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Sequential Completeness for $\top$-Quasi-Uniform Spaces and a Fixed Point Theorem

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Abstract: We define sequential completeness for $\top$-quasi-uniform spaces using Cauchy pair $\top$-sequences. We show that completeness implies sequential completeness and that for $\top$-uniform spaces with countable $\top$-uniform bases, completeness and sequential completeness are equivalent. As an illustration of the applicability of the concept, we give a fixed point theorem for certain contractive self-mappings in a $\top$-uniform space. This result yields, as a special case, a fixed point theorem for probabilistic metric spaces.

Keywords: $\top$-sequence; $\top$-quasi-uniform space; completeness; sequential completeness; fixed point theorem; probabilistic metric space

MSC: 54A40; 54E15; 54E70

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1. Introduction

Recently, Yue and Fang [1] studied completeness and completion for $\top$-quasi-uniform spaces with the help of pair $\top$-filters. Completeness and completion are important problems in topology and uniform spaces and their extensions to the lattice-valued case [2], such as $\top$-quasi-uniform spaces, are a natural setting for treating such questions. A further important application of completeness in uniform spaces are fixed point theorems for self-mappings, usually extending the famous Banach contraction principle [3].

A self-mapping $\varphi : X \longrightarrow X$ in a metric space (X, d) is called a contraction if there is a constant $\alpha \in [0, 1)$ such that $d(\varphi(x), \varphi(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. If the metric space is complete, then such a contraction has a unique fixed point $a = \varphi(a) \in X$ and this fixed point can be approximated by the sequence $(x, \varphi(x), \varphi(\varphi(x)), \ldots)$ for an arbitrary $x \in X$.

Applications of this principle are abundant in mathematics, for example, in the fields of differential or integral equations or in numerical analysis. For a recent textbook we refer to [4] where also applications in mathematics as well as to "real-world" problems are given. Further applications of fixed-point theorems outside mathematics can, for example, be found, among others, in the fields of psychology or biology [5,6].

In the realm of applications of Banach's contraction principle, it is sufficient to require the convergence of Cauchy sequences, that is, we need only require so-called sequential completeness of the space.

In this paper, we address the definition of sequential completeness of $\top$-quasi uniform spaces with the help of the recently introduced $\top$-sequences, [7]. $\top$-sequences are special instances of $\top$-nets and it was shown in [7] that $\top$-nets provide an alternative tool to $\top$-filters for studying convergence in lattice-valued topology. Our definition can be characterized in a similar way as completeness in [1], using pair $\top$-filters with countable $\top$-bases. On the one hand, this underlines the appropriateness of the definition based on $\top$-sequences. On the other hand, it shows that complete $\top$-quasi-uniform spaces are sequentially complete. Therefore, sequentially completing a $\top$-quasi-uniform space is trivial in the sense that any completion would do and the problem of constructing a completion was solved in [1]. We give, as an illustration of the applicability of the concept of sequential completeness, a fixed point theorem for $\top$-uniform spaces, generalizing a corresponding result of Taylor for uniform spaces [8]. The more general lattice-valued viewpoint of $\top$-uniform spaces encompasses probabilistic metric spaces and allows us to derive a fixed point theorem for probabilistic metric spaces from our result.

2. Preliminaries

In this paper, we will consider *commutative and integral quantales* $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice with distinct top and bottom elements $\top \neq \perp$, $(L, *)$ is a commutative semigroup with the top element of L as the unit, that is, $\alpha * \top = \alpha$ for all $\alpha \in L$, and $*$ is distributive over arbitrary joins, i.e., $(\bigvee_{i \in J} \alpha_i) * \beta = \bigvee_{i \in J} (\alpha_i * \beta)$ for all $\alpha_i, \beta \in L$, $i \in J$, see for example [9].

The *well-below relation* $\lhd$ in a complete lattice $(L, \leq)$ is defined by $\alpha \lhd \beta$, if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. This relation is sometimes called the *totally below relation*, see [9]. For more details and results on lattices, we refer to [10].

In a quantale, we can define an *implication* by $\alpha \to \beta = \bigvee \{\delta \in L \,:\, \delta * \alpha \leq \beta\}$. Then $\delta \leq \alpha \to \beta$ if and only if $\delta * \alpha \leq \beta$, i.e., $\to$ is the residuum in the quantale.

Sometimes we additionally require that the top element of L is approximable by a sequence, in the sense that there is a sequence $(\alpha_k)_{k \in \mathbb{N}}$ in L with the properties

(1) $\perp \neq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots$
(2) $\alpha_k \lhd \top$ for all $k \in \mathbb{N}$ and
(3) $\bigvee_{k \in \mathbb{N}} \alpha_k = \top$.

We call such a quantale $\top$-*approximable* and the sequence $(\alpha_k)_{k \in \mathbb{N}}$ a $\top$-*approximating sequence*. We note that for a $\top$-approximating sequence we have

$$\top = \bigvee_{k \in \mathbb{N}} \alpha_k * \bigvee_{l \in \mathbb{N}} \alpha_l \leq \bigvee_{k,l \in \mathbb{N}} (\alpha_{\max\{k,l\}} * \alpha_{\max\{k,l\}}) \leq \bigvee_{k \in \mathbb{N}} (\alpha_k * \alpha_k).$$

As a consequence, for all $B \subseteq L$, $\bigvee B = \top$ implies $\bigvee_{\beta \in B} \beta * \beta = \top$. For if $\bigvee B = \top \rhd \alpha_k$, there is $\beta_k \in B$ such that $\beta_k \geq \alpha_k$ and hence, $\bigvee_{\beta \in B} \beta * \beta \geq \bigvee_{k \in \mathbb{N}} \alpha_k * \alpha_k = \top$.

Typical examples are $\mathsf{L} = ([0,1], \leq, *)$ with a left-continuous t-norm on $[0,1]$ or *Lawvere's quantale* $\mathsf{L} = ([0,\infty], \geq, +)$. Another example is given by the *quantale of distance distribution functions* $\mathsf{L} = (\Delta^+, \leq, *)$, where Δ^+ is the set of all distance distribution functions $\varphi : [0,\infty] \longrightarrow [0,1]$ which are left-continuous in the sense that $\varphi(x) = \sup_{y<x} \varphi(y)$ for all $x \in [0,\infty]$ and $*$ is a *sup-continuous triangle function*, see [11,12]. It is shown in [11] that $(\Delta^+, \leq, *)$ is a commutative and integral quantale. To see that it is $\top$-approximable, we consider, for $0 \leq \delta \leq \infty$ and $0 < \epsilon \leq 1$, the distance distribution functions $\varphi_{\delta,\epsilon}$ defined by

$$\varphi_{\delta,\epsilon}(x) = \begin{cases} 0 & 0 \leq x \leq \delta \\ \epsilon & \delta < x \leq \infty \end{cases}$$

Then $\varphi_{0,1}$ is the top-element in Δ^+. We consider the sequence $\alpha_k = \frac{1}{k}$, $k = 1,2,3,\ldots$ and define $\varphi_k = \varphi_{\alpha_k, 1-\alpha_k}$. It is not difficult to see that the sequence $\varphi_1, \varphi_2, \ldots$ is a $\top$-approximating sequence.

An *L-set in X*, or, more precise, an *L-subset of X*, is a mapping $a : X \longrightarrow L$ and we denote the set of L-sets in X by L^X. We denote a constant L-set with value $\alpha \in L$ also by α. For $A \subseteq X$ we write $\top_A$ for the L-set on X defined by $\top_A(x) = \top$ if $x \in A$ and $= \perp$ otherwise. For $a \in L^X$, $b \in L^Y$ and a mapping $\varphi : X \longrightarrow Y$ we define $\varphi(a) \in L^Y$ by $\varphi(a)(y) = \bigvee_{\varphi(x)=y} a(x)$ for $y \in Y$, and $\varphi^{\leftarrow}(b) = b \circ \varphi \in L^X$. The lattice operations are extended pointwisely from L to L^X.

For L-sets u, v in $X \times X$ we define $u^{-1} \in L^{X \times X}$ by $u^{-1}(x,y) = u(y,x)$ for all $x, y \in X$ and $u \circ v \in L^{X \times X}$ by $u \circ v(x,y) = \bigvee_{z \in X} u(x,z) * u(z,y)$ for all $x, y \in X$.

For $a, b \in L^X$ we denote $[a, b] = \bigwedge_{x \in X}(a(x) \to b(x))$. The relation $[\cdot,\cdot] : L^X \times L^X \longrightarrow L$ is sometimes called the *fuzzy inclusion order* [13]. We collect some of the properties that we will need later.

Lemma 1. *Let $a, a', b, b', c \in L^X$, $d \in L^Y$, $u_1, u_2, v_1, v_2 \in L^{X\times X}$, and $\varphi : X \longrightarrow Y$ be a mapping. Then*

(i) $a \le b$ *if and only if* $[a, b] = \top$;
(ii) $a \le a'$ *implies* $[a', b] \le [a, b]$ *and* $b \le b'$ *implies* $[a, b] \le [a, b']$;
(iii) $[a, c] \wedge [b, c] = [a \vee b, c]$;
(iv) $[\varphi(a), d] = [a, \varphi^{\leftarrow}(d)]$;
(v) $[u_1, v_1] * [u_2, v_2] \le [u_1 \circ u_2, v_1 \circ v_2]$.

Proof. We only prove (v), see also [14] for $* = \wedge$. We have

$$
\begin{aligned}
[u_1, v_1] * [u_2, v_2] &\le \bigwedge_{x,y,s,t\in X} ((u_1(x,y) \to v_1(x,y)) * (u_2(s,t) \to v_2(s,t))) \\
&\le \bigwedge_{x,y,s,t\in X} ((u_1(x,y) * u_2(s,t)) \to (v_1(x,y) * v_2(s,t))) \\
&\le \bigwedge_{x,y,t\in X} ((u_1(x,y) * u_2(y,t)) \to (v_1(x,y) * v_2(y,t))) \\
&\le \bigwedge_{x,y,t\in X} ((u_1(x,y) * u_2(y,t)) \to (v_1 \circ v_2(x,t))) \\
&\le \bigwedge_{x,t\in X} \left((\bigvee_{y\in X} u_1(x,y) * u_2(y,t)) \to (v_1 \circ v_2(x,t)) \right) \\
&\le \bigwedge_{x,t\in X} ((u_1 \circ u_2(x,t)) \to (v_1 \circ v_2(x,t))) \;=\; [u_1 \circ u_2, v_1 \circ v_2].
\end{aligned}
$$

□

Definition 1 ([1,15])**.** *A subset $\mathbb{F} \subseteq L^X$ is called a* $\top$-filter *(on X) if*

(TF1) for all $b \in \mathbb{F}$, $\bigvee_{x\in X} b(x) = \top$;
(TF2) $a, b \in \mathbb{F}$ *implies* $a \wedge b \in \mathbb{F}$;
(TF3) $\bigvee_{b\in\mathbb{F}}[b, c] = \top$ *implies* $c \in \mathbb{F}$.

We denote the set of all $\top$-filters on X by $\mathsf{F}_{\mathsf{L}}^{\top}(X)$.

Example 1. *For $x \in X$, $[x] = \{a \in L^X : a(x) = \top\}$ is a $\top$-filter, the* point $\top$-filter of x. *More generally, for an L-set $a \in L^X$ with $a(x) = \top$ for some $x \in X$, then $[a] = \{b \in L^X : a \le b\}$ is a $\top$-filter.*

Definition 2 ([1,15])**.** *A subset $\mathbb{B} \subseteq L^X$ is called a* $\top$-filter base *(on X) if*

(TB1) for all $b \in \mathbb{B}$, $\bigvee_{x\in X} b(x) = \top$;
(TB2) $a, b \in \mathbb{B}$ *implies* $\bigvee_{c\in\mathbb{B}}[c, a \wedge b] = \top$.

For a $\top$-filter base $\mathbb{B}$, $[\mathbb{B}] = \{a \in L^X : \bigvee_{b\in\mathbb{B}}[b, a] = \top\}$ is the $\top$-filter generated by $\mathbb{B}$.

For a filter $\mathcal{F}$ on X, $\mathbb{B}_{\mathcal{F}} = \{\top_F : F \in \mathcal{F}\}$ is a $\top$-filter base and we have $f \in [\mathbb{B}_{\mathcal{F}}]$ if and only if $\bigvee_{F\in\mathcal{F}} \bigwedge_{x\in F} f(x) = \top$.

It is well-known that, for a $\top$-filter $\mathbb{F} \in \mathsf{F}_L^{\top}(X)$ and a mapping $\varphi : X \longrightarrow Y$, the set $\mathbb{B} = \{\varphi(a) : a \in \mathbb{F}\}$ is a $\top$-filter base on Y and we denote $\varphi(\mathbb{F})$ the generated $\top$-filter on Y, the *image of* $\mathbb{F}$ *under* φ, see [15].

For a $\top$-filter on Y and a mapping $\varphi : X \longrightarrow Y$ the set $\{\varphi^{\leftarrow}(b) : b \in \mathbb{F}\}$ is a $\top$-filter base if and only if $\bigvee_{y\in\varphi(X)} b(y) = \top$ for all $b \in \mathbb{F}$. In this case we denote the generated

$\top$-filter by $\varphi^{\leftarrow}(\mathbb{F})$ and call it the *preimage of* $\mathbb{F}$ *under* φ. If $M \subseteq X$, for the embedding mapping $i_M : M \longrightarrow X$, $i_M(x) = x$ for all $x \in M$, we denote for a $\top$-filter $\mathbb{F}$ on X, the preimage $i_M^{\leftarrow}(\mathbb{F}) = \mathbb{F}_M$. It is a $\top$-filter on M if and only if $\bigvee_{x\in M} b(x) = \top$ for all $b \in \mathbb{F}$ and in this case we have $\mathbb{F}_M = \{a|_M : a \in \mathbb{F}\}$ with the restrictions $a|_M = a \circ i_M$. Furthermore, we denote for $\mathbb{F} \in \mathbb{F}_L^{\top}(M)$, $i_M(\mathbb{F}) = [\mathbb{F}]$ and we have $[\mathbb{F}]_M = \mathbb{F}$. If $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$ and $\mathbb{F}_M$ exists, then $\mathbb{F} \leq [\mathbb{F}_M]$.

If $\mathbb{F} \in \mathbb{F}_L^{\top}(X), \mathbb{G} \in \mathbb{F}_L^{\top}(Y)$, then $\{f \otimes g : f \in \mathbb{F}, g \in \mathbb{G}\}$ is a $\top$-filter base. Here $f \otimes g(x,y) = f(x) * g(y)$ for all $(x,y) \in X \times Y$. We denote the generated $\top$-filter on $X \times Y$ by $\mathbb{F} \otimes \mathbb{G}$, see [16].

