

Article

Main Curvatures Identities on Lightlike Hypersurfaces of Statistical Manifolds and Their Characterizations

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Abstract: In this study, some identities involving the Riemannian curvature invariants are presented on lightlike hypersurfaces of a statistical manifold in the Lorentzian settings. Several inequalities characterizing lightlike hypersurfaces are obtained. These inequalities are also investigated on lightlike hypersurfaces of Lorentzian statistical space forms.

Keywords: lightlike hypersurface; statistical manifold; dual connections; chen-like inequalities; curvatures

MSC: 53C15; 53C25; 53C40



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1. Introduction

The concept of statistical manifolds is currently a new and attractive topic in differential geometry. It has many application areas such as neural networks, machine learning, artificial intelligence, and black holes [1–4]. A statistical structure on a Riemannian manifold was initially defined by S. Amari [5] in 1985, as follows:

A Riemannian manifold $(\tilde{M}, \tilde{g})$ with a Riemannian metric $\tilde{g}$ and the Levi–Civita connection $\tilde{\nabla}^0$ is called a statistical manifold if there exists a pair of torsion-free affine connections $(\tilde{\nabla}, \tilde{\nabla}^*)$, such that the following relation is satisfied for any tangent vector fields X, Y and Z on $\tilde{M}$

$$\tilde{g}(X, \tilde{\nabla}_Z^* Y) = Z\tilde{g}(X, Y) - \tilde{g}(\tilde{\nabla}_Z X, Y), \quad (1)$$

where

$$\tilde{\nabla}^0 = \frac{1}{2}(\tilde{\nabla} + \tilde{\nabla}^*). \quad (2)$$

There exist qualified papers dealing with statistical manifolds and their submanifolds admitting various differentiable structures. In this sense, the geometry of hypersurfaces of statistical manifolds was presented by H. Furuhashi in Refs. [6,7]. Statistical manifolds admitting contact structures or complex structures and their submanifolds were investigated in Refs. [8,9]. Statistical manifolds involving Riemannian submersions were also studied in Refs. [10,11]. In degenerate cases, lightlike hypersurfaces of statistical manifolds were introduced by the first author and M. M. Tripathi [12]. In addition, statistical lightlike hypersurfaces were studied by V. Jain, A. P. Singh, and R. Kumar in Ref. [13].

Besides these facts, many sharp inequalities on submanifolds in various ambient spaces are obtained from isometric immersions. The main inequalities on submanifolds of real space forms were initially established by B.-Y. Chen [14–17]. Later, these inequalities

have drawn the attention of many authors due to their interesting characterizations for the theory of submanifolds. Moreover, some inequalities involving curvature-like tensors were studied by M. M. Tripathi [18] and Chen-like inequalities and their characterizations were presented in submanifolds of statistical manifolds in Refs. [19–22].

In this paper, first of all, by examining the commonly known important submanifold types such as totally geodesic, totally umbilical, and minimal lightlike hypersurfaces with respect to the Levi–Civita connection, some relations are obtained. Then, various results are found by computing the curvature tensor fields such as the statistical sectional curvature, statistical screen Ricci curvature, and statistical screen scalar curvature. Finally, with the help of these curvature relations, various inequalities are established for hypersurfaces of statistical manifolds. The equality cases are also discussed.

2. Preliminaries

Let $(\tilde{M}, \tilde{g})$ be an $(m + 2)$ -dimensional Lorentzian manifold and (M, g) be a hypersurface of $(\tilde{M}, \tilde{g})$ with the induced metric g from $\tilde{g}$. If g is degenerate, then M is called a lightlike (null or degenerate) hypersurface. For a lightlike hypersurface (M, g) of $(\tilde{M}, \tilde{g})$, there exists a non-zero vector field ξ on M such that

$$g(\xi, X) = 0, \quad (3)$$

for all $X \in \Gamma(TM)$. Here the vector field ξ is called a null vector [23–25]. The radical or the null space $\text{Rad } T_x M$ at each point $x \in M$ is defined as

$$\text{Rad } T_x M = \{\xi \in T_x M : g_x(\xi, X) = 0, \forall X \in \Gamma(TM)\}. \quad (4)$$

The dimension of $\text{Rad } T_x M$ is called the nullity degree of g . We recall that the nullity degree of g for a lightlike hypersurface is equal to 1. Since g is degenerate and any null vector being orthogonal to itself, the normal space $T_x M^\perp$ is a null subspace. In addition, we have

$$\text{Rad } T_x M = T_x M^\perp. \quad (5)$$

The complementary vector bundle $S(TM)$ of $\text{Rad } TM$ in TM is called the screen bundle of M . We note that any screen bundle is non-degenerate. Therefore, we can write the following decomposition:

$$TM = \text{Rad } TM \perp S(TM). \quad (6)$$

Here $\perp$ denotes the orthogonal-direct sum. The complementary vector bundle $S(TM)^\perp$ of $S(TM)$ in $\tilde{M}$ is called the screen transversal bundle. Since $\text{Rad } TM$ is a lightlike subbundle of $S(TM)^\perp$, there exists a unique local section N of $S(TM)^\perp$ such that we have

$$\tilde{g}(N, N) = 0, \quad \tilde{g}(\xi, N) = 1. \quad (7)$$

Note that N is transversal to M and $\{\xi, N\}$ is a local frame field of $S(TM)^\perp$. Thus, there exists a line subbundle $\text{ltr}(TM)$ of $\tilde{M}$. This set is called the lightlike transversal bundle, locally spanned by N [23,24].

Let $\tilde{\nabla}^0$ be the Levi–Civita connection of $\tilde{M}$. The Gauss and Weingarten formulas are given

$$\tilde{\nabla}_X^0 Y = \nabla_X^0 Y + B^0(X, Y)N, \quad (8)$$

$$\tilde{\nabla}_X^0 N = -A_N^0 X + \tau^0(X)N \quad (9)$$

for any $X, Y \in \Gamma(TM)$. Here ∇^0 is the induced linear connection on TM , B^0 is the second fundamental form on TM , A_N^0 is the shape operator on TM , and τ^0 is a 1-form on TM .

A lightlike hypersurface M is called totally geodesic if $B^0 = 0$. If there exists a $\lambda \in \mathbb{R}$ at every point of M such that

$$B^0(X, Y) = \lambda g(X, Y),$$

for all $X, Y \in \Gamma(TM)$ then M is called totally umbilical [23].

