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Exponential Convergence to Equilibrium for Solutions of the Homogeneous Boltzmann Equation for Maxwellian Molecules

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Abstract: This paper is concerned with the spatially homogeneous Boltzmann equation, with the assumption of Maxwellian interaction. We consider initial data that belong to a small neighborhood of the equilibrium, which is a Maxwellian distribution. We prove that the solution remains in another small neighborhood with the same center and converges to this equilibrium exponentially fast, with an explicit quantification.

Keywords: Boltzmann equation; linearized Boltzmann collision operator; Maxwellian molecules; Maxwellian density function; neighborhood of equilibrium; spatially homogeneous models

MSC: 76P05; 82B40; 82C40; 82D05

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1. Introduction

This paper deals with large-time behavior of solutions to the *spatially homogeneous Boltzmann equation for Maxwellian molecules*. Besides Section 1.2 below, see [1–3] for an exhaustive, detailed treatment of the Boltzmann equation for diluted gases, whose solution $f(\cdot, t)$ provides the expected number of particles that lie in a specific subset of the phase space at each time $t \geq 0$. In this context, relaxation to equilibrium of solutions to Boltzmann-like equations is at the core of the kinetic theory of gases, since the work of Boltzmann himself. This kind of research is greatly enhanced by knowledge of the rate of approach to the limiting distribution, and even more so by a precise bound on the error in approximating the equilibrium for any fixed instant t. A solid physical motivation for this research rests on the fact that the validity of the Boltzmann equation breaks for very large times, and it is therefore crucial to obtain quantitative information on the time scale of the convergence to equilibrium, in order to show that this time scale is much smaller than that of the validity of the model. Ultimately, such information provides an analytical basis with the second principle of thermodynamics for a statistical physics model of a gas out of equilibrium. Furthermore, to add other physical motivations, we recall that any quantitative estimate of the speed of approach to the Maxwellian equilibrium gives information about the rate of decrease of Boltzmann's H-functional, defined as $H(t) := \int_{\mathbb{R}^3} f(\mathbf{v}, t) \log(f(\mathbf{v}, t)) \mathrm{d}\mathbf{v}$. Now, a classical statement by Lorentz [4,5] shows that, for a gas not far from equilibrium, the variation $\mathrm{d}H$ of the H-functional is proportional to the change in entropy $\mathrm{d}S$. Therefore, explicit estimates in other physical quantities of interest can be achieved by exploiting well-known results for the change of entropy.

Another strong motivation that makes the spatially homogeneous Boltzmann equation for Maxwellian molecules an appealing mathematical model, still studied in current times, rests on its mathematical tractability. In fact, it is also used as a toy model to understand some mathematical properties of more complex equations (such as the inhomogeneous Boltzmann equation), for which the same properties prove to be unsolved questions. See, e.g., [6–10].

Coming back to the contents of the present paper, it aims to provide a short, self-contained proof that initial data belonging to a suitable neighborhood of the Maxwellian

equilibrium produce solutions that remain in a slightly larger neighborhood, and converge exponentially quickly to that very same equilibrium with an explicit rate. Of course, due to the importance of the subject, this question has been tackled for a long time in many works, so the conclusions stated in this paper are not completely novel. However, as one can infer by reading the most significant papers on this subject (see, e.g., [11–14] for the specific situation of the Maxwellian molecules), the mathematical complexity of any proof hitherto conceived is so overwhelming as to overshadow the actual physical contents of the relaxation phenomenon. An exception may be represented by the noteworthy paper [15], which reconsiders the original Boltzmann's physical argument based on the celebrated *H-theorem* through a strengthened version of it, known as *Cercignani's conjecture* . However, this work also contains many highly technical parts that require some effort for the reader to understand. Therefore, it would be desirable to have an easy, straightforward proof of the above-mentioned convergence result, involving minimal mathematical tools and being comprehensible without too much effort. This represents the primarily aim of this work.

Historically, the main lines that were conceived to quantify the convergence to equilibrium for Boltzmann-equation solutions were based on: (A) linearization of the original non-linear equation, (see [16,17]); (B) quantitative strengthening of Boltzmann's H-theorem (see [15,18]); (C) Fourier transform estimates (see [11,12]); and (D) probabilistic techniques related to the central limit theorem (see [19–22]). Apropos of the first line, it shares the main advantage of explicitly highlighting the so-called linearized collision operator. See Formula (7) below. This operator possesses a discrete spectrum with non-positive eigenvalues, the least negative of which—denoted by Λ_b throughout the paper—represents the optimal rate of exponential convergence to equilibrium. This optimality issue, conjectured long ago in [17,22], was definitively proven some years ago in [13,14,23,24] for a broad class of initial data by exploiting probabilistic techniques as in line (D). See also [25] for a more probabilistic view. In any case, the main drawback of strategy (A) rests on the fact that the linearized equation can describe the behavior of those solutions that start and remain forever in a suitable neighborhood of the Maxwellian equilibrium. This restriction once again enhances the results obtained in [13,14,23,24], which are valid for initial data that can be very far from the equilibrium.

