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Non-Existence of Real Hypersurfaces with Parallel Structure Jacobi Operator in $S^6(1)$

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Abstract: It is well known that the sphere $S^6(1)$ admits an almost complex structure J which is nearly Kähler. If M is a hypersurface of an almost Hermitian manifold with a unit normal vector field N, the tangent vector field $\xi = -JN$ is said to be characteristic or the Reeb vector field. The Jacobi operator with respect to ξ is called the structure Jacobi operator, and is denoted by $l = R(\cdot, \xi)\xi$, where R is the curvature tensor on M. The study of Riemannian submanifolds in different ambient spaces by means of their Jacobi operators has been highly active in recent years. In particular, many recent results deal with questions around the existence of hypersurfaces with a structure Jacobi operator that satisfies conditions related to their parallelism. In the present paper, we study the parallelism of the structure Jacobi operator of real hypersurfaces in the nearly Kähler sphere $S^6(1)$. More precisely, we prove that such real hypersurfaces do not exist.

Keywords: real hypersurface; structure Jacobi operator; hopf hypersurface

MSC: 53B25; 53B35

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1. Introduction

If an almost Hermitian manifold $(\widetilde{M}, g, \widetilde{\nabla}, J)$ has a parallel almost complex structure J, i.e., $\widetilde{\nabla} J = 0$, then it is a Kähler manifold. If, however, a weaker condition holds, that is, if the tensor $G(X,Y) = (\widetilde{\nabla}_X J)Y$ is skew-symmetric, the manifold is nearly Kähler. It was shown in [1] that nearly Kähler manifolds are locally Riemannian products of six-dimensional nearly Kähler manifolds, certain homogenous nearly Kähler spaces, and twistor spaces over quaternionic Kähler manifolds with positive scalar curvature, endowed with the canonical nearly Kähler metric. It was proved in [2] that the only homogeneous six-dimensional nearly Kähler manifolds are the following compact 3-symmetric spaces: the nearly Kähler six-dimensional sphere $S^6(1)$, the manifold $S^3 \times S^3$, the projective space CP^3, and the flag manifold $SU(3)/U(1) \times U(1)$, where the last three are not endowed with their standard metrics.

The Jacobi operator with respect to a tangent vector field X of a Riemannian manifold M with the curvature tensor R is provided by $R(\cdot, X)X$, and is inspired in a natural way by the well-known differential equation of Jacobi fields along geodesics. In particular, if M is a hypersurface of an almost Hermitian manifold with a unit normal vector field N, the tangent vector field $\xi = -JN$ is said to be characteristic, or the Reeb Vector field. In this particular setting, the Jacobi operator with respect to ξ is called the structure Jacobi operator, and is denoted by $l = R(\cdot, \xi)\xi$.

The study of Riemannian submanifolds in different ambient spaces by means of their Jacobi operators has been active in recent years. One of the reasons for this is that the conditions provided in terms of the structure Jacobi operator generate larger families then the analogue conditions provided in terms of the Riemannian tensor. In particular, many recent results deal with questions of the existence of hypersurfaces with structure Jacobi operators that satisfy conditions related to their parallelism. In [3], the real hypersurfaces of the complex space form with a Ricci tensor and structure Jacobi operator parallel with respect to the Reeb vector field were classified, while in [4] it was shown that the hypersurfaces of the complex space form with a structure Jacobi operator symmetric along the Reeb flow and commuting with the shape operator is a Hopf hypersurface. In [5], the classification of the hypersurfaces in complex two-plane Grasmannians with a structure Jacobi operator commuting with any other Jacobi operator or with the normal Jacobi operator was provided, and in [6] it was shown that there are no Hopf real hypersurfaces in complex two-plane Grasmannians with parallel structure operators. In [7], the class of real hypersurfaces in non-flat complex space forms with generalized ξ-parallel structure Jacobi operators was classified. In [8], the non-existence of the particular class of Hopf hypersurfaces in complex two-plane Grasmannians was provided. The non-existence of real hypersurfaces in non-flat complex space forms with structure Jacobi operators of the Lie–Codazzi type was proven in [9]. In [10], the non-existence of real hypersurfaces in non-flat complex space forms with recurrent structure Jacobi operators was shown. In particular, most of the known results deal with Kähler manifolds, where the parallelism of the almost complex structure somewhat simplifies the calculations. Here, we want to initiate a similar line of research with respect to the hypersurfaces in nearly Kähler manifolds, in particular, the homogeneous six-dimensional sphere $S^6(1)$, in terms of its structure Jacobi operator. We prove the following non-existence theorem.

Theorem 1. *There exist no real hypersurfaces with parallel structure Jacobi operators in $S^6(1)$.*

We note that the skew symmetry of the tensor G imposes a somewhat different approach to analizing hypersurfaces in nearly Kähler manifolds compared to the one in Kähler manifolds, necessitating the construction of a suitable moving frame along the hypersurface.

2. Preliminaries

We denote by $\langle , \rangle$ and g, respectively, the standard Euclidean metric in the space $\mathbb{R}^7$ and the metric on $S^6(1)$ induced by $\langle , \rangle$. The corresponding Levi-Civita connections we denote by D and $\bar{\nabla}$, respectively.

We will briefly recall the construction of the almost complex structure of $S^6(1)$. Namely, one can regard the space $\mathbb{R}^7$ as the space of purely imaginary Cayley numbers $\mathcal{O}$ and use the Cayley multiplication to introduce a vector cross product in $\mathbb{R}^7$, in the following way

$$u \times v = \frac{1}{2}(uv - vu).$$

This cross product is well defined in the space $\mathbb{R}^7$. Moreover, if we denote by $e_1, \dots, e_7$ an orthonormal basis of $\mathbb{R}^7$ then we have the following multiplication table.

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

Any orthonormal basis or frame that satisfies the relations of this table is called a G_2 basis or a G_2 frame. Then, for an arbitrary point $p \in S^6(1)$ and $X \in T_pS^6(1)$ we define a (1,1)-tensor field J by

$$J_pX = p \times X.$$

Then, one can show that J is an almost complex structure which is, moreover, Hermitian and nearly Kähler.

Let M be a Riemannian submanifold of the nearly Kähler sphere $S^6(1)$ Then, the $(2,1)$-tensor field G on $S^6(1)$ defined by $G(X,Y) = (\bar{\nabla}_X J)Y$, where $\bar{\nabla}$ is the Levi–Civita connection on $S^6(1)$, is skew-symmetric and satisfies

$$G(X, JY) + JG(X,Y) = 0, \qquad g(G(X,Y),Z) + g(G(X,Z),Y) = 0.$$

Moreover, following [11], we have

$$(\bar{\nabla}G)(X,Y,Z) = g(X,Z)JY - g(X,Y)JZ - g(JY,Z)X, \tag{1}$$

for the arbitrary vector fields X, Y, Z tangent to $S^6(1)$.