Let $\{E_1, \dots, E_m, \xi\}$ be a local orthonormal basis of $\Gamma(TM)$, where $\{E_1, \dots, E_m\}$ is a local orthonormal basis of $\Gamma(S(TM))$. Then M is called minimal if [23]

$$\frac{1}{m} \text{trace}(B^0) = \frac{1}{m} \sum_{i=1}^m B^0(E_i, E_i) = 0. \quad (10)$$

3. Lightlike Hypersurfaces of a Statistical Manifold

Let $(\tilde{M}, \tilde{g})$ be a Lorentzian manifold. If there exists a torsion free connection $\tilde{\nabla}$ satisfying the following:

$$(\tilde{\nabla}_X \tilde{g})(Y, Z) = (\tilde{\nabla}_Y \tilde{g})(X, Z) \quad (11)$$

for all $X, Y, Z \in \Gamma(T\tilde{M})$ then $(\tilde{g}, \tilde{\nabla})$ is called a statistical structure. So, $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ is a statistical manifold [6].

For a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$, the dual of $\tilde{\nabla}$, denoted by $\tilde{\nabla}^*$, is defined by the following identity:

$$\tilde{g}(X, \tilde{\nabla}_Z^* Y) = Z\tilde{g}(X, Y) - \tilde{g}(\tilde{\nabla}_Z X, Y). \quad (12)$$

It is easy to check that $\tilde{\nabla}^*$ is torsion free. If $\tilde{\nabla}^0$ is the Levi-Civita connection of $\tilde{g}$, then we can write

$$\tilde{\nabla}^0 = \frac{1}{2}(\tilde{\nabla} + \tilde{\nabla}^*). \quad (13)$$

Let (M, g) be a lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. From (6) and (7), the Gauss and Weingarten formulas with respect to dual connections can be expressed

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad (14)$$

$$\tilde{\nabla}_X N = -A_N^* X + \tau^*(X)N \quad (15)$$

and

$$\begin{aligned} \tilde{\nabla}_X^* Y &= \nabla_X^* Y + B^*(X, Y)N, \\ \tilde{\nabla}_X^* N &= -A_N X + \tau(X)N \end{aligned} \quad (16)$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(\text{ltr } TM)$, where $\nabla_X Y$, $\nabla_X^* Y$, $A_N X$, $A_N^* X \in \Gamma(TM)$. Here, ∇ , ∇^* are called the induced connections on M , B and B^* are called the second fundamental forms, A_N and A_N^* are called the Weingarten mappings with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively. Using (7) in (14) and (15), we have

$$B(X, Y) = \tilde{g}(\tilde{\nabla}_X Y, \xi), \quad \tau^*(X) = \tilde{g}(\tilde{\nabla}_X N, \xi) \quad (17)$$

and

$$B^*(X, Y) = \tilde{g}(\tilde{\nabla}_X^* Y, \xi), \quad \tau(X) = \tilde{g}(\tilde{\nabla}_X^* N, \xi). \quad (18)$$

From the Gauss and Weingarten formulas, it is clear that both the induced connections ∇ and ∇^* are symmetric. In addition, both the second fundamental forms B and B^* are symmetric and bilinear, called the imbedding curvature tensors of the submanifold for $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively. We note that a lightlike hypersurface of a statistical manifold does not need to be a statistical manifold with respect to ∇ and ∇^* [12].

Let P denotes the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (6). For any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad } TM)$ we can write

$$\nabla_X PY = \bar{\nabla}_X PY + \bar{h}(X, PY), \quad (19)$$

$$\nabla_X \xi = -\bar{A}_\xi^t X + \bar{\nabla}_X^t \xi. \quad (20)$$

Here, $\bar{\nabla}_X PY$ and $\bar{A}_\xi X$ belong to $\Gamma(S(TM))$, $\bar{\nabla}$ and $\bar{\nabla}^t$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(\text{Rad } TM)$, respectively. The tensor fields $\bar{h}$ and $\bar{A}$ are called the screen second fundamental form and the screen shape operator of $S(TM)$ respectively. If we define

$$C(X, PY) = g(\bar{h}(X, PY), N), \quad (21)$$

$$\varepsilon(X) = g(\bar{\nabla}_X^t \xi, N), \quad X, Y \in \Gamma(TM), \quad (22)$$

then we can easily prove that

$$\varepsilon(X) = -\tau(X). \quad (23)$$

Therefore, we can write from (19), (20) and (23) that

$$\nabla_X PY = \bar{\nabla}_X PY + C(X, PY)\xi, \quad (24)$$

$$\nabla_X \xi = -\bar{A}_\xi X - \tau(X)\xi \quad (25)$$

for all $X, Y \in \Gamma(TM)$. Here $C(X, PY)$ is called the local screen fundamental form of $S(TM)$.

Similarly, the relations of induced dual objects on $S(TM)$ with respect to $\bar{\nabla}$ are given by

$$\nabla_X^* PY = \bar{\nabla}_X^* PY + C^*(X, PY)\xi, \quad (26)$$

$$\nabla_X^* \xi = -\bar{A}_\xi^* X - \tau^*(X)\xi, \quad X, Y \in \Gamma(TM). \quad (27)$$

Using (25), (27) and Gauss–Weingarten formulas, the relationship between induced geometric objects are given by

$$B(X, \xi) + B^*(X, \xi) = 0, \quad g(A_N X + A_N^* X, N) = 0, \quad (28)$$

$$C(X, PY) = g(A_N X, PY), \quad C^*(X, PY) = g(A_N^* X, PY). \quad (29)$$

As a result of (28), we obtain that the second fundamental forms B and B^* are not degenerate. Additionally, due to $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are dual connections, we obtain

$$B(X, Y) = g(\bar{A}_\xi^* X, Y) + B^*(X, \xi)\tilde{g}(Y, N), \quad (30)$$

$$B^*(X, Y) = g(\bar{A}_\xi X, Y) + B(X, \xi)\tilde{g}(Y, N). \quad (31)$$

Putting $X = \xi$ in (30), (31) and using the fact that $\bar{A}_\xi, \bar{A}_\xi^*$ are $S(TM)$ -valued tensor fields, we get

$$\bar{A}_\xi^* \xi + \bar{A}_\xi \xi = 0. \quad (32)$$

Definition 1. A hypersurface M is a screen locally conformal lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ if there exist non-vanishing smooth functions φ and φ^* on M such that

$$A_N = \varphi \bar{A}_\xi^*, \quad A_N^* = \varphi^* \bar{A}_\xi. \quad (33)$$

In particular, M is called screen homothetic if φ and φ^* are non-zero constants.

Definition 2. Any hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ is called [26,27]:

1. totally geodesic with respect to $\tilde{\nabla}$ if $B = 0$;
2. totally geodesic with respect to $\tilde{\nabla}^*$ if $B^* = 0$;
3. totally tangentially umbilical with respect to $\tilde{\nabla}$ if there exists a smooth function k such that $B(X, Y) = kg(X, Y)$ for all $X, Y \in \Gamma(TM)$;
4. totally tangentially umbilical with respect to $\tilde{\nabla}^*$ if there exists a smooth function k^* such that $B^*(X, Y) = k^*g(X, Y)$, for any $X, Y \in \Gamma(TM)$;

5. totally normally umbilical with respect to $\tilde{\nabla}$ if there exists a smooth function k such that $A_N^* X = kX$ for any $X, Y \in \Gamma(TM)$;
6. totally normally umbilical with respect to $\tilde{\nabla}^*$ if there exists a smooth function k^* such that $A_N X = k^* X$ for all $X, Y \in \Gamma(TM)$, where k^* is smooth function.