As announced above, this paper will consider only initial data that are close to the Maxwellian equilibrium, in order to avoid mathematical complications. By re-adapting and improving previous arguments contained in [16,17,26], our main results—stated below as Theorems 1 and 2—will state the above-mentioned exponential convergence to equilibrium with respect to the *total variation distance* (i.e., the distance induced by the L^1 norm), at the rate $\Lambda_b/2$. Although not coinciding with the sharpest rate, this rate is of the same scale as the optimal one, and fulfills the physical demand of providing a time scale for the convergence to equilibrium that is much smaller than the one of the validity of the model. Similar arguments can be also found in [27], which, although providing exponential convergence at optimal rates, does not cover the case of the Maxwellian molecules.

1.1. List of Symbols

To better understand the following treatment, we provide here a list of the main symbols adopted to describe both the Boltzmann equation and the related analysis.

$f(\cdot, t)$ = probability density function of the velocity-variable, at time t;
Q_b = collision operator, mapping any two densities into a real number;
$\mathbf{v}, \mathbf{w}$ = pre-collisional velocities;
$\mathbf{v}_*, \mathbf{w}_*$ = post-collisional velocities;
$\boldsymbol{\omega}$ = collision (solid) angle;
b = Maxwellian collision kernel, a function from $(-1, 1)$ to $\mathbb{R}$;
u_{S^2} = uniform probability measure on S^2;
$M_{\mathbf{v}_0,\sigma}$ = Maxwellian distribution with parameters $\mathbf{v}_0 \in \mathbb{R}^3$ and $\sigma > 0$;

$$
\begin{aligned}
M &= M_{\mathbf{0},1};\\
L_b &= \text{linearized collision operator};\\
\Lambda_b &= \text{least negative eigenvalue of } L_b;\\
\mathcal{H} &= \mathrm{L}^2(\mathbb{R}^3, M(\mathbf{x})\mathrm{d}\mathbf{x});\\
(\cdot,\cdot)_* &= \text{scalar product on } \mathcal{H};\\
||\cdot||_* &= \text{norm on } \mathcal{H}, \text{induced by } (\cdot,\cdot)_*;\\
\mathcal{N}_\delta &= \text{closed ball in } \mathcal{H} \text{ with radius } \delta > 0, \text{centered at the origin of } \mathcal{H}.
\end{aligned}
$$

1.2. The Equation and Its Linearization

The equation under study is the homogeneous Boltzmann equation for Maxwellian molecules, which describes a spatially homogeneous dilute gas composed of a very large number of like particles. See [1–3] for an exhaustive, detailed treatment of the Boltzmann model. The locution "Maxwellian molecules" means that each collision is influenced by a repulsive force proportional to r^{-5} (r standing for the distance between two colliding particles), since this peculiar situation was firstly studied by Maxwell himself. See [28]. In case of the absence of external forces, the equation reads

$$\frac{\partial}{\partial t} f(\mathbf{v}, t) = Q_b[f(\cdot, t), f(\cdot, t)](\mathbf{v}) \tag{1}$$

with $(\mathbf{v}, t)$ in $\mathbb{R}^3 \times (0, +\infty)$. A solution of (1), $f(\cdot, t)$, is required to be a probability density function (pdf) in the first variable at each instant t, the physical meaning being as follows:

$$\int_A f(\mathbf{v}, t)\mathrm{d}\mathbf{v} = \frac{\text{number of particles with velocity in } A \text{ at time } t}{\text{total number of particles}} \qquad (A \in \mathscr{B}(\mathbb{R}^3)) \,.$$

The *collision operator* Q_b is defined for every pair (φ, ψ) of real-valued functions in $\mathrm{L}^1(\mathbb{R}^3)$ through the relation

$$Q_b[\varphi, \psi](\mathbf{v}) := \int_{\mathbb{R}^3} \int_{S^2} [\varphi(\mathbf{v}_*)\psi(\mathbf{w}_*) \; - \; \varphi(\mathbf{v})\psi(\mathbf{w})] b\left(\frac{\mathbf{w} - \mathbf{v}}{|\mathbf{w} - \mathbf{v}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{w} \tag{2}$$

where u_{S^2} stands for the uniform probability measure on the unit sphere S^2, embedded in $\mathbb{R}^3$. Moreover, the *post-collisional velocities* $\mathbf{v}_*$ and $\mathbf{w}_*$ must obey the conservation of momentum and kinetic energy, that is

