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On the Property of Linear Autonomy for Symmetries of Fractional Differential Equations and Systems

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Abstract: The problem of finding Lie point symmetries for a certain class of multi-dimensional nonlinear partial fractional differential equations and their systems is studied. It is assumed that considered equations involve fractional derivatives with respect to only one independent variable, and each equation contains a single fractional derivative. The most significant examples of such equations are time-fractional models of processes with memory of power-law type. Two different types of fractional derivatives, namely Riemann–Liouville and Caputo, are used in this study. It is proved that any Lie point symmetry group admitted by equations or systems belonging to considered class consists of only linearly-autonomous point symmetries. Representations for the coordinates of corresponding infinitesimal group generators, as well as simplified determining equations are given in explicit form. The obtained results significantly facilitate finding Lie point symmetries for multi-dimensional time-fractional differential equations and their systems. Three physical examples illustrate this point.

Keywords: fractional differential equation; Lie point symmetry group; linearly autonomous symmetry

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1. Introduction

The classical Lie group analysis [1–4] is a powerful mathematical technique for finding exact solutions and studying symmetry properties of nonlinear ordinary and partial differential equations. Over the last four decades, several new directions in symmetry analysis have been introduced, such as Lie–Bäcklund symmetries [5] and higher-order symmetries [6], approximate symmetries [7], renormgroup symmetries [8], non-local symmetries [9], and some others [10]. Based on the invariance principle, various methods of modern group analysis have been proposed for investigating symmetry properties of integro-differential equations [11], difference equations [12], and equations with functional derivatives [13].

Recently, it is shown that fractional differential equations (FDEs) [14–18] can also be studied by using the theory of transformation groups. In [19], invariant solutions corresponding to the one-parameter Lie group of scaling transformations have been constructed for the fractional diffusion-wave equation. This is one of the first papers in this research area. The problem of group prolongation to fractional derivatives is firstly studied in [20], and the corresponding prolongation formula for infinitesimal group generator has been obtained there in an explicit form. In [21], the Lie point symmetry group classification problem for FDEs has been solved for the first time. A constructive algorithm for finding conservation laws for partial FDEs by using their point symmetries have been proposed in [22]. A systematic description of recent results in Lie group analysis of ordinary and partial FDEs can be found in [23–25]. In addition, some methods of modern group analysis have been extended to FDEs. So, first examples of nonlocal symmetries for FDEs have been constructed in [26]. A theoretical framework of potential symmetries for the time-fractional

PDEs has been established in [27]. An approach to finding approximate symmetries for FDEs in which the order of fractional differentiation has a small deviation from the nearest integer has been proposed in [28,29], and an algorithm for constructing approximate conservation laws for such equations has been presented in [30]. Moreover, in [31] it is shown that higher-order symmetries and corresponding recursion operators can be calculated for linear partial FDEs.

At present, numerous symmetries, invariant solutions, and conservation laws have been obtained for wide classes of FDEs describing various anomalous processes and phenomena (see, e.g., [32–41] and references therein). Nevertheless, finding symmetries of FDEs is a more complex problem than that for integer-order differential equations. Therefore, nowadays, symmetry properties of multi-dimensional FDEs and systems of FDEs are much less investigated. There is a relatively small number of papers devoted to these topics (we mention here only some recent papers [42–53], see also brief overview in the last section of [24]). However, there is a practically significant class of multi-dimensional FDEs that consists of equations with a single fractional derivative (usually, time-fractional derivatives are used in such equations). In this paper, it is shown that symmetries of such FDEs have a definite structure and therefore can be found more easily.

In [54], Ovsyannikov introduced a notion of *x-autonomous* Lie group of transformations. The property of x-autonomy means that all independent variables are transformed independently of dependent variables. Numerous integer-order partial differential equations and systems of such equations have only the symmetry groups that possess the property of x-autonomy. In particular, in [54] the necessary and sufficient conditions are established for a system of quasilinear first-order partial differential equations to admit an only x-autonomy transformation group. Notions of autonomous point transformations and autonomous symmetries are also used in the higher and generalized symmetry approaches for investigating integrable discrete and continuous models [55–57]. If, additionally, transformations of all dependent variables are linear with respect to these variables then such x-autonomous transformation group is called a *linearly autonomous* transformation group [58]. The property of linear autonomy for symmetry groups of FDEs is firstly discussed in [59] (see also [23,24]). In [27,53], it was proved that local symmetries of systems of multi-dimensional FDEs with the Riemann–Liouville fractional derivatives of orders $0 < \alpha_\mu < 1$ have the structure that corresponds to the property of linear autonomy. In this paper, systems of multi-dimensional FDEs with the Riemann–Liouville and the Caputo fractional derivatives of arbitrary orders $\alpha_\mu \in \mathbb{R}_+ \backslash \mathbb{N}$ are considered, and a new approach is proposed for proving that such systems may have only linearly autonomous symmetries.

This paper is organized as follows. Section 2 contains necessary notations and definitions. In Section 3, the theorem is proved that the system of FDEs, each of which involves a single fractional derivative of the Riemann–Liouville type, may has only linearly autonomous symmetries, and corresponding simplified system of determining equations is presented. Additionally, several specific cases of the systems of considered type are discussed. In Section 4, the similar results are presented for the systems of FDEs with the Caputo fractional derivatives, as well as for the systems with both the Riemann–Liouville and Caputo derivatives. The applicability of obtained theorems is illustrated in Section 5 by several examples of nonlinear FDEs.

2. Notations and Preliminaries

Let $\{x^0, x^1, \dots, x^n\}$ be the set of $n+1$ independent variables, and let $\{u^1, u^2, \dots, u^m\}$ be the set of m dependent variables that are functions of $x^0, \dots, x^n$. For convenience, we introduce the vectors $x = (x^1, \dots, x^n)$ and $u = (u^1, \dots, u^m)$ (note that x^0 is not included in x). We will consider systems of m fractional differential equations

$$ {}_0\mathcal{D}_{x^0}^{\alpha_\mu}(u^\mu) = F_\mu(x^0, x, u, u_{(1)}, \dots, u_{(r)}), \qquad \mu = 1, \dots, m, \tag{1} $$

where ${}_0\mathcal{D}^{\alpha_\mu}_{x^0}(u^\mu)$ is a fractional derivative of order $\alpha_\mu \in \mathbb{R}_+ \backslash \mathbb{N}$. In (1), we use the notations of differential algebra (see, e.g., [4]):

$$u_{(1)} = \{u^\mu_{i_1}\},\ u_{(2)} = \{u^\mu_{i_1 i_2}\},\ \ldots,\ u_{(r)} = \{u^\mu_{i_1 i_2 \ldots i_r}\},$$

where $i_1, \ldots, i_r = 0, \ldots, n$, $\mu = 1, \ldots, m$, and

$$u^\mu_{i_1} = D_{i_1}(u^\mu),\ \ u^\mu_{i_1 i_2} = D_{i_2}(u^\mu_{i_1}) = D_{i_2} D_{i_1}(u^\mu),\ \ \ldots,$$
$$u^\mu_{i_1 i_2 \ldots i_r} = D_{i_r}(u^\mu_{i_1 i_2 \ldots i_{r-1}}) = D_{i_r} D_{i_{r-1}} \ldots D_{i_1}(u^\mu),\ \ldots .$$

Here and in what follows, $D_i \equiv D_{x^i}$ denotes the operator of total differentiation with respect to x^i:

$$D_i = \frac{\partial}{\partial x^i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{i i_1} \frac{\partial}{\partial u^\mu_{i_1}} + u^\mu_{i i_1 i_2} \frac{\partial}{\partial u^\mu_{i_1 i_2}} + \cdots ,$$

and summation over repeated indices is implied.

In this paper, the Riemann–Liouville and Caputo left-sided partial fractional derivatives (see, e.g., [16]) will be used in (1) as the fractional derivative ${}_0\mathcal{D}^{\alpha_\mu}_{x^0}(u^\mu)$.

The left-sided Riemann–Liouville fractional derivative is defined by

$${}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}(u^\mu)(x^0, x) = \frac{1}{\Gamma(N_\mu + 1 - \alpha_\mu)} \frac{\partial^{N_\mu + 1}}{\partial (x^0)^{N_\mu + 1}} \int_0^{x^0} \frac{u^\mu(s, x)}{(x^0 - s)^{\alpha_\mu - N_\mu}} ds, \tag{2}$$

and the left-sided Caputo fractional derivative reads

$${}^{C}_{0}D^{\alpha_\mu}_{x^0}(u^\mu)(x^0, x) = \frac{1}{\Gamma(N_\mu + 1 - \alpha_\mu)} \int_0^{x^0} \frac{1}{(x^0 - s)^{\alpha_\mu - N_\mu}} \frac{\partial^{N_\mu + 1} u^\mu(s, x)}{\partial s^{N_\mu + 1}} ds, \tag{3}$$

where $N_\mu = [\alpha_\mu]$, and $\Gamma(z)$ is the gamma function.

