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An Imperfect Repair Model with Delayed Repair under Replacement and Repair Thresholds

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Abstract: Based on the extended geometric process, a repair replacement model of a degradation system is studied, in which the delayed repair time depends on the working time after the last repair. Replacement and repair thresholds describe when the system will be replaced and when the system can be repaired, respectively. Two kinds of replacement policies are studied. One policy is jointly determined by the moment of the Nth failure and the first hitting time of the working time after the last repair for the replacement threshold, and the system is replaced, whichever occurs first; the other is the special case of the first policy, and the system is replaced when the working time after the last repair first hits the replacement threshold. The exact expressions of the long-run average cost rate are obtained. The optimal policies exist and can be ascertained by numerical methods. Finally, numerical examples are presented to demonstrate the application of the results obtained in the paper.

Keywords: extended geometric process; replacement policy; delayed repair; replacement threshold; repair threshold

MSC: 60K10

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1. Introduction

In 1988, the geometric process repair model (GPRM) was first introduced by Lam [1,2]. Since then, it has been widely studied, and many extended models have been proposed [3]. For example, the geometric process is generalized to the extended geometric process [4], threshold geometric process [5], doubly geometric process [6], phase-type geometric process [7], α-series process [8], and so on. The GPRM is suitable for describing the phenomenon whereby "the successive working times of the system after repair become shorter and shorter, while the consecutive repair times of the system after failure become longer and longer" [1,2]. However, a system after repair does not always degrade successively in practice [9]. As Zhang & Wang [10] say, "a serious failure may lead to the deteriorating of the system, while a slight failure can be eliminated, so that the system is not degenerative". Zhang & Wang [4] proposed an extended geometric process repair model (EGPRM) and explicitly expressed the average reward rate. Zhang & Wang [11] obtained different optimal replacement policies to minimize the average cost rate, maximize the average availability rate, and optimize the tradeoff model of the average cost rate and the average availability rate. Zhang & Wang [10] proposed an EGPRM considering a repair-replacement problem for a cold standby repairable system with two different components and one repairman, and the optimal replacement policy based on the failure number of Component 2 was given by minimizing the average cost rate of the system. Considering the repairman having multiple vacations, Wang et al. [12] proposed an EGPRM and explicitly expressed the long-run average cost rate based on the failure number of the component. From the above literature, it can be found that although the EGPRM has more parameters than the GPRM, the EGPRM is closer to the reality than the GPRM, and the parameters can be estimated by the statistical EM (i.e., Expectation-Maximization) algorithm [11].

The above papers are all focused on the replacement policy N, i.e., the system is replaced when the number of failures of the system reaches N. It is much more reasonable to consider bivariate or multi-variant repair and replacement policies from the consideration of practice, and they have also been widely studied. Zhang [13] generalized Lam's work by introducing a bivariate replacement policy (T, N). Wang & Zhang [14] proposed a bivariate replacement policy (L, N) based on the fixed-length interval of the preventive repair, and the preventive repair number of the system. Wang & Zhang [15] proposed a bivariate replacement policy based on system reliability and failure number of the system (R, N). Chang et al. [16] considered a bivariate replacement policy (n, T) based on the life age and the number of Type-I and Type-II failures. Sheu et al. [17] extended the bivariate replacement policy (T, N) to two trivariate replacement policies (K, N, T) and (N, S, T), in which K is a tolerance limit of failure and S is the lower working age.

Since the working time after a lot of maintenance will become shorter and shorter, the maintenance will become more and more frequent, incurring more and more maintenance costs; therefore, when the working time after the last repair is too short, it is not a wise choice to repair the system, and the best option is to replace it. Aiming to resolve this issue, Dong et al. [18] proposed a bivariate replacement policy in which the system is replaced whenever the working time reaches T or at the first hitting time of the working time after repair concerning the working time threshold, whichever occurs first. In this paper, we will propose an EGPRM and further consider a new bivariate replacement policy (N_{τ_1}, N), i.e., the system is replaced when the working time after the last repair is not longer than the replacement threshold τ_1, or if the system is repaired N times, whichever occurs first. There are many applications of the EGPRM and the replacement policy (N_{τ_1}, N). For example, water filters are devices that remove impurities from water through good physical barriers and chemical or biological processes. The filter cartridge is the core device of water filters, and most of the water filters are made up of multi-stage filter cartridges that are arranged in order of precision from low to high. Interceptions such as rust, sand, colloid, and other impurities are deposited inside the filter cartridges; thus, they need to be regularly manually disassembled and washed, or parts of the filter cartridges need to be replaced (for example, the filter cartridge with low precision) to guarantee the normal operation of the machine. In general, when the speed of filtering water is lower than a certain critical value, which can be viewed as a failure, we wash the filter cartridges or replace part of the filter cartridges, which can be considered an imperfect repair. With the increase in cleaning times or the number of replacements of parts of the filter cartridges, the water filter's efficiency becomes lower and lower. In other words, the working time after the last repair becomes shorter and shorter; when the working time of a water filter is too short for it to be worth repairing, it is best to replace all the filter cartridges at the same time, which means that a renewal cycle is completed. Moreover, when the number of repairs reaches a fixed threshold, we will also replace all the filter cartridges simultaneously. Thus, considering a tradeoff between the work efficiency and the cost, it is necessary to study the optimal repair and replacement policy to minimize the long run average cost rate.

The delayed repair is common for complex repair systems due to several practical factors [19]. One reason is that the system failure cannot be detected in time [20–25]. Another reason is that the delayed repair time may be due to a maintenance resources mobilization (e.g., maintenance crew, spare parts, tools) [26,27], which will be considered in the paper. For example, Zhang [27] pointed out that it is impossible to repair the system immediately if the repairman is on holiday. The random procurement lead time is considered by Yu et al. [26], if the Nth failure of the old system occurs too early, the replacement has to wait until the desired spare part is delivered. There is a threshold for the system to be repaired immediately, which is called the repair threshold in this paper; when the working time after the last repair is longer than the time used for a maintenance resources mobilization, there is no delayed repair time. Otherwise, it needs to wait until the duration after the last repair reaches the repair threshold. Obviously, the delayed repair depends on the working time after the last repair, which is discussed in this paper; however, the delayed repair time and working time are assumed to be independent of each other in most early works [27].

To the authors' best knowledge, an extended geometric process repair model with imperfect repair considering replacement threshold and repair threshold has not been found, and Table 1 summarizes the current results on similar topics.

Table 1. A summary of the existing results on similar topics to this paper.

Literature	Models	Delayed Repair	Policy	Other Factors
Zhang [27]	GPRM	Independent of the working time; No cost	N	Average cost rate
Zhang & Wang [10]	EGPRM	Independent of the working time; No cost	N	Cold standby and average cost rate
Zhang & Wang [4]	EGPRM	Independent of the working time; No cost	N	Average reward rate
Zhang & Wang [11]	EGPRM	Independent of the working time; No cost	N	Average cost rate and average availability rate
Dong et al. [18]	GPRM	Dependent of the working time; a penalty proportional to the delayed repair time	(T, N_{τ_1})	System availability and average cost rate
Wang et al. [12]	EGPRM	Independent of the working time; No cost	N	The repairman has multiple vacation
This paper	EGPRM	Dependent of the working time; a penalty proportional to the delayed repair time	(N_{τ_1}, N), N_{τ_1}	Average cost rate

Remark 1. *In Table 1, N, (T, N_{τ_1}), (N_{τ_1}, N), and N_{τ_1} are all symbols of replacement policies, where N stands for the replacement policy under which the system is replaced when the number of failures of the system gets to N; (T, N_{τ_1}) stands for the replacement policy whenever the working age of the system reaches T or at the first hitting time of the working time after repair for the working time threshold τ, whichever occurs first; (N_{τ_1}, N) stands for the replacement policy under which the system is replaced when the working time is shorter than the replacement threshold τ_1, or if the system is repaired N times, whichever occurs first; N_{τ_1} stands for a replacement policy under which the system is replaced when the working time is shorter than the replacement threshold τ_1.*

The main contribution of this paper to the existing literature is as follows:

- A novel model for imperfect delayed repair is built by using extended geometric processes.
- Replacement and repair thresholds are involved.
- Two kinds of replacement policies (N_{τ_1}, N) and N_{τ_1} are considered.
- The explicit expressions of the long-run average cost rate are obtained.
- The existence of optimal policies is proved, and numerical examples are presented to demonstrate the application of the results obtained in the paper.