$$\mathbf{v} + \mathbf{w} = \; \mathbf{v}_* + \mathbf{w}_* \qquad \text{and} \qquad |\mathbf{v}|^2 + |\mathbf{w}|^2 = \; |\mathbf{v}_*|^2 + |\mathbf{w}_*|^2$$

and, consequently, can be parametrized by unit vectors $\boldsymbol{\omega}$ in S^2 according to

$$\begin{array}{rcl} \mathbf{v}_* & = \mathbf{v} & + \; [(\mathbf{w} - \mathbf{v}) \cdot \boldsymbol{\omega}] \, \boldsymbol{\omega} \\ \mathbf{w}_* & = \mathbf{w} & - \; [(\mathbf{w} - \mathbf{v}) \cdot \boldsymbol{\omega}] \, \boldsymbol{\omega} \end{array} \tag{3}$$

where $\cdot$ denotes the standard scalar product. The positive function b, called the *angular collision kernel*, is defined on $(-1, 1)$ and carries the information about any single collision at a microscopic level. We require this function to satisfy the symmetry conditions

$$b(x) = b(\sqrt{1 - x^2}) \frac{|x|}{\sqrt{1 - x^2}} = b(-x) \tag{4}$$

for any x in $(-1, 1)$, and the so-called *Grad angular cutoff*, here written as

$$\int_0^1 b(x)\mathrm{d}x = 1. \tag{5}$$

See Section 3 in Chapter 2A of [3] for further information about collision kernels.

Existence and uniqueness for solutions of (1) are well-understood questions, at least when (4) and (5) hold. More precisely, in [29], it is proven that, given a pdf f_0 as initial datum, the resulting Cauchy problem admits a unique solution $f(\cdot, t)$. More general results on existence and uniqueness are contained in [13]. However, stability properties of the solutions for large times are less trivial and less understood by far.

Another question of some relevance for its mathematical and physical implications is that there exist non-trivial stationary solutions of (1) that can be seen as possible equilibrium distributions. Within the class of all pdfs on $\mathbb{R}^3$, these stationary solutions are exactly the *Maxwellian pdfs*, given by

$$M_{\mathbf{v}_0,\sigma}(\mathbf{v}) := \left(\frac{1}{2\pi\sigma^2}\right)^{3/2} \exp\left\{-\frac{1}{2\sigma^2}|\mathbf{v}-\mathbf{v}_0|^2\right\}$$

where $(\mathbf{v}_0, \sigma)$ varies in $\mathbb{R}^3 \times (0, +\infty)$. From a physical point of view, $\mathbf{v}_0$ stands for the mean velocity, while σ^2 represents, up to a physical constant, the thermodynamic temperature. Other relevant properties are collected in Chapter VIII of [2].

Now, we provide a self-contained treatment of the linearization procedure, a subject which is still scattered—sometimes with discordant notation—in different sources. See [16,30,31]. Without loss of generality, we assume that the initial datum f_0 satisfies

$$\int_{\mathbb{R}^3} \mathbf{v} f_0(\mathbf{v}) \mathrm{d}\mathbf{v} = \mathbf{0} \quad \text{and} \quad \int_{\mathbb{R}^3} |\mathbf{v}|^2 f_0(\mathbf{v}) \mathrm{d}\mathbf{v} = 3\,. \tag{6}$$

Then, from the conservation of momentum and kinetic energy, we have

$$\begin{aligned} \int_{\mathbb{R}^3} \mathbf{v} f(\mathbf{v}, t) \mathrm{d}\mathbf{v} &= \int_{\mathbb{R}^3} \mathbf{v} f_0(\mathbf{v}) \mathrm{d}\mathbf{v} \\ \int_{\mathbb{R}^3} |\mathbf{v}|^2 f(\mathbf{v}, t) \mathrm{d}\mathbf{v} &= \int_{\mathbb{R}^3} |\mathbf{v}|^2 f_0(\mathbf{v}) \mathrm{d}\mathbf{v} \end{aligned}$$

for every t in $[0, +\infty)$. Moreover, under (6), the above conservations are preserved in the limit, and the relative Maxwellian equilibrium turns out to be $M_{\mathbf{0},1}$, which will be simply indicated as M.