From (6) and (7), we can consider a local quasi-orthonormal field $\{E_1, \dots, E_m, \xi, N\}$ of frames of $\tilde{M}$ along M where $\{E_1, \dots, E_m\}$ is an orthonormal basis of $\Gamma(S(TM))$. Then, the mean curvature H with respect to $\tilde{\nabla}$ is defined by

$$H = \frac{1}{m} \sum_{i=1}^m B(E_i, E_i). \quad (34)$$

The hypersurface M is called minimal with respect to $\tilde{\nabla}$ if $H = 0$ at every point of M . Now, we will give some examples of lightlike hypersurfaces of statistical manifolds:

Example 1. Let $(\mathbb{R}_1^4, \tilde{g})$ be a 4-dimensional semi-Euclidean space with signature $(-, +, +, +)$ of the canonical basis $(\partial_1, \dots, \partial_4)$. Consider a hypersurface M in $(\mathbb{R}_1^4, \tilde{g})$ defined by

$$\{(u, v + w, u, v - w) : u, v, w \in \mathbb{R}\}.$$

Then it is easy to check that M is a lightlike hypersurface such that

$$\begin{aligned} \text{Rad}(TM) &= \text{Span}\{\xi = \partial_1 + \partial_3\}, \quad \text{ltr}(TM) = \text{Span}\{N = -\frac{1}{2}(\partial_1 - \partial_3)\}, \\ S(TM) &= \text{Span}\{W_1 = \partial_2 + \partial_4, W_2 = \partial_2 - \partial_4\}. \end{aligned}$$

By direct calculations we obtain M as a totally geodesic lightlike hypersurface with respect to the Levi-Civita connection.

Now let us define an affine connection $\tilde{\nabla}$ as follows:

$$\tilde{\nabla}_{W_1} W_1 = \tilde{\nabla}_{W_2} W_2 = \xi \quad \text{and} \quad \tilde{\nabla}_{W_1} W_2 = \tilde{\nabla}_{W_2} W_1 = 0.$$

Using (13) we obtain

$$\tilde{\nabla}_{W_1}^* W_1 = \tilde{\nabla}_{W_2}^* W_2 = -\xi \quad \text{and} \quad \tilde{\nabla}_{W_1}^* W_2 = \tilde{\nabla}_{W_2}^* W_1 = 0.$$

Using (4), (12) and the above calculations, one can choose

$$\tilde{\nabla}_X \xi = \tilde{\nabla}_\xi X = \tilde{\nabla}_X^* \xi = \tilde{\nabla}_\xi^* X = 0 \quad \text{and} \quad \tilde{\nabla}_X N = \tilde{\nabla}_N X = \tilde{\nabla}_X^* N = \tilde{\nabla}_N^* X = 0,$$

for any $X \in \Gamma(TM)$.

Therefore $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are dual connections. Here one can easily see that these connections are torsion free and $\tilde{\nabla} \tilde{g} \neq 0$. Hence, from the definition statistical manifold, we see that $(\mathbb{R}_1^4, \tilde{g}, \tilde{\nabla}, \tilde{\nabla}^*)$ is a statistical manifold and M is totally geodesic with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$.

Example 2. Let us consider a lightlike M in R_1^4 defined by $\{(t, t \cos u, t \sin u, w) : t, w \in \mathbb{R}, u \in [0, 2\pi]\}$. Then we have

$$\begin{aligned} \text{Rad}(TM) &= \text{Span}\{\xi = \partial_1 + \cos u \partial_2 + \sin u \partial_3\}, \\ \text{ltr}(TM) &= \text{Span}\{N = \frac{1}{2}(-\partial_1 + \cos u \partial_2 + \sin u \partial_3)\}, \\ S(TM) &= \text{Span}\{W_1 = -t \sin u \partial_2 + t \cos u \partial_3, W_2 = \partial_4\}. \end{aligned}$$

By straightforward computation we have

$$\begin{aligned}\tilde{\nabla}_{W_1}^0 W_1 &= -t \cos u \partial_2 - t \sin u \partial_3, \\ \tilde{\nabla}_{W_1}^0 N &= -\frac{1}{2} \sin u \partial_2 + \frac{1}{2} \cos u \partial_3, \\ \tilde{\nabla}_{\xi}^0 W_1 &= \tilde{\nabla}_{W_1}^0 \xi = -\sin u \partial_2 + \cos u \partial_3, \\ \tilde{\nabla}_{W_1}^0 W_2 &= \tilde{\nabla}_{W_2}^0 W_1 = \tilde{\nabla}_{W_2}^0 W_2 = \tilde{\nabla}_{W_2}^0 N = \tilde{\nabla}_{W_2}^0 \xi = 0.\end{aligned}$$

Let us define an affine connection $\tilde{\nabla}$ as follows:

$$\begin{aligned}\tilde{\nabla}_{W_1} W_1 &= -2t \cos u \partial_2 - 2t \sin u \partial_3, \\ \tilde{\nabla}_{W_2} W_2 &= \frac{2}{t} \partial_1, \quad \tilde{\nabla}_{W_1} W_2 = \tilde{\nabla}_{W_2} W_1 = 0, \\ \tilde{\nabla}_{W_1} \xi &= -2 \sin u \partial_2 + 2 \cos u \partial_3, \quad \tilde{\nabla}_{W_2} \xi = 0.\end{aligned}$$

Then we obtain

$$\begin{aligned}\tilde{\nabla}_{W_1}^* W_1 &= 0, \quad \tilde{\nabla}_{W_2}^* W_2 = -\frac{2}{t} \partial_1, \\ \tilde{\nabla}_{W_1}^* W_2 &= \tilde{\nabla}_{W_2}^* W_1 = 0, \\ \tilde{\nabla}_{W_1}^* \xi &= \tilde{\nabla}_{W_2}^* \xi = 0.\end{aligned}$$

Therefore, $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are dual connections and $(\mathbb{R}_1^4, \tilde{g}, \tilde{\nabla}, \tilde{\nabla}^*)$ is a statistical Lorentzian manifold. It is clear that

$$B(X, Y) = -\frac{2}{t} g(X, Y),$$

for any $X, Y \in \Gamma(TM)$. Thus, we say that M is totally tangentially umbilical with respect to $\tilde{\nabla}$ and $k = -\frac{2}{t}$.

Further examples such as totally normally umbilical, minimal, and screen conformal lightlike hypersurfaces of a statistical manifold could be derived.

The following results are well known for lightlike hypersurfaces of statistical manifolds.

