(dm)^2}(c_2-d_2). \end{aligned}$$

Thus, applying (8), we conclude that

$$|a_3| \leq \frac{4\alpha^2x^2(\Gamma(2+\nu)\Phi_{\nu,d}(m))^2}{m^{2\nu}(dm)^2(1+\gamma)^2} + \frac{2|\alpha|x\Gamma(3+\nu)\Phi_{\nu,d}(m)}{(1+2\gamma+2\beta)m^{\nu}(dm)^2}.$$

This completes the proof of the Theorem. □

Making use of the values of a_2^2 and a_3, we prove the following Fekete–Szegö inequality for functions in the class $\mathfrak{G}_{\Sigma}^{\alpha}(x,\gamma,\beta)$.

Theorem 2. *Let $f \in \Sigma$ given by (2) belong to the class $\mathfrak{G}_{\Sigma}^{\alpha}(x,\gamma,\beta)$. Then,*

$$\left|a_3 - \eta a_2^2\right| \le \begin{cases} \frac{|\alpha| x \Gamma(3+\nu)\Phi_{\nu,d}(m)}{(1+2\gamma+2\beta)m^\nu (dm)^2}, & |\eta - 1| \le \delta \\ \frac{8\alpha^2 x^3 (\Gamma(2+\nu))^2 \left(\Phi_{\nu,d}(m)\right)^2 (1-\eta)}{\left[4\alpha x^2(1+2\gamma+2\beta)\frac{1}{\Gamma(3+\nu)}(\Gamma(2+\nu))^2\Phi_{\nu,d}(m) - (1+\gamma)^2 m^\nu (2(1+\alpha)x^2 - 1)\right] m^\nu (dm)^2}, & |\eta - 1| \ge \delta, \end{cases}$$

where

$$\delta = \left|1 - \frac{\Gamma(3+\nu)(1+\gamma)^2 m^\nu (2(1+\alpha)-1)}{4(1+2\gamma+2\beta)\alpha x^2 (\Gamma(2+\nu))^2 \Phi_{\nu,d}(m)}\right|.$$

Proof. From (34) and (35)

$$\begin{aligned} & a_3 - \eta a_2^2 \\ &= (1-\eta)\frac{\left[C_1^\alpha(x)\right]^3 (c_2+d_2)(\Gamma(2+\nu))^2 (\Phi_{\nu,d}(m))^2}{2\left[\frac{(1+2\gamma+2\beta)}{\Gamma(3+\nu)}(\Gamma(2+\nu))^2 \Phi_{\nu,d}(m)\left[C_1^\alpha(x)\right]^2 - (1+\gamma)^2 m^\nu C_2^\alpha(x)\right] m^\nu (dm)^2} \\ &+ \frac{C_1^\alpha(x)\Gamma(3+\nu)\Phi_{\nu,d}(m)}{2(1+2\gamma+2\beta)m^\nu (dm)^2}(c_2 - d_2) \\ &= C_1^\alpha(x)\left[h(\eta) + \frac{\Gamma(3+\nu)\Phi_{\nu,d}(m)}{2(1+2\gamma+2\beta)m^\nu (dm)^2}\right] c_2 \\ &+ C_1^\alpha(x)\left[h(\eta) - \frac{\Gamma(3+\nu)\Phi_{\nu,d}(m)}{2(1+2\gamma+2\beta)m^\nu (dm)^2}\right] d_2, \end{aligned}$$

where

$$h(\eta) = \frac{\left[C_1^\alpha(x)\right]^2 (c_2+d_2)(\Gamma(2+\nu))^2 (\Phi_{\nu,d}(m))^2 (1-\eta)}{2\left[(1+2\gamma+2\beta)\frac{1}{\Gamma(3+\nu)}(\Gamma(2+\nu))^2 \Phi_{\nu,d}(m)\left[C_1^\alpha(x)\right]^2 - (1+\gamma)^2 m^\nu C_2^\alpha(x)\right] m^\nu (dm)^2}.$$

Then, in view of (8), we conclude that

$$\left|a_3 - \eta a_2^2\right| \le \begin{cases} \frac{\Gamma(3+\nu)\Phi_{\nu,d}(m)\left|C_1^\alpha(x)\right|}{2(1+2\gamma+2\beta)m^\nu (dm)^2}, & 0 \le |h(\eta)| \le \frac{\Gamma(3+\nu)\Phi_{\nu,d}(m)}{2(1+2\gamma+2\beta)m^\nu (dm)^2}, \\ 2\left|C_1^\alpha(x)\right| |h(\eta)|, & |h(\eta)| \ge \frac{\Gamma(3+\nu)\Phi_{\nu,d}(m)}{2(1+2\gamma+2\beta)m^\nu (dm)^2}. \end{cases}$$

Which completes the proof of Theorem 2. □

3. Corollaries and Consequences

Corresponding essentially to Examples 1–3, Theorems 1 and 2 yield the following corollaries.