A central role will be played throughout this work by the so-called *linearized collision operator* L_b, defined by

$$\begin{aligned} L_b[h](\mathbf{v}) &:= \int_{\mathbb{R}^3} \int_{S^2} M(\mathbf{w}) \left[h(\mathbf{v}_*) + h(\mathbf{w}_*) - h(\mathbf{v}) - h(\mathbf{w})\right] \times \\ &\times b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|} \cdot \boldsymbol{\omega}\right) u_{S^2}(\mathrm{d}\boldsymbol{\omega}) \mathrm{d}\mathbf{w}\,. \end{aligned} \tag{7}$$

The introduction of L_b can be justified as follows. By the substitution

$$f(\mathbf{v}, t) = M(\mathbf{v})(1 + h(\mathbf{v}, t))\,, \tag{8}$$

Equation (1) changes into a new equation for h, which reads

$$\frac{\partial}{\partial t} h(\mathbf{v}, t) = L_b[h(\cdot, t)](\mathbf{v}) + R_b[h(\cdot, t), h(\cdot, t)](\mathbf{v}) \tag{9}$$

where R_b is defined by

$$R_b[\varphi, \psi](\mathbf{v}) := \frac{1}{M(\mathbf{v})} Q_b[M\varphi, M\psi](\mathbf{v})\,. \tag{10}$$

At this stage, the function h can be thought of as a sort of "remainder", which becomes smaller and smaller when t increases. Therefore, if the contribution of the quadratic operator R_b in (9) becomes negligible with respect to that given by L_b, then the spectral properties

of L_b could provide quantitative information about the rapidity of convergence of h to the null function. This insight, which may lead to the desired conclusion of quantifying the convergence to equilibrium, actually sums up the very content of this paper and will be formalized in a rigorous way in its sequel. Moreover, it is crucial to point out that this strategy works only under some restrictions on f_0, to be specified as well. As to the mentioned spectral analysis of L_b, it can be a very difficult task if based on the natural domain of L_b, namely the space of functions $h : \mathbb{R}^3 \to \mathbb{R}$, which can be written as

$$\frac{f(\cdot)}{M(\cdot)} - 1 \,,$$

when f is any pdf on $\mathbb{R}^3$. A remarkable idea in [17] consists in the introduction of the Hilbert space $\mathcal{H} := \mathrm{L}^2(\mathbb{R}^3, M(\mathbf{x})\mathrm{d}\mathbf{x})$ as a new domain for L_b, a device to makes computations feasible, since there is a Fourier basis for $\mathcal{H}$ that diagonalizes L_b. To complete the necessary notation, introduce

$$(\varphi, \psi)_* := \int_{\mathbb{R}^3} \varphi(\mathbf{x})\psi(\mathbf{x})M(\mathbf{x})\mathrm{d}\mathbf{x}$$

and

$$||\,\varphi\,||_* := \sqrt{(\varphi, \varphi)_*}$$

to denote the scalar product and the norm of $\mathcal{H}$, respectively, and

$$\mathcal{N}_\delta := \{h \in \mathcal{H} \mid \; ||\,h\,||_* \leq \delta\}$$

to indicate the ball of radius δ centered at the origin. The kernel of L_b coincides with the five-dimensional linear subspace $\mathrm{span}\{1, v_1, v_2, v_3, |\mathbf{v}|^2\}$ generated by the *collisional invariants*. The orthogonal complement, in $\mathcal{H}$, of the kernel of L_b will be indicated by $\mathcal{H}_0$.

Since

$$\begin{aligned} \int_{\mathbb{R}^3} \mathbf{v} f(\mathbf{v}, t)\mathrm{d}\mathbf{v} &= \int_{\mathbb{R}^3} \mathbf{v} M(\mathbf{v})\mathrm{d}\mathbf{v} \\ \int_{\mathbb{R}^3} |\mathbf{v}|^2 f(\mathbf{v}, t)\mathrm{d}\mathbf{v} &= \int_{\mathbb{R}^3} |\mathbf{v}|^2 M(\mathbf{v})\mathrm{d}\mathbf{v} \end{aligned}$$

for every t in $[0, +\infty)$, it follows that if $h(\cdot, t)$ belongs to $\mathcal{H}$, then it is in the subspace $\mathcal{H}_0$ for all t too. On the new domain, the linear operator L_b is self-adjoint and negative with a discrete set of eigenvalues, the least negative of which, denoted by Λ_b, represents the *spectral gap*. A precise analysis is contained in [31], where it is also shown that

$$\Lambda_b = -2 \int_0^1 x^2(1 - x^2)b(x)\mathrm{d}x$$

and, for every φ in $\mathcal{H}_0$,

$$\big(L_b[\varphi], \varphi\big)_* \leq \Lambda_b \; ||\,\varphi\,||_*^2 \;. \tag{11}$$

This spectral gap has been considered as a reference value for the rate of exponential convergence of $f(\cdot, t)$ to M in the original equation, but this claim long held out as an unproved conjecture.