The remainder of the paper is organized as follows: the problem definition is introduced in Section 2. In Section 3, the exact expressions of the long-run average cost rate under the policy (N_{τ_1}, N) and its special case (policy N_{τ_1}) are derived and optimal policies are proved. Section 4 provides numerical examples to show that optimal replacement policies $N^*_{\tau_1}$ and $(N_{\tau_1}, N)^*$ are existent and unique. Finally, conclusions are given in Section 5.

2. Problem Definition

In this paper, we study a repairable system based on the extended geometric process, and the basic assumptions about the replacement model are given as follows:

Assumption 1. *Initially, the system is new.*

Assumption 2. *The system degrades geometrically with probability q_n and does not degrade with probability $p_n = 1 - q_n$ at the nth repair for n = 1,2, ...*

Let X_n be the working time after the $(n-1)$th maintenance and $\{X_n, n = 1,2, \ldots\}$ be a non-increasing process, where X_1 is a new system's working time. Thus, $\{X_n, n = 1,2, \ldots\}$ constitutes an extended geometric process with the cumulative distribution function

$$F_n(t) = p_{n-1}F_{n-1}(t) + q_{n-1}F_{n-1}(at) \tag{1}$$

where $p_n + q_n = 1$, $0 \le p_n \le 1$, (n = 1,2, ...), $a > 1$ and $t \ge 0$. Furthermore, we assume that $E[X_1] = \lambda > 0$.

Let Y_n be the repair time after the nth failure, and $\{Y_n, n = 1,2, \ldots\}$ forms an extended geometric process, which has the cumulative distribution function

$$G_n(t) = p_{n-1}G_{n-1}(t) + q_{n-1}G_{n-1}(bt) \tag{2}$$

where $0 < b < 1$. Assume $EY_1 = \eta \ge 0$, and $\eta = 0$ implies that it is negligible for the repair time.

Assumption 3. *X_n and Y_n are independent of each other, n = 1,2, ...*

Assumption 4. *The system is subjected to self-announcing failures, i.e., system failures can be detected simultaneously. Assume a replacement threshold τ_1 exists, i.e., when the working time X_n is no longer than τ_1, the system will be replaced. Furthermore, we assume the system is replaced immediately, and the replacement takes negligible time.*

Assumption 5. *There exists the time for a maintenance resources mobilization, which takes time τ_2, i.e., the repair threshold.*

If $0 \le \tau_2 \le \tau_1$, the system can always be repaired immediately when the working time after the last repair is longer than τ_1. If $0 \le \tau_1 \le \tau_2$, there exist three cases for the working time X_n $(n = 1, 2, \ldots)$: (a) $0 \le X_n \le \tau_1$, the system will be replaced immediately; (b) $\tau_1 < X_n \le \tau_2$, the system will be repaired, but it needs to wait until the duration after the $(n-1)$th repair reaches τ_2, i.e., there exists the delayed repair; (c) $X_n > \tau_2$, the system will be repaired immediately. In the paper, we focus on the case of $0 \le \tau_1 \le \tau_2$, because τ_2 does not work for the case of $0 \le \tau_2 \le \tau_1$.

Assumption 6. *The cost rate of the repair is c_f; thus, $c_f Y_n$ is the cost of a repair when the system is repaired immediately after the nth failure. Moreover, we assume that there is a penalty because of the delayed repair, and the cost is in proportion to the length of the delayed repair time; thus, the cost of a repair is $c_f Y_n + c_d Z_n$, when the system is not repaired at once, where Z_n is the wait time after the nth failure, c_d is the penalty cost rate during the wait for repair state, and*

$$Z_n = \begin{cases} \tau_2 - X_n, & \tau_1 < X_n \le \tau_2 \\ 0, & \text{others} \end{cases} \tag{3}$$

The fixed replacement cost is c_r.

Assumption 7. *The replacement policy (N_{τ_1}, N) is used, i.e., the system is replaced at the first hitting time of the working time X_n concerning the replacement threshold τ_1 or at the moment of the Nth failure, whichever occurs first.*

Remark 2. *For the replacement policy (N_{τ_1}, N), if we let $N \to \infty$, i.e., we do not consider the number of repairs, we can obtain the replacement policy N_{τ_1}, i.e., the system is replaced at the first hitting time of the working time $X_n$for the replacement threshold τ_1, where $N_{\tau_1} = \min\{n | X_n \le \tau_1, n = 1, 2, \ldots\}$. In the paper, we also discuss the replacement policy N_{τ_1}in detail.*

Remark 3. *For the replacement policy (N_{τ_1}, N), if we let $\tau_1 = 0$, i.e., we do not consider the replacement threshold, we can obtain the replacement policy N, i.e., the system is replaced when the number of repair reaches N, which has been discussed by many authors under different situations, see, for example, Lam* [1,2]*, Zhang* [27]*, Zhang & Wang* [4,10,11]*, and references therein.*

A renewal cycle is defined as the time interval between the installation of a new system and the first replacement or a time interval between two consecutive replacements. A sample path of the system is illustrated in Figure 1. Our objective is to find the optimal replacement policies to minimize the long-run average cost rate of the system under two policies, (N_{τ_1}, N) and N_{τ_1}.

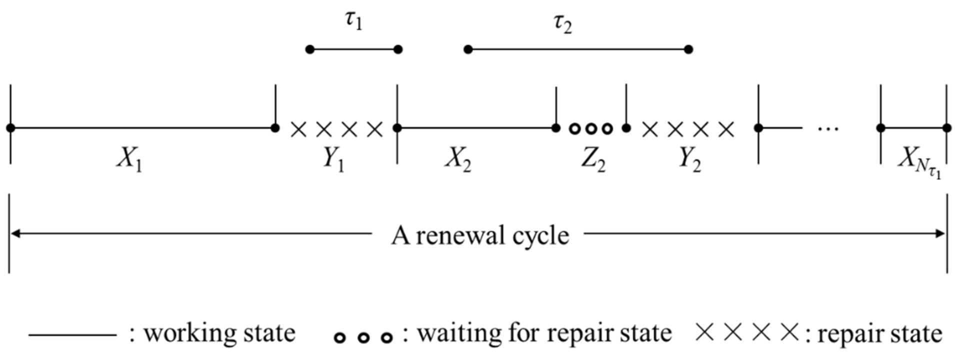

Figure 1. A possible sample path of the system.

3. Optimization Model Development

Let T_1 denote the first replacement time, and let T_n $(n \geq 2)$ denote the interval between the $(n-1)$th replacement and the nth replacement. Obviously, $\{T_n, n = 1, 2, \ldots\}$ forms a renewal process. According to the renewal theorem, the long-run average cost rate is given by

$$\frac{\text{the expected reward incurred in a renewal cycle}}{\text{the expected length of a renewal cycle}} \tag{4}$$

where a renewal cycle is a time between two consecutive replacements. In practice, the repair and replacement policies are mainly accounted for with regard to the cost or the availability consideration, aiming to find the optimal policies which minimize (or maximize) the long-run average cost rate (or availability) [28–36]. In the following, we will concentrate on the optimal replacement policies to minimize the long-run average cost rate under two kinds of replacement policies, (N_{τ_1}, N) and N_{τ_1}.

3.1. Long-Run Average Cost Rate under Replacement Policy (N_{τ_1}, N)

Under the replacement policy (N_{τ_1}, N), the system is replaced at the time of the Nth failure or at the first hitting time (N_{τ_1}) of X_n for τ_1, whichever occurs first; therefore, the length of a renewal cycle L is a random variable, and we have

$$\begin{aligned} L &= \left(U_{N_{\tau_1}} + V_{N_{\tau_1}-1} + W_{N_{\tau_1}-1}\right) I(N_{\tau_1} \leq N) + (U_N + V_{N-1} + W_{N-1}) I(N_{\tau_1} > N) \\ &= \left(\sum_{i=1}^{N_{\tau_1}} X_i + \sum_{i=1}^{N_{\tau_1}-1} Y_i + \sum_{i=1}^{N_{\tau_1}-1} Z_i\right) I(N_{\tau_1} \leq N) + \left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i + \sum_{i=1}^{N-1} Z_i\right) I(N_{\tau_1} > N), \end{aligned} \tag{5}$$

where $U_{N_{\tau_1}}$ (or U_N) is the working time during a cycle based on policy N_{τ_1} (or N); $V_{N_{\tau_1}-1}$ (or V_{N-1}) is the repair time during a cycle based on policy N_{τ_1} (or N); $W_{N_{\tau_1}-1}$ (or W_{N-1}) is the wait time during a cycle under policy N_{τ_1} (or N); define $\sum_{i=1}^{0} \equiv 0$; $I(A)$ is an indicator function of event A, i.e., $I(A) = 1$, if A occurs; otherwise $I(A) = 0$.