Proposition 1. *Let $M \subseteq X, N \subseteq Y$, and $\mathbb{F} \in \mathbb{F}_L^{\top}(M)$ and $\mathbb{G} \in \mathbb{F}_L^{\top}(N)$. Then $[\mathbb{F} \otimes \mathbb{G}] = [\mathbb{F}] \otimes [\mathbb{G}]$.*

Proof. We have with Proposition 3.11 in [16], $[\mathbb{F} \otimes \mathbb{G}] = (i_M \times i_N)(\mathbb{F} \otimes \mathbb{G}) = i_M(\mathbb{F}) \otimes i_N(\mathbb{G}) = [\mathbb{F}] \otimes [\mathbb{G}]$. $\square$

Proposition 2. *Let $M \subseteq X, N \subseteq Y$ and $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$ and $\mathbb{G} \in \mathbb{F}_L^{\top}(Y)$. If $\mathbb{F}_M$ and $\mathbb{G}_N$ exist then $(\mathbb{F} \otimes \mathbb{G})_{M\times N}$ exists and is $\leq \mathbb{F}_M \otimes \mathbb{G}_N$.*

Proof. We first note that for $f \in \mathbb{F}$ and $g \in \mathbb{G}$ we have $\bigvee_{(x,y)\in M\times N} f(x) * g(y) = \bigvee_{x\in M} f(x) * \bigvee_{y\in N} g(y) = \top * \top = \top$, because $\mathbb{F}_M, \mathbb{G}_N$ exist. Hence, we conclude for $d \in \mathbb{F} \otimes \mathbb{G}$,

$$\begin{aligned} \top &= \bigvee_{f\in\mathbb{F}, g\in\mathbb{G}} [f \otimes g, d] \leq \bigvee_{f\in\mathbb{F}, g\in\mathbb{G}} \bigwedge_{(x,y)\in M\times N} ((f(x) * g(y)) \to d(x,y)) \\ &\leq \bigvee_{f\in\mathbb{F}, g\in\mathbb{G}} \Big(\bigvee_{(x,y)\in M\times N} ((f(x) * g(y)) \to \Big(\bigvee_{(x,y)\in M\times N} d(x,y) \Big)\Big) = \bigvee_{(x,y)\in M\times N} d(x,y), \end{aligned}$$

and $(\mathbb{F} \otimes \mathbb{G})_{M\times N}$ exists. Let now $a \in (\mathbb{F} \otimes \mathbb{G})_{M\times N}$. Then there exists $d \in \mathbb{F} \otimes \mathbb{G}$ such that $a = d|_{M\times N}$. We conclude

$$\begin{aligned} \top &= \bigvee_{f\in\mathbb{F}, g\in\mathbb{G}} [f \otimes g, d] \\ &\leq \bigvee_{f\in\mathbb{F}, g\in\mathbb{G}} \bigwedge_{(x,y)\in M\times N} ((f(x) * g(y)) \to a(x,y)) \\ &= \bigvee_{f\in\mathbb{F}, g\in\mathbb{G}} [f|_M \otimes g|_N, a] \end{aligned}$$

and we have $a \in \mathbb{F}_M \otimes \mathbb{G}_N$. $\square$

For $\top$-filters $\mathcal{U}, \mathcal{V} \in \mathsf{F}_L^{\top}(X \times X)$ we define $\mathcal{U}^{-1} \in \mathsf{F}_L^{\top}(X \times X)$ as the $\top$-filter with $\top$-filter base $\{u^{-1} : u \in \mathcal{U}\}$ and $\mathcal{U} \circ \mathcal{V} \in \mathsf{F}_L^{\top}(X \times X)$ to be the $\top$-filter with $\top$-filter base $\{u \circ v : u \in \mathcal{U}, v \in \mathcal{V}\}$, provided $\bigvee_{x,y\in X} u \circ v(x,y) = \top$ for all $u \in \mathcal{U}, v \in \mathcal{V}$, see [17].

Definition 3 ([7])**.** *A mapping $s : \mathbb{N} \longrightarrow X \times L^*$, $n \longmapsto (s_X(n), s_L(n))$ is called a* $\top$-sequence *if $\bigvee_{k\geq n} s_L(k) = \top$ for all $n \in \mathbb{N}$. Here, $L^* = L \setminus \{\perp\}$.*

$\top$-sequences therefore consist of two ordinary sequences, the one $s_X : \mathbb{N} \longrightarrow X$ a sequence in X, and the other $s_L : \mathbb{N} \longrightarrow L$ a sequence in L. This latter sequence needs to satisfy the two conditions $s_L(n) \neq \perp$ for all $n \in \mathbb{N}$ and $\bigvee_{k\geq n} s_L(k) = \top$ for all $n \in \mathbb{N}$. This yields straightaway an abundance of examples. e.g., for a sequence $(x_n)_{n\in\mathbb{N}}$ in X and the constant sequence $\alpha_n = \top$ for all $n \in \mathbb{N}$, then $s : \mathbb{N} \longrightarrow X \times L^*$ defined by $s_X(n) = x_n$ and $s_L(n) = \top$ for all $n \in \mathbb{N}$ is a $\top$-sequence. Similarly, for a $\top$-approximating sequence $(\alpha_1, \alpha_2, \alpha_3, \dots)$, the definitions $s_X(n) = x_n$ and $s_L(n) = \alpha_n$ define a $\top$-sequence.

The concept of a $\top$-sequence is a special instance of the concept of a $\top$-net, see [7].

Proposition 3 ([7]). *For a a $\top$-sequence $s : \mathbb{N} \longrightarrow X \times L^*$, the L-sets $d_n = \bigvee_{k \geq n} s_L(k) \wedge \top_{s_X(k)}$ form a $\top$-base of a $\top$-filter $\mathbb{F}_s$ and we have*

$$a \in \mathbb{F}_s \iff \bigvee_{n \in \mathbb{N}} \bigwedge_{k \geq n} (s_L(k) \to a(s_X(k))) = \top.$$

For a $\top$-sequence $s : \mathbb{N} \longrightarrow X \times L^*$, $n \longmapsto (s_X(n), s_L(n))$ and a mapping $\varphi : X \longrightarrow Y$ we denote $\varphi(s) : \mathbb{N} \longrightarrow Y \times L^*$, $n \longmapsto (\varphi(s_X(n)), s_L(n))$. Then $\varphi(\mathbb{F}_s) = \mathbb{F}_{\varphi(s)}$, see [7]. For the special case, $M \subseteq X$ and $i_M : M \longrightarrow X$ and a $\top$-sequence $s : \mathbb{N} \longrightarrow M \times L^*$, we denote $[s] = i_M(s) : \mathbb{N} \longrightarrow X \times L^*$ and we have $[\mathbb{F}_s] = \mathbb{F}_{[s]}$.

If $s : \mathbb{N} \longrightarrow X \times L^*$ is a $\top$-sequence in X with $s_X(n) \in M$ for all $n \in \mathbb{N}$, then we can consider s as a $\top$-sequence in M in a natural way. We write $s_M : \mathbb{N} \longrightarrow M \times L^*$ for this $\top$-sequence in M, with the defnitions $(s_M)_M(n) = s_X(n)$ and $(s_M)_L(n) = s_L(n)$ for all $n \in \mathbb{N}$. It is not difficult to see that $[s_M] = s$ and, consequently, $[\mathbb{F}_{(s_M)}] = \mathbb{F}_s$. Moreover, we have $\mathbb{F}_{(s_M)} = (\mathbb{F}_s)_M$. To see this, let first $a \in (\mathbb{F}_s)_M$. Then there exists $b \in \mathbb{F}_s$ with $b|_M = a$. Hence, $\top = \bigvee_{n \in \mathbb{N}} \bigwedge_{k \geq n} (s_L(k) \to b(s_X(k))) = \bigvee_{n \in \mathbb{N}} \bigwedge_{k \geq n} ((s_M)_L(k) \to a((s_M)_M(k)))$, as $s_X(k) \in M$ for all $k \in \mathbb{N}$. Therefore, $a \in \mathbb{F}_{(s_M)}$. Conversely, if $a \in \mathbb{F}_{(s_M)}$, then we define $a^* \in L^X$ by $a^*(x) = a(x)$ if $x \in M$ and $a^*(x) = \top$ if $x \in X \setminus M$. It is not difficult to show that $a^* \in \mathbb{F}_s$ and hence, $a^*|_M = a \in (\mathbb{F}_s)_M$.

Proposition 4. *Let $M \subseteq X$ and $s, t : \mathbb{N} \longrightarrow X \times L^*$ be $\top$-sequences in X with values $s_X(n), t_X(n) \in M$ for all $n \in \mathbb{N}$. If $\mathcal{U}$ is $\top$-filter on $X \times X$ with $\mathcal{U} \leq \mathbb{F}_t \otimes \mathbb{F}_s$ and, if $\mathcal{U}_{M \times M}$ exists, then $\mathcal{U}_{M \times M} \leq \mathbb{F}_{(t_M)} \otimes \mathbb{F}_{(s_M)}$.*

Proof. Let $u = v|_{M \times M} \in \mathcal{U}_M$ with $v \in \mathcal{U}$. Then there are $a \in \mathbb{F}_t$ and $b \in \mathbb{F}_s$ such that $a \otimes b \leq v$ and we have $a|_M \in (\mathbb{F}_t)_M = \mathbb{F}_{(t_M)}$ and $b|M \in (\mathbb{F}_s)_M = \mathbb{F}_{(s_M)}$. Clearly, we have $a|_M \otimes b|_M = (a \otimes b)_{M \times M}$ and hence, $a|_M \otimes b|_M \leq v|_{M \times M} = u$. Consequently, $u \in \mathbb{F}_{(t_M)} \otimes \mathbb{F}_{(s_M)}$. $\square$

Proposition 5. *Let $M \subseteq X$ and $s, t : \mathbb{N} \longrightarrow M \times L^*$ be $\top$-sequences in M. If $\mathcal{U}$ is a $\top$-filter in $X \times X$ such that $\mathcal{U}_{M \times M}$ exists and $\mathcal{U}_{M \times M} \leq \mathbb{F}_t \otimes \mathbb{F}_s$, then $\mathcal{U} \leq \mathbb{F}_{[t]} \otimes \mathbb{F}_{[s]}$.*

Proof. Let $u \in \mathcal{U}$. Then $u|_{M \times M} \in \mathcal{U}_{M \times M}$ and hence, there exist $a \in \mathbb{F}_t, b \in \mathbb{F}_s$ such that $a \otimes b \leq u|_{M \times M}$. We define $a_* \in L^X$ by $a_*(x) = a(x)$ for $x \in M$ and $a_*(x) = \bot$ else. In the same way, b_* is defined. Then $a_*|_M = a$ and $b_*|_M = b$ and we have $a_* \in [\mathbb{F}_t] = \mathbb{F}_{[t]}$ and $b_* \in \mathbb{F}_{[s]}$. As $a_* \otimes b_* \leq u$ we see that $u \in \mathbb{F}_{[t]} \otimes \mathbb{F}_{[s]}$. $\square$

A $\top$-sequence $t : \mathbb{N} \longrightarrow X \times L^*$ is a $\top$-*subsequence* of the $\top$-sequence $s : \mathbb{N} \longrightarrow X \times L^*$ if there is a strictly increasing mapping $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ such that $s_X \circ \phi = t_X$ and $s_L \circ \phi \geq t_L$. The concept of a $\top$-subsequence is a special instance of the concept of a $\top$-subnet [7]. If t is a $\top$-subsequence of the $\top$-sequence s, then $\mathbb{F}_t \geq \mathbb{F}_s$, see [7].

3. $\top$-Quasi-Uniform Spaces

Definition 4 ([1,15]). *A pair $(X, \mathcal{U})$ with a $\top$-filter $\mathcal{U} \in \mathsf{F}^{\top}_{\mathsf{L}}(X \times X)$ satisfying*

(TU1) for all $x \in X$, $\mathcal{U} \leq [(x, x)]$;

(TU2) $\mathcal{U} \leq \mathcal{U} \circ \mathcal{U}$ is called a $\top$-quasi-uniform space. *$(X, \mathcal{U})$ is called a* $\top$-uniform space *if additionally the axiom;*

(TU3) $\mathcal{U} \leq \mathcal{U}^{-1}$ is satisfied. A mapping $\varphi : (X, \mathcal{U}) \longrightarrow (X', \mathcal{U}')$ between the $\top$-quasi-uniform spaces $(X, \mathcal{U}), (X', \mathcal{U}')$ is called uniformly continuous *if $(\varphi \times \varphi)(\mathcal{U}) \geq \mathcal{U}'$. The category with the $\top$-quasi-uniform spaces as objects and the uniformly continuous mappings as morphisms is denoted by $\top\!-\!\mathsf{QUnif}$.*

For a set X, we define $\mathcal{D} = \{u \in L^{X \times X} : u(x, x) = \top\}$. Then the pair $(X, \mathcal{D})$ is a $\top$-uniform space, the *discrete* $\top$-*uniform space*. If we define the L-set $\top_D \in L^{X \times X}$

by $\top_D(x,y) = \top$ if $x = y$ and $\top_D(x,y) = \bot$ if $x \neq y$, then $\mathcal{D} = [\top_D]$. If we define $\mathcal{I} = \{\top_{X\times X}\}$, then $(X,\mathcal{I})$ is a $\top$-uniform space, the *indiscrete* $\top$*-uniform space*.

An L-quasi-metric space (X,d), where $d : X \times X \longrightarrow L$ is an L-*quasi-metric* , i.e., it is *reflexive*, $d(x,x) = \top$ for all $x \in X$, and *transitive*, $d(x,y) * d(y,z) \leq d(x,z)$ for all $x,y,z \in X$, generates a "natural" $\top$-quasi-uniform space $(X,\mathcal{U}^d)$ with $\mathcal{U}^d = [d]$, see [17]. If the L-quasi-metric is *symmetric*, $d(x,y) = d(y,x)$ for all $x,y \in X$, then we speak of an L-metric space. For an L-metric space (X,d) the space $(X,[d])$ is a $\top$-uniform space.

Further examples arise from uniform spaces as we shall see after Proposition 8 and from probabilistic metric spaces, see Section 8.

Remark 1. *$\top$-quasi-uniform spaces are called probabilistic quasi-uniform spaces in [1], following the tradition of [18]. However, we would like to reserve the term "probabilistic" for the case where the quantale of distance distribution functions, Δ^+, is used. Then a probabilistic metric space is a Δ^+-metric space and has a "natural" underlying probabilistic uniform space, that is, a $\top$-uniform space defined by $\mathcal{U}^d = [d]$. Furthermore, the name is also in line with other names such as $\top$-quasi-uniform convergence spaces or $\top$-uniform limit spaces, [14,16,17]. Naturally, it would be even better to include the quantale in the name and to speak of $\top$-*L*-quasi-uniform spaces. However this seems to overload the nomenclature, and hence we refrain from it.*

The axioms of Definition 4 can be spelled out in the following form. For all $u \in \mathcal{U}$ and $x \in X$ we have (TU1) $u(x,x) = \top$; (TU2) $\bigvee_{v\in\mathcal{U}}[v \circ v, u] = \top$; (TU3) $u^{-1} \in \mathcal{U}$; and a mapping $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ is uniformly continuous iff $(\varphi \times \varphi)^{\leftarrow}(u') \in \mathcal{U}$ whenever $u' \in \mathcal{U}'$.

The category $\top-\mathsf{QUnif}$ allows initial constructions. We first need some preparations. We say that a subset $\mathbb{C} \subseteq L^X$ has the *finite intersection property* if for all finite subsets $\{c_1, c_2, \ldots, c_n\} \subseteq \mathbb{C}$ we have $\bigvee_{x\in X}(c_1 \wedge c_2 \wedge \cdots \wedge c_n)(x) = \top$. We call such a subset a $\top$*-filter subbase*.

Proposition 6. *For a $\top$-filter subbase $\mathbb{C} \subseteq L^X$, the set of all finite intersections of L-sets in $\mathbb{C}$, $[\mathbb{C}] = \{c_1 \wedge c_2 \wedge \cdots \wedge c_n \ : \ c_k \in \mathbb{C}\ \forall k = 1,2,\ldots,n; n \in \mathbb{N}\}$ is a $\top$-filter base.*

Proof. (TB1) is the finite intersection property. (TB2) is obvious as $b_1 \wedge b_2 \in [\mathbb{C}]$ for $b_1, b_2 \in [\mathbb{C}]$. $\square$

We say that $\mathbb{C}$ is a $\top$*-filter subbase* of the $\top$-filter base $\mathbb{B}$ if $[\mathbb{C}] = \mathbb{B}$.