Proposition 1 ([12]). Let (M, g) be a lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then the following assertions are equivalent:

1. M is totally geodesic with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$.
2. $\bar{A}_{\xi} X = \bar{A}_{\xi}^* X = 0$ for all $X \in \Gamma(TM)$.
3. $\nabla_X g + \nabla_X^* g = 0$ for all $X \in \Gamma(TM)$.
4. $\nabla_X \xi + \nabla_X^* \xi \in \Gamma(\text{Rad } TM)$ for all $X \in \Gamma(TM)$.

Theorem 1 ([12]). Let (M, g) be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then, M is totally tangentially umbilical with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$ if and only if there exists a smooth function ρ such that the following equation satisfies

$$\bar{A}_{\xi}^* X + \bar{A}_{\xi} X = \rho X \quad (35)$$

for all $X \in \Gamma(TM)$.

Proposition 2 ([12]). Let (M, g) be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. If M is totally normally umbilical with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$. Then we have

$$C(X, PY) + C^*(X, PY) = 0 \quad (36)$$

for all $X, Y \in \Gamma(TM)$.

Proposition 3 ([12]). Let (M, g) be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then, M is screen locally conformal if and only if

$$C(X, Y) + C^*(X, Y) = \rho(B(X, Y) + B^*(X, Y)), \quad X, Y \in \Gamma(S(TM)),$$

where ρ is non-vanishing smooth functions on M .

Theorem 2. Let (M, g) be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then M is totally geodesic with respect to the Levi-Civita connection $\tilde{\nabla}^0$ if and only if $B = -B^*$.

Theorem 3. Let (M, g) be a screen homothetic lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then M is totally normally umbilical if and only if M is totally geodesic with respect to the Levi-Civita connection $\tilde{\nabla}^0$.

Proof. Suppose that M is totally normally umbilical. From Proposition 2 we have $C = -C^*$. If we consider this fact and M is screen homothetic in Theorem 2, we see that M is totally geodesic with respect to the Levi-Civita connection $\tilde{\nabla}^0$. The converse part of proof could be given similarly. $\square$

Theorem 4. Let (M, g) be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. If M is totally umbilical with respect to $\tilde{\nabla}^0$ then the following relation satisfies:

$$\overline{A}_{\xi}^* X = -\overline{A}_{\xi} X \quad (37)$$

for any $X \in \Gamma(S(TM))$.

Proof. From (8) and (13), we can write $B^0 = \frac{1}{2}(B + B^*)$. Using the fact that M is totally umbilical with respect to $\tilde{\nabla}^0$, there exists a smooth function λ such that $B^0(X, Y) = \lambda g(X, Y)$ for any $X, Y \in \Gamma(TM)$. Now if we choose X and Y as orthonormal, then we have

$$B(X, X) = 2\lambda - B^*(X, X), \quad \text{and} \quad B(X, Y) = -B^*(X, Y). \quad (38)$$

From (30), (31) and (38), we obtain

$$B(X, Y) = g(\overline{A}_{\xi}^* X, Y) + B(X, \xi)$$

and

$$B^*(X, Y) = g(\overline{A}_{\xi} X, Y) - B(X, \xi)$$

which imply (37). $\square$

Corollary 1. Let (M, g) be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. If M is totally umbilical with respect to $\tilde{\nabla}^0$ then M can not be totally tangentially umbilical with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$.

Proof. Assume that M is totally tangentially umbilical with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$. From Theorems 1 and 4, we get $\rho = 0$, which is a contradiction. Thus, M can not be totally tangentially umbilical with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$. $\square$

Corollary 2. Let (M, g) be totally umbilical with respect to $\tilde{\nabla}^0$. Then we have the following statements:

1. M is totally tangentially umbilical with respect to $\tilde{\nabla}$ if and only if $B^* = 0$;
2. M is totally tangentially umbilical with respect to $\tilde{\nabla}^*$ if and only if $B = 0$.

Corollary 3. Let (M, g) be a screen conformal lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. If M is totally umbilical with respect to $\tilde{\nabla}^0$, then the following relation holds:

$$A_N = -\frac{\varphi}{\varphi^*} A_N^*. \quad (39)$$

Proposition 4. Let (M, g) be minimal with respect to the Levi-Civita connection $\tilde{\nabla}^0$ and the set $\{E_1, \dots, E_m\}$ be an orthonormal basis of $\Gamma(S(TM))$. Then we have the following relation:

$$\text{trace } \overline{A}_\xi^* + \text{trace } \overline{A}_\xi + \sum_{i=1}^n (B(E_i, \xi) + B^*(E_i, \xi)) = 0. \quad (40)$$

Proof. If (M, g) is minimal with respect to $\tilde{\nabla}^0$, then we have

$$\sum_{i=1}^n B^0(E_i, E_i) = \frac{1}{2} \sum_{i=1}^n (B(E_i, E_i) + B^*(E_i, E_i)) = 0,$$

which shows that

$$\sum_{i=1}^n (B(E_i, E_i) + B^*(E_i, E_i)) = 0. \quad (41)$$

The proof is straightforward from (30), (31) and (41). $\square$

4. Main Curvature Relations

Let (M, g) be a lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Denote the curvature tensors with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$ by $\tilde{R}$ and $\tilde{R}^*$, respectively. Using the Gauss and Weingarten formulas for $\tilde{\nabla}$ and $\tilde{\nabla}^*$, we obtain

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z - B(Y, Z)A_N^*X + B(X, Z)A_N^*Y \\ &\quad + (B(Y, Z)\tau^*(X) - B(X, Z)\tau^*(Y))N \\ &\quad + ((\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z))N, \end{aligned} \quad (42)$$

and

$$\begin{aligned} \tilde{R}^*(X, Y)Z &= R^*(X, Y)Z - B^*(Y, Z)A_N X + B^*(X, Z)A_N Y \\ &\quad + (B^*(Y, Z)\tau(X) - B^*(X, Z)\tau(Y))N \\ &\quad + ((\nabla_X^* B^*)(Y, Z) - (\nabla_Y^* B^*)(X, Z))N, \end{aligned} \quad (43)$$

where R and R^* are the curvature tensor with respect to ∇ and ∇^* , respectively.