Corollary 1. *Let $f \in \Sigma$ given by (2) belong to the class $\mathfrak{G}_\Sigma^\alpha(x,\gamma)$. Then,*

$$|a_2| \le \frac{2|\alpha| x \sqrt{2|\alpha| x}\,\Gamma(2+\nu)\Phi_{\nu,d}(m)}{\sqrt{\left|\left[2x^2 \Psi_{\nu,d}(m,\alpha,\gamma) + \alpha(1+\gamma)^2 m^\nu\right] m^\nu (dm)^2\right|}},$$

$$|a_3| \le \frac{4\alpha^2 x^2 (\Gamma(2+\nu)\Phi_{\nu,d}(m))^2}{m^{2\nu}(dm)^2 (1+\gamma)^2} + \frac{2|\alpha| x \Gamma(3+\nu)\Phi_{\nu,d}(m)}{(1+2\gamma)m^\nu (dm)^2},$$

and

$$\left|a_3 - \eta a_2^2\right| \le$$

$$\begin{cases} \frac{|\alpha|x\Gamma(3+\nu)\Phi_{\nu,d}(m)}{(1+2\gamma)m^{\nu}(cm)^2}, & |\eta-1| \leq \tau \\ \frac{8\alpha^2x^3(\Gamma(2+\nu))^2\left(\Phi_{\nu,d}(m)\right)^2(1-\eta)}{\left[4\alpha x^2(1+2\gamma)\frac{1}{\Gamma(3+\nu)}(\Gamma(2+\nu))^2\Phi_{\nu,d}(m)-(1+\gamma)^2m^{\nu}(2(1+\alpha)x^2-1)\right]m^{\nu}(dm)^2}, & |\eta-1| \geq \tau, \end{cases}$$

where

$$\tau = \left|1 - \frac{\Gamma(3+\nu)(1+\gamma)^2m^{\nu}(2(1+\alpha)-1)}{4(1+2\gamma)\alpha x^2(\Gamma(2+\nu))^2\Phi_{\nu,d}(m)}\right|$$

and

$$\Psi_{\nu,d}(m,\alpha,\gamma) = \frac{2(1+2\gamma)}{\Gamma(3+\nu)}(\Gamma(2+\nu))^2\Phi_{\nu,d}(m)\alpha^2 - (1+\gamma)^2m^{\nu}\alpha(1+\alpha).$$

Corollary 2. *Let $f \in \Sigma$ given by (2) belong to the class $\mathfrak{G}^{\alpha}_{\Sigma}(x)$. Then,*

$$|a_2| \leq \frac{|\alpha|x\sqrt{2|\alpha|x}\Gamma(2+\nu)\Phi_{\nu,d}(m)}{\sqrt{\left|\left[\left[\frac{6}{\Gamma(3+\nu)}(\Gamma(2+\nu))^2\Phi_{\nu,d}(m)\alpha^2 - 4m^{\nu}\alpha(1+\alpha)\right]x^2 + 2\alpha m^{\nu}\right]m^{\nu}(dm)^2\right|}},$$

$$|a_3| \leq \frac{\alpha^2x^2(\Gamma(2+\nu)\Phi_{\nu,d}(m))^2}{m^{2\nu}(dm)^2} + \frac{2|\alpha|x\Gamma(3+\nu)\Phi_{\nu,d}(m)}{3m^{\nu}(dm)^2},$$

and

$$\left|a_3 - \eta a_2^2\right| \leq$$
$$\begin{cases} \frac{|\alpha|x\Gamma(3+\nu)\Phi_{\nu,d}(m)}{3m^{\nu}(cm)^2}, & |\eta-1| \leq \left|1 - \frac{\Gamma(3+\nu)m^{\nu}(2(1+\alpha)-1)}{3\alpha x^2(\Gamma(2+\nu))^2\Phi_{\nu,d}(m)}\right| \\ \frac{2\alpha^2x^3(\Gamma(2+\nu))^2\left(\Phi_{\nu,d}(m)\right)^2(1-\eta)}{\left[3\alpha x^2\frac{1}{\Gamma(3+\nu)}(\Gamma(2+\nu))^2\Phi_{\nu,d}(m)-m^{\nu}(2(1+\alpha)x^2-1)\right]m^{\nu}(dm)^2}, & |\eta-1| \geq \left|1 - \frac{\Gamma(3+\nu)m^{\nu}(2(1+\alpha)-1)}{3\alpha x^2(\Gamma(2+\nu))^2\Phi_{\nu,d}(m)}\right|. \end{cases}$$