2. Main Results

The first result deals with existence and uniqueness of solutions to Equation (1), given a pdf f_0 on $\mathbb{R}^3$ as an initial datum. The novel contribution consists in proving that, if f_0 belongs to a suitable $\mathcal{H}$-neighborhood of the Maxwellian equilibrium, then the solution $f(\cdot, t)$ remains in a slightly larger $\mathcal{H}$-neighborhood of the same equilibrium at all times $t > 0$.

Theorem 1. *Let (4) and (5) be in force and let* $\delta := |\Lambda_b|/16$. *Given any pdf* f_0 *satisfying (6), if*

$$\frac{f_0(\cdot) - M(\cdot)}{M(\cdot)} \in \mathcal{N}_\delta \tag{12}$$

is valid, then there exists a unique solution $f(\cdot, t)$ *to Equation (1) such that*

$$\frac{f(\cdot, t) - M(\cdot)}{M(\cdot)} \in \mathcal{N}_{2\delta} \tag{13}$$

holds true at all times $t > 0$.

Now, the most important problem connected with the long-time behavior of the solutions of (1) is the quantification of the rate of convergence to equilibrium. As said in the introduction, the first technique introduced to pursue this goal was based on a linearization of the non-linear equation (1) and on the spectral analysis of the resulting linearized collision operator. The main difficulty of this strategy consists in the fact that the spectral properties of L_b, viewed as an operator on $\mathcal{H}$, are not directly connected with the properties of the solution of the non-linear Equation (1), where it would be more natural to consider the L^1 distance. This metric mismatch is now fixed by the above Theorem 1, where it is actually shown that the solution $f(\cdot, t)$ never abandons the larger $\mathcal{H}$-neighborhood of the Maxwellian equilibrium. This technical achievement drastically simplifies the ensuing mathematical arguments (see the next section) and leads to the following statement which, although not new compared to the main results in [13,14,23,24], provides, within a very tight formulation, an exponential rate of convergence to equilibrium of the same time-scale as the optimal one.

Theorem 2. *Let the assumptions of Theorem 1 be in force. Then, under (12),*

$$||f(\cdot, t) - M(\cdot)||_1 := \int_{\mathbb{R}^3} |f(\mathbf{v}, t) - M(\mathbf{v})| \mathrm{d}\mathbf{v} \;\leq\; C_* e^{\frac{1}{2}\Lambda_b t} \tag{14}$$

is valid for all $t > 0$ *with*

$$C_* := \left(\frac{1}{||h(\cdot, 0)||_*} + \frac{2}{\Lambda_b} \right)^{-2}.$$

Finally, with reference to the main motivations explained at the beginning, it is worth indicating how Theorem 2 is applied in the proof of Theorem 2.1 of [14]. The problem that one must tackle therein reduces to the case in which $f_0(\mathbf{v}) = \prod_{i=1}^3 g_{\sigma_i}(v_i)$, where $g_\sigma(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2\sigma^2} x^2\}$ and $\sum_{i=1}^3 \sigma_i^2 = 3$. A solution can be condensed as follows.

Proposition 1. *Assume that* $f_0(\mathbf{v}) = \prod_{i=1}^3 g_{\sigma_i}(v_i)$, *with* $\sum_{i=1}^3 \sigma_i^2 = 3$. *Then, Condition (12) is fulfilled whenever*

$$\sigma_i^2 \in \left[1 - \frac{\sqrt{42 + \delta^2}}{21 + \delta^2}\delta, 1 + \frac{\sqrt{42 + \delta^2}}{21 + \delta^2}\delta \right]$$

holds for $i = 1, 2, 3$, *with* $\delta := |\Lambda_b|/16$.

3. Proofs

This section is split into three subsections. In the first one, we prove Theorem 1. The second deals with the proof of Theorem 2. Finally, we prove Proposition 1 in the last subsection.

3.1. Proof of Theorem 1

Here, the validity of (13) is derived the study of Equation (9). Existence and uniqueness are tackled according to an approach rather different from the classical one presented

in [29], which requires new proofs. Following [26], after fixing the initial datum h_0 in $\mathcal{H}_0$, the solution of the Cauchy problem, resulting from (9) and this initial condition, is meant as an element of $\mathrm{C}([0,\infty);\mathcal{H}_0)\cap\mathrm{C}^1([0,\infty);\mathcal{H}_0)$.