Proposition 7. *Let $\mathbb{C} \subseteq L^X$ be a $\top$-filter subbase.*

(1) If for all $c \in \mathbb{C}$, $\bigvee_{d\in\mathbb{C}}[d \circ d, c] = \top$, then for all $e \in [\mathbb{C}]$, $\bigvee_{b\in[\mathbb{C}]}[b \circ b, e] = \top$.

(2) If for all $c \in \mathbb{C}$, $\bigvee_{d\in\mathbb{C}}[d, c^{-1}] = \top$, then for all $e \in [\mathbb{C}]$, $\bigvee_{b\in[\mathbb{C}]}[b, e^{-1}] = \top$.

Proof. (1) Let $e \in [\mathbb{C}]$. Then $e = c_1 \wedge c_2 \wedge \cdots \wedge c_n$ with $c_1, c_2, \ldots, c_n \in \mathbb{C}$. We then have

$$
\begin{aligned}
\top &= \bigvee_{d_1\in\mathbb{C}} [d_1 \circ d_1, c_1] * \bigvee_{d_2\in\mathbb{C}} [d_2 \circ d_2, c_2] * \cdots * \bigvee_{d_n\in\mathbb{C}} [d_n \circ d_n, c_n] \\
&\leq \bigvee_{d_1,d_2,\ldots,d_n\in\mathbb{C}} \bigvee_{d_1\in\mathbb{C}} [d_1 \circ d_1, c_1] \wedge \bigvee_{d_2\in\mathbb{C}} [d_2 \circ d_2, c_2] \wedge \cdots \wedge \bigvee_{d_n\in\mathbb{C}} [d_n \circ d_n, c_n] \\
&\leq \bigvee_{d_1,d_2,\ldots,d_n\in\mathbb{C}} [(d_1 \wedge d_2 \wedge \cdots \wedge d_n) \circ (d_1 \wedge d_2 \wedge \cdots \wedge d_n), c_1] \wedge \cdots \\
&\qquad \cdots \wedge [(d_1 \wedge d_2 \wedge \cdots \wedge d_n) \circ (d_1 \wedge d_2 \wedge \cdots \wedge d_n), c_n] \\
&\leq \bigvee_{d_1,d_2,\ldots,d_n\in\mathbb{C}} [(d_1 \wedge d_2 \wedge \cdots \wedge d_n) \circ (d_1 \wedge d_2 \wedge \cdots \wedge d_n), c_1 \wedge c_2 \wedge \cdots \wedge c_n] \\
&\leq \bigwedge_{b\in[\mathbb{C}]} [b \circ b, e].
\end{aligned}
$$

(2) $e = c_1 \wedge c_2 \wedge \cdots \wedge c_n$ implies $e^{-1} = c_1^{-1} \wedge c_2^{-1} \wedge \cdots \wedge c_n^{-1}$ and the proof is similar to (1). $\square$

Proposition 8. *For a $\top$-filter base $\mathbb{B}$ on $X \times X$ with the properties*

(TBU1) for all $x \in X$ and all $b \in \mathbb{B}$, $b(x,x) = \top$;
(TBU2) for all $c \in \mathbb{B}$, $\bigvee_{b \in \mathbb{B}} [b \circ b, c] = \top$, the generated $\top$-filter $[\mathbb{B}]$ is a $\top$-quasi-uniformity on X. If additionally
(TBU3) for all $c \in \mathbb{B}$, $\bigvee_{b \in \mathbb{B}} [b, c^{-1}] = \top$ is valid, then the generated $\top$-filter $[\mathbb{B}]$ is a $\top$-uniformity on X.

Proof. (TU1) We have for $u \in [\mathbb{B}]$ by (TBU1) $\top = \bigvee_{b \in \mathbb{B}} [b, u] \leq \bigvee_{b \in \mathbb{B}} \bigwedge_{z \in X} (b(z,z) \to u(z,z)) \leq u(x,x)$ for all $x \in X$.

(TU2) For $u \in [\mathbb{B}]$ we have, using (TBU2), $\top = \bigvee_{b \in \mathbb{B}} [b, u] = \bigvee_{b \in \mathbb{B}} \bigvee_{d \in \mathbb{B}} [d \circ d, b] * [b, u] \leq \bigvee_{d \in \mathbb{B}} [d \circ d, u] \leq \bigvee_{v \in [\mathbb{B}]} [v \circ v, u]$.

(TU3) For $u \in \mathbb{U}$ we conclude with (TBU3), $\top = \bigvee_{b \in \mathbb{B}} [b, u] = \bigvee_{b \in \mathbb{B}} [b^{-1}, u^{-1}] = \bigvee_{b \in \mathbb{B}} \bigvee_{d \in \mathbb{B}} [d, b^{-1}] * [b^{-1}, u^{-1}] \leq \bigvee_{d \in \mathbb{B}} [d, u^{-1}]$ and hence, $u^{-1} \in [\mathbb{B}]$. $\square$

We call a $\top$-filter base satisfying (TB1) and (TB2) a $\top$-*quasi-uniform base* and a $\top$-filter base satisfying (TB1), (TB2) and (TB3) a $\top$-*uniform base*. Similarly, we speak of a $\top$-*quasi-uniform subbase* $\mathbb{C}$ if the $\top$-filter base $[\mathbb{C}]$ is a $\top$-quasi-uniform base and we call it a $\top$-*uniform subbase* if additionally $[\mathbb{C}]$ satisfies (TBU3).

It is not difficult to see that for a (quasi-)uniform space $(X, \mathbb{U})$, where $\mathbb{U}$ is the filter of entourages, $\top_{\mathbb{U}} = \{\top_U \; : \; U \in \mathbb{U}\}$ is a $\top$-(quasi-)uniform base and we have $u \in [\top_{\mathbb{U}}]$ if and only if $\bigvee_{U \in \mathbb{U}} \bigwedge_{(x,y) \in U} u(x,y) = \top$. Generally, if $\mathbb{B}$ is a uniform base of the uniformity $\mathbb{U}$, then $\top_{\mathbb{B}} = \{\top_B \; : \; B \in \mathbb{B}\}$ is a $\top$-uniform base. In this way, a metric space generates a $\top$-uniformity via its "natural" uniform base $\{U(\epsilon) \; : \; \epsilon > 0\}$ with $U(\epsilon) = \{(x,y) \in X \times X \; : \; d(x,y) < \epsilon\}$.

Proposition 9. *Let $(X, \mathcal{U}), (Y, \mathcal{V})$ be $\top$-quasi-uniform spaces and $\mathbb{B}$ be a $\top$-quasi-uniform base for $\mathcal{V}$ and $\mathbb{C}$ be a $\top$-quasi-uniform subbase for $\mathbb{B}$. For a mapping $\psi : X \longrightarrow Y$ the following assertions are equivalent.*

(1) $\psi : (X, \mathcal{U}) \longrightarrow (Y, [[\mathbb{C}]])$ is continuous.
(2) For all $c \in \mathbb{C}$ we have $(\psi \times \psi)^{\leftarrow}(c) \in \mathcal{U}$.

Proof. The one direction is obvious as $\mathbb{C} \subseteq [\mathbb{B}]$. For the other direction, let $v \in [\mathbb{B}]$. Then

$$\begin{aligned} \top &= \bigvee_{c_1, c_2, \ldots, c_n \in \mathbb{C}} [c_1 \wedge c_2 \wedge \cdots \wedge c_n, v] \\ &\leq \bigvee_{c_1, c_2, \ldots, c_n \in \mathbb{C}} [(\psi \times \psi)^{\leftarrow}(c_1 \wedge c_2 \wedge \cdots \wedge c_n), (\psi \times \psi)^{\leftarrow}(v)] \\ &= \bigvee_{c_1, c_2, \ldots, c_n \in \mathbb{C}} [(\psi \times \psi)^{\leftarrow}(c_1) \wedge (\psi \times \psi)^{\leftarrow}(c_2) \wedge \cdots \wedge (\psi \times \psi)^{\leftarrow}(c_n), (\psi \times \psi)^{\leftarrow}(v)] \\ &\leq \bigvee_{u \in \mathbb{U}} [u, (\psi \times \psi)^{\leftarrow}(v)], \end{aligned}$$

and hence, $(\psi \times \psi)^{\leftarrow}(v) \in \mathcal{U}$. $\square$

Now let $(X_j, \mathcal{U}_j)$ be $\top$-quasi-uniform spaces for $j \in J$,let X be a set and for all $j \in J$, let $\varphi_j : X \longrightarrow (X_j, \mathcal{U}_j)$. Then the set

$$\mathbb{C} = \{(\varphi_j \times \varphi_j)^{\leftarrow}(u_j) \; : \; u_j \in \mathcal{U}_j, j \in J\}$$

is a $\top$-quasi-uniform subbase. First we note that

$$(\varphi_{j_1} \times \varphi_{j_1})^{\leftarrow}(u_{j_1}) \wedge \cdots \wedge (\varphi_{j_k} \times \varphi_{j_n})^{\leftarrow}(u_{j_n}))(x,x)$$

$$= u_{j_1}(\varphi_{j_1}(x), \varphi_{j_1}(x)) \wedge \cdots \wedge u_{j_n}(\varphi_{j_n}(x), \varphi_{j_n}(x)) = \top,$$

from which the finite intersection property and (TBU1) for $[\mathbb{C}]$ follows. Furthermore, for $c = (\varphi_j \times \varphi_j)^{\leftarrow}(u_j) \in \mathbb{C}$ we have

$$\begin{aligned}
(\varphi_j \times \varphi_j)^{\leftarrow}(u_j) \circ (\varphi_j \times \varphi_j)^{\leftarrow}(u_j)(x,y) &= \bigvee_{z \in X} u_j(\varphi_j(x), \varphi_j(z)) * u_j(\varphi_j(z), \varphi_j(y)) \\
&\leq \bigvee_{z_j \in X_j} u_j(\varphi_j(x), z_j) * u_j(\varphi_j(z), z_j) \\
&= (\varphi_j \times \varphi_j)^{\leftarrow}(u_j \circ u_j)(x,y).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\bigvee_{b \in [\mathbb{C}]} [b \circ b, c] &\geq \bigvee_{v_j \in \mathcal{U}_j} [(\varphi_j \times \varphi_j)^{\leftarrow}(v_j) \circ (\varphi_j \times \varphi_j)^{\leftarrow}(v_j), c] \\
&\geq \bigvee_{v_j \in \mathcal{U}_j} [(\varphi_j \times \varphi_j)^{\leftarrow}(v_j \circ v_j), (\varphi_j \times \varphi_j)^{\leftarrow}(u_j)] \geq \bigvee_{v_j \in \mathcal{U}_j} [v_j \circ v_j, u_j] = \top,
\end{aligned}$$

and we have (TUB2) for $[\mathbb{C}]$.

It is clear that all $\varphi_j : (X, [[\mathbb{C}]]) \longrightarrow (X_j, \mathcal{U}_j)$ are uniformly continuous. If $(Y, \mathcal{V})$ is a $\top$-quasi-uniform space and $\psi : Y \longrightarrow X$ is a mapping, and $\varphi_j \circ \psi : (Y, \mathcal{V}) \longrightarrow (X_j, \mathcal{U}_j)$ is uniformly continuous for all $j \in J$, then also $\psi : (Y, \mathcal{V}) \longrightarrow (X, [[\mathbb{C}]])$ is uniformly continuous. This follows with $((\varphi_j \circ \psi) \times (\varphi_j \times \psi))^{\leftarrow}(u_j) = (\psi \times \psi)^{\leftarrow}((\varphi_j \times \varphi_j^{\leftarrow}(u_j))$ from Proposition 9.

Finally, we point out that if all $(X_j, \mathcal{U}_j)$ are $\top$-uniform spaces, then also $(X, [[\mathbb{C}]])$ is a $\top$-uniform space. To this end, we show (TUB3) for $[\mathbb{C}]$. Let $c = (\varphi_j \times \varphi_j)^{\leftarrow}(u_j) \in \mathbb{C}$ with $u_j \in \mathcal{U}_j$. Then $u_j^{-1} \in \mathcal{U}_j$ and hence, $c^{-1} = ((\varphi_j \times \varphi_j) \leftarrow (u_j))^{-1} = (\varphi_j \times \varphi_j)^{\leftarrow}(u_j^{-1}) \in \mathbb{C}$. We conclude that $\bigvee_{b \in \mathbb{C}} [b, c^{-1}] = \top$, which shows (TUB3) for $[\mathbb{C}]$.

Putting everything together we can state the main result of this section.

Theorem 1. *The categories* $\top-\mathsf{QUnif}$ *and* $\top-\mathsf{Unif}$ *are topological categories in the definition of* [19].

Later we will consider *subspaces* for subsets $M \subseteq X$ of a $\top$-quasi-uniform space $(X, \mathcal{U})$. These are defined as initial constructions for the embedding mapping $i_M : M \longrightarrow (X, \mathcal{U})$. It is not difficult to show that the initial $\top$-quasi uniformity on M is given by $\mathcal{U}_M = \{u|_{M \times M} : u \in \mathcal{U}\}$, which is the trace of $\mathcal{U}$ on $M \times M$. However, we avoid the more cumbersome notation $\mathcal{U}_{M \times M}$. We therefore call $(M, \mathcal{U}_M)$ a subspace of $(X, \mathcal{U})$.

4. Cauchy Pair $\top$-Filters and Cauchy Pair $\top$-Nets

Definition 5 ([1])**.** *Let* $(X, \mathcal{U})$ *be a* $\top$*-quasi-uniform space and* $\mathbb{F}, \mathbb{G} \in \mathbb{F}_L^{\top}(X)$*.*

(1) *The pair* $(\mathbb{F}, \mathbb{G})$ *is called a* pair $\top$-filter *if for all* $f \in \mathbb{F}, g \in \mathbb{G}$, $\bigvee_{z \in X} f(z) * g(z) = \top$.
(2) *The pair* $\top$*-filter* $(\mathbb{F}, \mathbb{G})$ *is called* convergent to x *if* $\mathbb{F} \geq \mathcal{U}(x, \cdot)$ *and* $\mathbb{G} \geq \mathcal{U}(\cdot, x)$*.*
(3) *The pair* $\top$*-filter* $(\mathbb{F}, \mathbb{G})$ *is called a* Cauchy pair $\top$-filter *if* $\mathcal{U} \leq \mathbb{G} \otimes \mathbb{F}$*, that is, if for all* $u \in \mathcal{U}$, $\bigvee_{g \in \mathbb{G}, f \in \mathbb{F}} [g \otimes f, u] = \top$.

We note that Yue and Fang [1] used the opposite order on the set of $\top$-filters and demanded $\mathcal{U} \leq \mathbb{F} \otimes \mathbb{G}$. We showed in [17], Remark 7.8, that $\mathbb{F} \otimes [x] \geq \mathcal{U}$ if and only if $\mathbb{F} \geq \mathcal{U}(\cdot, x)$ and $[x] \otimes \mathbb{F} \geq \mathcal{U}$ if and only if $\mathbb{F} \geq \mathcal{U}(x, \cdot)$.