The purpose of this paper is to study general properties of the Lie point symmetries for the system (1). The infinitesimal generator of the corresponding Lie point symmetry group is given by

$$X = \xi^i(x^0, x, u) \frac{\partial}{\partial x^i} + \eta^\mu(x^0, x, u) \frac{\partial}{\partial u^\mu}, \tag{4}$$

where $i = 0, \ldots, n$, $\mu = 1, \ldots, m$. As it is usual in Lie group analysis, we will assume that all coordinates ξ^i and η^μ belong to C^∞ class with respect to all their variables.

The necessary condition for X being a symmetry of (1) leads to the following system of determining equations [24]:

$$X_{(\alpha_\mu)} {}_0\mathcal{D}^{\alpha_\mu}_{x^0}(u^\mu) - X_{(r)} F_\mu(x^0, x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \tag{5}$$

whenever u satisfies (1). Here

$$X_{(\alpha_\mu)} = \xi^i \frac{\partial}{\partial x^i} + \eta^\mu \frac{\partial}{\partial u^\mu} + \zeta^\mu_{(\alpha_\mu)} \frac{\partial}{\partial\, {}_0\mathcal{D}^{\alpha_\mu}_{x^0}(u^\mu)},$$

$$X_{(r)} = \xi^i \frac{\partial}{\partial x^i} + \eta^\mu \frac{\partial}{\partial u^\mu} + \zeta^\mu_i \frac{\partial}{\partial u^\mu_i} + \ldots + \zeta^\mu_{i_1 \ldots i_r} \frac{\partial}{\partial u^\mu_{i_1 \ldots i_r}}$$

are the so-called α_μth- and rth-order prolongations of the generator X defined by (4).

The functions ζ^μ are given by the prolongation formulae (see [23])

$$\zeta^\mu_{(\alpha_\mu)} = {}_0\mathcal{D}^{\alpha_\mu}_{x^0}(W^\mu) + \xi^0 D_{x^0}\left({}_0\mathcal{D}^{\alpha_\mu}_{x^0}(u^\mu)\right) + \xi^j\, {}_0\mathcal{D}^{\alpha_\mu}_{x^0}(u^\mu_j), \tag{6}$$

$$\zeta^\mu_{i_1 \ldots i_s} = D_{i_1 \ldots i_s}(W^\mu) + \xi^0 D_{i_1 \ldots i_s}(u^\mu_0) + \xi^j D_{i_1 \ldots i_s}(u^\mu_j), \quad s = 1, \ldots, r, \tag{7}$$

where $W^\mu = \eta^\mu - \xi^0 u_0^\mu - \xi^j u_j^\mu$ and $j = 1, \dots, n$. In [26], it was proved that the following additional condition should also be fulfilled:

$$\xi^0(x^0, x, u)|_{x^0=0} = 0. \tag{8}$$

Thus, the coordinates of the generator (4) can be found as a solution of the system (5) coupled with the condition (8).

The generator X of a Lie point symmetry group admitted by a FDE is called a *linearly autonomous symmetry* (see [58]) if $\xi^i_{u^\mu} = 0$, $\eta^\mu_{u^\nu u^\lambda} = 0$ for all $i = 0, \dots, n$ and $\mu, \nu, \lambda = 1, \dots, m$. It can be written in the form

$$X = \xi^i(x^0, x)\frac{\partial}{\partial x^i} + \left[\eta^\mu_{(0)}(x^0, x) + \eta^\mu_{(1)\nu}(x^0, x)u^\nu\right]\frac{\partial}{\partial u^\mu}. \tag{9}$$

The corresponding symmetry group is called a *linearly autonomous symmetry group*. Further, we prove that the systems (1) with the Riemann–Liouville and Caputo fractional derivatives may have only linearly autonomous symmetries.

3. Systems with the Riemann–Liouville Fractional Derivatives

Let us start from the consideration of the system (1) with the Riemann–Liouville fractional derivatives. The following theorem holds.

Theorem 1. *Let ${}_0\mathcal{D}^{\alpha_\mu}_{x^0}$ be the Riemann–Liouville fractional differential operator ${}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}$ defined by (2). Then the Lie point symmetry group admitted by the system (1) consists of only linearly autonomous symmetries (9) with*

$$\begin{gathered}
\xi^0 = \phi(x)(x^0)^2 + \psi(x)x^0,\ \ \xi^j = \theta^j(x),\ \ \eta^\mu_{(1)\mu} = (\alpha_\mu - 1)\phi(x)x^0 + \varphi_\mu(x), \\
\eta^\mu_{(1)\nu} = \begin{cases} \sum\limits_{p=0}^{l_{\mu\nu}} \omega^\mu_{p\nu}(x)(x^0)^p, & \alpha_\mu = \alpha_\nu + l_{\mu\nu}, \\ 0, & \alpha_\mu \neq \alpha_\nu + l_{\mu\nu}, \end{cases} \quad l_{\mu\nu} \in \mathbb{N}\cup\{0\},\ \ \nu \neq \mu,
\end{gathered} \tag{10}$$

where $j = 1, \dots, n$, $\mu, \nu = 1, \dots, m$. The functions $\phi(x)$, $\psi(x)$, $\theta^j(x)$, $\varphi_\mu(x)$, $\omega^\mu_{p\nu}(x)$, $\eta^\mu_{(0)}(x^0, x)$ are to be found from the system of determining equations

$$\begin{aligned}
&{}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}(\eta^\mu_{(0)}) + \left[\varphi_\mu - \alpha_\mu\psi - (1 + \alpha_\mu)\phi x^0\right]F_\mu - X_{(r)}F_\mu \\
&\qquad + \textstyle\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{m} \sum_{p=0}^{l_{\mu\nu}} \sum_{k=0}^{p} \binom{p}{k}\frac{\Gamma(\alpha_\mu+1)(x^0)^{p-k}}{\Gamma(\alpha_\mu-k+1)}\omega^\mu_{p\nu} D^{l_{\mu\nu}-k}_{x^0}(F_\nu) = 0,
\end{aligned} \tag{11}$$

where $\binom{p}{k}$ is a binomial coefficient.

Proof. A proposed approach for proving process is based on the following formal representation. We rewrite the functions $\eta^\mu(x^0, x, u)$ in the form

$$\eta^\mu(x^0, x, u) = \eta^\mu_{(0)}(x^0, x) + \eta^\mu_{(1)\nu}(x^0, x, u)u^\nu, \quad \mu, \nu = 1, \dots, m.$$

Note that this equality does not correspond to the property of linear autonomy since the functions $\eta^\mu_{(1)\nu}(x^0, x, u)$ depend on u. Thus, it is not assumed here that the functions $\eta^\mu(x^0, x, u)$ are linear in u^ν.

By using (6) and the equality $D_{x^0}\left({}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}\right) = {}^{RL}_{\ 0}D^{\alpha_\mu+1}_{x^0}$ (see, e.g., [14,16]), the first term in the left hand side of (5) can be written as

$$\begin{aligned}
X_{(\alpha_\mu)}\,{}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}(u^\mu) &= \zeta^\mu_{(\alpha_\mu)} \\
&= {}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}\left(\eta^\mu_{(0)} + \eta^\mu_{(1)\nu}u^\nu - \xi^0 u^\mu_0 - \xi^j u^\mu_j\right) + \xi^0\,{}^{RL}_{\ 0}D^{\alpha_\mu+1}_{x^0}(u^\mu) + \xi^j\,{}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}(u^\mu_j),
\end{aligned} \tag{12}$$

where $u_0^\mu = \partial u^\mu/\partial x^0$.