Corollary 3. *Let $f \in \Sigma$ given by (2) belong to the class $\widetilde{\mathfrak{G}}^{\alpha}_{\Sigma}(x,\gamma)$. Then,*

$$|a_2| \leq \frac{2|\alpha|x\sqrt{2|\alpha|x}\Gamma(2+\nu)\Phi_{\nu,d}(m)}{\sqrt{\left|\left[2x^2\Psi_{\nu,d}(m,\alpha,\gamma,1/2) + \alpha(1+\gamma)^2m^{\nu}\right]m^{\nu}(dm)^2\right|}},$$

and

$$|a_3| \leq \frac{4\alpha^2x^2(\Gamma(2+\nu)\Phi_{\nu,d}(m))^2}{m^{2\nu}(dm)^2(1+\gamma)^2} + \frac{|\alpha|x\Gamma(3+\nu)\Phi_{\nu,d}(m)}{(1+\gamma)m^{\nu}(dm)^2},$$

where

$$\Psi_{\nu,d}(m,\alpha,\gamma,1/2) = \frac{4(1+\gamma)}{\Gamma(3+\nu)}(\Gamma(2+\nu))^2\Phi_{\nu,d}(m)\alpha^2 - (1+\gamma)^2m^{\nu}\alpha(1+\alpha).$$

Corollary 4. *Let $f \in \Sigma$ given by (2) belong to the class $\widetilde{\mathfrak{G}}^{\alpha}_{\Sigma}(x,\gamma)$. Then,*

$$\left|a_3 - \eta a_2^2\right| \leq$$
$$\begin{cases} \frac{|\alpha|x\Gamma(3+\nu)\Phi_{\nu,d}(m)}{2(1+\gamma)m^{\nu}(dm)^2}, & |\eta-1| \leq \delta \\ \frac{8\alpha^2x^3(\Gamma(2+\nu))^2\left(\Phi_{\nu,d}(m)\right)^2(1-\eta)}{\left[8\alpha x^2(1+\gamma)\frac{1}{\Gamma(3+\nu)}(\Gamma(2+\nu))^2\Phi_{\nu,d}(m)-(1+\gamma)^2m^{\nu}(2(1+\alpha)x^2-1)\right]m^{\nu}(dm)^2}, & |\eta-1| \geq \delta, \end{cases}$$

where

$$\delta = \left|1 - \frac{\Gamma(3+\nu)(1+\gamma)^2 m^\nu (2(1+\alpha)-1)}{8(1+\gamma)\alpha x^2(\Gamma(2+\nu))^2 \Phi_{\nu,d}(m)}\right|.$$

Remark 1. *The results presented in this paper would lead to various other new results for the classes* $\mathfrak{G}^{1}_{\Sigma}(x,\gamma,\beta)$ *for Chebyshev Polynomials and* $\mathfrak{G}^{\frac{1}{2}}_{\Sigma}(x,\gamma,\beta)$ *for Legendre Polynomials.*

4. Conclusions

In our present investigation, we have introduced a new class $\mathfrak{G}^{\alpha}_{\Sigma}(x,\gamma,\beta)$ of normalized analytic and bi-univalent functions associated with the Miller–Ross-type Poisson distribution series. For functions belonging to this class, we have derived the estimates of the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ and the Fekete–Szegö functional problems. Furthermore, the results for the subclasses $\mathfrak{G}^{\alpha}_{\Sigma}(x,\gamma)$, $\mathfrak{G}^{\alpha}_{\Sigma}(x)$ and $\tilde{\mathfrak{G}}^{\alpha}_{\Sigma}(x,\gamma)$, which are defined in Examples 1–3, respectively, are associated with the Miller–Ross-type Poisson distribution series.

Author Contributions: Conceptualization, A.A., B.A.F. and T.M.S.; Data curation, A.A., B.A.F. and T.M.S.; Formal analysis, A.A., B.A.F. and T.M.S.; Funding acquisition, A.A. and T.M.S.; Investigation, A.A., B.A.F. and T.M.S.; Methodology, A.A., B.A.F. and T.M.S.; Resources, T.M.S.; Software, B.A.F. and T.M.S.; Writing—original draft, A.A., B.A.F. and T.M.S.; Writing—review & editing, A.A., B.A.F. and T.M.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Deanship of Scientific Research at Umm Al-Qura University, grant number [22UQU4350561DSR02], and the APC was funded by the Deanship of Scientific Research at Umm Al-Qura University.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data were used to support this study.

Acknowledgments: The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant Code: (22UQU4350561DSR02).

Conflicts of Interest: The authors declare no conflict of interest.

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