We denote by ∇ and $\nabla^{\perp}$ the Levi–Civita connection of M and the normal connection induced from $\bar{\nabla}$ in the normal bundle $T^{\perp}M$ of M in $S^6(1)$, respectively. Then, the formulas of Gauss and Weingarten are respectively provided by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

where X, Y are tangent, N is a normal vector field on M, and h and A_N are the second fundamental form and the shape operator with respect to the section N, respectively. The second fundamental form and the shape operator are related by $g(h(X,Y),\xi) = g(AX,Y)$. In addition, for the tangent vector fields X, Y, Z, and W we have the following Gauss and Codazzi equations

$$\begin{aligned} R(X,Y,Z,W) = {} & g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ & + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \end{aligned} \tag{2}$$

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{3}$$

where we denote by R the Riemannian curvature tensor of M, and consider that $(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y)$.

We denote by N the unit normal vector field of M and by $\xi = -JN$ the corresponding Reeb vector field. We denote by $\eta(X) = g(X,\xi)$ a 1-form on M. For a vector field X tangent to M, we set $JX = \phi(X) + \eta(X)N$, where $\phi(X)$ is the tangential component of JX. It then follows that ϕ is a $(1,1)$ tensor field on M and that (ϕ, ξ, η, g) defines an almost contact metric structure on M, that is,

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \eta \circ \phi = 0, \qquad \phi(\xi) = 0,$$

and $g(\phi X, \phi Y) = (X,Y) - \eta(X)\eta(Y)$ for X, Y tangent to M.

Let $\mathcal{D} = \operatorname{Ker} \eta = \{X \in \ TM \mid \eta(X) = 0\}$. Then, $\mathcal{D}$ is a four-dimensional smooth distribution on M, which is J-invariant.

A real hypersurface of an almost Hermitian manifold is Hopf if ξ is principal, that is, if $A\xi = \alpha\xi$ for a certain function α on the submanifold. Recall, see [12], that, for a Hopf hypersurface in $S^6(1)$, the function α is a constant. Namely, from (3), by taking $X = \xi$, we obtain that

$$(\xi\alpha)g(\xi,Y) + \alpha g(\nabla\xi, Y) - g(A\nabla_Y\xi, Y) = (Y\alpha) - g(A\nabla_Y\xi, \xi).$$

Hence, $Y\alpha = (\xi\alpha)g(\xi,Y)$.Note that, since there are no 4-dimensional almost complex submanifolds in $S^6(1)$, see [13], the distibution $\mathcal{D}$ is not integrable and there exist vector fields $X, Y \in \mathcal{D}$ such that $[X,Y]$ has a non-vanishing component in direction of ξ. Then $0 = [X,Y]\alpha = (\xi\alpha)g([X,Y],\xi)$ yielding $\xi\alpha = 0$. Consequently, α is a constant.

The classification of the Hopf hypersurfaces of the sphere $S^6(1)$ is well known; see [12]. Such hypersurfaces are either totally geodesic spheres or tubes around almost complex curves. Hence, every Hopf hypersurface in $S^6(1)$ has exactly one, two, or three distinct principal curvatures at each point. The umbilical varieties are open subsets of geodesic hyperspheres with a principal curvature α of multiplicity 5, while the non-umbilical varieties are open subsets of tubes around almost complex curves. If M is an open part of a tube around a totally geodesic almost complex curve in $S^6(1)$, then M has exactly two distinct principal curvatures, namely, α of multiplicity 3 and μ of multiplicity 2. Alternatively, if M is an open part of a tube around an almost complex curve of type (I), (II), or (III), then it has three distinct principal curvatures, α of multiplicity 3 and μ and λ of multiplicity 1; see [14].

3. The Moving Frame for Hypersurfaces in $S^6(1)$

Here, we present one of the more convenient moving frames to work with and the relationship between the connection coefficients in it; for details, see [15]. We also refer readers to [16–18].

For each unit vector field $E_1 \in \mathcal{D}$, let $E_2 = JE_1$, $E_3 = G(E_1, \xi)$, $E_4 = JE_3$. Then, the set $\{E_1, E_2, E_3, E_4, E_5 = \xi\}$ is a local orthonormal frame on M; see [15]. Moreover, the following holds.

Lemma 1 ([15])**.** *For the previously defined orthonormal frame, the following relations hold:*

$$\begin{array}{llll} G(E_1,E_2)=0, & G(E_1,E_3)=-\xi, & G(E_1,E_4)=N, & G(E_1,\xi)=E_3, \\ G(E_1,N)=-E_4, & G(E_2,E_3)=-N, & G(E_2,E_4)=\xi, & G(E_2,\xi)=-E_4, \\ G(E_2,N)=-E_3, & G(E_3,E_4)=0, & G(E_3,\xi)=-E_1, & G(E_3,N)=E_2, \\ G(E_4,\xi)=E_2, & G(E_4,N)=E_1. & & \end{array} \tag{4}$$

Note that such a moving frame is not uniquely determined and depends on the choice of the vector field $E_1 \in \mathcal{D}$.

For one such frame, let us denote by

$$g_{ij}^k = \langle D_{E_i}E_j, E_k\rangle, \qquad h_{ij} = \langle D_{E_i}E_j, N\rangle, \qquad 1 \le i,j,k \le 5. \tag{5}$$

The connection D is metric and the second fundamental formsymmetric, meaning that we have $g_{ij}^k = -g_{ik}^j$ and $h_{ij} = h_{ji}$.

Lemma 2. *For the previously defined coefficients, we have*

$$\begin{array}{llll} g_{12}^3=-g_{11}^4, & g_{12}^4=g_{11}^3, & h_{11}=-g_{12}^5, & h_{12}=g_{11}^5, \\ g_{22}^3=-g_{21}^4, & g_{22}^4=g_{21}^3, & g_{22}^5=-g_{11}^5, & h_{22}=g_{21}^5, \\ g_{32}^3=-g_{31}^4, & g_{32}^4=g_{31}^3, & h_{13}=1-g_{32}^5, & h_{23}=g_{31}^5, \\ g_{42}^3=-g_{41}^4, & g_{42}^4=g_{41}^3, & h_{14}=-g_{42}^5, & h_{24}=-1+g_{41}^5, \\ g_{52}^3=-1-g_{51}^4, & g_{52}^4=g_{51}^3, & h_{15}=-g_{52}^5, & h_{25}=g_{51}^5, \\ g_{32}^5=2+g_{14}^5, & g_{42}^5=-g_{13}^5, & g_{31}^5=-g_{24}^5, & g_{41}^5=2+g_{23}^5, \\ h_{33}=-g_{43}^5, & h_{34}=g_{33}^5, & g_{44}^5=-g_{33}^5, & h_{44}=g_{43}^5, \\ h_{35}=-g_{54}^5, & h_{45}=g_{53}^5. & & \end{array}$$

Proof. The above Lemma follows from (4) and the relation

$$\bar\nabla_X(JY) = G(X,Y) + J(\bar\nabla_X Y)$$

by taking $X \in \{E_1, \ldots, \xi\}$ and $Y \in \{E_1, \ldots, \xi, N\}$.