To start, let $\mathscr{T}^t$ denote the semigroup of linear operators on $\mathcal{H}_0$ sending an element g onto the solution $\mathscr{T}^t[g]$ of the evolution equation

$$\frac{\partial}{\partial t}h(\mathbf{v},t)=L_b[h(\cdot,t)](\mathbf{v})\,.$$

It is well-known that $\mathscr{T}^t$ admits a characterization in the form of exponential semigroup $\exp\{tL_b\}$. The basic properties of L_b, collected, for example, in [31], guarantee that $\mathscr{T}^t[g]$ is actually an element of $\mathcal{H}_0$ whenever g is in the same space. Following general references on abstract differential equations such as [32,33], one finds that the solution of (9) admits the representation

$$h(\mathbf{v},t)=\mathscr{T}^t[h_0](\mathbf{v})+\int_0^t\mathscr{T}^{t-s}[R_b[h(\cdot,s),h(\cdot,s)]](\mathbf{v})\mathrm{d}s \tag{15}$$

which lends itself to be interpreted as a fixed-point problem.

Now, a first preliminary fact, which follows from (11), is that

$$||\,\mathscr{T}^t[g]\,||_*\leq e^{\Lambda_b t}\,||\,g\,||_* \tag{16}$$

for every g in $\mathcal{H}_0$ and all t in $[0,+\infty)$. Another preliminary fact is encompassed in the inequality

$$|(R_b[\varphi,\psi],\rho)_*|\;\leq\;2\,||\,\varphi\,||_*||\,\psi\,||_*||\,\rho\,||_* \tag{17}$$

which is valid for every φ,ψ, and ρ in $\mathcal{H}$. A direct consequence of (17) is

$$||\,R_b[\varphi,\psi]\,||_*\leq 2\,||\,\varphi\,||_*||\,\psi\,||_*\;. \tag{18}$$

To prove (17), it can be observed that the quantity

$$\begin{aligned}(R_b[\varphi,\psi],\rho)_*&=\int_{\mathbb{R}^3}\rho(\mathbf{v})Q_b[M\varphi,M\psi](\mathbf{v})\mathrm{d}\mathbf{v}\\
&=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}\rho(\mathbf{v})\varphi(\mathbf{v}_*)\psi(\mathbf{w}_*)M(\mathbf{v}_*)M(\mathbf{w}_*)b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right)u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}\\
&-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}\rho(\mathbf{v})\varphi(\mathbf{v})\psi(\mathbf{w})M(\mathbf{v}_*)M(\mathbf{w}_*)b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right)u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}\end{aligned} \tag{19}$$

is decomposed as a difference of two terms. The former, which reads

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}\left[\rho(\mathbf{v})M^{1/2}(\mathbf{v}_*)M^{1/2}(\mathbf{w}_*)b^{1/2}\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right)\right]\times\\
\times&\left[\varphi(\mathbf{v}_*)\psi(\mathbf{w}_*)M^{1/2}(\mathbf{v}_*)M^{1/2}(\mathbf{w}_*)b^{1/2}\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right)\right]u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}\,,\end{aligned}$$

can be bounded from above, by means of the Cauchy–Schwarz inequality, by

$$\begin{aligned}&\left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}\rho^2(\mathbf{v})M(\mathbf{v}_*)M(\mathbf{w}_*)b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right)u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}\right]^{1/2}\times\\
\times&\left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}\varphi^2(\mathbf{v}_*)\psi^2(\mathbf{w}_*)M(\mathbf{v}_*)M(\mathbf{w}_*)b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right)u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}\right]^{1/2}.\end{aligned}$$

Since $M(\mathbf{v}_*)M(\mathbf{w}_*) = M(\mathbf{v})M(\mathbf{w})$ and $\int_{S^2} b(\mathbf{u}\cdot\boldsymbol{\omega})u_{S^2}(\mathrm{d}\boldsymbol{\omega}) = 1$ for every $\mathbf{u}$ in S^2, it follows that

$$\left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}\rho^2(\mathbf{v})M(\mathbf{v}_*)M(\mathbf{w}_*)b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right)u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}\right]^{1/2} = ||\,\rho\,||_* \ .$$

Moreover, since $(\mathbf{v},\mathbf{w}) \mapsto (\mathbf{v}_*,\mathbf{w}_*)$ is a linear isometry of $\mathbb{R}^6$ for every $\boldsymbol{\omega}$ in S^2 and

$$b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right) = b\left(\frac{\mathbf{w}_*-\mathbf{v}_*}{|\mathbf{w}_*-\mathbf{v}_*|}\cdot\boldsymbol{\omega}\right),$$

the change-of-variable theorem yields

$$\begin{aligned}&\left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}\varphi^2(\mathbf{v}_*)\psi^2(\mathbf{w}_*)M(\mathbf{v}_*)M(\mathbf{w}_*)b\left(\frac{\mathbf{w}-\mathbf{v}}{|\mathbf{w}-\mathbf{v}|}\cdot\boldsymbol{\omega}\right)u_{S^2}(\mathrm{d}\boldsymbol{\omega})\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}\right]^{1/2}\\ =\ &||\,\varphi\,||_*||\,\psi\,||_*\,,\end{aligned}$$