The expected length of a renewal cycle follows

$$E[L] = E\left[\left(\sum_{i=1}^{N_{\tau_1}} X_i + \sum_{i=1}^{N_{\tau_1}-1} Y_i + \sum_{i=1}^{N_{\tau_1}-1} Z_i\right) I(N_{\tau_1} \le N)\right] + E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i + \sum_{i=1}^{N-1} Z_i\right) I(N_{\tau_1} > N)\right] \tag{6}$$

Computing expectations by conditioning, the first term of Equation (6) becomes

$$\begin{aligned} & E\left[\left(\sum_{i=1}^{N_{\tau_1}} X_i + \sum_{i=1}^{N_{\tau_1}-1} Y_i + \sum_{i=1}^{N_{\tau_1}-1} Z_i\right) I(N_{\tau_1} \le N)\right] \\ &= E\left[E\left[\left(\sum_{i=1}^{N_{\tau_1}} X_i + \sum_{i=1}^{N_{\tau_1}-1} Y_i + \sum_{i=1}^{N_{\tau_1}-1} Z_i\right) I(N_{\tau_1} \le N) | N_{\tau_1}\right]\right] \\ &= \sum_{k=1}^{N}\left[E\left[\left(\sum_{i=1}^{N_{\tau_1}} X_i + \sum_{i=1}^{N_{\tau_1}-1} Y_i + \sum_{i=1}^{N_{\tau_1}-1} Z_i\right) | N_{\tau_1} = k\right] P\{N_{\tau_1} = k\}\right] \\ &= \sum_{k=1}^{N}\left[\left(\sum_{i=1}^{k} E[X_i] + \sum_{i=1}^{k-1} E[Y_i] + \sum_{i=1}^{k-1} E[Z_i]\right) P\{N_{\tau_1} = k\}\right]. \end{aligned} \tag{7}$$

From the results of Zhang & Wang (2017), since $\{X_n, n = 1, 2, \cdots\}$ and $\{Y_n, n = 1, 2, \cdots\}$ are extended geometric processes, we have

$$E[X_i] = \lambda \prod_{m=1}^{i-1}\left(p_m + \frac{q_m}{a}\right) = \lambda \prod_{m=1}^{i-1} r_m \tag{8}$$

$$E[Y_i] = \eta \prod_{m=1}^{i-1}\left(p_m + \frac{q_m}{b}\right) = \eta \prod_{m=1}^{i-1} h_m \tag{9}$$

where $r_m = p_m + \frac{q_m}{a}$, $h_m = p_m + \frac{q_m}{b}$, $E[X_{i+1}] = r_i E[X_i]$ and $E[Y_{i+1}] = h_i E[Y_i]$.

The expression of Z_i follows from Equation (3) that

$$\begin{aligned} E[Z_i] &= \int_{\tau_1}^{\tau_2} (\tau_2 - t) dF_i(t) \\ &= \int_{\tau_1}^{\tau_2} \tau_2 dF_i(t) - \int_{\tau_1}^{\tau_2} t dF_i(t) \\ &= \tau_2 F_i(\tau_2) - \tau_2 F_i(\tau_1) - tF_i(t)|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} F_i(t) dt \\ &= \tau_2 F_i(\tau_2) - \tau_2 F_i(\tau_1) - \tau_2 F_i(\tau_2) + \tau_1 F_i(\tau_1) + \int_{\tau_1}^{\tau_2} F_i(t) dt \\ &= \int_{\tau_1}^{\tau_2} F_i(t) dt - (\tau_2 - \tau_1) F_i(\tau_1). \end{aligned} \tag{10}$$

By substituting Equations (8)–(10) into Equation (7), we have

$$\begin{aligned} & E\left[\left(\sum_{i=1}^{N_{\tau_1}} X_i + \sum_{i=1}^{N_{\tau_1}-1} Y_i + \sum_{i=1}^{N_{\tau_1}-1} Z_i\right) I(N_{\tau_1} \le N)\right] \\ &= \sum_{k=1}^{N}\left[\left(F_k(\tau_1)\prod_{i=1}^{k-1}\overline{F}_i(\tau_1)\right)\left(\sum_{i=1}^{k}\left(\lambda\prod_{m=1}^{i-1} r_m\right) + \sum_{i=1}^{k-1}\left(\eta\prod_{m=1}^{i-1} h_m\right) + \sum_{i=1}^{k-1}\left(\int_{\tau_1}^{\tau_2} F_i(t)dt - (\tau_2-\tau_1)F_i(\tau_1)\right)\right)\right], \end{aligned} \tag{11}$$

where, define $\prod_{i=1}^{0} \equiv 1$; $P\{N_{\tau_1} = k\} = F_k(\tau_1)\prod_{i=1}^{k-1}\overline{F}_i(\tau_1)$ follows from the fact that $\{N_{\tau_1} = k\}$ is the event that the working time after the nth repair is longer than τ_1 for $n \le k-1$, and it is not longer than τ_1 after the $(k-1)$th repair.

Similar to the calculation of the first term of Equation (6), for the second term of Equation (6), we have

$$\begin{aligned} &E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i + \sum_{i=1}^{N-1} Z_i\right) I(N_{\tau_1} > N)\right] \\ &= \prod_{i=1}^{N} \overline{F}_i(\tau_1)\left(\sum_{i=1}^{N}\left(\lambda \prod_{m=1}^{i-1} r_m\right) + \sum_{i=1}^{N-1}\left(\eta \prod_{m=1}^{i-1} h_m\right) + \sum_{i=1}^{N-1}\left(\int_{\tau_1}^{\tau_2} F_i(t)dt - (\tau_2-\tau_1)F_i(\tau_1)\right)\right). \end{aligned} \tag{12}$$

Therefore, according to Equations (11) and (12), Equation (6) becomes

$$\begin{aligned} E[L] = &\sum_{k=1}^{N}\left[\left(F_k(\tau_1)\prod_{i=1}^{k-1}\overline{F}_i(\tau_1)\right)\left(\sum_{i=1}^{k}\left(\lambda \prod_{m=1}^{i-1} r_m\right) + \sum_{i=1}^{k-1}\left(\eta \prod_{m=1}^{i-1} h_m\right) + \sum_{i=1}^{k-1}\left(\int_{\tau_1}^{\tau_2} F_i(t)dt - (\tau_2-\tau_1)F_i(\tau_1)\right)\right)\right] \\ &+\prod_{i=1}^{N} \overline{F}_i(\tau_1)\left(\sum_{i=1}^{N}\left(\lambda \prod_{m=1}^{i-1} r_m\right) + \sum_{i=1}^{N-1}\left(\eta \prod_{m=1}^{i-1} h_m\right) + \sum_{i=1}^{N-1}\left(\int_{\tau_1}^{\tau_2} F_i(t)dt - (\tau_2-\tau_1)F_i(\tau_1)\right)\right). \end{aligned} \tag{13}$$

Since

$$C(\tau_1, N) = \left(c_f \sum_{i=1}^{N_{\tau_1}-1} Y_i + c_d \sum_{i=1}^{N_{\tau_1}-1} Z_i\right) I(N_{\tau_1} \le N) + \left(c_f \sum_{i=1}^{N-1} Y_i + c_d \sum_{i=1}^{N-1} Z_i\right) I(N_{\tau_1} > N) + c_r \tag{14}$$