For $X = E_1, Y = E_1$, we obtain

$$g_{12}^3 = -g_{11}^4, \qquad g_{12}^4 = g_{11}^3, \qquad h_{11} = -g_{12}^5, \qquad h_{12} = g_{11}^5.$$

For $X = E_2, Y = E_1$, we have

$$g_{22}^3 = -g_{21}^4, \qquad g_{22}^4 = g_{21}^3, \qquad g_{22}^5 = -h_{12} = -g_{11}^5, \qquad h_{22} = g_{21}^5.$$

The other relations follow in a similar way. □

Lemma 3. *The differentiable functions (5) satisfy*

$$\begin{aligned} &g_{52}^5 = g_{11}^2 + g_{13}^4, && g_{51}^5 = -g_{21}^2 - g_{23}^4, && g_{54}^5 = g_{31}^2 + g_{33}^4, \\ &g_{53}^5 = -g_{41}^2 - g_{43}^4, && h_{55} = -g_{51}^2 - g_{53}^4. \end{aligned} \tag{6}$$

Proof. By taking $X = E_1, Y = E_1, Z = E_3$ in (1), we obtain that $g_{52}^5 = g_{11}^2 + g_{13}^4$. Similarly, by taking $X = E_2, Y = E_1, Z = E_3$, we obtain $g_{51}^5 = -g_{21}^2 - g_{23}^4$ and for $X = E_3$, $Y = E_1$, $Z = E_3$, we obtain $g_{54}^5 = g_{31}^2 + g_{33}^4$. Finally, for $X = E_4, Y = E_1, Z = E_3$ and $X = \xi, Y = E_1$, $Z = E_3$, respectively, we have $g_{53}^5 = -g_{41}^2 - g_{43}^4$ and $h_{55} = -g_{51}^2 - g_{53}^4$ which completes the proof. □

If the hypersurface M is not Hopf, then the vector field $A\xi$ has a non-vanishing projection $A(\xi) - g(A\xi, \xi)\xi$ on $\mathcal{D}$ and therefore, there is a unique smooth vector field $E_1 \in \mathcal{D}$ such that $A(\xi) - g(A\xi, \xi)\xi = \beta E_1, \beta > 0$. If M is a Hopf hypersurface then we have that $A\xi = \alpha\xi$. Thus, for any smooth vector field $E_1 \in \mathcal{D}$ we may write $A\xi = 0 \cdot E_1 + \alpha\xi$. Hence, regardless of the case there exists a smooth vector field $E_1 \in \mathcal{D}$ and differentiable functions α and β such that

$$A\xi = \beta E_1 + \alpha\xi. \tag{7}$$

Because the components of $A\xi$ in the direction of E_2, E_3, E_4vanish, we have

$$\begin{aligned} &g_{13}^4 = -g_{11}^2 - \beta, && g_{23}^4 = -g_{21}^2, && g_{33}^4 = -g_{31}^2, \\ &g_{43}^4 = -g_{41}^2, && g_{53}^4 = -g_{51}^2 - \alpha. \end{aligned}$$

Now, we can use the Gauss equations to obtain further relations between the coefficients. In the following Lemma, we list those that we directly use in further calculations.

Lemma 4. *For the coefficients (5), the following relations hold:*

$$\begin{aligned} \xi(g_{11}^5) &= 1 + (g_{11}^5)^2 + g_{12}^5 g_{21}^5 - g_{13}^5 g_{24}^5 + g_{12}^5 g_{51}^2 + g_{21}^5 g_{51}^2 + g_{13}^5 g_{51}^3 - g_{24}^5 g_{51}^3 \\ &\quad + 2g_{51}^4 + g_{23}^5 g_{51}^4 + g_{14}^5(2 + g_{23}^5 + g_{51}^4) - g_{12}^5\alpha + g_{11}^2\beta - \beta^2, \\ E_2(g_{13}^5) &= -g_{11}^5 g_{21}^3 + g_{12}^5 g_{21}^4 + g_{21}^2 g_{23}^5 + g_{21}^3 g_{33}^5 - g_{12}^5 g_{41}^2 - g_{21}^5 g_{41}^2 - g_{13}^5 g_{41}^3 \\ &\quad + g_{24}^5 g_{41}^3 - 2g_{41}^4 - g_{23}^5 g_{41}^4 - g_{14}^5(g_{21}^2 + g_{41}^4) + g_{21}^4 g_{43}^5 + g_{13}^5\beta + g_{24}^5\beta + E_4(g_{11}^5), \\ \xi(g_{13}^5) &= g_{13}^5 g_{33}^5 + g_{14}^5 g_{43}^5 - g_{14}^5 g_{51}^2 + g_{23}^5 g_{51}^2 + g_{11}^5(g_{13}^5 - g_{51}^3) - \alpha + g_{33}^5 g_{51}^3 \\ &\quad + g_{43}^5 g_{51}^4 + g_{12}^5(1 + g_{23}^5 + g_{51}^4) - 2g_{14}^5\alpha + g_{11}^4\beta, \\ \xi(g_{14}^5) &= -g_{14}^5 g_{33}^5 + g_{13}^5 g_{34}^5 + g_{13}^5 g_{51}^2 + g_{24}^5 g_{51}^2 + g_{12}^5(g_{24}^5 - g_{51}^3) + g_{34}^5 g_{51}^3 \\ &\quad + g_{11}^5(g_{14}^5 - g_{51}^4) - g_{33}^5 g_{51}^4 + 2g_{13}^5\alpha - g_{11}^3\beta, \\ \xi(g_{21}^5) &= 2g_{51}^3 + 2g_{23}^5 g_{51}^3 + g_{24}^5(3 + 2g_{51}^4) + g_{11}^5(-2g_{51}^2 + \alpha) + g_{21}^2\beta, \\ E_1(g_{23}^5) &= -g_{11}^2 g_{13}^5 + g_{14}^5 g_{21}^2 + g_{11}^5 g_{21}^3 - g_{11}^3 g_{21}^5 - g_{21}^2 g_{23}^5 - g_{11}^2 g_{24}^5 - g_{21}^3 g_{33}^5 \\ &\quad - g_{11}^4(g_{11}^5 + g_{33}^5) + g_{11}^3 g_{43}^5 - g_{21}^4 g_{43}^5 - g_{21}^4 g_{12}^5 - 2g_{24}^5\beta + E_2(g_{13}^5), \\ E_2(g_{23}^5) &= -g_{13}^5 g_{21}^2 - g_{11}^5 g_{21}^4 - g_{21}^3 g_{21}^5 - g_{21}^2 g_{24}^5 - g_{21}^4 g_{33}^5 + 2g_{11}^5 g_{41}^2 - 2g_{41}^3 \end{aligned}$$

