mathematics

Article

Yamaguchi -Noshiro Type Bi-Univalent Functions Associated with Sălăgean-Erdély–Kober Operator

Asma Alharbi [1,†], Gangadharan Murugusundaramoorthy [2,†] and Sheza. M. El-Deeb [3,*,†,‡]

[1] Department of Mathematics, College of Science and Arts, ArRass, Qassim University, Buraidah 51452, Saudi Arabia; ao.alharbi@qu.edu.sa
[2] School of Advanced Sciences, Vellore Institute of Technology, Vellore 632014, TN, India; gms@vit.ac.in or gmsmoorthy@yahoo.com
[3] Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt
* Correspondence: shezaeldeeb@yahoo.com
† These authors contributed equally to this work.
‡ Current address: Department of Mathematics, College of Science and Arts, Al-Badaya, Qassim University, Buraidah 51911, Saudi Arabia.

Abstract: We defined two new subclasses of analytic bi-univalent function class Σ, in the open unit disk related with the Sălăgean–Erdély–Kober operator. The bounds on initial coefficients $|a_2|$, $|a_3|$ and $|a_4|$ for the functions in these new subclasses of Σ are investigated. Using the estimates of coefficients a_2, a_3, we also discuss the Fekete-Szegö inequality results for the function classes defined in this paper. Relevant connections of these results, presented here as corollaries, are new and not studied in association with Sălăgean-Erdély–Kober operator for the subclasses defined earlier.

Keywords: univalent functions; analytic functions; bi-univalent functions; Sălăgean operator; Erdély–Kober fractional-order derivative; coefficient bounds

MSC: 30C45; 30C50; 30C55

Citation: Alharbi, A.; Murugusundaramoorthy, G.; El-Deeb, S.M. Yamaguchi -Noshiro Type Bi-Univalent Functions Associated with Sălăgean-Erdély–Kober Operator. *Mathematics* **2022**, *10*, 2241. https://doi.org/10.3390/math10132241

Academic Editors: Valer-Daniel Breaz, Ioan-Lucian Popa, Carlo Bianca and Pasquale Vetro

Received: 24 May 2022
Accepted: 23 June 2022
Published: 26 June 2022

1. Introduction

Let $\mathcal{A}$ denote the class of holomorphic functions in open unit disk

$$\Delta := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

be given by the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta), \tag{1}$$

satisfying normalization conditions (see [1])

$$f(0) = f'(0) - 1 = 0,$$

normalization $f(0) = 0$ geometrically amounts to only a translation of the image domain, and $f'(0) = 1$ corresponds to the rotation of the image domain.The subclass of $\mathcal{A}$ consisting of all univalent functions f in Δ is denoted by $\mathcal{S}$.

Let $f_1,\ f_2 \in \mathcal{A}$ and be assumed as

$$f_1(z) = \sum_{n=0}^{\infty} a_{n,1} z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} a_{n,2} z^n \quad (z \in \Delta),$$

are convoluted $f_1 * f_2$ if the product is defined as

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z) \quad (z \in \Delta).$$

Recently, several authors have contributed to the growth of fractional calculus (differentiation and integration of arbitrary orders). Fractional calculus often find its applications in the field of engineering, such as capacitor theory, electrode–electrolyte interface models, feedback amplifiers, generalized voltage dividers, fractional order models of neurons, the electric conductance of biological systems, fitting experimental data, medical, and memory characteristics [2–6]. Fractional derivative operators, which are frequently defined through fractional integral operators, help in gathering useful information about the progress of the resources and processes involved in the phenomena. Many fractional derivative operators, such as the Riemann–Liouville fractional derivative operator associated with hypergeometric type function, the Caputo (CF) and Erdélyi–Kober (EK) fractional operators have been proposed and studied extensively in the literature. The Riemann–Liouville (R–L) fractional integral operator of order $\varepsilon > 0$, which is one of the most used and studied (see [2–7]) operators, is given by:

$$\mathcal{V}^{\varepsilon}_{\epsilon^+} f(t) = \frac{1}{\Gamma(\varepsilon)} \int_{\epsilon}^{t} (t-\kappa)^{\varepsilon-1} f(\kappa) d\kappa, t > \epsilon.$$

First, we recall the following differential operators:
In 1983, Sălăgean [8] introduced differential operator $\mathcal{D}^m : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned} &\mathcal{D}^0 f(z) = f(z),\ \mathcal{D}^1 f(z) = \mathcal{D} f(z) = z f'(z), \\ &\mathcal{D}^m f(z) = \mathcal{D}(\mathcal{D}^{m-1} f(z))' = z(\mathcal{D}^{m-1} f(z))',\ m \in \mathbb{N} = \{1,2,3,\dots\}. \end{aligned}$$

We note that

$$\mathcal{D}^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n,\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{2}$$

Let the integral operator be

$$I^{\varrho}_{\ell} : \mathcal{A} \to \mathcal{A}$$

by

$$I^{\varrho}_{\ell} f(z) = \begin{cases} \frac{\ell+1}{\varrho} z^{1-\frac{\ell+1}{\varrho}} \int_0^z t^{\frac{\ell+1}{\varrho}-2} f(t) dt; & \varrho \neq 0, \\ f(z), & \varrho = 0. \end{cases}$$