Proposition 10. *Let* $(X, \mathcal{U})$ *be a* $\top$*-quasi-uniform space and* $(\mathbb{F}, \mathbb{G}), (\mathbb{F}', \mathbb{G}')$ *be pair* $\top$*-filters on* X*.*

(TCP1) for all $x \in X$, $([x], [x])$ *is a Cauchy pair* $\top$*-filter;*
(TCP2) If $(\mathbb{F}, \mathbb{G})$ *is a Cauchy pair* $\top$*-filter and if* $\mathbb{F}' \geq \mathbb{F}$ *and* $\mathbb{G}' \geq \mathbb{G}$*, then* $(\mathbb{F}', \mathbb{G}')$ *is a Cauchy pair* $\top$*-filter.*

(TCP3) *If* $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}')$ *are Cauchy pair* $\top$*-filters and if* $\mathbb{F}\vee\mathbb{G}'$ *and* $\mathbb{F}'\vee\mathbb{G}$ *exist, then* $(\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}')$ *is a Cauchy pair* $\top$*-filter.*

Proof.

(TCP1) For $f,g\in[x]$ we have $\bigvee_{z\in X} f(z)*g(z)\geq f(x)*g(x)=\top*\top=\top$. Hence, $([x],[x])$ is a pair $\top$-filter. From (TU1) we get $[x]\otimes[x]=[(x,x)]\geq\mathcal{U}$ and $([x],[x])$ is a Cauchy pair $\top$-filter.

(TCP2) Clearly, $\mathbb{G}'\otimes\mathbb{F}'\geq\mathbb{G}\otimes\mathbb{F}\geq\mathcal{U}$.

(TCP3) Obviously, $(\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}')$ is a pair $\top$-filter. As $\mathbb{F}\vee\mathbb{G}'$ exists we have, using (TU2), $\mathcal{U}\leq\mathcal{U}\circ\mathcal{U}\leq(\mathbb{G}\otimes\mathbb{F})\circ(\mathbb{G}'\otimes\mathbb{F}')\leq\mathbb{G}\otimes\mathbb{F}'$ and, similarly, we obtain $\mathcal{U}\leq\mathbb{G}'\otimes\mathbb{F}$. Hence, using Proposition 3.10 [16], we obtain $\mathcal{U}\leq(\mathbb{G}\otimes\mathbb{F})\wedge(\mathbb{G}\otimes\mathbb{F}')\wedge(\mathbb{G}'\otimes\mathbb{F})\wedge(\mathbb{G}'\otimes\mathbb{F}')=(\mathbb{G}\wedge\mathbb{G}')\otimes(\mathbb{F}\wedge\mathbb{F}')$.

□

Proposition 11. *Let* $(X,\mathcal{U})$ *be a* $\top$*-quasi-uniform space and* $(\mathbb{F},\mathbb{G})$ *be a pair* $\top$*-filter on* X*. Then* $(\mathbb{F},\mathbb{G})$ *is convergent to* x *if and only if* $(\mathbb{F}\wedge[x],\mathbb{G}\wedge[x])$ *is a Cauchy pair* $\top$*-filter.*

Proof. If $(\mathbb{F}\wedge[x],\mathbb{G}\wedge[x])$ is a Cauchy pair $\top$-filter, then $\mathcal{U}\leq(\mathbb{G}\wedge[x])\otimes([x]\wedge\mathbb{F})\leq[x]\otimes\mathbb{F}$ and $\leq\mathbb{G}\otimes[x]$, which means that $(\mathbb{F},\mathbb{G})$ is convergent to x. Conversely, if $(\mathbb{F},\mathbb{G})$ is convergent to x, then $(\mathbb{G}\wedge[x])\otimes([x]\wedge\mathbb{F})=(\mathbb{G}\otimes\mathbb{F})\wedge([x]\otimes\mathbb{F})\wedge(\mathbb{G}\otimes[x])\wedge([x]\otimes[x])\geq\mathcal{U}$ and $(\mathbb{F}\wedge[x],\mathbb{G}\wedge[x])$ is a Cauchy pair $\top$-filter. □

Proposition 12. *Let* $(X,\mathcal{U})$ *be a* $\top$*-quasi-uniform space,* $x\in X$ *and* $(\mathbb{F},\mathbb{G}),(\mathbb{F}',\mathbb{G}')$ *be pair* $\top$*-filters on* X*. If* $(\mathbb{F},\mathbb{G})$ *is a Cauchy pair* $\top$*-filter,* $\mathbb{F}'\geq\mathbb{F}$ *and* $\mathbb{G}'\geq\mathbb{G}$ *and* $(\mathbb{F}',\mathbb{G}')$ *is convergent to* x*, then also* $(\mathbb{F},\mathbb{G})$ *is convergent to* x*.*

Proof. We note that by (TCP2), also $(\mathbb{F}',\mathbb{G}')$ is a Cauchy pair $\top$-filter. From $\top=\bigvee_{z\in X} f(z)*g(z)\leq\bigvee_{z\in X} f(z)\wedge g(z)$ for all $f\in\mathbb{F}'$ and all $g\in\mathbb{G}'$ we conclude that $\mathbb{F}'\vee\mathbb{G}'$ exists and hence, also $\mathbb{F}\vee(\mathbb{G}'\wedge[x])$ and $(\mathbb{F}'\wedge[x])\vee\mathbb{G}$ exist. As $[x]\otimes\mathbb{F}'\geq\mathcal{U}$ and $\mathbb{G}'\otimes[x]\geq\mathcal{U}$ we see that $(\mathbb{F}'\wedge[x],\mathbb{G}'\wedge[x])$ is a Cauchy pair $\top$-filter. With (TCP3) we conclude that $((\mathbb{F}'\wedge[x])\wedge\mathbb{F},(\mathbb{G}'\wedge[x])\wedge\mathbb{G})=(\mathbb{F}\wedge[x],\mathbb{G}\wedge[x])$ is a Cauchy pair $\top$-filter which means $[x]\otimes\mathbb{F}\geq\mathcal{U}$ and $\mathbb{G}\otimes[x]\geq\mathcal{U}$, i.e., $(\mathbb{F},\mathbb{G})$ converges to x. □

Definition 6. *Let* $s=(s_X,s_L)$ *and* $t=(t_X,t_L)$ *be* $\top$*-sequences in* X*. We call* (s,t) *a* pair $\top$-sequence *if there is a common* $\top$*-subsequence* $r=(r_X,r_L)$ *of* s *and* t*.*

Clearly, if $s:\mathbb{N}\longrightarrow X\times L^*$ is a $\top$-sequence and t is a $\top$-subsequence of s, then (s,t) is a pair $\top$-sequence.

Proposition 13. *Let* s,t *be* $\top$*-sequences in* X*. Then* (s,t) *is a pair* $\top$*-sequence if and only if* $(\mathbb{F}_s,\mathbb{F}_t)$ *is a pair* $\top$*-filter.*

Proof. First let (s,t) be a pair $\top$-sequence and let r be a common subsequence. We consider the $\top$-bases $\mathbb{B}_s$, $\mathbb{B}_t$ and $\mathbb{B}_r$ of $\mathbb{F}_s$, $\mathbb{F}_t$ and $\mathbb{F}_r$, respectively, with elements the "tails" of s,t,r and we denote these "tails" by $d^s_n=\bigvee_{k\geq n} s_L(k)\wedge\top_{s_X(k)}$, $d^t_m=\bigvee_{k\geq m} t_L(k)\wedge\top_{t_X(k)}$ and $d^r_l=\bigvee_{k\geq p} r_L(k)\wedge\top_{r_X(k)}$. Let $d^s_n\in\mathbb{B}_s\subseteq\mathbb{F}_s$ and $d^t_m\in\mathbb{B}_t\subseteq\mathbb{F}_t$. As $\mathbb{F}_r\geq\mathbb{F}_s,\mathbb{F}_t$ there is $d^r_l\in\mathbb{B}_r$ such that $d^r_l\leq d^s_n,d^t_m$. We then have $\bigvee_{z\in X} d^s_n(z)*d^t_m(z)\geq\bigvee_{z\in X} d^r_l(z)*d^r_l(z)=\top$.

Now let $f \in \mathbb{F}_s$ and $g \in \mathbb{F}_t$. Then $\top = \bigvee_{n\in\mathbb{N}}[d^s_n, f]$ and $\top = \bigvee_{m\in\mathbb{N}}[d^t_m, f]$ and we conclude

$$\begin{aligned}\top = \top * \top &\leq \bigvee_{n\in\mathbb{N}}\bigvee_{m\in\mathbb{N}}\bigwedge_{x\in X}(d^s_n(x) \to f(x)) * (d^t_m(x) \to g(x))\\ &\leq \bigvee_{n\in\mathbb{N}}\bigvee_{m\in\mathbb{N}}\left((\bigvee_{x\in X} d^s_n(x) * d^t_m(x)) \to (\bigvee_{x\in X} f(x) * g(x))\right)\\ &= \bigvee_{x\in X} f(x) * g(x),\end{aligned}$$

and hence, $(\mathbb{F}_s, \mathbb{F}_t)$ is a pair $\top$-filter.

Conversely, let $(\mathbb{F}_s, \mathbb{F}_t)$ be a pair $\top$-filter. Then $\bigvee_{z\in X} f(z) * g(z) = \top$ for all $f \in \mathbb{F}_s$ and all $g \in \mathbb{F}_t$. We consider a $\top$-approximating sequence $(\alpha_1, \alpha_2, \dots)$. Since $\bigvee_{z\in X} d^s_1(z) * d^t_1(z) = \bigvee_{z\in X}\bigvee_{p\geq 1}\bigvee_{q\geq 1}(s_L(p) \wedge \top_{s_X(p)}(z)) * (t_L(q) \wedge \top_{t_X(q)}(z)) = \top \rhd \alpha_1$, we find $z_1 \in X$ and $p_1, q_1 \geq 1$ such that $s_L(p_1) \geq \alpha_1, t_L(q_1) \geq \alpha_1, s_X(p_1) = z_1$ and $t_X(p_1) = z_1$.

Similarly, from $\bigvee_{z\in X} d^s_{p_1+1}(z) * d^t_{q_1+1}(z) = \top \rhd \alpha_2$ we find $z_2 \in X$ and $p_2 \geq p_1 + 1$, $q_2 \geq q_1 + 1$ such that $s_L(p_2) \geq \alpha_2, t_L(q_2) \geq \alpha_2, s_X(p_2) = z_2, t_X(p_2) = z_2$. Going on like this we obtain a $\top$-sequence r defined by $r_X(k) = z_k$, $r_L(k) = \alpha_k$ for $k = 1, 2, 3, \dots$. We define $\phi(k) = p_k$ and $\psi(k) = q_k$ and observe $s_X \circ \phi(k) = s_X(p_k) = r_X(k), s_L \circ \phi(k) = s_L(p_k) \geq \alpha_k = r_L(k)$ and $t_X \circ \psi(k) = t_X(q_k) = r_X(k), t_L \circ \psi(k) = t_L(q_k) \geq \alpha_k = r_L(k)$. For $k_1 < k_2$ we have $\phi(k_1) = p_{k_1} < p_{k_1} + 1 \leq p_{k_2} = \phi(k_2)$ and ϕ is strictly increasing. Similarly, ψ is strictly increasing and hence, r is a common $\top$-subsequence of s and t. $\square$

Definition 7. *Let $(X, \mathcal{U})$ be a $\top$-quasi-uniform space. A pair $\top$-sequence (s,t) is called a* Cauchy pair $\top$-sequence *if for all $u \in \mathcal{U}$ we have*

$$\bigvee_{m,n\in\mathbb{N}}\bigwedge_{k\geq n, l\geq m}((t_L(k) * s_L(l)) \to u(t_X(k), s_X(l))) = \top.$$

It is sufficient to work with $\top$-quasi-uniform bases here.

Proposition 14. *Let $(X, \mathcal{U})$ be a $\top$-quasi-uniform space, $\mathbb{B}$ be a $\top$-quasi-uniform base for $\mathcal{U}$ and (s,t) be a pair $\top$-sequence. Then (s,t) is a Cauchy pair $\top$-sequence if and only if for all $b \in \mathbb{B}$ we have*

$$\bigvee_{m,n\in\mathbb{N}}\bigwedge_{k\geq n, l\geq m}((t_L(k) * s_L(l)) \to b(t_X(k), s_X(l))) = \top.$$

Proof. One implication is obvious as $\mathbb{B} \subseteq \mathcal{U}$. Now let the condition be satisfied for all $b \in \mathbb{B}$ and let $u \in \mathcal{U}$. Then $\bigvee_{b\in\mathbb{B}}[b, u] = \top$. We consider a $\top$-approximating sequence $\alpha_1 \leq \alpha_2 \leq \dots$. Then there is $b_p \in \mathbb{B}$ such that $\alpha_p \leq [b_p, u]$. We fix $n, m \in \mathbb{N}$ and $k \geq n, l \geq m$. Then

$$\begin{aligned}&\alpha_p * \big((t_L(k) * s_L(l)) \to b_p(t_X(k), s_X(l))\big)\\ \leq\ &\big((t_L(k) * s_L(l)) \to b_p(t_X(k), s_X(l))\big) * \big(b_p(t_X(k), s_X(l)) \to u(t_X(k), s_X(l))\big)\\ \leq\ &(t_L(k) * s_L(l)) \to u(t_X(k), s_X(l)).\end{aligned}$$

Hence, we conclude

$$\begin{aligned}\alpha_p &= \alpha_p * \bigvee_{m,n\in\mathbb{N}}\bigwedge_{k\geq n, l\geq m}((t_L(k) * s_L(l)) \to b(t_X(k), s_X(l)))\\ &\leq \bigvee_{m,n\in\mathbb{N}}\bigwedge_{k\geq n, l\geq m}\alpha_p * ((t_L(k) * s_L(l)) \to b(t_X(k), s_X(l)))\\ &\leq \bigvee_{m,n\in\mathbb{N}}\bigwedge_{k\geq n, l\geq m}((t_L(k) * s_L(l)) \to u(t_X(k), s_X(l))).\end{aligned}$$

Taking the join for all α_p yields the result. □

Proposition 15. *Let $(X, \mathcal{U})$ be a $\top$-quasi-uniform space. A pair $\top$-sequence (s,t) is a Cauchy pair $\top$-sequence if and only if $(\mathbb{F}_s, \mathbb{F}_t)$ is a Cauchy pair $\top$-filter.*

Proof. First let (s,t) be a Cauchy pair $\top$-sequence and let $u \in \mathcal{U}$. We consider the "tails" $d_n^t = \bigvee_{k \geq n} t_L(k) \wedge \top_{t_X(k)} \in \mathbb{F}_t$ and $d_m^s = \bigvee_{l \geq m} s_L(l) \wedge \top_{s_X(l)} \in \mathbb{F}_s$. Then

$$\begin{aligned}
\bigvee_{f \in \mathbb{F}_t, g \in \mathbb{F}_s} [f \otimes g, u] &\geq \bigvee_{n,m \in \mathbb{N}} [d_n^t \otimes d_m^s, u] \\
&= \bigvee_{m,n \in \mathbb{N}} \bigwedge_{x,y \in X} \bigwedge_{k \geq n, l \geq m} \Big((t_L(k) \top_{t_X(k)}(x) * s_L(l) \top_{s_X(l)}(y)) \to u(x,y) \Big) \\
&= \bigvee_{m,n \in \mathbb{N}} \bigwedge_{x,y \in X} \bigwedge_{k \geq n, l \geq m} \Big((t_L(k) * s_L(l) \top_{s_X(l)}) \to u(t_X(k), s_X(l)) \Big) \;=\; \top.
\end{aligned}$$

Hence, $u \in \mathbb{F}_t \otimes \mathbb{F}_s$ and we have $\mathcal{U} \leq \mathbb{F}_t \otimes \mathbb{F}_s$.
Conversely, if $\mathcal{U} \leq \mathbb{F}_t \otimes \mathbb{F}_s$ then for $u \in \mathcal{U}$ we have

$$\top = \bigvee_{f \in \mathbb{F}_t, g \in \mathbb{F}_s} [f \otimes g, u] \leq \bigvee_{f \in \mathbb{F}_t, g \in \mathbb{F}_s} ((f(t_X(k)) * g(s_X(l))) \to u(t_X(k), s_X(l)))$$

for all $n, m \in \mathbb{N}$ and all $k \geq n, l \geq m$.