$$
\begin{aligned}
&-2g_{23}^5g_{41}^3-2g_{24}^5g_{41}^4+g_{213}g_{435}+E_4(g_{215}),\\
\xi(g_{23}^5)={}&-g_{33}^5+g_{23}^5g_{33}^5+g_{24}^5g_{43}^5+g_{43}^5g_{51}^3-g_{24}^5g_{51}^2-g_{21}^5g_{51}^3-g_{33}^5g_{51}^4\\
&+g_{13}^5(g_{21}^5-g_{51}^2)-g_{11}^5(1+g_{23}^5+g_{51}^4)-2g_{24}^5\alpha+g_{21}^4\beta,\\
E_1(g_{24}^5)={}&-g_{11}^2g_{14}^5-g_{13}^5g_{21}^2+g_{12}^5g_{21}^3+g_{11}^5g_{21}^4-g_{11}^4g_{21}^5+g_{11}^2g_{23}^5-g_{21}^2g_{24}^5\\
&+g_{11}^3(g_{11}^5-g_{33}^5)+g_{21}^4g_{33}^5-g_{11}^4g_{34}^5-g_{21}^3g_{34}^5+\beta+2g_{23}^5\beta+E_2(g_{14}^5),\\
E_2(g_{24}^5)={}&-g_{14}^5g_{21}^2+g_{11}^5g_{21}^3-g_{21}^4g_{21}^5+g_{21}^2g_{23}^5-2g_{11}^5g_{31}^2+2g_{31}^3+2g_{23}^5g_{31}^3\\
&+2g_{24}^5g_{31}^4-g_{21}^3g_{33}^5-g_{21}^4g_{34}^5-E_3(g_{21}^5),\\
E_4(g_{24}^5)={}&-g_{13}^5g_{31}^2-g_{24}^5g_{31}^2-g_{21}^5g_{31}^3-g_{11}^5g_{31}^4-g_{31}^4g_{33}^5-g_{14}^5g_{41}^2+g_{23}^5g_{41}^2\\
&+g_{11}^5g_{41}^3-g_{33}^5g_{41}^3-g_{21}^5g_{41}^4-g_{34}^5g_{41}^4+g_{31}^3g_{43}^5-E_3(g_{23}^5),\\
\xi(g_{24}^5)={}&-g_{24}^5g_{33}^5-g_{34}^5+g_{23}^5g_{34}^5+g_{14}^5(g_{21}^5-g_{51}^2)+g_{23}^5g_{51}^2-g_{33}^5g_{51}^3+\alpha\\
&+g_{11}^5(-g_{24}^5+g_{51}^3)-g_{21}^5g_{51}^4-g_{34}^5g_{51}^4+2g_{23}^5\alpha-g_{21}^3\beta,\\
E_2(g_{33}^5)={}&2g_{21}^4+g_{14}^5g_{21}^4+g_{21}^4g_{23}^5+g_{21}^3g_{24}^5+g_{24}^5g_{31}^2+g_{21}^5g_{31}^3+g_{11}^5g_{31}^4\\
&+g_{13}^5(-g_{21}^3+g_{31}^2)+g_{31}^4g_{33}^5-g_{21}^2g_{34}^5-g_{21}^2g_{43}^5-g_{31}^3g_{43}^5+E_3(g_{23}^5),\\
\xi(g_{33}^5)={}&3-g_{13}^5g_{24}^5+g_{34}^5g_{43}^5-g_{34}^5g_{51}^2-g_{43}^5g_{51}^2-g_{13}^5g_{51}^3+g_{24}^5g_{51}^3+(g_{33}^5)^2\\
&+2g_{51}^4+g_{23}^5(3+g_{51}^4)+g_{14}^5(1+g_{23}^5+g_{51}^4)-2g_{34}^5\alpha-g_{43}^5\alpha+g_{31}^4\beta,\\
E_1(g_{34}^5)={}&-2g_{11}^3(1+g_{14}^5)+2g_{11}^4g_{24}^5-g_{13}^5g_{31}^2-g_{24}^5g_{31}^2+g_{12}^5g_{31}^3+g_{11}^5g_{31}^4\\
&+2g_{11}^2g_{33}^5+g_{31}^4g_{33}^5-g_{31}^3g_{34}^5+3g_{33}^5\beta+E_3(g_{14}^5),\\
E_2(g_{34}^5)={}&-2(1+g_{14}^5)g_{21}^3+2g_{21}^4g_{24}^5+g_{14}^5g_{31}^2-g_{23}^5g_{31}^2-g_{11}^5g_{31}^3+g_{21}^5g_{31}^4\\
&+2g_{21}^2g_{33}^5+g_{31}^3g_{33}^5+g_{31}^4g_{34}^5+E_3(g_{24}^5),\\
E_2(g_{43}^5)={}&-2g_{13}^5g_{21}^4-2g_{21}^3(1+g_{23}^5)+2g_{21}^2g_{33}^5+g_{13}^5g_{41}^2+g_{24}^5g_{41}^2+g_{21}^5g_{41}^3\\
&+g_{11}^5g_{41}^4+g_{33}^5g_{41}^4-g_{41}^3g_{43}^5+E_4(g_{23}^5).
\end{aligned}
$$

Proof. By taking $X=E_1, Y=\xi, Z=E_1, W=\xi$ into the Gauss Equation (2), we obtain that

$$
\begin{aligned}
\xi(g_{11}^5)={}&1+(g_{11}^5)^2+g_{12}^5g_{21}^5-g_{13}^5g_{24}^5+g_{12}^5g_{51}^2+g_{21}^5g_{51}^2+g_{13}^5g_{51}^3-g_{24}^5g_{51}^3\\
&+2g_{51}^4+g_{23}^5g_{51}^4+g_{14}^5(2+g_{23}^5+g_{51}^4)-g_{12}^5\alpha+g_{11}^2\beta-\beta^2.
\end{aligned}
$$

Similarly, the second and third equations are obtained by taking $X=E_1$, $Y=E_4$, $Z=E_1$, $W=\xi$ and $X=E_1, Y=\xi, Z=E_3, W=\xi$ into (2). The other equalities follow in a similar way. □

Now, by taking the results of Lemma 4 and calculating the Gauss equation for different choices of the vector fields appearing in it, we can obtain the covariant derivatives of coefficients in various directions. Here, we omit most of the obtained expressions due to the length of the list. However, we note that by taking $(X,Y,Z,W)=(E_2,E_3,E_3,E_4)$, $(X,Y,Z,W)=(E_2,E_4,E_3,E_4)$, and $(X,Y,Z,W)=(E_3,E_4,E_3,E_4)$ in (2), we obtain the last three equations, respectively, in the following lemma as a result.