The differential operator is

$$D^{\varrho}_{\ell} : \mathcal{A} \to \mathcal{A}$$

by

$$D^{\varrho}_{\ell} f(z) = \begin{cases} \frac{\varrho}{\ell+1} z^{2-\frac{\ell+1}{\varrho}} \frac{d}{dz}\left(z^{\frac{\ell+1}{\varrho}-1} f(z) \right); & \varrho \neq 0, \\ f(z), & \varrho = 0. \end{cases}$$

For $\ell > -1; \varrho > 0$, and $m \in \mathbb{Z} = \{0, \pm1, \pm2, \pm3, \cdots\}$; let

$$D^m_{\ell,\varrho} : \mathcal{A} \to \mathcal{A}$$

by

$$D^m_{\ell,\varrho} f(z) = \begin{cases} D^{\varrho}_{\ell} \mathcal{D}^{m-1}_{\ell,\varrho} f(z) & m \in \mathbb{Z}^+; \\ I^{\varrho}_{\ell} \mathcal{D}^{m+1}_{\ell,\varrho} f(z), & m \in \mathbb{Z}^-; \\ f(z), & m = 0. \end{cases}$$

The following Erdély–Kober type ([9] (Section 5)) integral operator is used throughout this paper:

Definition 1. ***Erdély–Kober operator (EK):*** *Let for $\vartheta > 0, a, c \in \mathbb{C}$, be such that $\Re(c-a) \geq 0$, and $\mathcal{R}(a) > -\vartheta$ an Erdély–Kober type integral operator is*

$$\mathcal{V}_{\vartheta}^{a,c} : \mathcal{A} \to \mathcal{A}$$

given by

$$\mathcal{V}_{\vartheta}^{a,c} f(z) = \frac{\Gamma(c+\vartheta)}{\Gamma(a+\vartheta)} \frac{1}{\Gamma(c-a)} \int_0^1 (1-t)^{c-a-1} t^{a-1} f(zt^{\vartheta}) dt, \vartheta > 0. \tag{3}$$

For $f \in \mathcal{A}$ assumed by (1), by simple computation, we have

$$\begin{aligned} \mathcal{V}_{\vartheta}^{a,c} f(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma(c+\vartheta)\Gamma(a+n\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+n\vartheta)} a_n z^n \quad (z \in \Delta) \\ &= z + \sum_{n=2}^{\infty} \Upsilon_{\vartheta}^{a,c}(n) a_n z^n \quad (z \in \Delta), \end{aligned} \tag{4}$$

where

$$\Upsilon_{\vartheta}^{a,c}(n) = \frac{\Gamma(c+\vartheta)\Gamma(a+n\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+n\vartheta)}.$$

By fixing $c = a$, we obtain

$$\mathcal{V}_{\vartheta}^{a,a} f(z) = f(z). \tag{5}$$

Some applications and properties of the E–K fractional integral operator can be found in [2–7,10] and references therein. Now, we exploit the concept of the Sălăgean-Erdély–Kober fractional-order derivative(SEK) to define a new unified class of bi-univalent functions.

Definition 2. ***Sălăgean-Erdély–Kober operator (SEK):*** *For $m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$; $\vartheta > 0, \Re(c-a) \geq 0, \Re(a) > -\vartheta; \ell > -1; \varrho > 0$ and $f \in \mathcal{A}$ is assumed by (1), we have*

$$\begin{aligned} \Xi_{\vartheta,m,\ell}^{a,c,\varrho} f(z) &= \frac{\Gamma(c+\vartheta)}{\Gamma(a+\vartheta)} \frac{1}{\Gamma(c-a)} \int_0^1 (1-t)^{c-a-1} t^{a-1} D_{\ell,\varrho}^m f(zt^{\vartheta}) dt, \vartheta > 0 \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(c+\vartheta)\Gamma(a+n\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+n\vartheta)} \left[1 + \frac{\varrho(n-1)}{\ell+1}\right]^m a_n z^n \quad (z \in \Delta) \\ &= z + \sum_{n=2}^{\infty} \Upsilon_{\vartheta,m,\ell}^{a,c,\varrho} a_n z^n, \qquad (z \in \Delta), \end{aligned} \tag{6}$$

where

$$\Upsilon_n = \Upsilon_{\vartheta,m,\ell}^{a,c,\varrho}(n) = \frac{\Gamma(c+\vartheta)\Gamma(a+n\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+n\vartheta)} \left[1 + \frac{\varrho(n-1)}{\ell+1}\right]^m. \tag{7}$$

Particularly,

$$\Upsilon_2 = \Upsilon_{\vartheta,m,\ell}^{a,c,\varrho}(2) = \left[1 + \frac{\varrho}{\ell+1}\right]^m \frac{\Gamma(c+\vartheta)\Gamma(a+2\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+2\vartheta)}. \tag{8}$$

$$\Upsilon_3 = \Upsilon_{\vartheta,m,\ell}^{a,c,\varrho}(3) = \left[1 + \frac{2\varrho}{\ell+1}\right]^m \frac{\Gamma(c+\vartheta)\Gamma(a+3\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+3\vartheta)}. \tag{9}$$