Similarly, the expected cost in a renewal cycle is given by

$$\begin{aligned} E[C(\tau_1, N)] = &\sum_{k=1}^{N}\left[\left(F_k(\tau_1)\prod_{i=1}^{k-1}\overline{F}_i(\tau_1)\right)\left(c_f\sum_{i=1}^{k-1}\left(\eta \prod_{m=1}^{i-1} h_m\right) + c_d\sum_{i=1}^{k-1}\left(\int_{\tau_1}^{\tau_2} F_i(t)dt - (\tau_2-\tau_1)F_i(\tau_1)\right)\right)\right] \\ &+\prod_{i=1}^{N} \overline{F}_i(\tau_1)\left(c_f\sum_{i=1}^{N-1}\left(\eta \prod_{m=1}^{i-1} h_m\right) + c_d\sum_{i=1}^{N-1}\left(\int_{\tau_1}^{\tau_2} F_i(t)dt - (\tau_2-\tau_1)F_i(\tau_1)\right)\right) + c_r. \end{aligned} \tag{15}$$

Therefore, according to Equations (13) and (15), the long-run average cost rate $C_2(\tau_1, N)$ under policy (N_{τ_1}, N) is given by

$$C_2(\tau_1, N) = \frac{\left\{\begin{aligned} &\sum_{k=1}^{N}\left[\left(F_k(\tau_1)\prod_{i=1}^{k-1}\overline{F}_i(\tau_1)\right)\left(c_f\sum_{i=1}^{k-1}\left(\eta \prod_{m=1}^{i-1} h_m\right) + c_d\sum_{i=1}^{k-1}\left(\int_{\tau_1}^{\tau_2} F_i(t)dt - (\tau_2-\tau_1)F_i(\tau_1)\right)\right)\right] \\ &+\prod_{i=1}^{N} \overline{F}_i(\tau_1)\left(c_f\sum_{i=1}^{N-1}\left(\eta \prod_{m=1}^{i-1} h_m\right) + c_d\sum_{i=1}^{N-1}\left(\int_{\tau_1}^{\tau_2} F_i(t)dt - (\tau_2-\tau_1)F_i(\tau_1)\right)\right) + c_r \end{aligned}\right\}}{\left\{\begin{aligned} &\sum_{k=1}^{N}\left[\left(F_k(\tau_1)\prod_{i=1}^{k-1}\overline{F}_i(\tau_1)\right)\left(\sum_{i=1}^{k}\left(\lambda \prod_{m=1}^{i-1} r_m\right) + \sum_{i=1}^{k-1}\left(\eta \prod_{m=1}^{i-1} h_m\right) + \sum_{i=1}^{k-1}\left(\int_{\tau_1}^{\tau_2} F_i(t)dt - (\tau_2-\tau_1)F_i(\tau_1)\right)\right)\right] \\ &+\prod_{i=1}^{N} \overline{F}_i(\tau_1)\left(\sum_{i=1}^{N}\left(\lambda \prod_{m=1}^{i-1} r_m\right) + \sum_{i=1}^{N-1}\left(\eta \prod_{m=1}^{i-1} h_m\right) + \sum_{i=1}^{N-1}\left(\int_{\tau_1}^{\tau_2} F_i(t)dt - (\tau_2-\tau_1)F_i(\tau_1)\right)\right) \end{aligned}\right\}} \tag{16}$$

Specially, if $p_i = p$ and $q_i = q = 1 - p$, $i = 1, 2, \ldots$, we can obtain the special case of the EGPRM by substituting $r_i = p + a^{-1}q$ and $h_i = p + b^{-1}q$ $(i = 1, 2, \ldots)$ into Equation (16), i.e., the result is that the system degrades geometrically with constant probability q and does not degrade at each maintenance with constant probability p. Furthermore, if $p_i = 0$, $i = 1, 2, \ldots$, we can get the GPRM by substituting $r_i = a^{-1}$ and $h_i = b^{-1}$ $(i = 1, 2, \ldots)$ into Equation (16), i.e., the result in the case that the system after repair degrades successively.

There exists the minimum long-run average cost rate. Equation (16) is a bivariate function of τ_1 and N. When N is fixed, it is a function of τ_1. For example, if $N = n$, $C_2(\tau_1, N)$ will become

$$C_2(\tau_1, N) = C_{2,n}(\tau_1),\ n = 1, 2, \ldots.$$

Thus, if n is fixed, we can minimize $C_{2,n}(\tau_1^*)$ analytically or numerically to get optimal τ_{1n}^*, i.e., when $N = 1, 2, \cdots, n, \ldots$, $\tau_{11}^*, \tau_{12}^*, \cdots, \tau_{1n}^*, \ldots$ are obtained respectively, such that the corresponding $C_{2,1}(\tau_{11}^*)$, $C_{2,2}(\tau_{12}^*)$, $\ldots$, $C_{2,n}(\tau_{1n}^*)$, $\ldots$ is minimized. Because of the finiteness of the total lifetime span for the repairable system, the minimum long-run average cost rate can be confirmed. Therefore, the minimum long-run average cost rate based on $C_{2,1}(\tau_{11}^*), C_{2,2}(\tau_{12}^*), \ldots, C_{2,n}(\tau_{1n}^*), \ldots$ can be obtained; thus, we have

$$C_2\big((N_{\tau_1}, N)^*\big) = \min_N \left[\min_{\tau_1} C_2(\tau_1, N)\right]$$

Furthermore, we can also obtain the optimal policy from another angle, i.e.,

$$C_2\big((N_{\tau_1}, N)^*\big) = \min_{\tau_1} \left[\min_{N} C_2(\tau_1, N)\right]$$

3.2. Special Cases

Some special cases are summarized as follows.

Case 1. Policy N_{τ_1}.

If we let $N \to \infty$, the policy (N_{τ_1}, N) becomes the policy N_{τ_1}, and it follows from Equation (16) that the long-run average cost rate under the policy N_{τ_1} is given by

$$C_1(\tau_1) = \frac{c_f \sum\limits_{k=1}^{\infty}\left(\eta \prod\limits_{i=1}^{k-1} h_i \prod\limits_{i=1}^{k} \overline{F}_i(\tau_1)\right) + c_d \sum\limits_{k=1}^{\infty}\left(\left(\int\limits_{\tau_1}^{\tau_2} F_k(t)dt - (\tau_2-\tau_1)F_k(\tau_1)\right)\prod\limits_{i=1}^{k}\overline{F}_i(\tau_1)\right) + c_r}{\sum\limits_{k=1}^{\infty}\left(\lambda \prod\limits_{i=1}^{k-1} r_i \prod\limits_{i=1}^{k-1} \overline{F}_i(\tau_1)\right) + \sum\limits_{k=1}^{\infty}\left(\eta \prod\limits_{i=1}^{k-1} h_i \prod\limits_{i=1}^{k} \overline{F}_i(\tau_1)\right) + \sum\limits_{k=1}^{\infty}\left(\left(\int\limits_{\tau_1}^{\tau_2} F_k(t)dt - (\tau_2-\tau_1)F_k(\tau_1)\right)\prod\limits_{i=1}^{k}\overline{F}_i(\tau_1)\right)} \tag{17}$$

Especially, if the system degrades geometrically with constant probability q and does not degrade at each maintenance with constant probability p, i.e., $p_i = p$ and $q_i = q = 1-p$, $i = 1, 2, \ldots$, we can obtain the special case of the EGPRM by substituting $r_i = p + a^{-1}q$ and $h_i = p + b^{-1}q$, $(i = 1, 2, \ldots)$ into Equation (17). Furthermore, if the system after maintenance degrades successively, i.e., $p_i = 0$, $i = 1, 2, \ldots$, we can get the GPRM by substituting $r_i = a^{-1}$ and $h_i = b^{-1}$, $(i = 1, 2, \ldots)$ into Equation (17).

The optimal τ_1, which minimizes $C_1(\tau_1)$, exists. Because $C_1(\tau_1)$ is a continuous function on the interval $[0, \tau_2]$, from the extreme value theorem, there must exist an optimal τ_1 which minimizes $C_1(\tau_1)$. Moreover, the optimal policies can be obtained by numerical methods under some conditions, and the optimal policies under different conditions are unique in the following numerical examples.

Furthermore,

$$C_2\big((N_{\tau_1}, N)^*\big) = \min_{\tau_1}\left[\min_N C_2(\tau_1, N)\right] \le \min_{\tau_1}[C_2(\tau_1, \infty)] = C_1(\tau_1^*)$$

therefore, the optimal policy $(N_{\tau_1}, N)^*$ is better than the optimal policy $N_{\tau_1}^*$.

Case 2. $\tau_1 = 0$.