Lemma 5. *The functions (5), α, and β satisfy*

$$
\begin{aligned}
E_2(\alpha)&=(-3g_{21}^5+g_{51}^2+\alpha)\beta,\\
E_3(\alpha)&=(3g_{24}^5+g_{51}^3)\beta,\\
E_4(\alpha)&=-(5+3g_{23}^5-g_{51}^4)\beta,\\
E_1(\beta)&=g_{13}^5-2g_{11}^5g_{51}^2+2g_{51}^3+2g_{14}^5g_{51}^3-2g_{13}^5g_{51}^4+g_{11}^5\alpha-\xi(g_{12}^5),\\
E_2(\beta)&=2(g_{11}^5)^2+g_{14}^5+2g_{12}^5g_{21}^5+g_{23}^5+2g_{14}^5g_{23}^5-2g_{13}^5g_{24}^5-g_{12}^5\alpha+g_{21}^5\alpha+g_{11}^2\beta,\\
E_3(\beta)&=-g_{11}^5(3+2g_{14}^5)-2g_{12}^5g_{24}^5+g_{33}^5+2g_{14}^5g_{33}^5-2g_{13}^5g_{34}^5-(g_{13}^5+g_{24}^5)\alpha+g_{11}^3\beta,
\end{aligned}
$$

$$E_4(\beta) = 2g_{11}^5 g_{13}^5 + g_{12}^5(3 + 2g_{23}^5) + 2g_{13}^5 g_{33}^5 + g_{43}^5 + 2g_{14}^5 g_{43}^5 - g_{14}^5\alpha + g_{23}^5\alpha + 2\alpha + g_{11}^4\beta,$$

$$0 = (3 + 2g_{14}^5)g_{21}^5 - 2(g_{11}^5 + g_{33}^5)g_{24}^5 + g_{34}^5(1 + 2g_{23}^5) + (g_{23}^5 - g_{14}^5 - 2)\alpha + (g_{31}^2 - g_{21}^3)\beta, \tag{8}$$

$$0 = g_{11}^5(3 + 2g_{23}^5) - g_{33}^5 - 2g_{23}^5 g_{33}^5 - 2g_{24}^5 g_{43}^5 + g_{24}^5\alpha + g_{13}^5(-2g_{21}^5 + \alpha) + (g_{41}^2 - g_{21}^4)\beta, \tag{9}$$

$$\begin{aligned} 0 = {} & 2g_{13}^5 g_{24}^5 - 4 - 3g_{23}^5 - g_{14}^5(3 + 2g23^5) - 2(g_{33}^5)^2 - 2g_{34}^5 g_{43^5} + g_{34}^5 \\ & - g_{43}^5\alpha + (g_{41}^3 - g_{31}^4)\beta = 0. \end{aligned} \tag{10}$$

4. Proof of the Main Theorem

Let $n_{ij}^k = g(\nabla_{E_i}(lE_j) - l(\nabla_{E_i}E_j), E_k)$. The condition of the parallel structure Jacobi operator is equivalent to $n_{ij}^k = 0, 1 \leq i, j, k \leq 5$.

From $n_{5i}^5 = 0, 1 \leq i \leq 4$, using Lemma 5, we have, respectively,

$$g_{11}^5\alpha\beta = 0, \qquad (1 + g_{21}^5\alpha)\beta = 0, \qquad g_{24}^5\alpha\beta = 0, \qquad (1 + g_{23}^5)\alpha\beta = 0. \tag{11}$$

We now treat the cases of Hopf and non-Hopf hypersurfaces separately.

Case 1: Suppose that M is a Hopf hypersurface, i.e., $\beta = 0$, ξ is an eigenvector field for the shape operator A, $A\xi = \alpha\xi$. Then, see Preliminaries, α is a constant.

We have from Lemma 5 and $E_i(\beta) = 0, i = 2, 3, 4$ that

$$\begin{aligned} & g_{14}^5 + g_{23}^5 + 2((g_{11}^5)^2 + g_{12}^5 g_{21}^5 + g_{14}^5 g_{23}^5 - g_{13}^5 g_{24}^5) + (g_{21}^5 - g_{12}^5)\alpha = 0, \\ & g_{33}^5 - g_{11}^5(3 + 2g_{14}^5) + 2(g_{14}^5 g_{33}^5 - g_{12}^5 g_{24}^5 - g_{13}^5 g_{34}^5) - (g_{13}^5 + g_{24}^5)\alpha = 0, \\ & g_{12}^5(3 + 2g_{23}^5) + 2g_{13}^5(g_{11}^5 + g_{33}^5) + g_{43}^5 + 2g_{14}^5 g_{43}^5 + (2 - g_{14}^5 + g_{23}^5)\alpha = 0. \end{aligned} \tag{12}$$

Recall that the principal curvatures are continuous and smooth functions, and that, around any point in M, there is a local orthonormal frame, consisting of principal directions, diagonalizing the shape operator. Note that in this case we have the freedom to choose the vector field $E_1 \in \mathcal{D}$. Therefore, let us take E_1 to be an eigenvector field for the shape operator A, corresponding to the eigenvalue that we may specify a few steps later. As $AE_1 = -g_{12}^5 E_1 + g_{11}^5 E_2 - (1 + g_{14}^5)E_3 + g_{13}^5 E_4$, we have

$$g_{11}^5 = 0, \quad g_{14}^5 = -1, \quad g_{13}^5 = 0,$$

thus, $\xi(g_{14}^5) = 0$, while from Lemma 4 we have

$$g_{33}^5 + g_{24}^5 g_{51}^2 + g_{12}^5(g_{24}^5 - g_{51}^3) + g_{34}^5 g_{51}^3 - g_{33}^5 g_{51}^4 = 0. \tag{13}$$

From $n_{15}^4 = 0$, we have

$$g_{23}^5 + 2(g_{33}^5)^2 + 2g_{34}^5 g_{43}^5 + g_{12}^5\alpha + g_{12}^5 g_{23}^5\alpha - g_{34}^5\alpha = 0,$$

thus, if we add this up with (10), we obtain

$$-1 + g_{12}^5(1 + g_{23}^5)\alpha - g_{43}^5\alpha = 0,$$

therefore, $\alpha \neq 0$. From

$$0 = n_{15}^3 = -(g_{12}^5 g_{24}^5 + g_{33}^5)\alpha,$$

we obtain $g_{33}^5 = -g_{12}^5 g_{24}^5$, thus, the second equation of (12) becomes $-g_{24}^5(g_{12}^5 + \alpha) = 0$.