Next, we use the equality

$$\xi^0 u_0^\mu = D_{x^0}(\xi^0 u^\mu) - D_{x^0}(\xi^0)u^\mu.$$

Since $\xi^0 \in C^\infty$ and $\xi^0|_{x^0=0} = 0$, the condition $(\xi^0 u^\mu)|_{x^0=0} = 0$ holds. Therefore, we can use the following property of the Riemann–Liouville fractional derivative [14,16]: if ${}^{RL}_{\ 0}D^{\alpha+1}_{x^0} f$ exists and $f|_{x^0=0} = 0$, ${}^{RL}_{\ 0}D^{\alpha+1}_{x^0} f = {}^{RL}_{\ 0}D^{\alpha}_{x^0}(D_{x^0} f)$. Thus, we have

$${}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}\left(D_{x^0}(\xi^0 u^\mu)\right) = {}^{RL}_{\ 0}D^{\alpha_\mu+1}_{x^0}(\xi^0 u^\mu).$$

Then, we use the generalized Leibnitz rule [14]

$${}^{RL}_{\ 0}D^{\beta}_{x^0}(fg) = \sum_{k=0}^{\infty}\binom{\beta}{k}\,{}^{RL}_{\ 0}D^{\beta-k}_{x^0}(f)\, D^k_{x^0}(g), \quad \beta > 0,$$

where ${}^{RL}_{\ 0}D^{\beta-k}_{t}(f)$ for $k < \beta$ is the Riemann–Lioville fractional derivative, and for $k > \beta$ it is the fractional integral

$${}^{RL}_{\ 0}D^{\beta-k}_{t}(f) \equiv {}_0I^{k-\beta}_t(f) = \frac{1}{\Gamma(k-\beta)}\int_0^t \frac{f(\tau,x)}{(t-\tau)^{\beta-k+1}}d\tau.$$

As a result, expression (12) takes the form

$$\begin{gathered}
\zeta^\mu_{(\alpha_\mu)} = {}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}(\eta^\mu_{(0)}) \\
+\left[\eta^\mu_{(1)\mu} - \alpha_\mu D_{x^0}(\xi^0)\right]{}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}(u^\mu) + \textstyle\sum_{k=1}^\infty \binom{\alpha_\mu}{k}{}^{RL}_{\ 0}D^{\alpha_\mu-k}_{x^0}(u^\mu)\, D^k_{x^0}(\eta^\mu_{(1)\mu}) \\
+\textstyle\sum_{\substack{\nu=1\\ \nu\neq\mu}}^m \left[\eta^\mu_{(1)\nu}\ {}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}(u^\nu) + \sum_{k=1}^\infty \binom{\alpha_\mu}{k}{}^{RL}_{\ 0}D^{\alpha_\mu-k}_{x^0}(u^\nu)\, D^k_{x^0}(\eta^\mu_{(1)\nu})\right] \\
-\textstyle\sum_{k=1}^\infty \binom{\alpha_\mu}{k+1}{}^{RL}_{\ 0}D^{\alpha_\mu-k}_{x^0}(u^\mu)\, D^{k+1}_{x^0}(\xi^0) - \sum_{k=1}^\infty \binom{\alpha_\mu}{k}{}^{RL}_{\ 0}D^{\alpha_\mu-k}_{x^0}(u^\mu_j)\, D^k_{x^0}(\xi^j).
\end{gathered} \tag{13}$$

One can eliminate all fractional derivatives ${}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}(u^\mu)$ from (13) by using the equations of system (1). Substituting the expression (13) into the system of determining Equation (5) yields

$$\begin{gathered}
{}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}(\eta^\mu_{(0)}) + \left[\eta^\mu_{(1)\mu} - \alpha_\mu D_{x^0}(\xi^0)\right]F_\mu + \textstyle\sum_{k=1}^\infty \binom{\alpha_\mu}{k}{}^{RL}_{\ 0}D^{\alpha_\mu-k}_{x^0}(u^\mu)\, D^k_{x^0}(\eta^\mu_{(1)\mu}) \\
+\textstyle\sum_{\substack{\nu=1\\ \nu\neq\mu}}^m \left[\eta^\mu_{(1)\nu}\ {}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}(u^\nu) + \sum_{k=1}^\infty \binom{\alpha_\mu}{k}{}^{RL}_{\ 0}D^{\alpha_\mu-k}_{x^0}(u^\nu)\, D^k_{x^0}(\eta^\mu_{(1)\nu})\right] \\
-\textstyle\sum_{k=1}^\infty \binom{\alpha_\mu}{k+1}{}^{RL}_{\ 0}D^{\alpha_\mu-k}_{x^0}(u^\mu)\, D^{k+1}_{x^0}(\xi^0) - \sum_{k=1}^\infty \binom{\alpha_\mu}{k}{}^{RL}_{\ 0}D^{\alpha_\mu-k}_{x^0}(u^\mu_j)\, D^k_{x^0}(\xi^j) \\
-X_{(r)}F_\mu = 0.
\end{gathered} \tag{14}$$

If $\alpha_\mu = \alpha_\nu + l_{\mu\nu}$ ($l_{\mu\nu} \in \mathbb{N}\cup\{0\}$), the fractional derivatives ${}^{RL}_{\ 0}D^{\alpha_\mu-k}_{x^0}(u^\nu)$ ($\nu \neq \mu$, $k = 0,\ldots,l_{\mu\nu}$) can also be eliminated from (14) by using (1). We have

$${}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}(u^\nu) = {}^{RL}_{\ 0}D^{\alpha_\nu+l_{\mu\nu}}_{x^0}(u^\nu) \equiv D^{l_{\mu\nu}}_{x^0}\left({}^{RL}_{\ 0}D^{\alpha_\nu}_{x^0}(u^\nu)\right) = D^{l_{\mu\nu}}_{x^0}(F_\nu),$$

and, similarly,

$${}^{RL}_{\ 0}D^{\alpha_\mu-k}_{x^0}(u^\nu) = {}^{RL}_{\ 0}D^{\alpha_\nu+l_{\mu\nu}-k}_{x^0}(u^\nu) \equiv D^{l_{\mu\nu}-k}_{x^0}\left({}^{RL}_{\ 0}D^{\alpha_\nu}_{x^0}(u^\nu)\right) = D^{l_{\mu\nu}-k}_{x^0}(F_\nu)$$

for $k = 1,\ldots,l_{\mu\nu}$. Otherwise, all these fractional derivatives should be considered as independent variables in the system (14).

Note that in the system (14) all unknown functions ξ^i, $\eta^\mu_{(0)}$, $\eta^\mu_{(1)\nu}$ $(i = 0,\dots,n,\ \mu,\ \nu = 1,\dots,m)$ depend only on x^0, x, u. Therefore, we can isolate the terms containing the fractional derivatives and integrals, and set each of them equal to zero.

The terms containing ${}_0^{RL}D^{\alpha_\mu - k}_{x^0}(u^\mu_i)$ lead to equations

$$D^k_{x^0}(\xi^j) = 0, \quad j = 1,\dots,n, \quad k = 1,2,\dots. \tag{15}$$

For $k = 1$ we have

$$\frac{\partial \xi^j}{\partial x^0} + \frac{\partial \xi^j}{\partial u^\mu} u^\mu_0 = 0,$$

where

$$\frac{\partial \xi^j}{\partial x^0} = 0, \qquad \frac{\partial \xi^j}{\partial u^\mu} = 0$$

for all $j = 1,\dots,n$ and $\mu = 1,\dots,m$. The solution of this system can be written as

$$\xi^j = \theta^j(x), \quad j = 1,\dots,n, \tag{16}$$

where $\theta^j(x)$ are arbitrary functions related to x only. It is easy to see that in view of (16) all Equation (15) are satisfied identically.

Similarly, the terms of (14) containing ${}_0^{RL}D^{\alpha_\mu - k}_{x^0}(u^\mu)$ lead to the infinite system of differential equations

$$\binom{\alpha_\mu}{k} D^k_{x^0}(\eta^\mu_{(1)\mu}) - \binom{\alpha_\mu}{k+1} D^{k+1}_{x^0}(\xi^0) = 0, \quad k = 1,2,\dots$$

or, after simplification,

$$(k+1)D^k_{x^0}(\eta^\mu_{(1)\mu}) - (\alpha_\mu - k)D^{k+1}_{x^0}(\xi^0) = 0, \quad k = 1,2,\dots. \tag{17}$$

Considering equations corresponding to $k = 1$ and $k = 2$, we can exclude ξ^0 and obtain the equation

$$D^2_{x^0}(\eta^\mu_{(1)\mu}) = 0.$$

This equation can be rewritten in an equivalent form

$$\frac{\partial^2 \eta^\mu_{(1)\mu}}{\partial (x^0)^2} + 2\frac{\partial^2 \eta^\mu_{(1)\mu}}{\partial x^0 \partial u^\nu} u^\nu_0 + \frac{\partial^2 \eta^\mu_{(1)\mu}}{\partial u^\nu \partial u^\lambda} u^\nu_0 u^\lambda_0 + \frac{\partial \eta^\mu_{(1)\mu}}{\partial u^\nu} u^\nu_{00} = 0,$$

which leads to the system

$$\frac{\partial^2 \eta^\mu_{(1)\mu}}{\partial (x^0)^2} = 0, \qquad \frac{\partial \eta^\mu_{(1)\mu}}{\partial u^\nu} = 0, \quad \mu,\nu = 1,\dots,m.$$