Since $F_k(0) = 0$ and $\overline{F}_k(0) = 1$ for any $k \in Z^+$, the policy N_{τ_1} is the case that the system will fail only when the working time reaches 0; therefore, Equation (17) becomes

$$C_1(0) = \frac{c_f \sum\limits_{k=1}^{\infty}\left(\eta \prod\limits_{i=1}^{k-1} h_i\right) + c_d \sum\limits_{k=1}^{\infty}\left(\int\limits_{0}^{\tau_2} F_k(t)dt\right) + c_r}{\sum\limits_{k=1}^{\infty}\left(\lambda \prod\limits_{i=1}^{k-1} r_i\right) + \sum\limits_{k=1}^{\infty}\left(\eta \prod\limits_{i=1}^{k-1} h_i\right) + \sum\limits_{k=1}^{\infty}\left(\int\limits_{0}^{\tau_2} F_k(t)dt\right)} \tag{18}$$

If $\tau_1 = 0$, the policy (N_{τ_1}, N) will become the policy N, and Equation (16) becomes

$$C_2(0,N) = \frac{c_f \sum_{i=1}^{N-1}\left(\eta \prod_{m=1}^{i-1} h_m\right) + c_d \sum_{i=1}^{N-1}\left(\int_0^{\tau_2} F_i(t)dt\right) + c_r}{\sum_{i=1}^{N}\left(\lambda \prod_{m=1}^{i-1} r_m\right) + \sum_{i=1}^{N-1}\left(\eta \prod_{m=1}^{i-1} h_m\right) + \sum_{i=1}^{N-1}\left(\int_0^{\tau_2} F_i(t)dt\right)} \tag{19}$$

Case 3. $\tau_1 = \tau_2 = 0$.
Equation (16) becomes

$$C_2(0,N) = \frac{c_f \sum_{i=1}^{N-1}\left(\eta \prod_{m=1}^{i-1} h_m\right) - \sum_{i=1}^{N}\left(\lambda \prod_{m=1}^{i-1} r_m\right) + c_r}{\sum_{i=1}^{N}\left(\lambda \prod_{m=1}^{i-1} r_m\right) + \sum_{i=1}^{N-1}\left(\eta \prod_{m=1}^{i-1} h_m\right)} \tag{20}$$

Moreover, if $\tau_1 = \tau_2 = 0$, $r_m = a^{-1}$ and $h_m = b^{-1}$, $m = 1, 2, \cdots$, Equation (20) becomes Lam's result [1,2], i.e.,

$$C_2(0,N) = \frac{c_f \eta \sum_{i=1}^{N-1} b^{i-1} - \lambda \sum_{i=1}^{N} a^{i-1} + c_r}{\lambda \sum_{i=1}^{N} a^{i-1} + \eta \sum_{i=1}^{N-1} b^{i-1}} \tag{21}$$

4. Numerical Example

In this section, numerical examples are provided to demonstrate the optimal replacement policies under two cases, respectively.

We assume that a new system's lifetime (X_1) and the first repair time (Y_1) follow the Gamma distributions with mean λ and η, respectively, that is

$$F_1(x) = 1 - e^{-2\lambda^{-1}x}(1 + 2\lambda^{-1}x) \text{ and } G_1(x) = 1 - e^{-2\eta^{-1}x}(1 + 2\eta^{-1}x),\ x \geq 0.$$

4.1. Long-Run Average Cost Rate under Policy N_{τ_1}

Firstly, we determine the optimal replacement threshold that minimizes the long-run average cost rate under policy N_{τ_1}, and three cases are considered. Algorithm 1, which can be adopted to compute an optimal threshold τ_1^*, is summarized as follows. This algorithm could be coded and calculated by MATLAB.

Algorithm 1 Long-Run Average Cost Rate under Policy N_{τ_1}

Input $\lambda, \eta, a, b, p, q, c_f, c_d, c_r, \tau_2$.
Step 1. Compute $C_1(\tau_1)$ as defined by Equation (17).
Step 2. Find the optimal threshold τ_1^* to minimize $C_1(\tau_1)$; output τ_1^* and $C_1(\tau_1^*)$.
Step 3. For $\tau_1 = 0$ to τ_2, compute $C_1(\tau_1)$ as defined by Equation (17).
Step 4. Plot $C_1(\tau_1)$ against threshold τ_1.
Stop.

Remark 4. *Since the plot of $C_1(\tau_1)$ against the replacement threshold τ_1 can be obtained, Steps 3 and 4 can be viewed as a verification of the global and unique optimal threshold obtained in Step 2.*

Moreover, the calculation of $C_1(\tau_1)$ involves infinite series, which should be calculated approximately by the nth partial sums of the series. The approximation precision is determined by n, and the corresponding approximation error can assess it via the following formula:

$$S = \max_{t \in (0,\tau_2)} |C_1(t, n+1) - C_1(t, n)|$$

where $C_1(t, n)$ is the long-run average cost rate in the case that the infinite series is approximated by the nth partial sums. Therefore, the approximation error can be reduced by increasing n. In the numerical examples, we choose suitable n to guarantee $S < 0.0001$.

Case 1. $0 < \tau_1^* < \tau_2$.

Let $\lambda = 80$, $\eta = 15$, $a = 1.05$, $b = 0.95$, $p = 0.45$, $q = 0.55$, $c_f = 10$, $c_d = 50$, $c_r = 1800$, $\tau_2 = 20$. Figure 2 is the plot of the long-run average cost rate $C_1(\tau_1)$ given by Equation (17). It can be found that $C_1(\tau_1)$ decreases in the interval [0, 12.0509] and increases in the interval [12.0509, 20]. The result means that the optimal $\tau_1^* = 12.0509$, and the minimum of the long-run average cost rate is $C_1(\tau_1^*) = 4.1864$, which is in accordance with the result obtained using the nonlinear programming method. In other words, based on the optimal policy, the replacement occurs when the working time reaches 12.0509 for the first time.

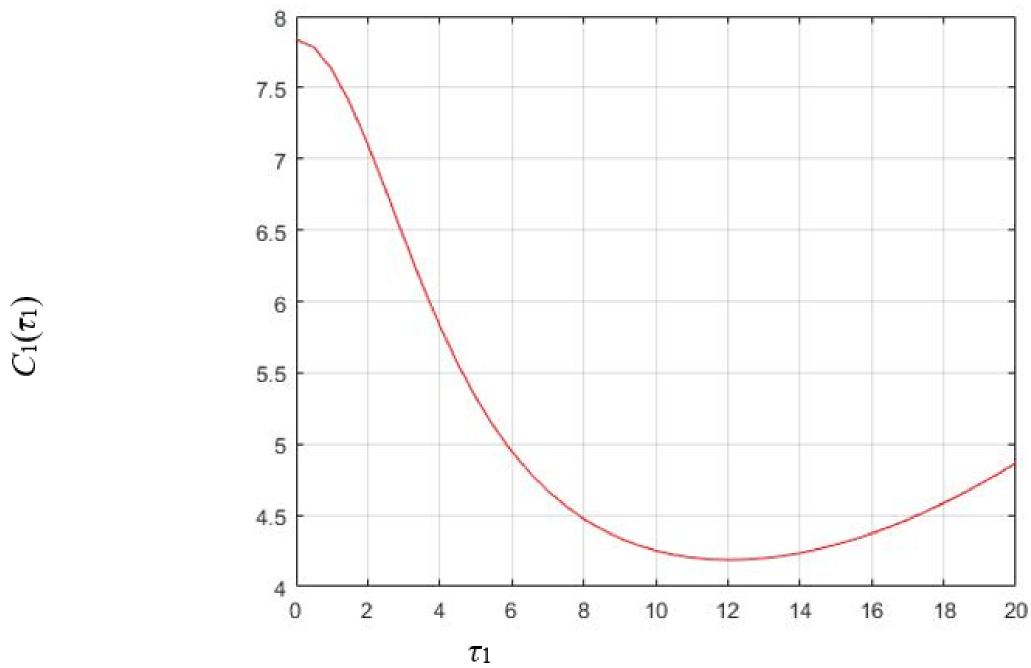

Figure 2. The plot of $C_1(\tau_1)$ against threshold τ_1, and $0 < \tau_1^* < \tau_2$.

Case 2. $\tau_1^* = 0$.