As $\alpha \neq 0$, the Hopf hypersurface M is not totally geodesic, and is therefore part of the tube around an almost complex surface. Therefore, there exists an eigenvalue for the shape operator A different from α. We may assume that the vector field E_1 is such that its corresponding eigenvalue $-g_{12}^5 \neq \alpha$. Straightforwardly, it follows that $g_{24}^5 = 0$ and

$g_{33}^5 = 0$. Then, from Lemma 4, $E_2(g_{33}^5) = 0$, $\xi(g_{33}^5) = 0$ and $E_4(g_{24}^5) = 0$. Thus, we obtain, respectively,

$$\begin{aligned} E_3(g_{23}^5) &= -g_{21}^4(1+g_{23}^5) - g_{21}^5 g_{31}^3 + g_{21}^2 g_{34}^5 + g_{21}^2 g_{43}^5 + g_{31}^3 g_{43}^5, \\ 2 + g_{51}^4 + g_{23}^5(2+g_{51}^4) &+ g_{34}^5(g_{43}^5 - g_{51}^2 - 2\alpha) - g_{43}^5(g_{51}^2 + \alpha) = 0, \\ g_{21}^4 + g_{21}^4 g_{23}^5 + g_{41}^2 + g_{23}^5 g_{41}^2 &- (g_{21}^5 + g_{34}^5) g_{41}^4 - g_{21}^2(g_{34}^5 + g_{43}^5) = 0. \end{aligned} \tag{14}$$

Now, Equation (8) and the third equation of (12)reduce to $g_{21}^5 = -(1+2g_{23}^5)g_{34}^5 + \alpha - g_{23}^5\alpha$ and $g_{43}^5 = g_{12}^5(3+2g_{23}^5) + (3+g_{23}^5)\alpha$. By subtracting (10) from the first equation of (12), we obtain $4(g_{12}^5+\alpha)(g_{34}^5+\alpha) = 0$; furthermore, $g_{34}^5 = -\alpha$. Then, Lemma 4 yields

$$\begin{aligned} 0 &= E_1(g_{34}^5) = g_{31}^3(g_{12}^5 + \alpha), \\ 0 &= E_2(g_{34}^5) = -(1+g_{23}^5)g_{31}^2 + g_{31}^4(g_{21}^5 - \alpha), \end{aligned} \tag{15}$$

and g_{31}^3=0. Therefore, (13) becomes $-g_{51}^3(g_{12}^5+\alpha) = 0$, so $g_{51}^3 = 0$. From (8) and (10) we obtain, respectively, $g_{21}^5 - (2+g_{23}^5)\alpha = 0$ and $-1 - g_{23}^5 + g_{43}^5\alpha - \alpha^2 = 0$; thus,

$$g_{21}^5 - \alpha = (1+g_{23}^5)\alpha = (g_{43}^5 - \alpha)\alpha^2. \tag{16}$$

From the previous, we have

$$\begin{aligned} &AE_1 = -g_{12}^5 E_1, \qquad && AE_2 = g_{21}^5 E_2 + (1+g_{23}^5)E_4, \qquad && AE_3 = \alpha E_3, \\ &AE_4 = (1+g_{23}^5)E_2 + g_{43}^5 E_4, \qquad && A\xi = \alpha\xi. \end{aligned} \tag{17}$$

If we assume that $g_{43}^5 = \alpha$, from (16) we have $1 + g_{23}^5 = 0$ and $g_{21}^5 = \alpha$, and because (17), we can find that E_2, E_3, E_4 and ξ are eigenvectors with eigenvalue α and that E_1 is an eigenvector with eigenvalue $-g_{12}^5 \neq \alpha$, which is a contradiction.

Thus, $g_{43}^5 \neq \alpha$. From (16), we have $g_{23}^5 = (g_{43}^5 - \alpha)\alpha - 1$ and $g_{21}^5 = (g_{43}^5 - \alpha)\alpha^2 + \alpha$. Now, the second and third equations of (14) and (15) become, respectively,

$$\begin{aligned} (g_{43}^5 - \alpha)(-g_{51}^2 + g_{51}^4\alpha) &= 0, \\ (g_{43}^5 - \alpha)(g_{21}^2 - \alpha(g_{21}^4 + g_{41}^2 - g_{41}^4\alpha)) &= 0, \\ (g_{43}^5 - \alpha)\alpha(-g_{31}^2 + g_{31}^4\alpha) &= 0, \end{aligned}$$

thus, $g_{51}^2 = g_{51}^4\alpha$, $g_{21}^2 = \alpha(g_{21}^4 + g_{41}^2 - g_{41}^4\alpha)$ and $g_{31}^2 = g_{31}^4\alpha$. From

$$0 = E_2(g_{23}^5 - (g_{43}^5 - \alpha)\alpha + 1) = (g_{43}^5 - \alpha)(g_{21}^3 - g_{41}^3\alpha)(1+\alpha^2),$$

using Lemma 4 we can obtain $g_{21}^3 = g_{41}^3\alpha$. Now, from

$$0 = n_{51}^4 = \alpha(g_{43}^5 - \alpha)(-1 + g_{12}^5\alpha),$$

we have $g_{12}^5\alpha = 1$. Finally, by putting this into

$$0 = n_{15}^4 = -1 - g_{12}^5\alpha^3 + g_{43}^5\alpha(-1 + g_{12}^5\alpha) = -1 - \alpha^2,$$

we obtain a contradiction.

Case 2: We now assume that M is not a Hopf hypersurface, i.e., that $\beta \neq 0$. Then, the second equation of (11) implies $\alpha \neq 0$; thus, from (11) we obtain

$$g_{11}^5 = 0, \qquad g_{21}^5\alpha = -1, \qquad g_{24}^5 = 0, \qquad g_{23}^5 = -1. \tag{18}$$

Further, from Lemma 4, we have

Lemma 6.