The statistical manifold $(\tilde{M}, \tilde{g})$ is called of constant curvature c if the following relation satisfies for any $X, Y, Z \in \Gamma(TM)$ [13]

$$\tilde{R}(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}. \quad (44)$$

Let $\Pi = \text{span}\{X, Y\}$ be a two dimensional non-degenerate plane of $T_x M$ at $x \in M$. The statistical sectional curvature of Π with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$ is defined respectively by Ref. [13]

$$\kappa(\Pi) \equiv \kappa(X, Y) = \frac{g(R(X, Y)Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2} \quad (45)$$

and

$$\kappa^*(\Pi) \equiv \kappa^*(X, Y) = \frac{g(R^*(X, Y)Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2}. \quad (46)$$

Suppose that ξ is a null vector of $T_x M$. A plane Π of $T_x M$ is a null plane if it contains ξ and X such that $g(\xi, X) = 0$ and $g(X, X) \neq 0$. Then, the statistical null sectional curvatures are given respectively by

$$\kappa^{null}(\Pi) \equiv \kappa^{null}(X, \xi) = \frac{g(R(X, \xi)\xi, X)}{g(X, X)} \quad (47)$$

and

$$\kappa^{*null}(\Pi) \equiv \kappa^{*null}(X, \xi) = \frac{g(R^*(X, \xi)\xi, X)}{g(X, X)}. \quad (48)$$

From the above equations and the Gauss–Weingarten formulas for M and $S(TM)$, one can obtain the following proposition:

Proposition 5. Let (M, g) be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then we have the following equalities for any $X, Y, Z, W \in \Gamma(TM)$:

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, PW) &= g(R(X, Y)Z, PW) - B(Y, Z)C^*(X, PW) \\ &\quad + B(X, Z)C^*(Y, PW), \end{aligned} \quad (49)$$

$$\begin{aligned} \tilde{g}(\tilde{R}^*(X, Y)Z, PW) &= g(R^*(X, Y)Z, PW) - B^*(Y, Z)C(X, PW) \\ &\quad + B^*(X, Z)C(Y, PW), \end{aligned} \quad (50)$$

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, \xi) &= B(Y, Z)\tau^*(X) - B(X, Z)\tau^*(Y) \\ &\quad + (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z), \end{aligned} \quad (51)$$

$$\begin{aligned} \tilde{g}(\tilde{R}^*(X, Y)Z, \xi) &= B^*(Y, Z)\tau(X) - B^*(X, Z)\tau(Y) \\ &\quad + (\nabla_X^* B^*)(Y, Z) - (\nabla_Y^* B^*)(X, Z), \end{aligned} \quad (52)$$

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, N) &= g(R(X, Y)Z, N) - B(Y, Z)g(A_N^* X, N) \\ &\quad + B(X, Z)g(A_N^* Y, N), \end{aligned} \quad (53)$$

$$\begin{aligned} \tilde{g}(\tilde{R}^*(X, Y)Z, N) &= g(R^*(X, Y)Z, N) - B^*(Y, Z)g(A_N X, N) \\ &\quad + B^*(X, Z)g(A_N Y, N), \end{aligned} \quad (54)$$

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)\xi, N) &= g(R(X, Y)\xi, N) - B(Y, \xi)g(A_N^* X, N) \\ &\quad + B(X, \xi)g(A_N^* Y, N), \end{aligned} \quad (55)$$

$$\begin{aligned} \tilde{g}(\tilde{R}^*(X, Y)\xi, N) &= g(R^*(X, Y)\xi, N) - B^*(Y, \xi)g(A_N X, N) \\ &\quad + B^*(X, \xi)g(A_N Y, N), \end{aligned} \quad (56)$$

where

$$\tilde{g}(\tilde{R}(X, Y)\xi, N) = C(Y, \bar{A}_\xi^* X) - C(X, \bar{A}_\xi^* Y) - 2d\tau(X, Y) \quad (57)$$

and

$$g(R^*(X, Y)\xi, N) = C^*(Y, \bar{A}_\xi^* X) - C^*(X, \bar{A}_\xi^* Y) - 2d\tau(X, Y). \quad (58)$$

Proposition 6. Let (M, g) be a lightlike hypersurface of a statistical Lorentzian space form $\tilde{M}(c)$. Then we have

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = B(Y, Z)\tau^*(X) - B(X, Z)\tau^*(Y) \quad (59)$$

for any $X, Y, Z \in \Gamma(TM)$

Proof. From (44) it follows that

$$\tilde{g}(\tilde{R}(X, Y)Z, \xi) = 0. \quad (60)$$

We obtain the claim of proposition by using the above relation in (51). $\square$

With similar arguments as in the proof of Lemma 1.8 in Ref. [28], we have the following lemma for the semi-Riemannian case:

Lemma 1. For any statistical manifold $\tilde{M}$, the following identities hold for any tangent vector fields X, Y, Z on $T\tilde{M}$.

1. $\tilde{R}(X, Y)Z = -\tilde{R}(Y, X)Z$ and $\tilde{R}^*(X, Y)Z = -\tilde{R}^*(Y, X)Z$.
2. $\tilde{g}(\tilde{R}(X, Y)Z, W) = -\tilde{g}(\tilde{R}^*(X, Y)W, Z)$.
3. $\tilde{R}(X, Y)Z + \tilde{R}(Y, Z)X + \tilde{R}(Z, X)Y = 0$.

We note that if $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ is of a constant curvature c with respect to $\tilde{\nabla}$, then it is also of a constant curvature c with respect to $\tilde{\nabla}^*$.

Proposition 7. Let (M, g) be a lightlike hypersurface of $\tilde{M}(c)$. Then we have for any unit vector $X \in \Gamma(S(TM))$ that

$$\kappa^{*null}(X, \xi) = B(X, \xi)C(\xi, X). \quad (61)$$

Here κ^{*null} denotes the null sectional curvature with respect to ∇^* .

Proof. Using (60) and the (ii) statement of Lemma 1, we get $\tilde{g}(\tilde{R}^*(X, Y)\xi, Z) = 0$. Using this fact in (50), we obtain

$$g(R^*(X, Y)\xi, Z) = B^*(Y, \xi)C(X, Z) - B^*(X, \xi)C(Y, Z). \quad (62)$$

Putting $Y = \xi$ and $X = Z$ in (62), it follows that

$$\kappa^{*null}(X, \xi) = B^*(\xi, \xi)C(X, X) - B^*(X, \xi)C(\xi, X). \quad (63)$$

From (28) and (63) we write

$$\kappa^{*null}(X, \xi) = B(X, \xi)C(\xi, X) - B(\xi, \xi)C(X, X). \quad (64)$$

If we put $X = Y = \xi$ in (30) and (31), we see that $B(\xi, \xi) = 0$. Hence we obtain the Equation (61) from (64). $\square$

Corollary 4. Let (M, g) be a screen homothetic lightlike hypersurface of $\tilde{M}(c)$. Then κ^{*null} and φ have the same signs.

Proof. Since (M, g) be a screen homothetic lightlike hypersurface, we get from Proposition 7 that

$$\kappa^{*null}(X, \xi) = \varphi[B(X, \xi)]^2 \quad (65)$$

for any unit vector $X \in \Gamma(S(TM))$. This identity shows that the sign of κ^{*null} and φ have the same signs. $\square$

Following the terminology used in Ref. [29], it is said to be any two vector field V and W are conjugate if $B(V, W) = 0$ [23].

Corollary 5. Let (M, g) be a screen homothetic lightlike hypersurface of $\tilde{M}(c)$. Then the null sectional curvature $\kappa^{*null}(X, \xi)$ vanishes for any $X \in \Gamma(S(TM))$ if and only if X and ξ are conjugate.