The solution of this system is given by

$$\eta^\mu_{(1)\mu} = \vartheta_\mu(x)x^0 + \varphi_\mu(x),$$

where $\vartheta_\mu(x)$ and $\varphi_\mu(x)$ are arbitrary functions related to x only. Substituting this solution into (17) with $k = 1$, we obtain the equations

$$D^2_{x^0}(\xi^0) = 2(\alpha_\mu - 1)^{-1}\vartheta_\mu(x), \quad \mu = 1,\dots,m.$$

Since the left-hand sides of all these equations are identical and do not depend on μ, the following representations hold: $\vartheta_\mu(x) = (\alpha_\mu - 1)\phi(x)$ $(\mu = 1, \ldots m)$, where $\phi(x)$ is an arbitrary function related to x only. Then, after expansion, we have

$$\frac{\partial^2 \xi^0}{\partial (x^0)^2} + 2\frac{\partial^2 \xi^0}{\partial x^0 \partial u^\mu} u_0^\mu + \frac{\partial^2 \xi^0}{\partial u^\mu \partial u^\nu} u_0^\mu u_0^\nu + \frac{\partial \xi^0}{\partial u^\mu} u_{00}^\mu = 2\phi(x)$$

whence

$$\frac{\partial^2 \xi^0}{\partial (x^0)^2} = 2\phi(x), \qquad \frac{\partial \xi^0}{\partial u^\mu} = 0, \quad \mu = 1, \ldots, m.$$

Integration of this system yields

$$\xi^0 = \phi(x)(x^0)^2 + \psi(x)x^0 + \rho(x)$$

with arbitrary $\psi(x)$ and $\rho(x)$. Taking into account the condition (8), we obtain $\rho(x) = 0$. Thus, we can write

$$\xi^0 = \phi(x)(x^0)^2 + \psi(x)x^0, \qquad \eta^\mu_{(1)\mu} = (\alpha_\mu - 1)\phi(x)x^0 + \varphi_\mu(x). \tag{18}$$

If $\alpha_\mu \neq \alpha_\nu + l_{\mu\nu}$, $l_{\mu\nu} \in \mathbb{N} \cup \{0\}$, the terms containing ${}^{RL}_{0}D^{\alpha_\mu}_{x^0}(u^\nu)$ $(\nu \neq \mu)$ lead to equations

$$\eta^\mu_{(1)\nu} = 0, \quad \nu \neq \mu, \quad \nu, \mu = 1, \ldots, m. \tag{19}$$

If $\alpha_\mu = \alpha_\nu + l_{\mu\nu}$, $l_{\mu\nu} \in \mathbb{N} \cup \{0\}$, the terms containing ${}^{RL}_{0}D^{\alpha_\mu - k}_{x^0}(u^\nu)$ with $k > l_{\mu\nu}$ give

$$D^k_{x^0}(\eta^\mu_{(1)\nu}) = 0, \quad \mu, \nu = 1, \ldots, m, \quad \nu \neq \mu, \quad k = l_{\mu\nu} + 1, l_{\mu\nu} + 2, \ldots.$$

The solutions of these equations can be written in the form

$$\eta^\mu_{(1)\nu} = \sum_{p=0}^{l_{\mu\nu}} \omega^\mu_{p\nu}(x)(x^0)^p, \tag{20}$$

where $\omega^\mu_{p\nu}(x)$ are arbitrary functions related to x only.

Thus, we obtain all representations from (10). The system of determining Equation (11) is obtained by substituting (10) into the remaining part of the system (14). $\square$

Now, let us consider some special cases of the system (1) which are widely encountered in practice. For example, if a time-fractional derivative is applied to multidimensional time-dependent vector fields (such as the flow velocity in hydrodynamics, the displacement in solid state mechanics, the electric field strength in electrodynamics, etc.) then all fractional derivatives in the system will have the same order. Another important case is a single partial FDE. Such equations are frequently used for modelling scalar fields in systems with memory and long-range interactions. In all the mentioned cases, the system of determining Equation (11) can be significantly simplified.

Case 1. Let $\alpha_\mu = \alpha = const$ for all $\mu = 1, \ldots, m$. Then, $l_{\mu\nu} = 0$ and $\eta^\mu_{(1)\nu} = \omega^\mu_\nu(x)$ for all $\nu = 1, \ldots, m$. The system of determining Equation (11) takes the form

$${}^{RL}_{0}D^{\alpha}_{x^0}(\eta^\mu_{(0)}) + \left[\varphi_\mu - \alpha\psi - (1+\alpha)\phi x^0\right] F_\mu - X_{(r)}F_\mu + \sum_{\substack{\nu=1 \\ \nu\neq\mu}}^{m} \omega^\mu_\nu F_\nu = 0. \tag{21}$$

If $0 < \alpha < 1$, this result coincides with that presented in [53]

Additionally, it is often the case that none of the functions F_μ $(\mu = 1, \dots, m)$ depend on the variables u_0^ν $(\nu = 1, \dots, m)$ and any their derivatives, i.e.,

$$\frac{\partial F_\mu}{\partial u^\nu_{0i_1\dots i_s}} = 0, \quad s = 0, \dots, r-1, \quad \mu, \nu = 1, \dots, m. \tag{22}$$

Nevertheless, the determining Equation (21) contains the variables $u^\nu_{0i_1\dots i_s}$ because the prolonged generator $X_{(r)}$ depends on the expressions $D_{i_1\dots i_s}(\xi^0 u_0^\mu) - \xi^0 D_{i_1\dots i_s}(u_0^\mu)$ $(s = 1, \dots, r)$. Isolating the terms containing $u^\nu_{0i_1\dots i_{r-1}}$ and setting all of them equal to zero, we obtain the system

$$\sum_{j=1}^{n}\left(1 + \sum_{s=1}^{r-1}\delta_{i_s j}\right)\frac{\partial F_\mu}{\partial u^\nu_{ji_1\dots i_{r-1}}} D_j(\xi^0) = 0, \tag{23}$$

where $\delta_{i_s j}$ is the Kronecker delta, $\mu, \nu = 1, \dots, m$, $j, i_1, \dots, i_{r-1} = 1, \dots, n$ and $i_1 \le i_2 \le \dots \le i_{r-1}$. The system (23) is a homogeneous linear system with respect to $D_j(\xi^0)$ $(j = 1, \dots, n)$. If the rank of coefficient matrix of this system is equal to n, (23) has only trivial solution, i.e., $\xi^0 = \xi^0(x^0)$. Taking into account the representation for ξ^0 from (18), we found

$$\psi(x) = C_1, \qquad \phi(x) = C_2, \tag{24}$$

where C_1 and C_2 are arbitrary constants. Then, (18) takes the form

$$\xi^0 = C_1 x^0 + C_2 (x^0)^2, \qquad \eta^\mu_{(1)\mu} = \varphi_\mu(x) + (\alpha - 1) C_2 x^0. \tag{25}$$

Case 2. Let the system (1) consists of a single equation (i.e., $\mu = 1$, $F_1 \equiv F$, $\alpha_1 \equiv \alpha$). Then,

$$\xi^0 = \psi(x) x^0 + \phi(x)(x^0)^2, \quad \xi^j = \theta^j(x), \quad \eta_{(1)} = \varphi(x) + (\alpha - 1)\phi(x) x^0,$$

and the determining equation reads

$${}^{RL}_{\ 0}D^{\alpha}_{x^0}(\eta_{(0)}) + [\varphi - \alpha\psi - (1+\alpha) t\phi] F - X_{(r)} F = 0.$$

Case 3. Theorem 1 is also applicable for a special type of systems of ordinary fractional differential equations. Let $x^0 = t$ be a single independent variable in the system (1). Then, $F_\mu = F_\mu(t, u, u_t, \dots)$ and (10) takes the form

$$\xi^0 = C_1 t + C_2 t^2, \quad \eta^\mu_{(1)\mu} = (\alpha_\mu - 1) C_2 t + C^\mu_\mu,$$

$$\eta^\mu_{(1)\nu} = \begin{cases} \sum\limits_{p=0}^{l_{\mu\nu}} C^\mu_{p\nu}\, t^p, & \alpha_\mu = \alpha_\nu + l_{\mu\nu}, \\ 0, & \alpha_\mu \ne \alpha_\nu + l_{\mu\nu}, \end{cases} \quad l_{\mu\nu} \in \mathbb{N} \cup \{0\}, \ \nu \ne \mu,$$

where $\mu, \nu = 1, \dots, m$, and C_1, C_2, C^μ_μ, $C^\mu_{p\nu}$ are arbitrary constants. The system of determining Equation (11) can be written as

$$\begin{aligned} &{}^{RL}_{\ 0}D^{\alpha_\mu}_t(\eta^\mu_{(0)}) + \left[C^\mu_\mu - \alpha_\mu C_1 - (1+\alpha_\mu) C_2 t\right] F_\mu - X_{(r)} F_\mu \\ &\qquad + \sum_{\substack{\nu=1\\ \nu\ne\mu}}^{m} \sum_{p=0}^{l_{\mu\nu}} \sum_{k=0}^{p} \binom{p}{k} \frac{\Gamma(\alpha_\mu + 1) t^{p-k}}{\Gamma(\alpha_\mu - k + 1)} C^\mu_{p\nu} D_t^{l_{\mu\nu} - k}(F_\nu) = 0. \end{aligned}$$

4. Systems with the Caputo Fractional Derivatives

Now, let us consider the system (1) with the Caputo fractional derivatives.