For $f \in \mathbb{F}_t$ we have $\bigvee_{n \in \mathbb{N}} \bigwedge_{k \geq n} (t_L(k) \to f(t_X(k))) = \top$ and, similarly, for $g \in \mathbb{F}_g$ we have $\bigvee_{m \in \mathbb{N}} \bigwedge_{l \geq m} (s_L(l) \to g(s_X(l))) = \top$. We conclude

$$\begin{aligned}
\top \;=\; \top * \top \quad &\leq \bigvee_{n,m \in \mathbb{N}} \bigwedge_{k \geq n, l \geq m} (t_L(k) \to f(t_X(k))) * (s_L(l) \to g(s_X(l))) \\
&\leq \bigvee_{n,m \in \mathbb{N}} \bigwedge_{k \geq n, l \geq m} ((t_L(k) * s_L(l)) \to (f(t_X(k)) * g(s_X(l)))).
\end{aligned}$$

We conclude from this

$$\begin{aligned}
\top \;&=\; \bigvee_{f \in \mathbb{F}_t, g \in \mathbb{F}_s} \left(\bigvee_{n,m \in \mathbb{N}} \bigwedge_{k \geq n, l \geq m} ((t_L(k) * s_L(l)) \to (f(t_X(k)) * g(s_X(l)))) \right) * [f \otimes g, u] \\
&\leq \bigvee_{f \in \mathbb{F}_t, g \in \mathbb{F}_s} \left(\bigvee_{n,m \in \mathbb{N}} \bigwedge_{k \geq n, l \geq m} ((t_L(k) * s_L(l)) \to (f(t_X(k)) * g(s_X(l)))) \right) * \\
&\qquad\qquad * ((f(t_X(k)) * g(s_X(l))) \to u(t_X(k), s_X(l))) \\
&\leq \bigvee_{f \in \mathbb{F}_t, g \in \mathbb{F}_s} \bigwedge_{k \geq n, l \geq m} ((t_L(k) * s_L(l)) \to u(t_X(k), s_X(l))).
\end{aligned}$$

□

5. Completeness in $\top$-QUnif

We review concepts and theory introduced by Fang and Yue [1], see also [20], and adapt it slightly to suit our needs. In particular we use as order relation on $\mathbb{F}_L^\top(X)$ the subsethood order, $\Phi \leq \Psi \iff \Phi \subseteq \Psi$, whereas in [1] the opposite order is used, and we identify for a one-point set $\{\bullet\}$ the sets $X \times \{\bullet\}$ and $\{\bullet\} \times X$ with X.

Let $(X, \mathcal{U})$ be a $\top$-quasi-uniform space and let $\Phi, \Psi \in \mathbb{F}_L^\top(X)$. We call Φ a *left-promodule* if $\Phi \leq \Phi \circ \mathcal{U}$ and we call Ψ a *right-promodule* if $\Psi \leq \mathcal{U} \circ \Psi$. Here, $\Phi \circ \mathcal{U}$ is the $\top$-filter on X with $\top$-filter base $\{\phi \circ u : \phi \in \Psi, u \in \mathcal{U}\}$ with $\phi \circ u(x) = \bigvee_{z \in X} \phi(z) * u(z,x)$. Similarly, $\mathcal{U} \circ \Psi$ is the $\top$-filter on X with $\top$-filter base $\{u \circ \psi : \psi \in \Psi, u \in \mathcal{U}\}$ with $u \circ \psi(x) = \bigvee_{z \in X} u(x,z) * \psi(z)$.

If Φ is a left-promodule and Ψ is a right-promodule, then we call Φ *left-adjoint to* Ψ (and Ψ *right-adjoint to* Φ), denoted by $\Phi \dashv \Psi$, if $\mathcal{U} \le \Psi \otimes \Phi$ and $\bigvee_{z \in X} \phi(z) * \psi(z) = \top$ for all $\phi \in \Phi, \psi \in \Psi$, i.e., if (Φ, Ψ) are a Cauchy pair $\top$-filter.

Proposition 16. *Let $(X, \mathcal{U})$ be a $\top$-quasi-uniform space and $\Phi, \Phi' \in \mathbb{F}_L^\top(X)$ be left-promodules and $\Psi, \Psi' \in \mathbb{F}_L^\top(X)$ be right-promodules and $\Phi \dashv \Psi$ and $\Phi' \dashv \Psi'$. If $\Phi \le \Phi'$ and $\Psi \le \Psi'$ then $\Phi = \Phi'$ and $\Psi = \Psi'$.*

Proof. We have $\Phi' \le \Phi' \circ \mathcal{U} \le \Phi' \circ (\Psi \otimes \Phi) \le \Phi' \circ (\Psi \otimes \Phi) = \Phi$, because for $\phi' \in \Phi'$, $\psi' \in \Psi'$ and $\phi \in \Phi$ we have

$$\phi' \circ (\psi' \otimes \phi)(x) = \bigvee_{z \in X} \phi'(z) * (\psi' \otimes \phi)(z, x) = \underbrace{\bigvee_{z \in X} \phi'(z) * \Psi'(z)}_{=\top} * \phi(x) = \phi(x).$$

Similarly, we can show $\Psi' = \Psi$. $\square$

Proposition 17. *Let $(X, \mathcal{U})$ be a $\top$-quasi-uniform space and $x \in X$. We denote $\mathcal{U}(x, \cdot)$ and $\mathcal{U}(\cdot, x)$ the $\top$-filters on X with $\top$-filter bases $\{u(x, \cdot) : u \in \mathcal{U}\}$ and $\{u(\cdot, x) : u \in \mathcal{U}\}$, respectively. Then $\mathcal{U}(x, \cdot)$ is a left-promodule and $\mathcal{U}(\cdot, x)$ is a right-promodule and $\mathcal{U}(x, \cdot) \dashv \mathcal{U}(\cdot, x)$.*

Proof. It is shown in [1] that $(\mathcal{U}(x, \cdot), \mathcal{U}(\cdot, x))$ are Cauchy pair $\top$-filters. We show that $\mathcal{U}(\cdot, x)$ is a right-promodule. Let $a \in \mathcal{U}(\cdot, x)$. For $u \in \mathcal{U}$ we note that $u \circ (u(\cdot, x))(y) = \bigvee_{z \in X} u(y, z) * u(\cdot, x)(z) = \bigvee_{z \in X} u(y, z) * u(z, x) = u \circ u(y, x) = (u \circ u)(\cdot, x)(y)$, i.e., we have $u \circ (u(\cdot, x)) = (u \circ u)(\cdot, x)$. Hence,

$$\top = \bigvee_{v \in \mathcal{U}} \bigwedge_{x, y \in X} (v \circ v(y, x) \to u(y, x)) = \bigvee_{v \in \mathcal{U}} \bigwedge_{x \in X} [v \circ v(\cdot, x) \to u(\cdot, x)].$$

We conclude

$$\top = \bigvee_{u \in \mathcal{U}} [u(\cdot, x), a] = \bigvee_{u \in \mathcal{U}} \bigvee_{v \in \mathcal{U}} [v \circ v(\cdot, x), u(\cdot, x)] * [u(\cdot, x), a] \le \bigvee_{v \in \mathcal{U}} [v \circ v(\cdot, x), a]$$

$$= \bigvee_{v \in \mathcal{U}} [v \circ (v(\cdot, x)), a] \le \bigvee_{v \in \mathcal{U}, d \in \mathcal{U}(\cdot, x)} [v \circ d, a]$$

and we have $a \in \mathcal{U} \circ \mathcal{U}(\cdot, x)$. Similarly, we can show that $\mathcal{U}(x, \cdot)$ is a left-promodule. $\square$

Definition 8 (cf. [1])**.** *The $\top$-quasi-uniform space $(X, \mathcal{U})$ is called* complete *if for all left-promodules Φ and for all right-promodules Ψ with $\Phi \dashv \Psi$ there exists $x_0 \in X$ such that $\Phi = \mathcal{U}(x_0, \cdot)$ and $\Psi = \mathcal{U}(\cdot, x_0)$.*

Yue and Fang [1] show that $(X, \mathcal{U})$ is complete if and only if each Cauchy pair $\top$-filter converges to some $x_0 \in X$. The key point is here that two $\top$-filters $\mathbb{F}, \mathbb{G}$ on X are adjoint left- and right-promodules, respectively, if and only if they are a *minimal* Cauchy pair $\top$-filter, that is, for any other Cauchy pair $\top$-filter $(\mathbb{H}, \mathbb{K})$ with $\mathbb{H} \le \mathbb{F}$ and $\mathbb{K} \le \mathbb{G}$ we have $\mathbb{F} = \mathbb{H}$ and $\mathbb{G} = \mathbb{K}$.

A special case arises for L-quasi-metric spaces (X, d). We review concepts and notation from [11]. Let (X, d) be an L-quasi-metric space. A mapping $\varphi : X \longrightarrow L$ is called an *order filter* if $\varphi \circ d \le \varphi$ and a mapping $\psi : X \longrightarrow L$ is called an *order ideal* if $d \circ \psi \le \psi$. Furthermore, for an order filter φ and an order ideal ψ we call φ *left-adjoint to* ψ (and write $\varphi \dashv \psi$) if $\psi(x) * \varphi(y) \le d(x, y)$ for all $x, y \in X$ and if $\bigvee_{z \in X} \varphi(z) * \psi(z) = \top$. Then (X, d) is called *complete* if and only if for all order filters φ and all order ideals ψ with $\varphi \dashv \psi$ there is $x_0 \in X$ such that $\varphi = d(x_0, \cdot)$ and $\psi = d(\cdot, x_0)$.

Theorem 2. *An L-quasi-metric space (X, d) is complete if and only if $(X, [d])$ is complete.*

Proof. Let (X, d) be complete and consider a left-promodule Φ and a right-promodule Ψ with $\Phi \dashv \Psi$ in $(X, [d])$, i.e., we have

(1) $\Phi \leq \Phi \circ [d]$ and $\Psi \leq [d] \circ \Psi$;
(2) $\bigvee_{z \in X} \phi(z) * \psi(z) = \top$ for all $\phi \in \Phi, \psi \in \Psi$;
(3) $[d] \leq \Psi \otimes \Phi$.

From (3), there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that $\psi \otimes \phi \leq d$. From (1), there exist $\phi_1 \in \Phi$ and $\psi_1 \in \Psi$ with $\phi_1 \circ d \leq \phi$ and $d \circ \psi_1 \leq \psi$. We define $\overline{\phi} = \phi_1 \circ d$ and $\overline{\psi} = d \circ \psi_1$. We note that $\overline{\phi} = \bigvee_{z \in X} \phi_1(z) * d(z, x) \geq \phi_1(x)$ for all $x \in X$ and similarly, $\overline{\psi} \geq \psi_1$ and hence, we have $\overline{\phi} \in \Phi$ and $\overline{\psi} \in \Psi$. Furthermore, from $d \circ d \leq d$ we infere

$$\overline{\phi} \circ d = (\phi_1 \circ d) \circ d = \phi_1 \circ (d \circ d) \leq \phi_1 \circ d = \overline{\phi}$$

and

$$d \circ \overline{\psi} = d \circ (d \circ \psi_1) = (d \circ d) \circ \psi_1 \leq d \circ \psi_1 = \overline{\psi},$$

that is, $\overline{\phi}$ is an order filter and $\overline{\psi}$ is an order ideal. In addition, $\overline{\psi}(x) * \overline{\phi}(y) \leq \psi(x) * \psi(y) \leq d(x, y)$ and $\bigvee_{z \in X} \overline{\psi}(z) * \overline{\phi}(z) = \top$ from (2). Hence, $\overline{\phi} \dashv \overline{\psi}$ and from the completeness of (X, d), there is $x_0 \in X$ such that $\overline{\phi} = d(x_0, \cdot)$ and $\overline{\psi} = d(\cdot, x_0)$. This implies $[d(x_0, \cdot)] \leq \Phi$ and $[d(\cdot, x_0)] \leq \Psi$. As also $[d(x_0, \cdot)] \dashv [d(\cdot, x_0)]$ we conclude from Proposition 16 that $[d(x_0, \cdot)] = \Phi$ and $[d(\cdot, x_0)] = \Psi$ and $(X, [d])$ is complete.

Now let $(X, [d])$ be complete and consider an order filter $\varphi : X \longrightarrow L$ and an order ideal $\psi : X \longrightarrow L$ with $\varphi \dashv \psi$. From $\bigvee_{z \in X} \varphi(z) * \psi(z) = \top$ we see that $\bigvee_{x \in X} \varphi(x) = \top$ and $\bigvee_{x \in X} \psi(x) = \top$. We define $\Phi = [\varphi]$ and $\Psi = [\psi]$. From $\varphi \circ d \leq \varphi$ we see that $\Phi \leq \Phi \circ [d]$ and from $d \circ \psi \leq \psi$ we obtain $\Psi \leq [d] \circ \Psi$. As $\psi \otimes \varphi \leq d$ we have $d \in \Psi \otimes \Phi$, i.e., $[d] \leq \Psi \otimes \Phi$ and for $a \in \Phi$, $b \in \Psi$ we have $\bigvee_{z \in X} a(z) * b(z) \geq \bigvee_{z \in X} \varphi(z) * \psi(z) = \top$. Hence, $\Phi \dashv \Psi$ and the completeness of $(X, [d])$ ensures the existence of $x_0 \in X$ s.t. $\Phi = [d(x_0, \cdot)]$ and $\Psi = [d(\cdot, x_0)]$, i.e., $\phi = d(x_0, \cdot)$ and $\psi = d(\cdot, x_0)$ and (X, d) is complete. $\square$

6. Sequential Completeness in $\top$-QUnif

Throughout this section we assume that the quantale $\mathsf{L} = (L \leq, *)$ is $\top$-approximable and we fix a $\top$-approximating sequence $(\alpha_1, \alpha_2, \alpha_3, \dots)$ in L.

We call a $\top$-quasi-uniform space $(X, \mathcal{U})$ *sequentially complete* if for all Cauchy pair $\top$-sequences (s, t) there exists $x_0 \in X$ such that (s, t) *converges to* x_0, i.e., such that for all $u \in \mathcal{U}$ we have

$$\bigvee_{n \in \mathbb{N}} \bigwedge_{k \geq n} (s_L(k) \to u(x_0, s_X(k))) = \top \text{ and } \bigvee_{n \in \mathbb{N}} \bigwedge_{k \geq n} (t_L(k) \to u(t_X(k), x_0)) = \top$$

A subset $M \subseteq X$ is called *sequentially complete* if the subspace $(M, \mathcal{U}_M)$ is sequentially complete.

So we have that $(X, \mathcal{U})$ is sequentially complete if and only if for all Cauchy pair $\top$-sequences (s, t) there exists $x_0 \in X$ such that $\mathbb{F}_s \geq \mathcal{U}(x_0, \cdot)$ and $\mathbb{F}_t \geq \mathcal{U}(\cdot, x_0)$.

As in Proposition 14 we can see that it is sufficient to check convergence for a $\top$-quasi-uniform base $\mathbb{B}$ of $\mathcal{U}$ only.

For the proof of the next two important results, it is essential that we allow grades $s_L(n)$ of our $\top$-sequences that need not equal the top element.