Let $\lambda = 80$, $\eta = 15$, $a = 5$, $b = 0.5$, $p = 0.05$, $q = 0.95$, $c_f = 10$, $c_d = 50$, $c_r = 800$, $\tau_2 = 20$, Figure 3 is the plot of the long-run average cost rate $C_1(\tau_1)$, which is given by Equation (17). We can find that $C_1(\tau_1)$ increases in the interval [0, 20], which means that the optimal $\tau_1^* = 0$, and the minimum of the long-run average cost rate is $C_1(0) = 10.5241$. In other words, the system will be replaced when the working time reaches 0 for the first time based on the optimal policy.

Case 3. $\tau_1^* = \tau_2$.

Let $\lambda = 120$, $\eta = 15$, $a = 1.05$, $b = 0.95$, $p = 0.45$, $q = 0.55$, $c_f = 10$, $c_d = 50$, $c_r = 300$, $\tau_2 = 20$; Figure 4 is the plot of the long-run average cost function $C_1(\tau_1)$ which is given by Equation (17). We can find that $C_1(\tau_1)$ decreases in the interval [0, 20], which means that the optimal $\tau_1^* = 20$, and the minimum of the long-run average cost rate is $C_1(20) = 1.8842$. That is to say, based on the optimal policy, the system will be replaced at the first time the working time hits 20.

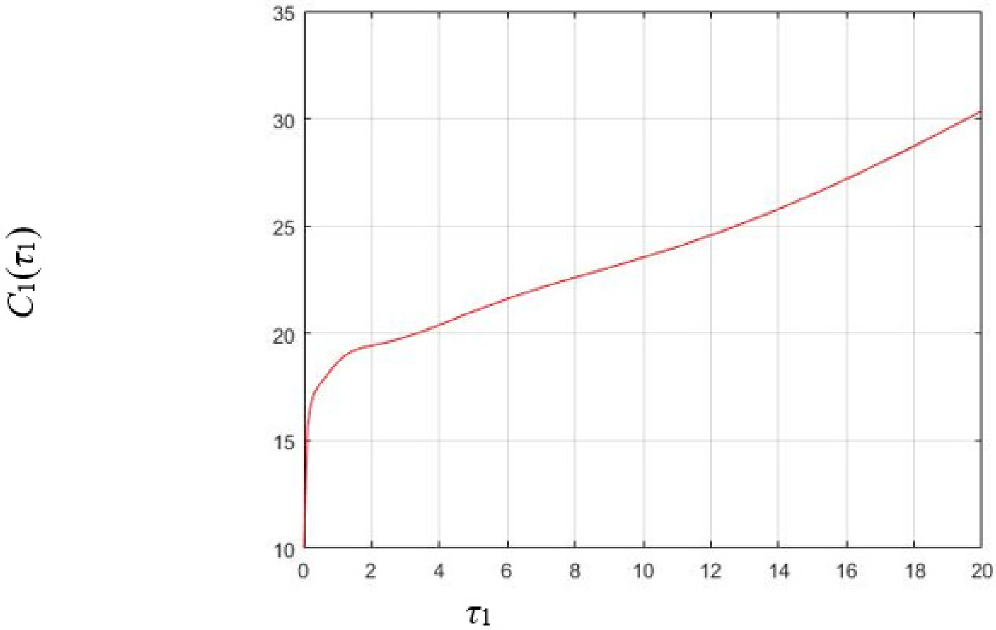

Figure 3. The plot of $C_1(\tau_1)$ against threshold τ_1, and $\tau_1^* = 0$.

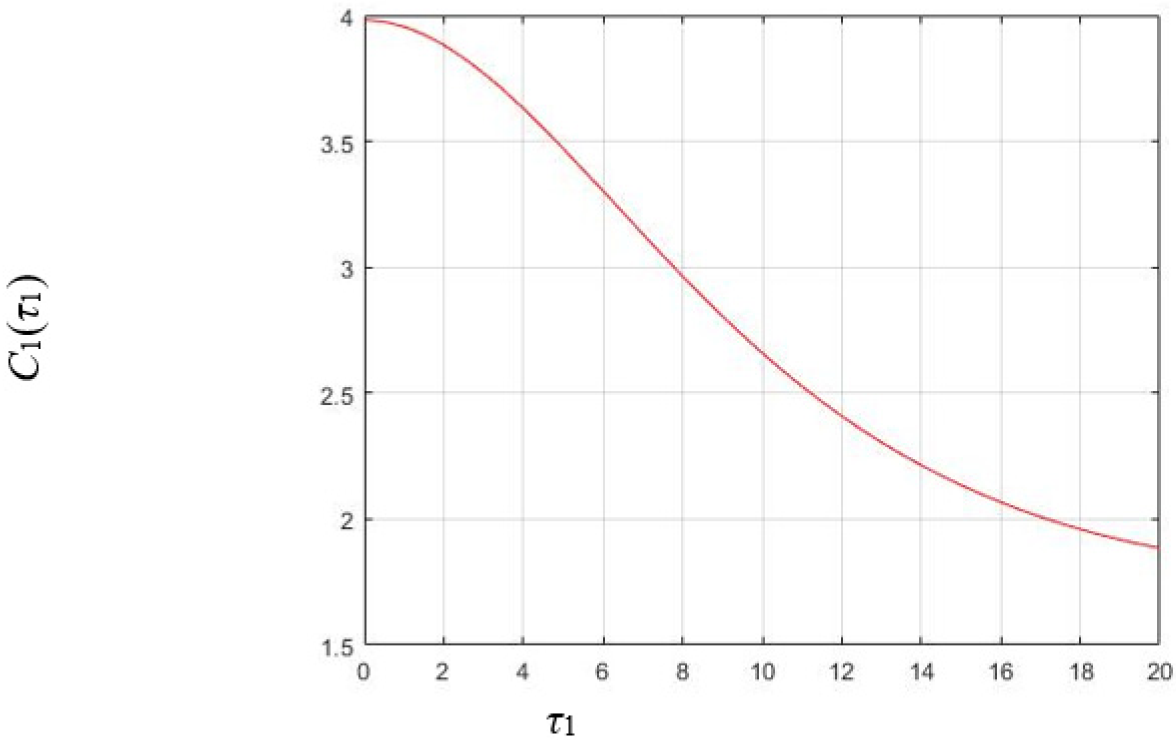

Figure 4. The plot of $C_1(\tau_1)$ against threshold τ_1, and $\tau_1^* = \tau_2$.

Secondly, we consider the influences of repair threshold τ_2 on the results under the policy N_{τ_1}. Let $\lambda = 80$, $\eta = 15$, $a = 1.05$, $b = 0.95$, $p = 0.45$, $q = 0.55$, $c_f = 10$, $c_d = 50$, $c_r = 1800$. The optimal τ_1^* and $C_1(\tau_1^*)$ for different values of τ_2 are tabulated in Table 2. According to Table 2, when τ_2 is small, optimal policy τ_1^* equals τ_2, which can be interpreted by the fact that $C_1(\tau_1)$ decreases on the interval $[0, \tau_2]$; when τ_2 is large enough, with the increase of τ_2, optimal policy τ_1^* and the corresponding long-run average cost rate increase gradually. The above analysis results are in line with our intuition.

Table 2. Optimal τ_1^* and $C_1(\tau_1^*)$ obtained for different values τ_2.

τ_2	τ_1^*	$C_1(\tau_1^*)$
3	3.0000	4.5377
5	5.0000	4.2703
8	8.0000	3.9904
10	9.8398	3.9459
15	10.8330	4.0076
20	12.0509	4.1864
25	13.4093	4.4771
30	14.8871	4.8706

In order to illustrate the effects of c_f on the long-run average cost function $C_1(\tau_1)$, curves for different values of c_f are put in the same figure. We choose c_f = 10, 15, 20, and 30, and the other parameter values are the same as those in Case 1. The curves are all depicted in Figure 5. From Figure 5, we can find that the higher c_f is, ceteris paribus, the higher the long-run average cost, which is in line with our intuition.