which is the desired bound for the former term under discussion. Then, since

$$\int_{S^2} b(\mathbf{u}\cdot\boldsymbol{\omega})u_{S^2}(\mathrm{d}\boldsymbol{\omega}) = 1$$

for every $\mathbf{u}$ in S^2, the latter term in (19) is equal to

$$\int_{\mathbb{R}^3} M(\mathbf{v})\varphi(\mathbf{v})\rho(\mathbf{v})\mathrm{d}\mathbf{v}\cdot\int_{\mathbb{R}^3} M(\mathbf{v})\psi(\mathbf{v})\mathrm{d}\mathbf{v} = (\varphi,\rho)_*\cdot(\psi,1)_*$$

and the Cauchy–Schwarz inequality gives

$$|(\varphi,\rho)_*\cdot(\psi,1)_*| \le ||\,\varphi\,||_*||\,\psi\,||_*||\,\rho\,||_* \ .$$

The proof of (17) follows from the combination of the upper bounds just obtained.

After these preliminaries, existence and uniqueness will be proved via a contraction mapping principle, as in [26].

The first step consists in the definition, for $T > 0$ of the Banach space $\mathbb{X} := \mathrm{C}([0,T];\mathcal{H}_0)$ endowed with the norm

$$|||\,x\,||| := \sup_{t\in[0,T]} ||\,x(t)\,||_* \ .$$

Then, the formula

$$Z_b[x] := \mathscr{T}^t[h_0] + \int_0^t \mathscr{T}^{t-s}[R_b[x(s),x(s)]]\mathrm{d}s$$

defines an operator on $\mathbb{X}$. Indeed, both h_0 and $x(s)$ belong to $\mathcal{H}_0$ and, consequently, $R_b[x(s),x(s)]$ and $\mathscr{T}^u[R_b[x(s),x(s)]]$ are again elements of $\mathcal{H}_0$, for every s,u in $[0,T]$. After setting

$$D := \{x\in\mathbb{X}\mid\ |||\,x\,||| \le |\Lambda_b|/8\}\,,$$

which is obviously a closed subset of $\mathbb{X}$, it can be proved that $Z_b(D)\subset D$ and that

$$|||\,Z_b[x]-Z_b[y]\,||| \le \frac{1}{2}\,|||\,x-y\,||| \tag{20}$$

for every x and y in D, provided that h_0 belongs to $\mathcal{N}_\delta$ with $\delta = |\Lambda_b|/16$.

The proof of the first claim is based on (16) and (18), which give

$$|||\,Z_b[x]\,||| \quad\le\quad \sup_{t\in[0,T]}\left[||\,h_0\,||_*\,e^{\Lambda_b t} + \int_0^t e^{\Lambda_b(t-s)}\,||\,R_b[x(s),x(s)]\,||_*\,\mathrm{d}s\right]$$

$$\leq \quad ||\, h_0 \,||_* \; + 2 \sup_{t\in[0,T]} \int_0^t e^{\Lambda_b (t-s)} \; ||\, x(s) \,||_*^2 \; \mathrm{d}s \; .$$

In view of the bound on h_0 and the fact that x is in D, the claim is proved by means of the inequality

$$|||\, Z_b[x] \,||| \leq \; \frac{1}{16}|\Lambda_b| \; + \; 2\frac{1}{64}|\Lambda_b|^2 \frac{1}{|\Lambda_b|} \; < \; \frac{1}{8}|\Lambda_b| \; .$$

To verify (20), the inequalities (16) and (18) can be used again to write

$$\begin{aligned} |||\, Z_b[x] - Z_b[y] \,||| \quad &\leq \quad \sup_{t\in[0,T]} \int_0^t e^{\Lambda_b (t-s)} \; ||\, R_b[x(s), x(s)] - R_b[y(s), y(s)] \,||_* \; \mathrm{d}s \\ &= \quad \sup_{t\in[0,T]} \int_0^t e^{\Lambda_b (t-s)} \; ||\, R_b[x(s) + y(s), x(s) - y(s)] \,||_* \; \mathrm{d}s \\ &\leq \quad \sup_{t\in[0,T]} \; 2 \int_0^t e^{\Lambda_b (t-s)} \; ||\, x(s) + y(s) \,||_* \cdot ||\, x(s) - y(s) \,||_* \; \mathrm{d}s \\ &\leq \quad 2\frac{1}{4}|\Lambda_b| \frac{1}{|\Lambda_b|} \; |||\, x - y \,||| \quad . \end{aligned}$$

Existence and uniqueness of the solution of (9) are now direct consequences of Theorem 1.1 in Chapter IV of [33], and this solution can be viewed as a map from $[0, T]$ into $\mathcal{H}_0$. Finally, since the above argument is independent of the choice of T, the solution can be extended to $[0, +\infty)$ and this proves the first part of Theorem 2. Indeed, the validity of (13) is nothing but the translation of the fact that the fixed-point problem represented by (15) admits a unique solution in D, rewritten, through Equation (8), in terms of $f(\cdot, t)$.