Theorem 3. *Let $(X, \mathcal{U})$ be a $\top$-quasi-uniform space. Then $(X, \mathcal{U})$ is sequentially complete if and only if for all Cauchy pair $\top$-filters $(\mathbb{F}, \mathbb{G})$ with countable $\top$-bases there is $x_0 \in X$ such that $\mathbb{F} \geq \mathcal{U}(x_0, \cdot)$ and $\mathbb{G} \geq \mathcal{U}(\cdot, x_0)$.*

Proof. First let $(X, \mathcal{U})$ be sequentially complete and let $\mathbb{B} = \{b_1, b_2, b_3, \dots\}$ be a countable $\top$-base of $\mathbb{F}$ and $\mathbb{C} = \{c_1, c_2, c_3, \dots\}$ be a countable $\top$-base of $\mathbb{G}$. It is not difficult to show that for a countable $\top$-base $\{b_1, b_2, \dots\}$ the L-sets $b_1, b_1 \wedge b_2, b_1 \wedge b_2 \wedge b_3, \dots$ again form a countable $\top$-base. Hence, we may assume $b_1 \geq b_2 \geq b_3 \geq \cdots$ and $c_1 \geq c_2 \geq c_3 \geq \cdots$.

We consider now a $\top$-approximating sequence $\alpha_1 \leq \alpha_2 \leq \cdots$. As $\mathbb{B} \subseteq \mathbb{F}$ and $\mathbb{C} \subseteq \mathbb{G}$ we conclude from $\alpha_1 \lhd \top = \bigvee_{z \in X} b_1(z) * c_1(z)$ that there exists $z_1 \in X$ such that $\alpha \leq b_1(z_1)$ and $\alpha_1 \leq c_1(z_1)$. Similarly, from $\alpha_2 \lhd \top = \bigvee_{z \in X} b_2(z) * c_2(z)$ we conclude that there exists $z_2 \in X$ such that $\alpha_2 \leq b_2(z_2) \leq b_1(z_2)$ and $\alpha_2 \leq c_2(z_2) \leq c_1(z_2)$. Going on like this, we obtain a sequence $z_1, z_2, z_3, \ldots$ with $\alpha_k \leq b_k(z_k) \leq b_{k-1}(z_k) \leq \cdots \leq b_1(z_k)$ and $\alpha_k \leq c_k(z_k) \leq c_{k-1}(z_k) \leq \cdots \leq c_1(z_k)$ for all $k = 1, 2, 3, \ldots$. Hence, we have

$$b_1, c_1 \geq \bigvee_{k \geq 1} \alpha_k \top_{z_k},\ b_2, c_2 \geq \bigvee_{k \geq 2} \alpha_k \top_{z_k}, \ldots$$

We define the $\top$-sequence $s = (s_X, s_L)$ by $s_X(k) = z_k$, $s_L(k) = \alpha_k$. Then $\mathbb{F} \leq \mathbb{F}_s$ and $\mathbb{G} \leq \mathbb{F}_s$. Trivially, (s, s) is a pair $\top$-sequence and we have $\mathcal{U} \leq \mathbb{G} \otimes \mathbb{F} \leq \mathbb{F}_s \otimes \mathbb{F}_s$, that is, (s, s) is a Cauchy pair $\top$-sequence. By sequential completeness, there is $x_0 \in X$ such that $\mathbb{F}_s \geq \mathcal{U}(x_0, \cdot)$ and $\mathbb{F}_s \geq \mathcal{U}(\cdot, x_0)$. From Proposition 12 we conclude that also $\mathbb{F} \geq \mathcal{U}(x_0, \cdot)$ and $\mathbb{G} \geq \mathcal{U}(\cdot, x_0)$.

For the converse, let (s, t) be a Cauchy pair $\top$-sequence. Then $(\mathbb{F}_s, \mathbb{F}_t)$ is a Cauchy pair $\top$-filter with countable bases and hence, there exists $x_0 \in X$ such that $\mathbb{F}_s \geq \mathcal{U}(x_0, \cdot)$ and $\mathbb{F}_t \geq \mathcal{U}(\cdot, x_0)$, that is, (s, t) converges to x_0. $\square$

Hence, a complete $\top$-quasi-uniform space is sequentially complete. For this reason, a "sequential completion" of a $\top$-quasi-uniform space can be obtained by a completion of the space, see [1] and no new contruction is needed.

We call a $\top$-quasi-uniform space $(X, \mathcal{U})$ *countable* if $\mathcal{U}$ has a countable $\top$-quasi-uniform base.

Proposition 18. *A countable $\top$-quasi-uniform space $(X, \mathcal{U})$ is complete if and only if it is sequentially complete.*

Proof. Sequential completeness is implied by completeness by Theorem 3. For the converse, let $(\mathbb{F}, \mathbb{G})$ be a Cauchy pair $\top$-filter and let $\{u_1, u_2, u_3, \ldots\}$ be a countable $\top$-uniform base of $\mathcal{U}$. From $\mathcal{U} \leq \mathbb{G} \otimes \mathbb{F}$ we find for for every $k = 1, 2, 3, \ldots$ L-sets $f_k \in \mathbb{F}$ and $g_k \in \mathbb{G}$ such that $g_k \otimes f_k \leq u_k$. We define $\mathbb{B} = \{f_1, f_1 \wedge f_2, f_1 \wedge f_2 \wedge f_3, \ldots\}$ and $\mathbb{C} = \{g_1, g_1 \wedge g_2, g_1 \wedge g_2 \wedge g_3, \ldots\}$ and we define $\mathbb{F}_{\mathbb{B}} = [\mathbb{F}]$ and $\mathbb{F}_{\mathbb{C}} = [\mathbb{C}]$. For $f \in \mathbb{F}_{\mathbb{B}}$ we have $\top = \bigvee_{k \in \mathbb{N}} [f_1 \wedge \cdots \wedge f_k, f]$, i.e., we have $f \in \mathbb{F}$ and hence, $\mathbb{F}_{\mathbb{B}} \leq \mathbb{F}$. Similarly, we see that $\mathbb{F}_{\mathbb{C}} \leq \mathbb{G}$. Let now $u \in \mathcal{U}$. Then

$$\top = \bigvee_{k \in \mathbb{N}} [u_k, u] \leq \bigvee_{k \in \mathbb{N}} [g_k \otimes f_k, u] \leq \bigvee_{k \in \mathbb{N}} [(g_1 \wedge \cdots \wedge g_k) \otimes (f_1 \wedge \cdots \wedge f_k), u],$$

and hence, $u \in \mathbb{F}_{\mathbb{C}} \otimes \mathbb{F}_{\mathbb{B}}$ and $(\mathbb{F}_{\mathbb{B}}, \mathbb{F}_{\mathbb{C}})$ is a Cauchy pair $\top$-filter with countable bases. As $(X, \mathcal{U})$ is sequentially complete, Theorem 3 implies that $x_0 \in X$ exists with $\mathbb{F}_{\mathbb{B}} \geq \mathcal{U}(x_0, \cdot)$ and $\mathbb{F}_{\mathbb{C}} \geq \mathcal{U}(\cdot, x_0)$. Hence, also $\mathbb{F} \geq \mathcal{U}(x_0, \cdot)$ and $\mathbb{G} \geq \mathcal{U}(\cdot, x_0)$ and $(X, \mathcal{U})$ is complete. $\square$

Remark 2. *For an* L*-quasi-metric space (X, d) the $\top$-quasi-uniform space $(X, [d])$ has the countable $\top$-quasi-uniform base $\{d\}$. Hence, for* L*-quasi-metric spaces, countable completeness and completeness are equivalent. This result was obtained in a different way in [21].*

Finally, we characterize sequentially complete subsets.

Proposition 19. *A subset $M \subseteq X$ of a $\top$-quasi-uniform space $(X, \mathcal{U})$ is sequentially complete if and only if for all Cauchy pair $\top$-sequences (s, t) with $s_X(k), t_X(k) \in M$ for all $k \in \mathbb{N}$ there exists $x_0 \in M$ such that $\mathbb{F}_s \geq \mathcal{U}(x_0, \cdot)$ and $\mathbb{F}_t \geq \mathcal{U}(\cdot, x_0)$.*

Proof. Let $M \subseteq X$ be sequentially complete and let (s, t) be a Cauchy pair $\top$-sequence with values $s_X(k), t_X(k) \in M$ for all $k = 1, 2, 3, \ldots$. Then $\mathbb{U} \leq \mathbb{F}_t \otimes \mathbb{F}_s$ and hence, $\mathcal{U}_M \leq \mathbb{F}_{(t_M)} \otimes \mathbb{F}_{(s_M)}$, that is, (s_M, t_M) is a Cauchy pair $\top$-sequence in $(M, \mathcal{U}_M)$. Hence, there exists $x_0 \in M$ such that $\mathbb{F}_{(s_M)} \geq \mathcal{U}_M(x_0, \cdot)$ and $\mathbb{F}_{(t_M)} \geq \mathcal{U}_M(\cdot, x_0)$. This is equivalent to $[x_0] \otimes \mathbb{F}_{(s_M)} \geq \mathcal{U}_M$

and $\mathbb{F}_{(t_M)} \otimes [x_0] \geq \mathcal{U}_M$. Hence, $[x_0] \otimes \mathbb{F}_s = [x_0] \otimes [\mathbb{F}_{(s_M)}] \geq [\mathcal{U}_M] \geq \mathcal{U}$ and $\mathbb{F}_t \otimes [x_0] = [\mathbb{F}_{(t_M)}] \otimes [x_0] \geq [\mathcal{U}_M] \geq \mathcal{U}$, that is, $\mathbb{F}_s \geq \mathcal{U}(x_0, \cdot)$ and $\mathbb{F}_t \geq \mathcal{U}(\cdot, x_0)$.

Conversely, let (s,t) be a Cauchy pair $\top$-sequence in the subspace $(M, \mathcal{U}_M)$. Then $\mathcal{U}_M \leq \mathbb{F}_t \otimes \mathbb{F}_s$ and hence, $\mathcal{U} \leq \mathbb{F}_{[t]} \otimes \mathbb{F}_{[s]}$ and $([s],[t])$ is a Cauchy pair $\top$-sequence in $(X, \mathcal{U})$ with values in M. Therefore, there exists $x_0 \in M$ such that $[x_0] \otimes \mathbb{F}_{[s]} \geq \mathcal{U}$ and $\mathbb{F}_{[t]} \otimes [x_0] \geq \mathcal{U}$. This implies $\mathcal{U}_M \leq [x_0] \otimes (\mathbb{F}_{[s]})_M = [x_0] \otimes \mathbb{F}_s$ and $\mathcal{U}_M \leq (\mathbb{F}_{[t]})_M \otimes [x_0] = [x_0] \otimes \mathbb{F}_t$, i.e., (s,t) converges to $x_0 \in M$ in $(M, \mathcal{U}_M)$. $\square$

7. A Fixed Point Theorem

In this section we consider $\top$-uniform spaces, that is, we assume that the "symmetry axiom" (TU3) is satisfied and we generalize a fixed point theorem established by Taylor [8] for uniform spaces. We point out that L is $\top$-approximable and we fix a $\top$-approximating sequence $(\alpha_k)_{k \in \mathbb{N}}$.

For a self-mapping $\varphi : X \longrightarrow X$ we define $\varphi^1 = \varphi$ and $\varphi^n = \varphi \circ \varphi^{n-1}$ for $n \geq 2$. For $u \in L^{X \times X}$ we denote $u^1 = u$ and $u^n = u \circ u^{n-1}$ for $n \geq 2$.

For a $\top$-uniform space $(X, \mathcal{U})$ we call $\mathbb{F} \in \mathsf{F}_L^\top(X)$ a *Cauchy $\top$-filter* if $\mathbb{F} \otimes \mathbb{F} \geq \mathcal{U}$, that is, if $(\mathbb{F}, \mathbb{F})$ is a Cauchy pair $\top$-filter. Similarly, we call a $\top$-sequence s a *Cauchy $\top$-sequence* if $\mathbb{F}_s \otimes \mathbb{F}_s \geq \mathcal{U}$. This is equivalent to $\top = \bigvee_{m,n \in \mathbb{N}} \bigwedge_{k \geq n, l \geq m} (s_L(k) * s_L(l) \rightarrow u(s_X(k), s_X(l)))$ for all $u \in \mathcal{U}$ and again it suffices to check this for a $\top$-uniform base $\mathbb{B}$ of $\mathcal{U}$. Finally, we call a $\top$-filter $\mathbb{F} \in \mathsf{F}_L^\top(X)$ *convergent to* $x \in X$ if $\mathbb{F} \geq \mathcal{U}(x, \cdot)$ and we call a $\top$-sequence s *convergent to* $x \in X$ if $\mathbb{F}_s$ is convergent to x. We note that for a $\top$-sequence $s : \mathbb{N} \longrightarrow X \times L^*$ converging to x, a $\top$-subsequence t also converges to x. This follows from $\mathbb{F}_t \geq \mathbb{F}_s \geq \mathcal{U}(x, \cdot)$.

Proposition 20. *A $\top$-uniform space $(X, \mathcal{U})$ is sequentially complete if and only if for all Cauchy $\top$-sequences $s = (s_X, s_L)$ there exists $x_0 \in X$ such that for all $u \in \mathcal{U}$ we have $\top = \bigvee_{n \in \mathbb{N}} \bigwedge_{k \geq n} (s_L(k) \rightarrow u(x_0, s_X(k)))$, that is, $\mathbb{F}_s \geq \mathcal{U}(x_0, \cdot)$.*

Proof. If $(X, \mathcal{U})$ is sequentially complete and s is a Cauchy $\top$-sequence, then (s,s) is a Cauchy pair $\top$-sequence and hence, there exists $x_0 \in X$ such that for all $u \in \mathcal{U}$ we have $\top = \bigvee_{n \in \mathbb{N}} \bigwedge_{k \geq n} (s_L(k) \rightarrow u(x_0, s_X(k)))$.

For the converse, let (s,t) be a Cauchy pair $\top$-sequence and consider a common $\top$-subsequence r of s,t. Then $\mathbb{F}_r \geq \mathbb{F}_s, \mathbb{F}_t$ and hence, $\mathbb{F}_r \otimes \mathbb{F}_r \geq \mathbb{F}_t \otimes \mathbb{F}_s \geq \mathcal{U}$, that is, r is a Cauchy $\top$-sequence. Hence, there exists $x_0 \in X$ such that $\mathbb{F}_r \geq \mathcal{U}(x_0, \cdot) = \mathcal{U}(\cdot, x_0)$, i.e., the Cauchy pair $\top$-filter $(\mathbb{F}_r, \mathbb{F}_r)$ converges to x_0. By Proposition 12 then also the Cauchy pair $\top$-sequence (s,t) converges to x_0. $\square$

We call a $\top$-uniform space $(X, \mathcal{U})$ a T2-space [15] if for all $x, y \in X$ with $x \neq y$ there is $u \in \mathcal{U}$ such that $u(x,y) \neq \top$. We note the following simple result.

Proposition 21. *A $\top$-uniform space $(X, \mathcal{U})$ is a T2-space if and only if convergent $\top$-filters (and hence, also convergent $\top$-sequences) have unique limits.*

Proof. First let $(X, \mathcal{U})$ be a T2-space and let the $\top$-filter $\mathbb{F}$ converge to x and y. Then $\mathbb{F} \otimes [x] \geq \mathcal{U}$ and $\mathbb{F} \otimes [y] = ([y] \otimes \mathbb{F})^{-1} \geq \mathcal{U}$ and hence, also $[(x,y)] = [x] \otimes [y] \geq \mathcal{U} \otimes \mathcal{U} \geq \mathcal{U}$. Therefore, for $u \in \mathcal{U}$ we have $u(x,y) = \top$ and $x = y$.