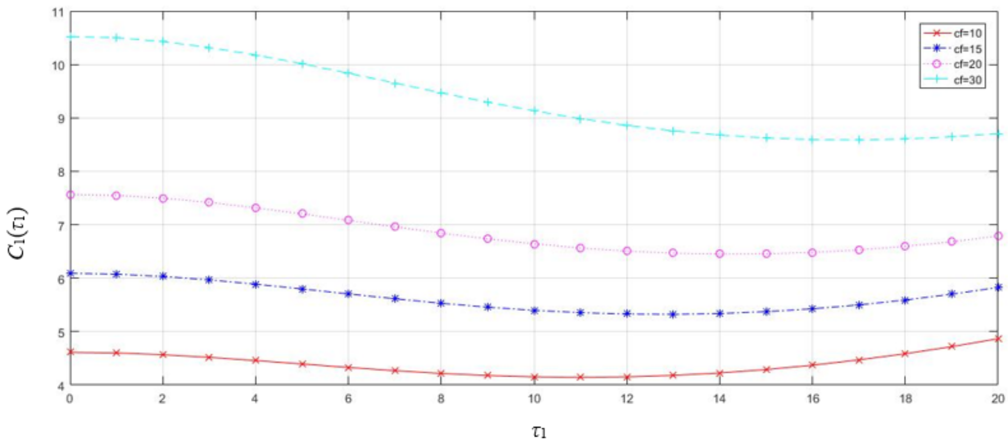

Figure 5. The plot of $C_1(\tau_1)$ for different values of c_f.

4.2. Long-Run Average Cost Rate under Policy (N_{τ_1}, N)

In the following, we determine the optimal replacement threshold and repair number that minimize the long-run average cost rate under the policy (N_{τ_1}, N). Algorithm 2, which can be adopted to compute optimal threshold and failure number (τ_1^*, N^*) by the numerical methods, is summarized as follows. This algorithm could be coded and calculated by MATLAB.

Algorithm 2 Long-Run Average Cost Rate under Policy (N_{τ_1}, N)

Input λ, η, a, b, p, q, c_f, c_d, c_r, τ_2.
Step 1. Compute $C_2(\tau_1, N)$ as defined by Equation (16).
Step 2. Find the optimal τ_1^* and N^* to minimize $C_2(\tau_1, N)$.
Step 3. Input n; for $N = 1$ to n, $\tau_1 = 0$ to τ_2, compute $C_2(\tau_1, N)$ as defined by Equation (16).
Step 4. Plot $C_2(\tau_1, N)$ against τ_1 and N.
Stop.

Let $\lambda = 80$, $\eta = 15$, $p = 0.3$, $q = 0.7$, $a = 1.2$, $b = 0.8$, $c_f = 10$, $c_d = 50$, $c_r = 1500$, $\tau_2 = 20$; Figure 6 is the plot of the long-run average cost rate function $C_2(t, N)$, given by Equation (16).

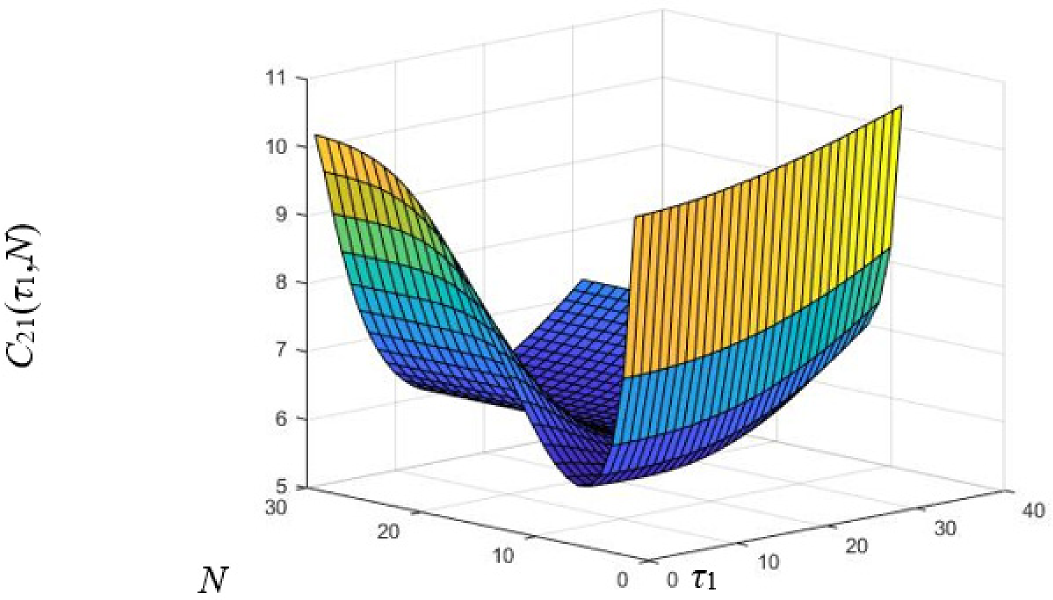

Figure 6. The plot of $C_2(\tau_1, N)$ against τ_1 and N.

The objective function is a bivariate function of N and τ_1, whereas N is a discrete variable and τ_1 is a continuous variable. One unit as the step size of τ_1 and N is adopted to obtain the plot of $C_2(\tau_1, N)$ versus (τ_1, N). By the searching procedure, we can find that $(\tau_1, N)^* = (8.5769, 7)$, which minimizes $C_2(\tau_1, N)$, i.e., $C_2(\tau_1^*, N^*) = C_2(8.5769, 7) = 5.7723$ is the optimal long-run average cost rate, which is in line with the result of the graphic display. The optimal long-run average cost rates for different values of N are shown in Table 3. From Table 3, we can obtain the same result of $(\tau_1, N)^* = (8.5769, 7)$.

Table 3. The optimal long-run average cost rates for different values of N.

N	2	3	4	5	6	7	8	9
τ_1^*	0	0	0	3.5386	6.4304	**8.5769**	10.3333	11.8186
$C_2((N_{\tau_1}, N)^*)$	9.9598	7.5453	6.5230	6.0469	5.8372	**5.7723**	5.7868	5.8386
N	10	11	12	13	14	15	16	17
τ_1^*	13.0631	14.0577	14.7908	15.2778	15.5656	15.7148	15.7812	15.8058
$C_2((N_{\tau_1}, N)^*)$	5.8999	5.9534	5.9918	6.0153	6.0277	6.0333	6.0354	6.0361

By using the same parameter values as those in Figure 6, we can also obtain the optimal $\tau_1^* = 13.5929$, and the minimum of the long-run average cost rate is $C_1(\tau_1^*) = 7.0073$. The comparison between the optimal policy $(N_{\tau_1}, N)^*$ and $N_{\tau_1}^*$ is given in Table 4. Obviously, $C_2((N_{\tau_1}, N)^*) = 5.7723 \leq C_1(\tau_1^*) = 7.0073$, i.e., the optimal policy $(N_{\tau_1}, N)^*$ is better than the optimal policy $N_{\tau_1}^*$.

Table 4. The comparison between the optimal policy $(N_{\tau_1}, N)^*$ and $N_{\tau_1}^*$.

N	**7**	**N**	∞
τ_1^*	**8.5769**	τ_1^*	**13.5929**
$C_2((N_{\tau_1}, N)^*)$	**5.7723**	$C_1(\tau_1^*)$	**7.0073**

5. Conclusions

The working time becomes shorter and shorter after many repairs are conducted for a repairable degradation system; when the working time of a system is too short for it to be worth repairing, it is best to replace it. Moreover, because of the delayed repair, the system cannot be repaired immediately, and the delayed repair time is dependent on the working time, whereas the delayed repair time is always assumed to be independent of the working time in most early works. In order to describe the above phenomena, a repair replacement model was developed using the extended geometric processes.

To minimize the long-run average cost rate, there are two replacement policies, one is based on the working time after the last repair, and the system is replaced when the working time first hits the replacement threshold; the other is a bivariate policy, and the system is replaced when the working time first hits the replacement threshold or when the Nth failure happens, whichever comes first. The explicit expressions of the long-run average cost rate under these two policies and some special cases can be easily used. The existence of optimal policies is proved, and numerical examples are presented to illustrate the application of the developed approach. Moreover, the optimal policy $(N_{\tau_1}, N)^*$ is proved to be better than the optimal policy $N^*_{\tau_1}$.

Furthermore, if we consider the reward, Lam's model [1,2] can be viewed as a special case of ours, in the case that $\tau_1 = \tau_2 = 0$.

According to the proposed model, the characteristics of the long-run average cost rate curves based on the real applications can be easily obtained, and the optimal replacement policies are suitable in practical applications.