3.2. Proof of Theorem 2

This subsection contains the proof of (14). Starting from the identity (8), Jensen's inequality entails that

$$||\, f(\mathbf{v}, t) - M(\mathbf{v}) \,||_1^2 \leq \; \int_{\mathbb{R}^3} \frac{\big(f(\mathbf{v}, t) - M(\mathbf{v})\big)^2}{M(\mathbf{v})} \mathrm{d}\mathbf{v} = ||\, h(\cdot, t) \,||_*^2 =: \theta(t) \; . \tag{21}$$

Now, taking the scalar product $(\cdot, \cdot)_*$ of both members of (9) with the solution $h(\cdot, t)$ of the same equation yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \big(h(\mathbf{v}, t), h(\mathbf{v}, t)\big)_* = \big(L_b[h(\cdot, t)], h(\mathbf{v}, t)\big)_* + \big(R_b[h(\cdot, t), h(\cdot, t)], h(\mathbf{v}, t)\big)_* \; .$$

Since $h(\cdot, t)$ belongs to $\mathcal{H}_0$ for every $t \geq 0$, the utilization of (11) and (17) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) \leq \Lambda_b \theta(t) + 2[\theta(t)]^{3/2} \; .$$

After setting $\vartheta(t) := \theta(t) e^{-\Lambda_b t}$, the above inequality becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t) \leq 2[\vartheta(t)]^{3/2} e^{-\Lambda_b t} = 2[\vartheta(t)]^{3/2} e^{\frac{1}{2}\Lambda_b t} \; .$$

where

$$-2\left[\frac{1}{\sqrt{\vartheta(t)}} - \frac{1}{\sqrt{\vartheta(0)}}\right] = \int_0^t [\vartheta(\tau)]^{-3/2} \vartheta'(\tau) \mathrm{d}\tau \leq 2 \int_0^t e^{\frac{1}{2}\Lambda_b \tau} \mathrm{d}\tau \leq \left(\frac{4}{-\Lambda_b}\right)$$

and, after some elementary algebra, one obtains

$$\vartheta(t) \leq \left[\frac{1}{\sqrt{\vartheta(0)}} + \frac{2}{\Lambda_b}\right]^{-2} = \left[\frac{1}{\sqrt{\theta(0)}} + \frac{2}{\Lambda_b}\right]^{-2} = \left[\frac{1}{||h(\cdot,0)||_*} + \frac{2}{\Lambda_b}\right]^{-2} = C_* \,. \tag{22}$$

Notice that (13) guarantees that C_* is a well-defined, strictly positive real constant. The combination of (22) with the definition of ϑ gives $\theta(t) \leq C_* e^{\Lambda_b t}$, which entails the desired conclusion.

3.3. Proof of Proposition 1

By direct computation, we obtain

$$\left|\left|\frac{f_0(\cdot) - M(\cdot)}{M(\cdot)}\right|\right|_*^2 \leq 3\frac{1}{\sigma_2\sqrt{2-\sigma_2^2}}\frac{1}{\sigma_3\sqrt{2-\sigma_3^2}}\left[\frac{1}{\sigma_1\sqrt{2-\sigma_1^2}} - 1\right] + 3\frac{1}{\sigma_3\sqrt{2-\sigma_3^2}}\left[\frac{1}{\sigma_2\sqrt{2-\sigma_2^2}} - 1\right] + 3\left[\frac{1}{\sigma_3\sqrt{2-\sigma_3^2}} - 1\right].$$

Then, by elementary algebra, we conclude that Condition (12) follows whenever

$$\sigma_i^2 \in \left[1 - \frac{\sqrt{42+\delta^2}}{21+\delta^2}\delta, 1 + \frac{\sqrt{42+\delta^2}}{21+\delta^2}\delta\right]$$

holds for $i = 1, 2, 3$.

4. Conclusions

In this paper, we have studied the problem of convergence to equilibrium for solutions to the spatially homogeneous Boltzmann equation for Maxwellian molecules. We have reconsidered an original strategy to prove a quantitative form of such convergence based on the linearization of the original non-linear equation. By re-formulating and improving previous results, we have provided a new proof that stands out, among the others, for its simplicity and shortness. Our results are valid for both initial data and solutions that remain in a suitable neighborhood of the equilibrium. In the end, we have obtained exponential convergence to equilibrium with a rate which is comparable, at the level of physical scales, with the sharpest one.

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