Now we recall the following result of V. Jain, A. P. Singh and R. Kumar (cf. Theorem 3.7 in Ref. [13]) as follows:

Theorem 5. Let (M, g) be a statistical lightlike hypersurface of an indefinite statistical manifold. Then the following statements are equivalent:

1. $\text{Rad } TM$ is a Killing distribution;
2. $\text{Rad } TM$ is a parallel distribution with respect to ∇ ;
3. $\bar{A}_\xi$ vanishes on $\Gamma(TM)$ for any $\xi \in \text{Rad } TM$.

From Proposition 7 and Theorem 5, we get the following result:

Corollary 6. Let (M, g) be a screen homothetic statistical lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla}, \tilde{\nabla}^*)$. The null sectional curvature κ^{*null} vanishes if and only if ξ is a Killing vector field.

Proposition 8. Let (M, g) be a lightlike hypersurface of $\tilde{M}(c)$. If M is totally geodesic with respect to $\tilde{\nabla}^0$ then we have

$$g(R(X, Y)Z, W) + g(R^*(X, Y)W, Z) = B(Y, Z)C^*(X, W) - B(X, Z)C^*(Y, W) - B(Y, W)C(X, Z) + B(X, W)C(Y, Z) \quad (66)$$

for any $X, Y, Z, W \in \Gamma(S(TM))$.

Proof. Using Theorem 2 and the statement ii) of Lemma 1 in (49) and (50), the proof of proposition is straightforward. $\square$

Corollary 7. Let (M, g) be a screen conformal lightlike hypersurface of $\tilde{M}(c)$. If M is totally geodesic with respect to $\tilde{\nabla}^0$ then

$$\kappa(X, Y) = \kappa^*(X, Y) \quad (67)$$

for any two linearly independent vector fields $X, Y \in \Gamma(S(TM))$.

Proof. From the definition of the Riemannian curvature tensor we write

$$R^*(X, Y)W = -R^*(Y, X)W \quad (68)$$

for any $X, Y, W \in \Gamma(S(TM))$. Hence, putting $Z = Y$ and $X = W$ in (66) and by choosing X and Y are unit vector field in $\Gamma(S(TM))$, we derive

$$\begin{aligned} \kappa(X, Y) - \kappa^*(X, Y) &= B(Y, Y)C^*(X, X) - B(X, Y)C^*(Y, X) \\ &\quad - B(Y, X)C(X, Y) + B(X, X)C(Y, Y). \end{aligned} \quad (69)$$

Since M is totally geodesic we have $C^* = -C$ and M is screen homothetic, we can write $C = \varphi B$. Using these facts in (69), we get

$$\kappa(X, Y) - \kappa^*(X, Y) = \varphi B(X, X)B(Y, Y) - \varphi B(Y, Y)B(X, X) = 0,$$

which is the claim of corollary. $\square$

Proposition 9. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. If M is totally geodesic with respect to $\tilde{\nabla}^0$ then we have

$$\tilde{g}(\tilde{R}^*(X, Y)\xi, N) = g(R^*(X, Y)\xi, N) - g(R(X, Y)\xi, N) \quad (70)$$

for any $X, Y \in \Gamma(S(TM))$.

Proof. Since $\tilde{M}$ is a Lorentzian space form, we have from (44) that $\tilde{g}(\tilde{R}(X, Y)\xi, N) = 0$ for any $X, Y \in \Gamma(S(TM))$. Considering this fact in (55), we derive

$$g(R(X, Y)\xi, N) = B(Y, \xi)g(A_N^*X, N) - B(X, \xi)g(A_N^*Y, N).$$

Since M is totally geodesic, we have $B = B^*$ and hence, from (28), we obtain $A_N^* = -A_N$. Therefore, we write

$$g(R(X, Y)\xi, N) = -B(Y, \xi)g(A_N X, N) + B(X, \xi)g(A_N Y, N). \quad (71)$$

Considering the right hand sides of (56) and (71), we obtain (70). $\square$

Corollary 8. Let (M, g) be a screen conformal lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. If M is totally geodesic with respect to $\tilde{\nabla}^0$ then we have

$$\tilde{g}(\tilde{R}^*(X, Y)\xi, N) = 0 \quad (72)$$

for any $X, Y \in \Gamma(S(TM))$.

Proof. By using (57) and (58) in (70), the proof is straightforward. $\square$

Proposition 10. Let (M, g) be a totally umbilical with respect to $\tilde{\nabla}^0$. If M is screen conformal then for any orthonormal vector pair (X, Y) in $\Gamma(S(TM))$, we have

$$\tilde{\kappa}(X, Y) = \kappa(X, Y) + \varphi^* \left(-2\lambda B(Y, Y) + B(X, X)B(Y, Y) - [B(X, Y)]^2 \right) \quad (73)$$

and

$$\tilde{\kappa}^*(X, Y) = \kappa^*(X, Y) + \varphi \left(-2\lambda B(X, X) + B(X, X)B(Y, Y) - [B(X, Y)]^2 \right). \quad (74)$$

Proof. Using the fact that M is screen conformal in (49), we derive

$$\tilde{\kappa}(X, Y) = \kappa(X, Y) - \varphi^* B(Y, Y)B^*(X, X) + \varphi^* B(X, Y)B^*(Y, X)$$

for any orthonormal vector pair (X, Y) in $\Gamma(S(TM))$. Since M is totally umbilical with respect to $\tilde{\nabla}^0$, we derive from (38) that

$$\tilde{\kappa}(X, Y) = \kappa(X, Y) - \varphi^* B(Y, Y)((2\lambda - B(X, X)) - \varphi^*[B(X, Y)]^2,$$

which implies (73). The proof of (74) could be shown in a similar way. $\square$

5. Some Main Inequalities

Let (M, g) be a $(m+1)$ -dimensional lightlike statistical hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla}, \tilde{\nabla}^*)$. We assume that $\{E_1, \dots, E_m, \xi\}$ is the basis of $\Gamma(TM)$, where $\{E_1, \dots, E_m\}$ is an orthonormal basis of $\Gamma(S(TM))$. We set $\pi_k = \text{Span}\{E_1, \dots, E_k\}$ as a k -dimensional non-degenerate sub-plane section. We denote the sectional curvature of the plane section spanned by E_i and E_j for any $i \neq j \in \{1, \dots, m\}$ by κ_{ij} .