As a generalization and development of the GPRM, EGPRM is more reasonable than GPRM, and there are still many aspects worthy of in-depth study. For example, EGPRM can be applied to maintenance problems for a multi-component repairable system, and different systems, including series systems, parallel systems or k-out-of-n: G systems or multiple failure modes, are also worth considering. In addition, the parameter estimation is very important, which is also of great interest to investigate.

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References

1. Lam, Y. A note on the optimal replacement problem. *Adv. Appl. Probab.* **1988**, *20*, 479–482. [CrossRef]
2. Lam, Y. Geometric processes and replacement problem. *Acta Math. Appl. Sin.* **1988**, *4*, 366–377. [CrossRef]
3. Wu, D.; Peng, R.; Wu, S.M. A review of the extensions of the geometric process, applications, and challenges. *Qual. Reliab. Eng. Int.* **2020**, *36*, 436–446. [CrossRef]
4. Zhang, Y.L.; Wang, G.J. An extended geometric process repair model with delayed repair and slight failure type. *Commun. Stat.-Theory Methods* **2017**, *46*, 427–437. [CrossRef]
5. Chan, J.S.K.; Yu, P.L.H.; Lam, Y.; Ho, A.P.K. Modelling SARS data using threshold geometric process. *Stat. Med.* **2006**, *25*, 1826–1839. [CrossRef]
6. Wu, S.M. Doubly geometric processes and applications. *J. Oper. Res. Soc.* **2018**, *69*, 66–77. [CrossRef]
7. Sarada, Y.; Shenbagam, R. On a random lead time and threshold shock model using phase-type geometric processes. *Appl. Stoch. Models Bus. Ind.* **2018**, *34*, 407–422. [CrossRef]
8. Braun, W.J.; Li, W.; Zhao, Y.Q. Properties of the geometric and related processes. *Nav. Res. Logist.* **2005**, *52*, 607–616. [CrossRef]
9. Sun, Q.Z.; Ye, Z.S.; Zhu, X. Managing component degradation in series systems for balancing degradation through reallocation and maintenance. *IISE Trans.* **2020**, *52*, 797–810. [CrossRef]

10. Zhang, Y.L.; Wang, G.J. An extended geometric process repair model for a cold standby repairable system with imperfect delayed repair. *Int. J. Syst. Sci. Oper. Logist.* **2016**, *3*, 163–175. [CrossRef]
11. Zhang, Y.L.; Wang, G.J. An extended geometric process repair model with imperfect delayed repair under different objective functions. *Commun. Stat.-Theory Methods* **2018**, *47*, 3204–3219. [CrossRef]
12. Wang, J.Y.; Ye, J.M.; Ma, Q.R.; Xie, P.F. An extended geometric process repairable model with its repairman having vacation. *Ann. Oper. Res.* **2022**, *311*, 401–415. [CrossRef]
13. Zhang, Y.L. A bivariate optimal replacement policy for a repairable system. *J. Appl. Probab.* **1994**, *31*, 1123–1127. [CrossRef]
14. Wang, G.J.; Zhang, Y.L. Optimal periodic preventive repair and replacement policy assuming geometric process repair. *IEEE Trans. Reliab.* **2006**, *55*, 118–122. [CrossRef]
15. Wang, G.J.; Zhang, Y.L. A bivariate mixed policy for a simple repairable system based on preventive repair and failure repair. *Appl. Math. Model.* **2009**, *33*, 3354–3359. [CrossRef]
16. Chang, C.C.; Sheu, S.H.; Chen, Y.L. A bivariate optimal replacement policy for a system with age-dependent minimal repair and cumulative repair-cost limit. *Commun. Stat.-Theory Methods* **2013**, *42*, 4108–4126. [CrossRef]
17. Sheu, S.H.; Chien, Y.H.; Chang, C.C.; Chiu, C.H. Optimal trivariate replacement policies for a deteriorating system. *Qual. Technol. Quant. Manag.* **2014**, *11*, 307–320. [CrossRef]
18. Dong, Q.L.; Cui, L.R.; Gao, H.D. A bivariate replacement policy for an imperfect repair system based on geometric processes. *Proc. Inst. Mech. Eng. Part O J. Risk Reliab.* **2019**, *233*, 670–681. [CrossRef]
19. Sun, Q.Z.; Ye, Z.S.; Chen, N. Optimal inspection and replacement policies for multi-unit systems subject to degradation. *IEEE Trans. Reliab.* **2018**, *47*, 401–413. [CrossRef]
20. Qiu, Q.A.; Maillart, L.; Prokopyev, O.; Cui, L.R. Optimal condition-based mission abort decisions. *IEEE Trans. Reliab.* **2022**, 1–18. [CrossRef]
21. Mendes, A.A.; Ribeiro, J.L.D.; Coit, D.W. Optimal time interval between periodic inspections for a two-component cold standby multistate system. *IEEE Trans. Reliab.* **2017**, *66*, 559–574. [CrossRef]
22. Wang, J.J.; Qiu, Q.A.; Wang, H.H. Joint optimization of condition-based and age-based replacement policy and inventory policy for a two-unit series system. *Reliab. Eng. Syst. Saf.* **2021**, *205*, 107251. [CrossRef]
23. Yang, L.; Chen, Y.; Qiu, Q.A.; Wang, J. Risk control of mission-critical systems: Abort decision-makings integrating health and age conditions. *IEEE T. Ind. Inform.* **2022**. [CrossRef]
24. Dong, Q.L.; Cui, L.R.; Si, S.B. Reliability and availability analysis of stochastic degradation systems based on bivariate Wiener processes. *Appl. Math. Model.* **2020**, *79*, 414–433. [CrossRef]
25. Dong, Q.L.; Cui, L.R. Reliability analysis of a system with two-stage degradation using Wiener processes with piecewise linear drift. *IMA J. Manag. Math.* **2021**, *32*, 3–19. [CrossRef]
26. Yu, M.M.; Tang, Y.H.; Wu, W.Q.; Zhou, J. Optimal order-replacement policy for a phase-type geometric process model with extreme shocks. *Appl. Math. Model.* **2014**, *38*, 4323–4332. [CrossRef]
27. Zhang, Y.L. A geometrical process repair model for a repairable system with delayed repair. *Comput. Math. Appl.* **2008**, *55*, 1629–1643. [CrossRef]
28. Zhang, M.M.; Ye, Z.S.; Xie, M. A condition-based maintenance strategy for heterogeneous populations. *Comput. Ind. Eng.* **2014**, *77*, 103–114. [CrossRef]
29. Xie, W.; Liao, H.T.; Jin, T.D. Maximizing system availability through joint decision on component redundancy and spares inventory. *Eur. J. Oper. Res.* **2014**, *237*, 164–176. [CrossRef]
30. Zhao, X.F.; Mizutani, S.; Nakagawa, T. Which is better for replacement policies with continuous or discrete scheduled times? *Eur. J. Oper. Res.* **2015**, *242*, 477–486. [CrossRef]
31. Liu, B.; Xie, M.; Xu, Z.G.; Kuo, W. An imperfect maintenance policy for mission-oriented systems subject to degradation and external shocks. *Comput. Ind. Eng.* **2016**, *102*, 21–32. [CrossRef]
32. Tsai, H.N.; Sheu, S.H.; Zhang, Z.G. A trivariate optimal replacement policy for a deteriorating system based on cumulative damage and inspections. *Reliab. Eng. Syst. Saf.* **2017**, *160*, 74–88. [CrossRef]
33. Zhao, X.; Guo, X.X.; Wang, X.Y. Reliability and maintenance policies for a two-stage shock model with self-healing mechanism. *Reliab. Eng. Syst. Saf.* **2017**, *172*, 185–194. [CrossRef]
34. Zhao, X.J.; Gaudoin, O.; Doyen, L.; Xie, M. Optimal inspection and replacement policy based on experimental degradation data with covariates. *IISE Trans.* **2019**, *51*, 322–336. [CrossRef]
35. Gao, K.Y.; Peng, R.; Qu, L.; Wu, S.M. Jointly optimizing lot sizing and maintenance policy for a production system with two failure modes. *Reliab. Eng. Syst. Saf.* **2020**, *202*, 106996. [CrossRef]
36. Chen, K.; Zhao, X.; Qiu, Q.A. Optimal task abort and maintenance policies considering time redundancy. *Mathematics* **2022**, *10*, 1360. [CrossRef]