Theorem 2. *Let* ${}_0\mathcal{D}^{\alpha_\mu}_{x^0}$ *be the Caputo fractional differential operator* ${}^C_0D^{\alpha_\mu}_{x^0}$ *defined by* (3). *Then, the Lie point symmetry group admitted by the system* (1) *consists of only linearly autonomous symmetries* (9) *with*

$$\begin{aligned} &\xi^0 = \psi(x)x^0, \quad \xi^j = \theta^j(x), \quad \eta^\mu_{(1)\mu} = \varphi_\mu(x), \\ &\eta^\mu_{(1)\nu} = \begin{cases} \omega^\mu_\nu(x), & \alpha_\mu = \alpha_\nu + l_{\mu\nu}, \\ 0, & \alpha_\mu \neq \alpha_\nu + l_{\mu\nu}, \end{cases} \quad l_{\mu\nu} \in \mathbb{N} \cup \{0\}, \ \nu \neq \mu, \end{aligned} \tag{26}$$

where $j = 1, \ldots, n$, $\mu, \nu = 1, \ldots, m$. *The functions* $\psi(x)$, $\theta^i(x)$, $\varphi_\mu(x)$, $\eta^\mu_{(1)\nu}(x)$, $\eta^\mu_{(0)}(x^0, x)$ *are to be found from the system of determining equations*

$${}^C_0D^{\alpha_\mu}_{x^0}(\eta^\mu_{(0)}) + [\varphi_\mu - \alpha_\mu\psi]F_\mu - X_{(r)}F_\mu + \sum_{\substack{\nu=1 \\ \nu\neq\mu}}^m \eta^\mu_{(1)\nu} D^{l_{\mu\nu}}_{x^0}(F_\nu) = 0. \tag{27}$$

Proof. It is well-known (see, e.g., [16]) that the Caputo and the Riemann–Liouville fractional derivatives are connected by the relation

$${}^C_0D^\alpha_t(u) = {}^{RL}_{\ 0}D^\alpha_t(u) - \sum_{q=0}^N \frac{t^{q-\alpha}}{\Gamma(q+1-\alpha)} \frac{\partial^q u}{\partial t^q}\Big|_{t=0},$$

where $N = [\alpha]$. By using this relation, the system (1) with the Caputo fractional derivatives can be rewritten in terms of the Riemann–Liouville fractional derivatives in the following way:

$${}^{RL}_{\ 0}D^{\alpha_\mu}_{x^0}(u^\mu) - \sum_{q=0}^{N_\mu} \frac{(x^0)^{q-\alpha_\mu} U^\mu_q(x)}{\Gamma(q+1-\alpha_\mu)} = F_\mu(x^0, x, u, u_{(1)}, \ldots, u_{(r)}), \tag{28}$$

where $\alpha_\mu \in \mathbb{R}_+ \backslash \mathbb{N}$, $\mu = 1, \ldots, m$, $N_\mu = [\alpha_\mu]$, and

$$U^\mu_q(x) = \frac{\partial^q u^\mu}{\partial (x^0)^q}\Big|_{x^0=0}, \quad q = 0, \ldots, N_\mu.$$

Each function $U^\mu_q(x)$ should be considered as an arbitrary function in the system (28). Its infinitesimal transformation is a particular case of the infinitesimal transformation of the function $\frac{\partial^q u^\mu}{\partial (x^0)^q}$ considering at $x^0 = 0$. In the Lie group analysis approach the functions U^μ_q can be considered as new variables, and symmetry group generator (4) can be prolonged on all these variables. The corresponding prolongation can be written as

$$\hat{X} = X + \eta^\mu[0]\frac{\partial}{\partial U^\mu_0} + \sum_{q=1}^{N_\mu} \zeta^\mu_{0q}[0]\frac{\partial}{\partial U^\mu_q},$$

where, for convenience, we introduce the notation $f[0] = f(x^0, x, u, \ldots)|_{x^0=0}$. In accordance with (7), we have

$$\zeta^\mu_{0q} = D^q_{x^0}(W^\mu) + \xi^0 D^{q+1}_{x^0}(u^\mu) + \xi^j D^q_{x^0}(u^\mu_j), \quad j = 1, \ldots, n.$$

The invariance principle for the system (28) leads to the following system of determining equations:

$$\begin{aligned} \zeta^\mu_{(\alpha_\mu)} - \tfrac{(x^0)^{-\alpha_\mu}}{\Gamma(1-\alpha_\mu)}\eta^\mu[0] - \xi^0 \textstyle\sum_{q=0}^{N_\mu} \tfrac{(x^0)^{q-\alpha_\mu-1}}{\Gamma(q-\alpha_\mu)} U^\mu_q& \\ - \textstyle\sum_{q=1}^{N_\mu} \tfrac{(x^0)^{q-\alpha_\mu}}{\Gamma(q+1-\alpha_\mu)}\zeta^\mu_{0q}[0] - X_{(r)}F_\mu &= 0, \end{aligned} \tag{29}$$

whenever u satisfies (28).

Taking into account the representation (13) and eliminating all fractional derivatives ${}^{RL}_{0}D^{\alpha_\mu}_{x^0}(u^\mu)$ by using the system (28), we can rewrite the system (29) in the form

$$\begin{gathered}
{}^{RL}_{0}D^{\alpha_\mu}_{x^0}(\eta^\mu_{(0)}) + \left[\eta^\mu_{(1)\mu} - \alpha_\mu D_{x^0}(\xi^0)\right]\left(F_\mu + \textstyle\sum_{q=0}^{N_\mu} \frac{(x^0)^{q-\alpha_\mu}U^\mu_q}{\Gamma(q+1-\alpha_\mu)}\right) \\
-\tfrac{(x^0)^{-\alpha_\mu}}{\Gamma(1-\alpha_\mu)}\left(\eta^\mu_{(0)}(0,x) + \eta^\mu_{(1)\nu}(0,x)U^\nu_0\right) - \textstyle\sum_{q=1}^{N_\mu}\frac{(x^0)^{q-\alpha_\mu}}{\Gamma(q+1-\alpha_\mu)}\zeta^\mu_{0q}[0] \\
-\xi^0\textstyle\sum_{q=0}^{N_\mu}\frac{(x^0)^{q-\alpha_\mu-1}}{\Gamma(q-\alpha_\mu)}U^\mu_q + \sum_{k=1}^{\infty}\binom{\alpha_\mu}{k}{}^{RL}_{0}D^{\alpha_\mu-k}_{x^0}(u^\mu)\,D^k_{x^0}(\eta^\mu_{(1)\mu}) \\
+\textstyle\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{m}\left[\eta^\mu_{(1)\nu}\;{}^{RL}_{0}D^{\alpha_\mu}_{x^0}(u^\nu) + \sum_{k=1}^{\infty}\binom{\alpha_\mu}{k}{}^{RL}_{0}D^{\alpha_\mu-k}_{x^0}(u^\nu)\,D^k_{x^0}(\eta^\mu_{(1)\nu})\right] \\
-\textstyle\sum_{k=1}^{\infty}\binom{\alpha_\mu}{k+1}{}^{RL}_{0}D^{\alpha_\mu-k}_{x^0}(u^\mu)\,D^{k+1}_{x^0}(\xi^0) - \sum_{k=1}^{\infty}\binom{\alpha_\mu}{k}{}^{RL}_{0}D^{\alpha_\mu-k}_{x^0}(u^\mu_i)\,D^k_{x^0}(\xi^i) \\
-X_{(r)}F_\mu = 0.
\end{gathered} \tag{30}$$