Now let convergent $\top$-filters have unique limits. If $u(x,y) = \top$ for all $u \in \mathcal{U}$, then $\mathcal{U} \leq [(x,y)] = [x] \otimes [y]$, that is, the point $\top$-filter $[y]$ converges to x. As $[y]$ converges also to y, we obtain $x = y$. $\square$

For an L-metric space (X, d) we have that $(X, [d])$ is a T2-space if and only if the L-metric is *separated*, i.e., if $d(x,y) = \top$ implies $x = y$.

Definition 9. *Let $(X, \mathcal{U})$ be a $\top$-uniform space, $\mathbb{B}$ be a $\top$-uniform base for $\mathcal{U}$ and $\varphi : X \longrightarrow X$ be a self-mapping.*

(1) *φ is called a $\mathbb{B}$-*contraction *if for all $u \in \mathbb{B}$ there exists $v \in \mathbb{B}$ such that $v \circ u \leq (\varphi \times \varphi)^{\leftarrow}(u)$.*

(2) *φ is called* asymptotically regular *if for all $u \in \mathcal{U}$ and all $x \in X$ we have $\top = \bigvee_{n_0 \in \mathbb{N}} \bigwedge_{n \geq n_0} u(\varphi^n(x), \varphi^{n+1}(x))$.*

(3) *$(X, \mathcal{U})$ is called* well-chained *if for all $x, y \in X$ and all $u \in \mathcal{U}$ we have $\bigvee_{n \in \mathbb{N}} u^n(x, y) = \top$.*

For a uniform space $(X, \mathbb{U})$ with uniform base $\mathbb{B}$, a $\mathbb{B}$-contraction in the definition of [8] is a $\top_{\mathbb{B}}$-contraction for the $\top$-uniform space $(X, \top_{\mathbb{U}})$.

Well-chainedness only depends on a $\top$-uniform base $\mathbb{B}$ of $\mathcal{U}$, that is, $(X, \mathcal{U})$ is well-chained if and only if for all $x, y \in X$ and all $b \in \mathbb{B}$ we have $\bigvee_{n \in \mathbb{N}} b^n(x, y) = \top$. To see this, let $x, y \in X$ and $u \in \mathcal{U}$ and $\alpha_k \leq \top = \bigvee_{b \in \mathbb{B}} [b, u]$. Then there is $b_k \in \mathbb{B}$ such that $\alpha_k \leq [b_k, u] \leq [b_k^n, u^n]$ and we conclude $\alpha_k = \alpha_k * \bigvee_{n \in \mathbb{N}} b_k^n(x, y) \leq \bigvee_{n \in \mathbb{N}} u^n(x, y)$. Taking the join over all α_k yields $\top = \bigvee_{n \in \mathbb{N}} u^n(x, y)$.

Furthermore, we note that a $\mathbb{B}$-contraction is uniformly continuous: For $u \in \mathbb{B}$ there is $v \in \mathbb{B}$ such that $v \circ u \leq (\varphi \times \varphi)^{\leftarrow}(u)$ and as $v \leq v \circ u$ this implies $(\varphi \times \varphi)^{\leftarrow}(u) \in \mathcal{U}$.

Remark 3 (L-metric case)**.**

(1) *For an L-metric space (X, d), the "natural" $\top$-uniform space $(X, [d])$ has a $\top$-uniform base $\mathbb{B} = \{d\}$ for $[d]$. A self-mapping $\varphi : X \longrightarrow X$ is a $\{d\}$-contraction if and only if $d \circ d \leq (\varphi \times \varphi)^{\leftarrow}(d)$. Noting that $d \leq d \circ d$ and, moreover, by transitivity also $d \leq d \circ d$, then φ is a $\{d\}$-contraction if and only if it is an expansive mapping, that is, if $d(x, y) \leq d(\varphi(x), \varphi(y))$ for all $x, y \in X$.*

(2) *For an L-metric space (X, d), noting again that $d^n = d$ for all $n \in \mathbb{N}$, $(X, [d])$ is well-chained if and only if for all $x, y \in X$ we have $d(x, y) = \top$. If (X, d) is separated, this is only possible for a one-point space.*

Both these properties point to the fact that, despite its simplicity, for an L-metric space (X, d) the space $(X, [d])$ may not be an ideal choice for an "L-metrically generated" $\top$-uniform space. However, we wish to point out at this point, that for Lawvere's quantale $\mathsf{L} = ([0, \infty], \geq, +)$, the metric uniformity of a metric space is not a $\top$-uniformity.

Lemma 2. *Let $(X, \mathcal{U})$ be a $\top$-uniform space, $\mathbb{B}$ be a $\top$-uniform base for $\mathcal{U}$ and $\varphi : X \longrightarrow X$ be a $\mathbb{B}$-contraction. If for $u, v \in \mathbb{B}$ we have $v \circ u \leq (\varphi \times \varphi)^{\leftarrow}(u)$, then for all $n \in \mathbb{N}$ we have $v^n \circ u \leq (\varphi^n \times \varphi^n)^{\leftarrow}(u)$.*

Proof. We use induction on n. The case $n = 1$ is just the assumption. Now assume that for a given $n \in \mathbb{N}$ we have for $u, v \in \mathbb{B}$ that $v^n \circ u \leq (\varphi^n \times \varphi^n)^{\leftarrow}(u)$. As φ is a $\mathbb{B}$-contraction, there exists $\widetilde{v} \in \mathbb{B}$ such that $\widetilde{v} \circ v \leq (\varphi \times \varphi)^{\leftarrow}(v)$ and from $v \leq \widetilde{v} \circ v$ we obtain $v \leq (\varphi \times \varphi)^{\leftarrow}(v) \leq (\varphi \times \varphi)^{\leftarrow}((\varphi \times \varphi)^{\leftarrow}(v)) = (\varphi^2 \times \varphi^2)^{\leftarrow}(v) \leq \cdots \leq (\varphi^n \times \varphi^n)^{\leftarrow}(v)$. We conclude $v^{n+1} \circ u = v \circ (v^n \circ u) \leq (\varphi^n \times \varphi^n)^{\leftarrow}(v) \circ (\varphi^n \times \varphi^n)^{\leftarrow}(u) \leq (\varphi^n \times \varphi^n)^{\leftarrow}(v \circ u) \leq (\varphi^{n+1} \times \varphi^{n+1})^{\leftarrow}(u)$. □

Proposition 22. *For a well-chained $\top$-uniform space $(X, \mathcal{U})$ with $\top$-uniform base $\mathbb{B}$, a $\mathbb{B}$-contraction $\varphi : X \longrightarrow X$ is asymptotically regular.*

Proof. Let $x \in X$ and let $b \in \mathbb{B}$. By Lemma 2 there exists $v \in \mathbb{B}$ such that $v^n \circ b \leq (\varphi^n \times \varphi^n)^{\leftarrow}(b)$ for all $n \in \mathbb{N}$. For $\alpha_k \lhd \top = \bigvee_{n \in \mathbb{N}} v^n(x, \varphi(x))$, choose $n_k \in \mathbb{N}$ such that $v^{n_k}(x, \varphi(x)) \geq \alpha_k$. Then for all $n \geq n_k$ we have $v^n(x, \varphi(x)) \geq \alpha_k$, and for $n \geq n_k$ we obtain

$$\alpha_k \leq v^n \circ b(x, \varphi(x)) \leq (\varphi^n \times \varphi^n)^{\leftarrow}(b)(x, \varphi(x)) = b(\varphi^n, \varphi^{n+1}(x)).$$

Let now $u \in \mathcal{U}$. We have, for $b \in \mathbb{B}$,

$$\begin{aligned}
[b,u] &= \bigwedge_{x,y\in X} (b(x,y) \to u(x,y)) \\
&\leq \bigwedge_{n\geq n_k} (b(\varphi^n(x), \varphi^{n+1}(x)) \to u(\varphi^n(x), \varphi^{n+1}(x))) \\
&\leq \bigwedge_{n\geq n_k} (\alpha_k \to u(\varphi^n(x), \varphi^{n+1}(x))) \\
&= \alpha_k \to \bigwedge_{n\geq n_k} u(\varphi^n(x), \varphi^{n+1}(x)) \\
&\leq \alpha_k \to \bigvee_{n_0\in\mathbb{N}} \bigwedge_{n\geq n_0} u(\varphi^n(x), \varphi^{n+1}(x))
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
[b,u] &\leq \bigwedge_{k\in\mathbb{N}} \left(\alpha_k \to \bigvee_{n_0\in\mathbb{N}} \bigwedge_{n\geq n_0} u(\varphi^n(x), \varphi^{n+1}(x)) \right) \\
&= (\bigvee_{k\in\mathbb{N}} \alpha_k) \to \bigvee_{n_0\in\mathbb{N}} \bigwedge_{n\geq n_0} u(\varphi^n(x), \varphi^{n+1}(x)) \\
&= \top \to \bigvee_{n_0\in\mathbb{N}} \bigwedge_{n\geq n_0} u(\varphi^n(x), \varphi^{n+1}(x)) \\
&= \bigvee_{n_0\in\mathbb{N}} \bigwedge_{n\geq n_0} u(\varphi^n(x), \varphi^{n+1}(x)).
\end{aligned}$$

Therefore, we get $\top = \bigvee_{b\in\mathbb{B}}[b,u] \leq \bigvee_{n_0\in\mathbb{N}} \bigwedge_{n\geq n_0} u(\varphi^n(x), \varphi^{n+1}(x))$ and φ is asymptotically regular. $\square$

Proposition 23. *Let $(X,\mathcal{U})$ be a $\top$-uniform space, $\mathbb{B}$ be a $\top$-uniform base for $\mathcal{U}$ and $\varphi : X \longrightarrow X$ be an asymptotically regular $\mathbb{B}$-contraction. Then for each $x \in X$, $s = (s_X, s_L)$, with $s_X(n) = \varphi^n(x)$ and $s_L(n) = \top$ for all $n \in \mathbb{N}$, is a Cauchy $\top$-sequence.*

Proof. Let $u \in \mathcal{U}$. Then $w = u \wedge u^{-1} \in \mathcal{U}$ and we have $w = w^{-1} \leq u$ and $\bigvee_{b\in\mathbb{B}}[b,w] = \top$. Let $\alpha_k \lhd \top$ and choose $b_k \in \mathbb{B}$ such that $\alpha_k * b_k \leq w$. As φ is a $\mathbb{B}$-contraction, for $b_k \in \mathbb{B}$ we may choose $v_k \in \mathbb{B}$ such that $v_k \circ b_k \leq (\varphi \times \varphi)^{\leftarrow}(b_k)$. As φ is asymptotically regular, for v_k we may choose $n_k \in \mathbb{N}$ such that for all $n \geq n_k$ we have $v_k(\varphi^{n-1}(x), \varphi^n(x)) \geq \alpha_k$. We fix $n \geq n_k$. We show that for $j = 0,1,2,\ldots$ we have

$$\alpha_k^j \leq b_k(\varphi^n(x), \varphi^{n+j}(x)).$$

This is clear for $j = 0$. Now assume that $\alpha_k^j \leq b_k(\varphi^n(x), \varphi^{n+j}(x))$. Then

$$\alpha_k^{j+1} \leq v_k \circ b_k(\varphi^{n-1}(x), \varphi^{n+j}(x)) \leq (\varphi \times \varphi)^{\leftarrow}(b_k)(\varphi^{n-1}(x), \varphi^{n+j}(x)) = b_k(\varphi^n(x), \varphi^{n+j+1}(x)),$$

which completes the proof by induction. We conclude that for all $n > n_k$ and all $j = 0,1,2,3,\ldots,$

$$\alpha_k * \alpha_k^j \leq \alpha_k * b_k(\varphi^n(x), \varphi^{n+j}(x)) \leq w(\varphi^n(x), \varphi^{n+j}(x))$$

Hence, we have for all $j = 0,1,2,3,\ldots,$

$$\top = \bigvee_{k\in\mathbb{N}} \alpha_k^{j+1} \leq \bigvee_{k\in\mathbb{N}} \bigwedge_{n\geq n_k} w(\varphi^n(x), \varphi^{n+j}(x)).$$

Now let $\alpha_p \lhd \top$. Then there exists $k_p \in \mathbb{N}$ such that for all $n \geq n_{k_p}$ we have $\alpha_p \leq w(\varphi^n(x), \varphi^{n+j}(x))$ for all $j = 0, 1, 2, 3, \ldots$. Therefore, $\alpha_p \leq w(\varphi^n(x), \varphi^l(x)) = w(\varphi^l(x), \varphi^n(x))$ for all $n_{k_p} \leq n \leq l$ and we conclude that

$$\begin{aligned} \alpha_p &\leq \bigwedge_{n,l \geq n_{k_p}} w(\varphi^n(x), \varphi^l(x)) \\ &\leq \bigvee_{n \in \mathbb{N}} \bigwedge_{k,l \geq n} w(\varphi^k(x), \varphi^l(x)) \\ &\leq \bigvee_{n,m \in \mathbb{N}} \bigwedge_{k \geq n, l \geq m} w(\varphi^n(x), \varphi^l(x)). \end{aligned}$$

Taking the join over all α_p this yields

$$\top = \bigvee_{n,m \in \mathbb{N}} \bigwedge_{k \geq n, l \geq m} w(\varphi^n(x), \varphi^l(x)) \leq \bigvee_{n,m \in \mathbb{N}} \bigwedge_{k \geq n, l \geq m} u(\varphi^n(x), \varphi^l(x))$$

and $s = (\varphi^n(x), \top)$ is a Cauchy $\top$-sequence. $\square$

In view of Proposition 23 a weaker completeness concept will be appropriate here. We call a $\top$-uniform space *weakly sequentially complete* if all Cauchy $\top$-sequences $s = (s_X, s_L)$ with $s_L(n) = \top$ for all $n \in \mathbb{N}$ converge to some point $x_0 \in X$.

Putting everything together, we obtain the desired fixed point theorem.

Theorem 4. *Let the $\top$-uniform space $(X, \mathcal{U})$ be weakly sequentially complete, well-chained and a T2-space . Furthermore, let $\mathbb{B}$ be a $\top$-uniform base for $\mathcal{U}$ and $\varphi : X \longrightarrow X$ be a $\mathbb{B}$-contraction. Then φ has a unique fixed point $a = \varphi(a)$ and for each $x \in X$, the $\top$-sequence s: $\mathbb{N} \longrightarrow X \times L^*$, with $s_X(n) = \varphi^n(x), s_L(n) = \top$ for all $n \in \mathbb{N}$, converges to a.*

Proof. From Proposition 22 we see that φ is asymptotically regular and Proposition 23 tells us that $(\varphi^n(x), \top)$ is a Cauchy $\top$-sequence. As $(X, \mathcal{U})$ is sequentially complete, this $\top$-sequence converges to a point $a \in X$. As φ is uniformly continuous, the $\top$-sequence $(\varphi^{n+1}, \top)$ converges to $\varphi(a)$ and as $(\varphi^{n+1}, \top)$ is a subsequence of $(\varphi^n, \top)$, it also converges to a. The T2-property implies $a = \varphi(a)$ and a is a fixed point of φ.