For $k = m$, $\pi_m = \text{Span}\{E_1, \dots, E_m\} = \Gamma(S(TM))$. In this case, the statistical screen Ricci curvature is given by

$$\text{Ric}_{S(TM)}(E_1) = \sum_{j=2}^m \kappa_{1j} = \kappa_{12} + \dots + \kappa_{1m}$$

and the statistical screen scalar curvature is given by

$$\sigma_{S(TM)}(p) = \sum_{i,j=1}^m \kappa_{ij}.$$

From (49), one can easily derive the following relation:

$$\begin{aligned}\sigma_{S(TM)}(p) &= \tilde{\sigma}_{S(TM)}(p) + \sum_{i,j=1}^m [B_{ii}C_{jj}^* - B_{ij}C_{ji}^*] \\ &= \tilde{\sigma}_{S(TM)}(p) + \sum_{i,j=1}^m B_{ii}C_{jj}^* - \frac{1}{2} \sum_{i,j=1}^m (B_{ij} + C_{ji}^*)^2 \\ &\quad + \frac{1}{2} \sum_{i,j=1}^m [(B_{ij})^2 + (C_{ij}^*)^2],\end{aligned}\quad (75)$$

where $B_{ij} = B(E_i, E_j)$, $C_{ij}^* = C^*(E_i, E_j)$, and

$$\tilde{\sigma}_{S(TM)}(p) = \sum_{i,j=1}^m \tilde{\kappa}_{ij}.$$

We can rewrite the last term of (75) as Ref. [30]

$$\sum_{i,j=1}^m [(B_{ij})^2 + (C_{ij}^*)^2] = \frac{1}{2} \left[\sum_{i,j=1}^m (B_{ij} + C_{ij}^*)^2 + \sum_{i,j=1}^m (B_{ij} - C_{ij}^*)^2 \right]. \quad (76)$$

Then, (75) and (76) give us

$$\begin{aligned}\sigma_{S(TM)}(p) &= \tilde{\sigma}_{S(TM)}(p) + \sum_{i,j=1}^m [B_{ii}C_{jj}^* - B_{ij}C_{ji}^*] \\ &= \tilde{\sigma}_{S(TM)}(p) + \sum_{i,j=1}^m B_{ii}C_{jj}^* - \frac{1}{2} \sum_{i,j=1}^m (B_{ij} + C_{ji}^*)^2 \\ &\quad + \frac{1}{4} \left[\sum_{i,j=1}^m (B_{ij} + C_{ij}^*)^2 + \sum_{i,j=1}^m (B_{ij} - C_{ij}^*)^2 \right] \\ &= \tilde{\sigma}_{S(TM)}(p) + \sum_{i,j=1}^m B_{ii}C_{jj}^* - \frac{1}{4} \sum_{i,j=1}^m (B_{ij} + C_{ji}^*)^2 \\ &\quad + \frac{1}{4} \sum_{i,j=1}^m (B_{ij} - C_{ij}^*)^2.\end{aligned}\quad (77)$$

Hence, we obtain

$$\begin{aligned}\sigma_{S(TM)}(p) &= \tilde{\sigma}_{S(TM)}(p) + mH(\text{trace}(A_N^*)) - \frac{1}{4} \sum_{i,j=1}^m (B_{ij} + C_{ji}^*)^2 \\ &\quad + \frac{1}{4} \sum_{i,j=1}^m (B_{ij} - C_{ij}^*)^2.\end{aligned}\quad (78)$$

Thus, we have the following results:

Theorem 6. Let (M, g) be a $(m+1)$ -dimensional lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then

$$\sigma_{S(TM)}(p) \leq \tilde{\sigma}_{S(TM)}(p) + mH(\text{trace}(A_N^*)) + \frac{1}{4} \sum_{i,j=1}^m (B_{ij} - C_{ij}^*)^2. \quad (79)$$

The equality holds for all $p \in M$ if and only if either M is a screen homothetic with $\varphi^* = -1$ or M is a totally geodesic with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$.

Proof. From (78), we clearly have the inequality (79). The equality case of this inequality holds for all $p \in M$ if and only if we have

$$\sum_{i,j=1}^m (B_{ij} + C_{ji}^*) = 0,$$

which implies that $B_{ij} = -C_{ji}^*$ for all $i, j \in \{1, \dots, m\}$ or $B_{ij} = C_{ji}^* = 0$. These show that M is a screen homothetic with $\varphi^* = -1$ or M is a totally geodesic with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$. $\square$

Theorem 7. Let (M, g) be a $(m+1)$ -dimensional lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then

$$\sigma_{S(TM)}(p) \geq \tilde{\sigma}_{S(TM)}(p) + mH(\text{trace}(A_N^*)) - \frac{1}{4} \sum_{i,j=1}^m (B_{ij} + C_{ij}^*)^2. \quad (80)$$

The equality holds for all $p \in M$ if and only if either M is a screen homothetic with $\varphi^* = 1$ or M is a totally geodesic with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$.

Proof. From (78), the proof of (80) is straightforward. The equality case of this inequality holds for all $p \in M$ if and only if we have

$$\sum_{i,j=1}^m (B_{ij} - C_{ji}^*) = 0,$$

which implies that $B_{ij} = C_{ji}^*$ for all $i, j \in \{1, \dots, m\}$ or $B_{ij} = C_{ji}^* = 0$. These relations show that M is a screen homothetic with $\varphi^* = 1$ or M is a totally geodesic with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$. $\square$

Remark 1. Note that both the equality cases of (79) and (80) hold at $p \in M$ if and only if p is a totally geodesic point with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$.

Now we give the following lemma for later uses:

Lemma 2. For an $(m+1)$ -dimensional lightlike statistical hypersurface M of a Lorentzian statistical space form $\tilde{M}(c)$, we have

$$\begin{aligned} g(R(X, Y)X, PW) &= c\{g(Y, Z)g(X, PW) - g(X, Z)g(Y, PW)\} \\ &\quad + B(Y, Z)C^*(X, PW) - B(X, Z)C^*(Y, PW) \end{aligned} \quad (81)$$

for any $X, Y, X, W \in \Gamma(TM)$.

Proof. From (44) and (49), the proof of lemma is straightforward. $\square$

Considering Lemma 2 in Theorem 6 and Theorem 7, we get the following results:

Corollary 9. Let M be an $(m+1)$ -dimensional lightlike hypersurface of $\tilde{M}(c)$. Then

$$\sigma_{S(TM)}(p) \leq m(m-1)c + mH(\text{trace}(A_N^*)) + \frac{1}{4} \sum_{i,j=1}^m (B_{ij} - C_{ij}^*)^2. \quad (82)$$

The equality holds for all $p \in M$ if and only if either M is a screen homothetic with $\varphi^* = -1$ or M is a totally geodesic with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$.

Corollary 10. Let M be an $(m + 1)$ -dimensional lightlike hypersurface of $\tilde{M}(c)$. Then

$$\sigma_{S(TM)}(p) \geq m(m-1)c + mH(\text{trace}(A_N^*)) - \frac{1}{4} \sum_{i,j=1}^m (B_{ij} + C_{ij}^*)^2. \quad (83)$$

The equality holds for all $p \in M$ if and only if either M is a screen homothetic with $\varphi^* = 1$ or M is a totally geodesic with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$.