Similarly to the proof of Theorem 1, we can isolate the terms containing identical fractional derivatives and integrals from the system (30) and set each of them equal to zero. Later calculations give the expressions (16), (18)–(20). Then $\zeta^\mu_{0q}[0]$ can be written as

$$\zeta^\mu_{0q}[0] = \left.\frac{\partial^q \eta^\mu_{(0)}}{\partial(x^0)^q}\right|_{x^0=0} + (\varphi_\mu - q\psi)U^\mu_q + (\alpha_\mu - q)q\phi U^\mu_{q-1} + \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{m}\sum_{p=0}^{l_{\mu\nu}}\frac{\Gamma(q+1)}{\Gamma(q+1-p)}\omega^\mu_{p\nu}U^\nu_{q-p}.$$

Since all U^μ_q are arbitrary functions, we can additionally isolate from (30) the terms containing these functions. The terms with $U^\mu_{N_\mu}$ lead to equations

$$\frac{(N_\mu+1)(x^0)^{N_\mu+1-\alpha_\mu}}{\Gamma(N_\mu+1-\alpha_\mu)}\phi = 0, \quad \mu = 1,\dots,m,$$

whence

$$\phi = 0. \tag{31}$$

Calculations show that in view of (31) all terms with U^μ_q are equal to zero identically.

The terms with U^ν_q $(\nu \neq \mu)$ lead to the following equation:

$$\sum_{p=0}^{l_{\mu\nu}}\sum_{q=0}^{N_\nu}\omega^\mu_{p\nu}\left[\frac{\Gamma(p+q+1)(x^0)^{p+q}U^\nu_q}{\Gamma(q+1)\Gamma(p+q+1-\alpha_\mu)} - \frac{\Gamma(q+1)(x^0)^q U^\nu_{q-p}}{\Gamma(q+p+1)\Gamma(q+1-\alpha_\mu)}\right] = 0.$$

It is easy to see that in this equation the term corresponding to $p = 0$ is equal to zero. Splitting this equation with respect to U^ν_p for each $\nu = 1,\dots,m$, $p = 1,\dots,l_{\mu\nu}$, and $(x^0)^q$ for each $q = 0,\dots,N_\nu$, we obtain

$$\omega^\mu_{p\nu} = 0, \quad p = 1,\dots,l_{\mu\nu}. \tag{32}$$

Thus, $\omega^\mu_{0\nu}(x) \equiv \omega^\mu_\nu(x)$ are the only functions that are not equal to zero.

Substituting (31) and (32) into (18) and (20), we obtain (26). The system (27) is obtained by substituting (26) into the remaining part of the system (30). □

Similarly to previous section, some special cases can be considered.

Case 1. Let $\alpha_\mu = \alpha = const$ for all $\mu = 1,\dots,m$. Then, $l_{\mu\nu} = 0$ for all $\nu = 1,\dots,m$, $\eta^\mu_{(1)\nu} = \omega^\mu_\nu(x)$, and the system of determining Equation (27) takes the form

$$ {}^{C}_{0}D^{\alpha}_{x^0}(\eta^\mu_{(0)}) + [\varphi_\mu - \alpha\psi]F_\mu - X_{(r)}F_\mu + \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{m}\omega^\mu_\nu F_\nu = 0. \tag{33}$$

If the equalities (22) are satisfied, similarly to (24) one can obtain $\psi(x) = C_1$ and $\xi^0 = C_1 x^0$, where C_1 is an arbitrary constant.

Case 2. Let the system (1) consists of a single equation with the Caputo fractional derivative (i.e., $\mu = 1$, $F_1 \equiv F$, $\alpha_\mu = \alpha$). Then, we have

$$\xi^0 = \psi(x)x^0, \quad \xi^i = \theta^i(x), \quad \eta_{(1)} = \varphi(x), \tag{34}$$

and

$$ {}_0^C D_{x^0}^{\alpha}(\eta_{(0)}) + [\varphi - \alpha\psi]F - X_{(r)}F = 0. \tag{35}$$

Case 3. Let $x^0 = t$ be a single independent variable. Then, the system (1) is a system of ordinary FDEs with $F_\mu = F_\mu(t, u, u_t, \ldots)$. In this case, we have

$$\xi = C_1 t, \quad \eta^\mu_{(1)\mu} = C^\mu_\mu,$$
$$\eta^\mu_{(1)\nu} = \begin{cases} C^\mu_\nu, & \alpha_\mu = \alpha_\nu + l_{\mu\nu}, \\ 0, & \alpha_\mu \neq \alpha_\nu + l_{\mu\nu}, \end{cases} \quad l_{\mu\nu} \in \mathbb{N} \cup \{0\}, \ \nu \neq \mu,$$

where $\mu, \nu = 1, \ldots, m$, and C_1, C^μ_ν are arbitrary constants. The system of determining Equation (27) takes the form

$$ {}_0^C D_t^{\alpha_\mu}(\eta^\mu_{(0)}) + \left[C^\mu_\mu - \alpha_\mu C_1\right] F_\mu - X_{(r)}F_\mu + \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^{m} \eta^\mu_{(1)\nu} D_t^{l_{\mu\nu}}(F_\nu) = 0. \tag{36}$$

Finally, we consider the system (1) involving both the Riemann–Liouville and Caputo fractional derivatives.

Theorem 3. *If in the system (1) ${}_0\mathcal{D}^{\alpha_\mu}_{x^0} = {}_0^{RL}D^{\alpha_\mu}_{x^0}$ for $\mu = 1, \ldots, m_0$, and ${}_0\mathcal{D}^{\alpha_\mu}_{x^0} = {}_0^{C}D^{\alpha_\mu}_{x^0}$ for $\mu = m_0 + 1, \ldots, m$, where $1 \leq m_0 \leq m - 1$, then the Lie point symmetry group of this system consists of only linearly-autonomous symmetries (9) with*

$$\xi^0 = \psi(x)x^0, \quad \xi^j = \theta^j(x), \quad \eta^\mu_{(1)\mu} = \varphi_\mu(x), \ \mu = 1, \ldots, m_0,$$
$$\eta^\mu_{(1)\nu} = \begin{cases} \sum\limits_{p=0}^{l_{\mu\nu}} \omega^\mu_{p\nu}(x)(x^0)^p, & \alpha_\mu = \alpha_\nu + l_{\mu\nu}, \quad l_{\mu\nu} \in \mathbb{N} \cup \{0\}, \quad \mu = 1, \ldots, m, \\ 0, & \alpha_\mu \neq \alpha_\nu + l_{\mu\nu}, \quad \nu \neq \mu, \qquad\qquad \nu = 1, \ldots, m_0, \end{cases}$$
$$\eta^\mu_{(1)\nu} = \begin{cases} \omega^\mu_{0\nu}(x), & \alpha_\mu = \alpha_\nu + l_{\mu\nu}, \quad l_{\mu\nu} \in \mathbb{N} \cup \{0\}, \quad \mu = m_0 + 1, \ldots, m, \\ 0, & \alpha_\mu \neq \alpha_\nu + l_{\mu\nu}, \quad \nu \neq \mu, \qquad\qquad \nu = m_0 + 1, \ldots, m, \end{cases}$$
$$\eta^\mu_{(1)\nu} = 0, \quad \begin{matrix} \mu = 1, \ldots, m_0, \\ \nu = m_0 + 1, \ldots, m; \end{matrix} \quad \text{and} \quad \begin{matrix} \mu = m_0 + 1, \ldots, m, \\ \nu = 1, \ldots, m_0. \end{matrix}$$

Here, $j = 1, \ldots, n$ and unknown functions $\eta^\mu_{(0)}(x^0, x)$, $\psi(x)$, $\theta^i(x)$, $\varphi_\mu(x)$, $\omega^\mu_{p\nu}(x)$ are to be found from the system of Equation (11) and (27), such that $\mu = 1, \ldots, m_0$ in (11) and $\mu = m_0 + 1, \ldots, m$ in (27).

In this theorem, we assume that all Riemann–Liouville fractional derivatives in the system (1) cannot be represented in terms of the Caputo fractional derivatives, i.e., derivatives

$$\left.\frac{\partial^q u^\mu}{\partial (x^0)^q}\right|_{x^0=0}$$

do not exist for all $q = 0, \ldots, N_\mu$, $\mu = 1, \ldots, m_0$. Then, the proof of Theorem 3 is based on combination of proofs of Theorems 1 and 2.