$$
\begin{aligned}
0 =&\xi(g_{11}^5) = 1 + g_{21}^5 g_{51}^2 + g_{13}^5 g_{51}^3 + g_{51}^4 + g_{14}^5(1 + g_{51}^4) + (g_{11}^2 - \beta)\beta \\
&+ g_{12}^5(g_{21}^5 + g_{51}^2 - \alpha), && (19)\\
0 =&E_1(g_{24}^5) = -g_{11}^2(1 + g_{14}^5) + (g_{12}^5 + g_{21}^5)g_{31}^2 + g_{13}^5 g_{31}^3 + g_{31}^4 + g_{14}^5 g_{31}^4 \\
&- g_{11}^3 g_{33}^5 - g_{11}^4(g_{21}^5 + g_{34}^5) + (2 + g_{14}^5)\beta, \\
0 =&E_4(g_{24}^5) = -g_{13}^5 g_{31}^2 - (1 + g_{14}^5)g_{41}^2 - g_{33}^5(g_{31}^4 + g_{41}^3) - g_{34}^5 g_{41}^4 + g_{31}^3 g_{43}^5 \\
&- g_{21}^5(g_{31}^3 + g_{41}^4), \\
0 =&\xi(g_{24}^5) = -2g_{34}^5 + g_{14}^5(g_{21}^5 - g_{51}^2) - g_{51}^2 - g_{33}^5 g_{51}^3 - (g_{21}^5 + g_{34}^5)g_{51}^4 \\
&- g_{21}^3\beta - \alpha, && (20)\\
0 =&E_1(g_{23}^5) = -g_{11}^4 g_{33}^5 - (g_{12}^5 + g_{21}^5)g_{41}^2 - (1 + g_{14}^5)g_{41}^4 + g_{11}^3(-g_{21}^5 + g_{43}^5) \\
&- g_{13}^5(g_{11}^2 + g_{41}^3 - \beta), \\
0 =&\xi(g_{23}^5) = g_{13}^5(g_{21}^5 - g_{51}^2) - g_{21}^5 g_{51}^3 + g_{43}^5 g_{51}^3 - g_{33}^5(2 + g_{51}^4) + g_{21}^4\beta.
\end{aligned}
$$

Then, from $E_2(g_{24}^5) = 0$ and $E_2(g_{23}^5) = 0$, we have

$$
\begin{aligned}
E_3(g_{21}^5) &= -(1 + g_{14}^5)g_{21}^2 - g_{21}^3 g_{33}^5 - g_{21}^4(g_{21}^5 + g_{34}^5), \\
E_4(g_{21}^5) &= g_{13}^5 g_{21}^2 + g_{21}^4 g_{33}^5 + g_{21}^3(g_{21}^5 - g_{43}^5).
\end{aligned} \tag{21}
$$

Now, by using Lemmas 4 and 5 and (21), from $E_i(g_{21}^5\alpha) = 0$, $i = 2, 3, 4$, we obtain

$$
\begin{aligned}
&E_2(g_{21}^5) = -(g_{21}^5)^2(3g_{21}^5 - g_{51}^2 - \alpha)\beta, \\
&g_{21}^5 g_{51}^3\beta - ((1 + g_{14}^5)g_{21}^2 + g_{21}^3 g_{33}^5 + g_{21}^4(g_{21}^5 + g_{34}^5))\alpha = 0, \\
&g_{13}^5 g_{21}^2\alpha + g_{21}^4 g_{33}^5\alpha + g_{21}^3(g_{21}^5 - g_{43}^5)\alpha + g_{21}^5(-2 + g_{51}^4)\beta = 0.
\end{aligned} \tag{22}
$$

Further, from $0 = n_{11}^5 = -g_{13}^5\alpha$ and $0 = n_{33}^5 = g_{33}^5$, we have $g_{13}^5 = 0$ and $g_{33}^5 = 0$; hence, the following holds.

Lemma 7.

$$
\begin{aligned}
0 &= E_2(g_{13}^5) = -(1 + g_{14}^5)g_{21}^2 + g_{21}^3 g_{33}^5 + g_{12}^5(g_{21}^4 - g_{41}^2) - g_{21}^5 g_{41}^2 - (1 + g_{14}^5)g_{41}^4 + g_{21}^4 g_{43}^5, \\
0 &= \xi(g_{13}^5) = -g_{51}^2 + g_{33}^5 g_{51}^3 + (g_{12}^5 + g_{43}^5)g_{51}^4 - \alpha + g_{11}^4\beta + g_{14}^5(g_{43}^5 - g_{51}^2 - 2\alpha), && (23)\\
0 &= E_2(g_{33}^5) = g_{21}^4 + g_{14}^5 g_{21}^4 + g_{21}^5 g_{31}^3 - g_{21}^2 g_{34}^5 - (g_{21}^2 + g_{31}^3)g_{43}^5, \\
0 &= \xi(g_{33}^5) = g_{51}^4(1 + g_{14}^5) + g_{34}^5(g_{43}^5 - g_{51}^2 - 2\alpha) - g_{43}^5(g_{51}^2 + \alpha) + g_{31}^4\beta. && (24)
\end{aligned}
$$

Lemma 8. *The coefficients in (5) satisfy*

$$
\begin{aligned}
g_{43}^5 - g_{34}^5(1 + (g_{43}^5)^2) - g_{51}^2 - g_{21}^5(3 + g_{14}^5 + g_{51}^4) - g_{145}(g_{51}^2 + g_{43}^5 g_{51}^4)& \\
+\alpha + (g_{34}^5 + g_{43}^5)(-g_{51}^4 + g_{43}^5(g_{51}^2 + \alpha)) - g_{31}^2\beta - g_{31}^4 g_{43}^5\beta &= 0, && (25)\\
1 + g_{14}^5 + g_{43}^5(g_{21}^5 + g_{34}^5) &= 0, \\
g_{41}^4(2g_{51}^2 - 2(g_{12}^5 + g_{43}^5)g_{51}^4 + \alpha + g_{14}^5(-2g_{43}^5 + 2g_{51}^2 + 3\alpha) - 2g_{11}^4\beta) &= 0, \\
-(1 + g_{14}^5)g_{41}^4\alpha &= 0, \\
-g_{11}^3 + (g_{21}^5 + g_{34}^5)g_{41}^4 + (g_{21}^2 - g_{31}^3)g_{43}^5 &= 0, \\
g_{14}^5 + g_{34}^5\alpha &= 0, \\
-g_{11}^4(1 + g_{14}^5) + (g_{34}^5 + g_{43}^5)(g_{11}^2 + \beta) &= 0, \\
g_{14}^5(1 + g_{43}^5\alpha) &= 0,
\end{aligned}
$$

$$(1+g_{14}^5)(g_{11}^2\alpha - g_{11}^4) = 0,$$
$$g_{21}^2(g_{34}^5 + g_{43}^5) = 0,$$
$$(1+g_{14}^5)(1+g_{21}^5 g_{43}^5)\alpha = 0,$$
$$-(g_{21}^5 + g_{34}^5)g_{41}^4\alpha = 0.$$

Proof. From (9), we have $(-g_{21}^4 + g_{41}^2)\beta = 0$, and therefore, $g_{41}^2 = g_{21}^4$.

Now, by taking $n_{43}^5 = 0$, we obtain the first relation of the Lemma.

Further, it follows that $(25) + g_{43}^5(24) + (8) - (20) = 0$, and we have $g_{43}^5 - (1+g_{14}^5)\alpha - g_{34}^5 g_{43}^5\alpha = 0$. If we multiply this by g_{21}^5 and use $g_{21}^5\alpha = -1$, we obtain the second relation of the Lemma.