For the next inequalities, we rewrite the second term of (77) as Ref. [30]

$$\sum_{i,j=1}^m B_{ii}C_{jj}^* = \frac{1}{2} \left\{ \left[\sum_{i,j=1}^m (B_{ii} + C_{jj}^*) \right]^2 - \left(\sum_{i=1}^m B_{ii} \right)^2 - \left(\sum_{j=1}^m C_{jj}^* \right)^2 \right\}. \quad (84)$$

On combining (77) and (84), we derive the following results:

Theorem 8. Let (M, g) be an $(m + 1)$ -dimensional lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then

$$\begin{aligned} \sigma_{S(TM)}(p) &\leq \tilde{\sigma}_{S(TM)}(p) + \frac{1}{2} [(\text{trace}(\tilde{A}))^2 - (\text{trace}(A_N^*))^2] \\ &\quad - \frac{1}{4} \sum_{i,j=1}^m (B_{ij} + C_{ji}^*)^2 + \frac{1}{4} \sum_{i,j=1}^m (B_{ij} - C_{ij}^*)^2, \end{aligned}$$

where

$$\tilde{A} = \begin{pmatrix} B_{11} + C_{11}^* & \cdots & B_{1m} + C_{m1}^* \\ B_{21} + C_{12}^* & \cdots & B_{2m} + C_{m2}^* \\ \vdots & \vdots & \vdots \\ B_{m1} + C_{1m}^* & \cdots & B_{mm} + C_{mm}^* \end{pmatrix}. \quad (85)$$

The equality holds for every point $p \in M$ if and only if M is minimal.

Corollary 11. Let M be an $(m + 1)$ -dimensional lightlike hypersurface of $\tilde{M}(c)$. Then

$$\begin{aligned} \sigma_{S(TM)}(p) &\leq m(m-1)c + \frac{1}{2} [(\text{trace}(\tilde{A}))^2 - (\text{trace}(A_N^*))^2] \\ &\quad - \frac{1}{4} \sum_{i,j=1}^m (B_{ij} + C_{ji}^*)^2 + \frac{1}{4} \sum_{i,j=1}^m (B_{ij} - C_{ij}^*)^2, \end{aligned}$$

where $\tilde{A}$ is given by (85). The equality holds for every point $p \in M$ if and only if M is minimal.

Finally, we give some inequalities on totally umbilical lightlike hypersurfaces with respect to the their Levi-Civita connections:

Theorem 9. Let (M, g) be an $(m + 1)$ -dimensional screen homothetic lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Suppose that (M, g) is totally umbilical with respect to $\tilde{\nabla}^0$. For any unit vector field X in $\Gamma(S(TM))$, we have

$$\text{Ric}_{S(TM)}(X) \geq \tilde{\text{Ric}}_{S(TM)}(X) + \varphi^* mH(2\lambda - B(X, X)) - 2\lambda \varphi^* B(X, X). \quad (86)$$

The equality case of (86) holds for all $X \in \Gamma(S(TM))$ if and only if M is totally umbilical with respect to $\tilde{\nabla}^*$.

Proof. From Proposition 10, we have

$$\tilde{\kappa}(E_1, E_j) = \kappa(E_i, E_j) + \varphi^* (-2\lambda B_{jj} + B_{11}B_{jj} - [B_{1j}]^2). \quad (87)$$

If we take trace in (87) then we get

$$\sum_{j=2}^m \tilde{\kappa}(E_1, E_j) = \sum_{j=2}^m \kappa(E_i, E_j) + \varphi^* \sum_{j=2}^m \left(-2\lambda B_{jj} + B_{11} B_{jj} - [B_{1j}]^2 \right).$$

Therefore, we obtain

$$\begin{aligned} \widetilde{\text{Ric}}_{S(TM)}(E_1) &= \text{Ric}_{S(TM)}(E_1) - 2\lambda \varphi^* \left(\sum_{j=1}^m B_{jj} - B_{11} \right) \\ &\quad + \varphi^* B_{11} \left(\sum_{j=1}^m B_{jj} - B_{11} \right) - \varphi^* \sum_{j=2}^m [B_{1j}]^2. \end{aligned} \quad (88)$$

Putting $X = E_1$, the proof of (86) is straightforward from (34) and (88).

The equality case of (89) holds for all unit vector fields X in $\Gamma(S(TM))$ if and only if $B(X, E_j) = 0$ for all $j \in \{1, \dots, m\}$. Thus, using the fact that B is bilinear we get $B(X, Y) = 0$ for any $X, Y \in \Gamma(S(TM))$. From the (ii) statement of Corollary 2 we get M as totally umbilical with respect to $\tilde{\nabla}^*$. $\square$

With a similar arguments as Theorem 9, we obtain the following theorem:

Theorem 10. Let (M, g) be an $(m+1)$ -dimensional screen homothetic lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Suppose that (M, g) is totally umbilical with respect to $\tilde{\nabla}^0$. For any unit vector field X in $\Gamma(S(TM))$, we have

$$\text{Ric}_{S(TM)}^*(X) \geq \widetilde{\text{Ric}}_{S(TM)}^*(X) + \varphi B(X, X)[2(m-2)\lambda - B(X, X) - mH]. \quad (89)$$

The equality case of (89) holds for all $X \in \Gamma(S(TM))$ if and only if M is totally umbilical with respect to $\tilde{\nabla}^*$.

Theorem 11. Let (M, g) be an $(m+1)$ -dimensional screen homothetic lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Suppose that (M, g) is totally umbilical with respect to $\tilde{\nabla}^0$. For all $p \in M$ we have

$$\sigma_{S(TM)}(p) \geq \tilde{\sigma}_{S(TM)}(p) + \varphi^* mH(2\lambda m - 2\lambda - mH) \quad (90)$$

with the equality holds if and only if M is totally umbilical with respect to $\tilde{\nabla}^*$.

Proof. Taking trace in (88) we get

$$\sigma_{S(TM)}(p) = \tilde{\sigma}_{S(TM)}(p) + \varphi^* mH(2\lambda m - 2\lambda - mH) + \varphi^* \sum_{i,j=1}^n [B_{ij}]^2, \quad (91)$$

which implies (90). From (91), the equality case of (90) satisfies if and only if $B_{ij} = 0$ for all $i, j \in \{1, \dots, m\}$. From the (ii) statement of Corollary 2 we get M is totally umbilical with respect to $\tilde{\nabla}^*$. $\square$

6. Future Works

In the last section, we obtained Chen-like inequalities of a lightlike hypersurface of statistical manifolds and Lorentzian statistical space forms including screen scalar and mean curvatures. We also considered equality cases. Many similar results can be seen in Refs. [31–34]. We can use these results for future projects, and give some characterizations of a lightlike hypersurface on a statistical manifold. The results stated here motivate further studies to obtain similar relationships for many kinds of invariants of similar nature for several statistical submersions. In particular, by introducing the curvature invariant $\delta(m_1, \dots, m_k)$ on lightlike hypersurfaces of a statistical manifold, we can obtain similar relationships for lightlike hypersurfaces and the equality cases can be discussed.

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