In a special case of $\alpha_\mu = \alpha$ $(\mu = 1, \ldots, m)$ the system of determining equations consists of the Equation (21) for $\mu = 1, \ldots, m_0$ and Equation (33) for $\mu = m_0 + 1, \ldots, m$.

A comparison of Theorems 2 and 3 with Theorem 1 leads to the important conclusion that symmetry groups of FDEs with the Riemann–Liouville fractional derivatives can be more various than those for FDEs with the Caputo fractional derivatives. For example, the projective group with the generator

$$X = (x^0)^2 \frac{\partial}{\partial x^0} + (\alpha_\mu - 1) x^0 u^\mu \frac{\partial}{\partial u^\mu},$$

that is of importance for finding blow-up invariant solutions and constructing non-trivial conservation laws, can be admitted by FDEs only with the Riemann–Liouville fractional derivatives. The main reason of such differences between the equations with different types of fractional derivatives is that the Caputo derivative ${}^{C}_{0}D^{\alpha}_{x^0}u$ does not tend to $D^{N}_{x^0}u$ as $\alpha \to N$, $N = [\alpha]$, whereas for the Riemann–Liouville derivative we have ${}^{RL}_{0}D^{\alpha}_{x^0}u \to D^{N}_{x^0}u$ as $\alpha \to N$. Note that ${}^{C}_{0}D^{\alpha}_{x^0}u \to D^{N+1}_{x^0}u$ and ${}^{RL}_{0}D^{\alpha}_{x^0}u \to D^{N+1}_{x^0}u$ as $\alpha \to N+1$. As a result, any FDE ${}^{RL}_{0}D^{\alpha}_{x^0}u = f$ with the Riemann–Liouville fractional derivative provides a continuous connection between the neighbouring integer-order differential equations $D^{N}_{x^0}u = f$ and $D^{N+1}_{x^0}u = f$, and inherit certain symmetry properties of both these equations.

5. Examples

To illustrate the applicability of theorems presented in previous sections, let us consider several examples. In all of them, the time variable t will be used as independent variable x^0.

Example 1. *Let us consider a 3D anomalous diffusion model with a nonlinear source term and time-fractional derivative of the Caputo type:*

$$ {}^{C}_{0}D^{\alpha}_{t}u = u_{xx} + u_{yy} + u_{zz} + u^{\sigma}, \qquad \sigma \neq 0, 1. \tag{37}$$

This equation can be considered as a time-fractional generalization of the basic 3D blow-up model. It follows from Theorem 2 and representations (34) that any symmetry of the Equation (37) should have the form

$$X = C_1 t \frac{\partial}{\partial t} + \theta^1(x,y,z)\frac{\partial}{\partial x} + \theta^2(x,y,z)\frac{\partial}{\partial y} + \theta^3(x,y,z)\frac{\partial}{\partial z} + \left[\eta^0(t,x,y,z) + \varphi(x,y,z)u\right]\frac{\partial}{\partial u},$$

where C_1 is an arbitrary constant. The determining Equation (35) takes the form

$$\begin{aligned} {}^{C}_{0}D^{\alpha}_{t}(\eta^0) + (\varphi - \alpha C_1)(u_{xx} + u_{yy} + u_{zz} + u^{\sigma}) \\ - \zeta_{xx} - \zeta_{yy} - \zeta_{zz} - \sigma u^{\sigma-1}(\eta^0 + \varphi u) = 0, \end{aligned} \tag{38}$$

where

$$\begin{aligned} \zeta_{xx} &= \eta^0_{xx} + \varphi_{xx}u + (2\varphi_x - \theta^1_{xx})u_x - \theta^2_{xx}u_y - \theta^3_{xx}u_z \\ &\quad + [\varphi + (\alpha-1)C_2 - 2\theta^1_x]u_{xx} - 2\theta^2_x u_{xy} - 2\theta^3_x u_{xz}, \\ \zeta_{yy} &= \eta^0_{yy} + \varphi_{yy}u - \theta^1_{yy}u_x + (2\varphi_y - \theta^2_{yy})u_y - \theta^3_{yy}u_z \\ &\quad - 2\theta^1_y u_{xy} + [\varphi + (\alpha-1)C_2 - 2\theta^2_y]u_{yy} - 2\theta^3_y u_{yz}, \\ \zeta_{zz} &= \eta^0_{zz} + \varphi_{zz}u - \theta^1_{zz}u_x - \theta^2_{zz}u_y + (2\varphi_z - \theta^3_{zz})u_z \\ &\quad - 2\theta^1_z u_{xz} - 2\theta^2_z u_{yz} + [\varphi + (\alpha-1)C_2 - 2\theta^3_z]u_{zz}. \end{aligned}$$

Isolating in (38) the terms containing first and second order partial derivatives of u with respect to x, y, z, as well as the term free of these variables, and setting each term equal to zero, we obtain the following system of equations:

$$\begin{aligned}
&\theta^1_x = \frac{\alpha}{2}C_1, && \theta^2_y = \frac{\alpha}{2}C_1, && \theta^3_z = \frac{\alpha}{2}C_1, \\
&\theta^1_y + \theta^2_x = 0, && \theta^1_z + \theta^3_x = 0, && \theta^2_z + \theta^3_y = 0, \\
&\theta^1_{yy} + \theta^1_{zz} = 2\varphi_x, && \theta^2_{xx} + \theta^2_{zz} = 2\varphi_y, && \theta^3_{xx} + \theta^3_{yy} = 2\varphi_z,
\end{aligned}$$

$${}^{C}_{0}D^{\alpha}_{t}(\eta^0) + (\varphi - \alpha C_1)u^\sigma = \eta^0_{xx} + \eta^0_{yy} + \eta^0_{zz} + (\varphi_{xx} + \varphi_{yy} + \varphi_{zz})u + \sigma u^{\sigma-1}\eta^0 + \sigma u^\sigma \varphi.$$

The solution of this system for $\sigma \neq 0, 1$ is given by

$$\begin{aligned}
&\theta^1 = \frac{\alpha}{2}C_1 x + C_5 y + C_6 z + C_2, && \theta^2 = \frac{\alpha}{2}C_1 y - C_5 x + C_7 z + C_3, \\
&\theta^3 = \frac{\alpha}{2}C_1 z - C_6 x - C_7 y + C_4, && \eta^0 = 0, \qquad \varphi = \frac{\alpha}{1-\sigma}C_1
\end{aligned}$$

with seven arbitrary constants C_i. Thus, the Equation (37) has seven linearly independent symmetries:

$$\begin{aligned}
&X_1 = \frac{2}{\alpha}t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \frac{2}{1-\sigma}u\frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \\
&X_4 = \frac{\partial}{\partial z}, \quad X_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad X_6 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad X_7 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}.
\end{aligned}$$

Example 2. *As an example of the system with the Riemann–Liouville fractional derivatives let us consider the system of nonlinear fractional equations for modeling one-dimensional fluid flow in inhomogeneous porous medium:*

$$\begin{aligned}
{}^{RL}_{0}D^{\alpha}_{t}u &= (u_x^2 + v_x^2)^\sigma(u_{xx} + v_{xx}), \\
{}^{RL}_{0}D^{\alpha}_{t}v &= (u_x^2 + v_x^2)^\sigma(v_{xx} - u_{xx}),
\end{aligned} \tag{39}$$

with $\sigma = \alpha/(1-\alpha)$ and $\alpha \in (0, 1)$.