Now, the sum of the third relation of Lemma 6 and the third relation of Lemma 7 and the first relation of Lemma 7, respectively, reduce to

$$\begin{aligned}
&-(g_{21}^5 + g_{34}^5)g_{41}^4 - g_{21}^2(g_{34}^5 + g_{43}^5) = 0,\\
&g_{43}^5(g_{21}^5 + g_{34}^5)g_{21}^2 + g_{43}^5(g_{21}^5 + g_{34}^5)g_{41}^4 + g_{21}^4(-g_{21}^5 + g_{43}^5) = 0.
\end{aligned} \tag{26}$$

If we multiply the first relation of (26) by g_{435} and add it to the second, we obtain

$$(g_{21}^5 - g_{43}^5)(-g_{21}^4 + g_{21}^2 g_{43}^5) = 0.$$

If we assume that $g_{21}^5 = g_{43}^5$, from the last relation of Lemma 6 we have $g_{21}^4 = 0$. On the other hand, if we assume that $g_{21}^4 = g_{21}^2 g_{43}^5$, first, by using the second relation of the Lemma we obtain $g_{21}^5 g_{51}^3\beta = 0$ from the first equation of (22), thus, $g_{51}^3 = 0$. Then, from the last equation of Lemma 6, we have $g_{21}^2 g_{43}^5\beta = 0$; thus, $0 = g_{21}^2 g_{43}^5 = g_{21}^4$. In both cases, we have $g_{21}^4 = 0$.

The third relation of the Lemma now follows directly from $n_{41}^2 = 0$.

The fourth relation is obtained as the sum of the third and the product of (23) by $2g_{41}^4$.

Note that the second relation of (26) has now become

$$(g_{21}^5 + g_{34}^5)(g_{21}^2 + g_{41}^4)g_{43}^5 = 0.$$

From the forth relation, using the second relation of the Lemma we have $0 = -(1+g_{14}^5)g_{41}^4 = (g_{21}^5 + g_{34}^5)g_{43}^5 g_{41}^4$; thus, from the second relation of (26), we obtain $0 = g_{43}^5(g_{21}^5 + g_{34}^5)g_{21}^2 = (1+g_{14}^5)g_{21}^2$. Now, the first relation of (22) becomes $g_{21}^5 g_{51}^3\beta = 0$, therefore, it holds that $g_{51}^3 = 0$.

Using the second relation of the Lemma along with $\beta \neq 0$, from $\xi(1 + g_{14}^5 + g_{43}^5(g_{21}^5 + g_{34}^5)) = 0$ we can obtain the fifth equation.

The sixth equation follows from $n_{25}^3 - (24) + g_{21}^5(23) = 0$ by taking $g_{21}^5\alpha = -1$.

Because $n_{13}^4 - (\beta + g_{11}^2)(2\cdot(24) + (10)) - g_{11}^4(2\cdot(20) - (8)) = 0$, we obtain the seventh equation.

The eighth equation is deduced from $n_{15}^4 - g_{14}^5((24) + (10)) + g_{12}^5(20) = 0$, and the ninth then follows from $g_{21}^5(n_{12}^1 + g_{11}^2(19) - g_{11}^4(2\cdot(20) - (8))) - g_{11}^2(n_{25}^1 + (20) - (8)) = 0$.

From $n_{24}^3 - g_{21}^2(2\cdot(24) + (10)) = 0$,$n_{25}^1 + (20) - (8) - g_{21}^5(n_{45}^1 - g_{43}^5((20) - (8))) = 0$ and $n_{43}^2 + g_{41}^4((19) + (24)) = 0$, respectively, we obtain $g_{21}^2(g_{34}^5 + g_{43}^5)\alpha = 0$, and the tenth, eleventh, and twelfth relations are obtained. □

Now, we consider the ninth relation of Lemma 8.

Case 2.1: We first assume that $g_{14}^5 = -1$; then, from the sixth relation of Lemma 8 we have $g_{34}^5 = -g_{21}^5$ and from the eight we have $g_{43}^5 = g_{21}^5$.

From Lemma 4 and $0 = \xi(g_{14}^5) = -g_{31}^3\beta$, we obtain $g_{31}^3 = 0$. From $2\cdot(20) - (8) = 0$ and the second relation of (22) we have, respectively,

$$-(g_{21}^3 + g_{31}^2)\beta = 0, \qquad g_{21}^5(-2 + g_{51}^4)\beta = 0,$$

thus, $g_{31}^2 = -g_{21}^3$ and $g_{51}^4 = 2$. The fifth relation of Lemma 8 then yields $g_{21}^5(g_{21}^2 - g_{31}^3) = 0$, thus, $g_{31}^3 = g_{21}^2$. As $g_{34}^5 + g_{21}^5 = 0$, from $E_2(g_{34}^5 + g_{21}^5) = 0$ and using Lemma 4, we obtain

$$(g_{21}^5)^2(-3g_{21}^5 + g_{51}^2 + \alpha)\beta = 0,$$

thus, we have $g_{51}^2 = 3g_{21}^5 - \alpha$. If we multiply (24) by two and add it to (10), we have $(g_{31}^4 + g_{41}^3)\beta = 0$, thus, $g_{41}^3 = -g_{31}^4$. Finally, if we add $2g_{31}^4\beta$ to (10), we obtain $2g_{21}^5(g_{21}^5 - \alpha) = 0$; thus, $g_{21}^5 = \alpha$. Now, from (18), we have $-1 = g_{21}^5\alpha = \alpha^2$, which is a contradiction. Therefore, $g_{14}^5 \neq -1$, $g_{34}^5 \neq -g_{21}^5$, and $g_{43}^5 \neq -g_{34}^5$.

Case 2.2: From the tenth relation of Lemma 8 we have $g_{21}^2 = 0$, while the twelfth yields $g_{41}^4 = 0$ and the eleventh $g_{43}^5 = \alpha$.

From the third relation of Lemma 6 and the fifth relation of Lemma 8 we have, respectively, $g_{31}^3(-g_{21}^5 + \alpha) = 0$ and $-g_{11}^3 - g_{31}^3\alpha = 0$. If $g_{21}^5 = \alpha$, we have a contradiction; thus, $g_{31}^3 = 0$ and $g_{11}^3 = 0$. Furthermore, from the eighth relation of Lemma 8, we have $g_{14}^5(1 + \alpha^2) = 0$; thus, $g_{14}^5 = 0$ and the second yields $1 + (g_{21}^5 + g_{34}^5)\alpha = 0$, meaning that $g_{34}^5 = 0$. Now, from the ninth and seventh equations of Lemma 8 we respectively obtain $g_{11}^4 = g_{11}^2\alpha$ and $\alpha\beta = 0$, which is a contradiction.

This completes the proof.

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