It remains to show that the fixed point is unique. Assume $a = \varphi(a)$ and $b = \varphi(b)$. Then for each $n \in \mathbb{N}$ we have $a = \varphi^n(a)$ and $b = \varphi^n(b)$. Let $u \in \mathbb{B}$ and choose $v \in \mathbb{B}$ such that $v \circ u \leq (\varphi \times \varphi)^{\leftarrow}(u)$. As $(X, \mathcal{U})$ is well-chained, for $\alpha_k \lhd \top$ we may choose $n_k \in \mathbb{N}$ such that $\alpha_k \leq v^{n_k}(a, b)$ and we have with Lemma 2,

$$\alpha_k \leq v^{n_k}(a, b) \leq v^{n_k} \circ u(a, b) \leq u(\varphi^{n_k}(a), \varphi^{n_k}(b)) = u(a, b).$$

Hence, $\top = \bigvee_{k \in \mathbb{N}} \alpha_k \leq u(a, b)$. This is true for any $u \in \mathbb{B}$ and the T2-property yields $a = b$. $\square$

8. A Fixed Point Theorem for Probabilistic Metric Spaces

In the sequel, for notation and concepts, we refer to [12,21]. We again denote Δ^+ the set of *distance distribution functions* $\varphi : [0, \infty] \longrightarrow [0, 1]$. The top element in Δ^+ is denoted by $\varepsilon_0 = \varphi_{0,1}$. Furthermore, we consider a fixed left-continuous *t-norm* $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ on $[0, 1]$, that is, a quantale operation on $[0, 1]$ and endow Δ^+ with the quantale operation $\circledast$ defined by $\varphi \circledast \psi(t) = \bigvee_{r+s=t} \varphi(s) * \psi(t)$ for all $t \geq 0$. A *probabilistic metric space* [12] (sometimes called a *Menger space*) is a set X together with a probabilistic metric $d : X \times X \longrightarrow \Delta^+$ with the properties

(PM1) for all $x \in X$, $d(x, x) = \varepsilon_0$;
(PM2) for all $x, y, z \in X$ and $s, t \in [0, \infty]$, $d(x, y)(s) * d(y, z)(t) \leq d(x, y)(s + t)$;

(PM3) for all $x, y \in X$, $d(x,y) = d(y,x)$. Hence, a probabilistic metric space is an L-metric space for the quantale $\mathsf{L} = (\Delta^+, \leq, \circledast)$. If the probabilistic metric satisfies

(PM4) $d(x,y) = \varepsilon_0$ implies $x = y$

Then we call the probabilistic metric space *separated*.

Let (X,d) be a probabilistic metric space. Following [1,15,18] we define a $\top$-uniform space for $\mathsf{L} = ([0,1], \leq, *)$ by the $\top$-uniform base $\mathbb{B}_d = \{d(\cdot,\cdot)(t) \ : \ t > 0\}$, and denote it by $(X, \mathcal{U}_d)$. So we have $u \in [0,1]^{X \times X}$ is in $\mathcal{U}_d$ if $1 = \bigvee_{t>0} [d(\cdot,\cdot)(t), u]$.

We note that [18] considers a more general situation and uses the set $\Delta^+(L)$ of distance distribution functions with values in L, $\varphi : [0,\infty] \longrightarrow L$, and considers L-probabilistic metric spaces. Our Δ^+ is then $\Delta^+([0,1])$. In this setting, it appears natural to associate a $\top$-uniformity (for the quantale $(L, \leq, *)$) for an L-probabilistic metric space.

Proposition 24. *A probabilistic metric space (X,d) is separated if and only if $(X,\mathcal{U}_d)$ is a T2-space.*

Proof. First let (X,d) be separated. If $u(x,y) = 1$ for all $u \in \mathcal{U}_d$, then for all $t > 0$ we have $d(x,y)(t) = 1$, i.e., $d(x,y) = \epsilon_0$ and hence, $x = y$.

Conversely, if $(X,\mathcal{U}_d)$ is a T2-space, let $d(x,y) = \epsilon_0$. Then for all $t > 0$ we have $d(x,y)(t) = 1$. For $u \in \mathcal{U}_d$ then $1 = \bigvee_{t>0}[d(\cdot,\cdot)(t), u] \leq \bigvee_{t>0}(d(x,y)(t) \to u(x,y)) = u(x,y)$. Hence, $x = y$ and the proof is complete. □

For a probabilistic metric space (X,d), a self-mapping $\varphi : X \longrightarrow X$ is called a *contraction mapping* [12,22] with *contraction constant* $0 < \alpha < 1$, if $d(x,y)(t/\alpha) \leq d(\varphi(x), \varphi(y))(t)$ for all $x, y \in X$ and all $t > 0$.

Proposition 25. *Let (X,d) be a probabilistic metric space. If $\varphi : X \longrightarrow X$ is a contraction mapping then φ is a $\mathbb{B}_d$-contraction.*

Proof. Let $x, y \in X$ and let $t > 0$. Define $s = t(1-\alpha)/\alpha > 0$ with the contraction constant $0 < \alpha < 1$. Then $s + t = t/\alpha$ and we have for any $z \in X$

$$d(x,z)(s) * d(z,y)(t) \leq d(x,y)(s+t) = d(x,y)(t/\alpha) \leq d(\varphi(x), \varphi(y)).$$

So for $d(\cdot,\cdot)(t) \in \mathbb{B}_d$ we choose $d(\cdot,\cdot)(s) \in \mathbb{B}_d$ and find $d(\cdot,\cdot)(s) \circ d(\cdot,\cdot)(t) \leq (\varphi \times \varphi)^{\leftarrow}(d(\cdot,\cdot)(t))$. □

The following concept is used in [22] for a fixed point theorem for probabilistic metric spaces. A probabilistic metric space (X,d) is called (ϵ,λ)-*chainable* [22] if for each $x, y \in X$ there exists a finite sequence $x = z_0, z_1, z_2, \ldots, z_{n-1}, z_n = y$ such that $d(z_k, z_{k+1})(\epsilon) > 1 - \lambda$ for all $k = 0, 1, 2, \ldots, n-1$.

Proposition 26. *Let (X,d) be a probabilistic metric space. Then $(X,\mathcal{U}_d)$ is well-chained if and only if (X,d) is (t,α)-chainable for all $t > 0$ and $0 < \alpha < 1$.*

Proof. Let $(X,\mathcal{U}_d)$ be well-chained. As for any $t > 0$, $d(\cdot,\cdot)(t) \in \mathcal{U}^d$ we have for $x, y \in X$,

$$1 = \bigvee_{n \in \mathbb{N}} (d(\cdot,\cdot)(t))^n(x,y) = \bigvee_{n \in \mathbb{N}} \bigvee_{x = z_0, z_1, \ldots, z_n = y} d(x,z_1)(t) * d(z_1,z_2)(t) * \cdots * d(z_{n-1},y)(t).$$

Hence, for $0 < \alpha < 1$ there is $n \in \mathbb{N}$ and a finite sequence $x = z_0, z_1, \ldots, z_n = y$ such that $1 - \alpha < d(x,z_1)(t) * d(z_1,z_2)(t) * \cdots * d(z_{n-1},y)(t) \leq d(z_k, z_{k+1})(t)$ for all $k = 0, 1, 2, \ldots, n-1$.

Now let for all $x, y \in X$, $t > 0$ and $0 < \alpha < 1$ there be a finite sequence $x = z_0, z_1, \ldots, z_n = y$ such that $1 - \alpha < d(z_k, z_{k+1})(t)$ for all $k = 0, 1, 2, \ldots, n-1$. Let $u \in \mathcal{U}_d$. Then $1 = \bigvee_{t>0}[d(\cdot,\cdot)(t), u]$ and hence, for $\alpha < 1$ there is $t_\alpha > 0$ such that $\alpha \leq [d(\cdot,\cdot)(t_\alpha), u]$.

We choose a finite sequence $x = z_0, z_1, \ldots, z_{n_\alpha} = y$ such that $\alpha < d(z_k, z_{k+1})(t_\alpha)$ for all $k = 0, 1, 2, \ldots, n_\alpha - 1$. Then

$$\begin{aligned}
\alpha &\leq \bigwedge_{x_1, x_2 \in X} (d(x_1, x_2)(t_\alpha) \to u(x_1, x_2)) \\
&\leq \bigwedge_{k=0}^{n_\alpha - 1} (d(z_k, z_{k+1})(t_\alpha) \to u(z_k, z_{k+1})) \\
&\leq \left(\bigvee_{k=0}^{n_\alpha - 1} (d(z_k, z_{k+1})(t_\alpha) \right) \to \left(\bigvee_{k=0}^{n_\alpha - 1} (u(z_k, z_{k+1}) \right) \\
&\leq \alpha \to \left(\bigvee_{k=0}^{n_\alpha - 1} (u(z_k, z_{k+1}) \right).
\end{aligned}$$

Hence, we obtain $\alpha * \alpha \leq u^{n_\alpha}(x, y)$ and finally $1 = \bigvee_{\alpha < 1} \alpha * \alpha \leq \bigvee_{\alpha < 1} u^{n_\alpha}(x, y) \leq \bigvee_{n \in \mathbb{N}} u^n(x, y)$ and $(X, \mathcal{U}_d)$ is well-chained. □

We shall therefore call a probabilistic metric space (X, d) *well-chained* if it is (t, α)-chainable for all $t > 0$ and $0 < \alpha < 1$.

A sequence $(x_1, x_2, \ldots.)$ in a probabilistic metric space (X, d) is called a *strong Cauchy sequence* [12] if for all $t > 0$ and $0 < \alpha < 1$ there exists $n_0 \in \mathbb{N}$ such that $(x_n, x_m) \in U(t, \alpha)$ for all $m, n \geq n_0$. It is called *strongly convergent to* $x_0 \in X$ [12] if for all $t > 0$ and $0 < \alpha < 1$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in N(x_0, t, \alpha)$ for all $n \geq n_0$. Here, for $t > 0$ it is defined $U(t, \alpha) = \{(x, y) \in X \times X \,:\, d(x, y)(t) > 1 - \alpha\}$ and $N(x_0, t, \alpha) = \{y \in X \,:\, d(x, y)(t) > 1 - \alpha\}$.

The probabilistic metric space (X, d) is called *complete* if every strong Cauchy sequence is strongly convergent to a point in X, [12].

Proposition 27. *A probabilistic metric space (X, d) is complete if and only if $(X, \mathcal{U}_d)$ is weakly sequentially complete.*

Proof. Let (X, d) be complete and let $s = (s_X, s_L)$ be a Cauchy $\top$-sequence in $(X, \mathcal{U}_d)$ with $s_L(n) = 1$ for all $n \in \mathbb{N}$. Then $\bigvee_{m,n \in \mathbb{N}} \bigwedge_{k \geq n, l \geq m} d(s_X(k), s_X(l))(t) = 1$ for all $t > 0$. Let $0 < \alpha < 1$. Then there exists $m, n \in \mathbb{N}$ such that for all $k \geq n, l \geq m$ we have $d(s_X(k), s_X(l)) > 1 - \alpha$ and hence, $(s_X(n))$ is a strong Cauchy sequence in (X, d). Therefore, there exists $x_0 \in X$ such that for all $t > 0$ and $0 < \alpha < 1$ there exists $n_0 \in \mathbb{N}$ such that $d(x_0, s_X(n))(t) > 1 - \alpha$, that is, we have $\bigvee_{n_0 \in \mathbb{N}} \bigwedge_{n \geq n_0} d(x_0, s_X(n))(t) = 1$ for all $t > 0$ and s converges to x_0 in $(X, \mathcal{U}_d)$.

Now let $(X, \mathcal{U}_d)$ be weakly sequentially complete and let (x_n) be a strong Cauchy sequence in (X, d). Then for all $t > 0$ and $0 < \alpha < 1$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $d(x_n, x_m) > 1 - \alpha$. Hence, for all $t > 0$ we have $\bigvee_{n_0 \in \mathbb{N}} \bigwedge_{n,m \geq n_0} d(x_n, x_m)(t) = 1$ and therefore $s = (s_X, s_L)$ defined by $s_X(n) = x_n, s_L(n) = \top$ is a Cauchy $\top$-sequence in $(X, \mathcal{U}_d)$. Hence, there exists $x_0 \in X$ such that for all $t > 0$ we have $\bigvee_{n_0 \in \mathbb{N}} \bigwedge_{n \geq n_0} (d(x_0, s_X(n))) = 1$ and we have for all $t > 0$ and $0 < \alpha < 1$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $d(x_0, x_n)(t) > 1 - \alpha$, that is, (x_n) is strongly convergent to x_0 in (X, d). □

We can finally formulate the desired fixed point theorem.

Theorem 5. *Let (X, d) be a separated, well-chained and complete probabilistic metric space and $\varphi : X \longrightarrow X$ be a contraction mapping. Then there is a unique $a \in X$ such that $a = \varphi(a)$ and for each $x \in X$ the sequence $(\varphi^n(x))_{n \in \mathbb{N}}$ strongly converges to a.*

Proof. The Propositions 24, 25 and 27 ensure that the $\top$-uniform space $(X, \mathcal{U}_d)$ is well-chained, weakly complete and is a T2-space. Furthermore, by Proposition 25, φ is a $\mathbb{B}_d$-contraction. Hence, Theorem 4 ensures that there exists a unique fixed point and that in $(X, \mathcal{U}_d)$, the $\top$-sequence $(\varphi^n(x), \top)$ converges to this fixed point for any point $x \in X$. Hence, the sequence $(\varphi^n(x))$ strongly converges to the fixed point in (X, d). $\square$

9. Conclusions

We defined sequential completeness for $\top$-quasi-uniform spaces and $\top$-uniform spaces using the recently introduced $\top$-sequences [7].

$\top$-uniform spaces, like related lattice-valued generalizations of uniform spaces, allow us to study different kinds of spaces, such as metric spaces, uniform spaces, probabilistic metric spaces, from one viewpoint. We illustrated the advantage of such a lattice-valued approach by proving a fixed point theorem for $\top$-uniform spaces – which demonstrates that the definition of sequential completeness is working and useful – which transforms in the subcase of probabilistic metric spaces to a fixed point theorem in that setting.

If we only take the fixed point Theorem 4, it seems that we can restrict the theory to ordinary sequences, identified with $\top$-sequences where the elements of the sequence all have maximum grade $\top$. However, in order to show that completeness [1] implies sequential completeness, we needed a characterization of sequential completeness by $\top$-filters with countable $\top$-bases. Such a characterization requires the use of our more general $\top$-sequences. As $\top$-sequences appear as a special instance of the concept of $\top$-net, this generality is even more justified as only if we allow grades of elements other than $\top$, do we get the usual "duality" between $\top$-filter convergence and $\top$-net convergence, see [7].

Results on completenes and fixed point theorems have a long history in the theory of probabilistic metric spaces, see e.g., [12,22]. Our fixed point theorem seems different from existing results in this realm, as we had to impose a connectedness condition on the space. This condition is "inherited" from the fixed point theorem for $\top$-uniform spaces, as our concept of contraction mapping is motivated by a definition of Taylor [8] for uniform spaces and is there, in the special case of metric spaces, weaker than the usual one. A similar connectedness property is also used in fixed point theorems for "local contractions" in probabilistic metric spaces [22]. It would be interesting to relate this connectedness condition to conditions such as the boundedness of the trajectories $\{\varphi^n(x), n \in \mathbb{N}\}$, see for example [12], that ensure the existence and uniqueness of fixed points of contraction mappings in the theory of probabilistic metric spaces.

In the theory of uniform spaces, Tarafdar [23] gave a definition of a contraction mapping that allows us to prove a fixed point theorem without further assumptions on the spaces (such as well-chainedness). In this way a closer analogue of Banach's contraction principle was obtained. The keypoint of his definition is the representation of a uniformity by a family of pseudometrics. It would therefore be useful to find a corresponding representation of $\top$-uniformities by families of L-metrics.

From a categorical point of view, the category of $\top$-uniform spaces is not completely satisfactory, as it does not possess "nice" function spaces, making it Cartesian closed. Hence, it makes sense to study completeness and sequential completeness also in the supercategory of $\top$-uniform limit spaces [17]. We will address this research question in the future.

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