In accordance with Theorem 1 and representations (25), any symmetry of the system (39) should have the form

$$X = \tau(t)\frac{\partial}{\partial t} + \theta(x)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,u,v)\frac{\partial}{\partial v}$$

with

$$\begin{aligned}
&\tau(t) = C_1 t + C_2 t^2, \\
&\eta^1(t,x,u,v) = \eta^1_{(0)}(t,x) + \varphi^1(x)u + (\alpha-1)C_2 t u + \omega_1(x)v, \\
&\eta^2(t,x,u,v) = \eta^2_{(0)}(t,x) + \varphi^2(x)v + (\alpha-1)C_2 t v + \omega_2(x)u,
\end{aligned}$$

where C_1 and C_2 are arbitrary constants. The system of determining Equation (21) takes the form

$$\begin{aligned}
{}^{RL}_{0}D^{\alpha}_{t}(\eta^1_{(0)}) &+ (\varphi_1 - \alpha C_1 - (1+\alpha)tC_2)(u_x^2 + v_x^2)^\sigma(u_{xx} + v_{xx}) + \omega_1(u_x^2 + v_x^2)^\sigma(v_{xx} - u_{xx}) \\
&- \sigma(u_x^2 + v_x^2)^{\sigma-1}(2u_x\zeta^1_1 + 2v_x\zeta^2_1)(u_{xx} + v_{xx}) - (u_x^2 + v_x^2)^\sigma(\zeta^1_{11} + \zeta^2_{11}) = 0,
\end{aligned}$$

$$\begin{aligned}
{}^{RL}_{0}D^{\alpha}_{t}(\eta^2_{(0)}) &+ (\varphi_2 - \alpha C_1 - (1+\alpha)tC_2)(u_x^2 + v_x^2)^\sigma(v_{xx} - u_{xx}) + \omega_2(u_x^2 + v_x^2)^\sigma(u_{xx} + v_{xx}) \\
&- \sigma(u_x^2 + v_x^2)^{\sigma-1}(2u_x\zeta^1_1 + 2v_x\zeta^2_1)(v_{xx} - u_{xx}) - (u_x^2 + v_x^2)^\sigma(\zeta^2_{11} - \zeta^1_{11}) = 0.
\end{aligned}$$

Here

$$\zeta_1^1 = \eta^1_{(0)x} + \varphi_1' u + (\varphi_1 + (\alpha-1)C_2 t)u_x + \omega_1' v + \omega_1 v_x - \theta' u_x,$$
$$\zeta_1^2 = \eta^2_{(0)x} + \varphi_2' v + (\varphi_2 + (\alpha-1)C_2 t)v_x + \omega_2' u + \omega_2 u_x - \theta' v_x,$$
$$\zeta_{11}^1 = \eta^1_{(0)xx} + \varphi_1'' u + \omega_1'' v + (2\varphi_1' - \theta'')u_x + 2\omega_1' v_x + (\varphi_1 + (\alpha-1)C_2 t - 2\theta')u_{xx} + \omega_1 v_{xx},$$
$$\zeta_{11}^2 = \eta^2_{(0)xx} + \omega_2'' u + \varphi_2'' v + 2\omega_2' u_x + (2\varphi_2' - \theta'')v_x + \omega_2 u_{xx} + (\varphi_2 + (\alpha-1)C_2 t - 2\theta')v_{xx}.$$

The solution of this system with $\sigma = \alpha/(1-\alpha)$ *is given by*

$$\theta = \frac{\alpha(1-\alpha)}{2}C_1 x + \alpha C_4 x + C_3, \quad \varphi_1 = (\alpha-1)C_2 t + C_4, \quad \varphi_2 = (\alpha-1)C_2 t + C_4,$$
$$\omega_1 = C_5, \quad \omega_2 = -C_5, \quad \eta^1_{(0)} = C_6 t^{\alpha-1}, \quad \eta^2_{(0)} = C_7 t^{\alpha-1}$$

with seven arbitrary constants C_i*. Hence, the system (39) has seven linearly independent Lie point symmetries*

$$X_1 = t\frac{\partial}{\partial t} + \frac{\alpha(1-\alpha)}{2}x\frac{\partial}{\partial x}, \quad X_2 = t^2\frac{\partial}{\partial t} + (\alpha-1)tu\frac{\partial}{\partial u} + (\alpha-1)tv\frac{\partial}{\partial v}, \quad X_3 = \frac{\partial}{\partial x},$$
$$X_4 = \alpha x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad X_5 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \quad X_6 = t^{\alpha-1}\frac{\partial}{\partial u}, \quad X_7 = t^{\alpha-1}\frac{\partial}{\partial v}.$$

Example 3. *As an example of the system involving both the Riemann–Liouville and the Caputo fractional derivatives, we consider a fractional generalization of one-dimensional nonlinear system of coupled thermoelastic equations with temperature-dependent material properties:*

$$\begin{aligned} {}^{RL}_{\ 0}D^{\alpha}_t \vartheta &= (\vartheta^{\beta}\vartheta_x)_x - \vartheta^{\sigma}u_{tx}, \\ {}^{C}_{0}D^{\alpha+1}_t u &= u_{xx} - \vartheta^{\sigma}\vartheta_x, \end{aligned} \tag{40}$$

with $\alpha \in (0,1)$ *and* $\beta, \sigma \neq 0$*. In (40)* u *is the displacement and* ϑ *is the temperature difference.*

In view of Theorem 3 one can conclude that any symmetry of this system should have the form

$$X = \psi(x)t\frac{\partial}{\partial t} + \theta(x)\frac{\partial}{\partial x} + (\eta^1_{(0)}(t,x) + \varphi_1(x)\vartheta)\frac{\partial}{\partial \vartheta} + (\eta^2_0(t,x) + \varphi_2(x)u)\frac{\partial}{\partial u}.$$

The corresponding system of determining equations can be written in the form

$$\begin{aligned} {}^{RL}_{\ 0}D^{\alpha}_t(\eta^1_{(0)}) + [\varphi_1 - \alpha\psi]\Big[(\vartheta^{\beta}\vartheta_x)_x - \vartheta^{\sigma}u_{tx}\Big] - \vartheta^{\beta}\zeta^1_{11} - 2\beta\vartheta^{\beta-1}\vartheta_x\zeta^1_1 + \vartheta^{\sigma}\zeta^2_{01} \\ - \Big(\eta^1_{(0)}(t,x) + \varphi_1(x)\vartheta\Big)\Big[\beta(\beta-1)\vartheta^{\beta-2}\vartheta_x^2 + \beta\vartheta^{\beta-1}\vartheta_{xx} - \sigma\vartheta^{\sigma-1}u_{tx}\Big] = 0, \end{aligned}$$

$$\begin{aligned} {}^{C}_{0}D^{\alpha+1}_t(\eta^2_{(0)}) + [\varphi_2 - (\alpha+1)\psi][u_{xx} - \vartheta^{\sigma}\vartheta_x] - \zeta^2_{11} \\ + \Big(\eta^1_{(0)}(t,x) + \varphi_1(x)\vartheta\Big)\sigma\vartheta^{\sigma-1}\vartheta_x + \vartheta^{\sigma}\zeta^1_1 = 0, \end{aligned}$$

where

$$\zeta_1^1 = \eta^1_{(0)x} + \varphi_1'\vartheta + (\varphi_1 - \theta')\vartheta_x - t\psi'\vartheta_t,$$
$$\zeta_{11}^1 = \eta^1_{(0)xx} + \varphi_1''\vartheta + (2\varphi_1' - \theta'')\vartheta_x + (\varphi_1 - 2\theta')\vartheta_{xx} - t\psi''\vartheta_t - 2t\psi'\vartheta_{tx},$$
$$\zeta_{01}^2 = \eta^2_{(0)tx} + (\varphi_2' - \psi')u_t + (\varphi_2 - \psi - \theta')u_{tx} - t\psi' u_{tt},$$
$$\zeta_{11}^2 = \eta^2_{(0)xx} + \varphi_2'' u + (2\varphi_2' - \theta'')u_x + (\varphi_2 - 2\theta')u_{xx} - t\psi'' u_t - 2t\psi' u_{tx}.$$

The solution of this system with respect to the functions $\psi(x)$, $\theta(x)$, $\varphi_1(x)$, $\varphi_2(x)$, $\eta^1_{(0)}(t,x)$ and $\eta^2_{(0)}(t,x)$ leads to the following result. The system (40) with arbitrary $\sigma \neq 0$ and $\beta \neq 0$ has four linearly independent infinitesimal symmetries

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial u}, \quad X_3 = t\frac{\partial}{\partial u}, \quad X_4 = x\frac{\partial}{\partial u},$$

and in the specific case of $\sigma = (1-\alpha)\beta/2$ there is an additional symmetry

$$X_5 = t\frac{\partial}{\partial t} + \frac{1+\alpha}{2}x\frac{\partial}{\partial x} + \frac{1}{\beta}\vartheta\frac{\partial}{\partial \vartheta} + \left(1+\frac{1}{\beta}\right)u\frac{\partial}{\partial u}.$$

6. Conclusions

The presented theorems significantly facilitate the finding of Lie point symmetry groups for certain classes of multi-dimensional FDEs with the Riemann–Liouville and Caputo fractional derivatives, as well as for systems of such equations. The obtained simplified determining equations can be solved using well-known algorithms of classical Lie group analysis of integer-order differential equations. Moreover, numerous computer algebra packages can be used for this purpose. It is sufficiently obvious that more wide classes of FDEs can admit only linearly autonomous Lie point symmetry groups. Finding such classes is an important problem of modern Lie group analysis.

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