$\Xi_{\vartheta,m,\ell}^{a,a,\varrho}$ includes various differential and integral operators, as illustrated below:

Remark 1. *By fixing $m = 0$, operator $\Xi^{a,a,\varrho}_{\vartheta,m,\ell} \equiv \mathcal{V}^{a,c}_{\vartheta}$, and suitably choosing parameters a, c, ϑ we obtain*

1. *For $a = \varsigma; c = \sigma + \varsigma\vartheta = 1$, and $\sigma \geq 0$; $\varrho > 1$ we obtain the operator $\mathcal{Q}^{\sigma}_{\varsigma}$ studied by Jung et al.* [11];
2. *For $a = \sigma - 1; c = \varsigma - 1$ and $\vartheta = 1$, with $\sigma; \varsigma \in \mathbb{C} \in \mathbb{Z}_0; \mathbb{Z}_0 = \{0; -1; -2; \cdots\}$ we obtain the operator $\mathcal{L}_{\sigma,\varsigma} f(z)$ studied by Carlson and Shafer* [12];
3. *For $a = \rho - 1; c = \ell$ and $\vartheta = 1$, with $\rho > 0; \ell > 1$, we obtain the operator $\mathcal{V}_{\rho,\ell}$ studied by Choi et al.* [13];
4. *For $a = \sigma; c = 0$ and $\vartheta = 1$, with $\sigma > -1$ we obtain the operator $\mathcal{D}^{\sigma}$ studied by Ruscheweyh* [14];
5. *For $a = 1; c = n \in \mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ and $\mu = 1$, we obtain the operator $\mathcal{V}_n$ studied in* [15,16];
6. *For $a = \varsigma; c = \varsigma + 1$ and $\vartheta = 1$; we obtain the Bernardi integral operator* [17] *denoted as $\mathcal{V}_{\varsigma,1}$;*
7. *For $a = 1; c = 2$ and $\vartheta = 1$, give the Libera integral operator* [18] *as $\mathcal{V}_{1,1} = \mathcal{I}$ and Livingston* [19].

Remark 2. *Let $m \in \mathbb{N}_0 = \{0, 1, 2, 3, \cdots .\}$ By fixing the values of a, c, ϑ as specified below, $\Xi^{a,c,\varrho}_{\vartheta,m,\ell}$ includes various operators as cited below:*

1. *By fixing $\ell = 0$ we have $\Xi^{a,a,\varrho}_{\vartheta,m,\ell} \equiv \mathcal{V}^{\varrho}_{\vartheta,m}$, Al-Oboudi operator* [20].
2. *Assuming $\varrho = 1; \ell = 0$ then $\Xi^{a,a,\varrho}_{\vartheta,m,\ell} \equiv \mathcal{V}^{m}_{\vartheta}$, Salagean operator* [8].
3. *Assuming $c = 0; \vartheta = 1$ then $\Xi^{a,0,\varrho}_{1,m,\ell} \equiv \mathcal{V}^{a,\ell}_{m,\varrho}$, Catas operator* [21].
4. *By fixing $\varrho = 1; \ell = \eta$ and $\Xi^{a,a,\varrho}_{\vartheta,-m,\ell} \equiv \mathcal{V}^{\eta+1}_{-m}$, Komatu operator* [22].

Fractional calculus operators have fruitfully been applied in obtaining, for example, the characterization properties, coefficient estimates, distortion inequalities, and convolutional structures of various subclasses of analytic functions. In this article, we study the subclasses of bi-univalent functions.

Bi-Univalent Functions Σ

The renowned Koebe one-quarter theorem (see [1]) asserts that the image of Δ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus, the inverse of $f \in \mathcal{A}$ is a univalent analytic function on the disk $\Delta_\rho := \{z : z \in \mathbb{C} \text{ and } |z| < \rho;\ \rho \geq \frac{1}{4}\}$. Consequently, for each function $f(z) = w \in \sigma$, there is an inverse function $f^{-1}(w)$ of $f(z)$ defined by

$$f^{-1}(f(z)) = z \quad (z \in \Delta),$$

and

$$f(f^{-1}(w)) = w \quad (w \in \Delta_\rho),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots. \tag{10}$$

A function $f \in \mathcal{A}$ is supposed to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ.

Let Σ denotes the class of bi-univalent functions in Δ given by (1). Note that the functions

$$f_1(z) = \frac{z}{1-z}, \qquad f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \qquad f_3(z) = -\log(1-z) \tag{11}$$

with their corresponding inverses

$$f_1^{-1}(w) = \frac{w}{1+w}, \qquad f_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}, \qquad f_3^{-1}(w) = \frac{e^{w}-1}{e^{w}} \tag{12}$$

are elements of Σ. The concept of bi-univalent analytic functions was introduced by Lewin [23] in 1967, and he showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [24] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [25] , on the other hand, showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. The coefficient estimate problem for each of the following Taylor–Maclaurin coefficients:

$$|a_n| \quad (n \in \mathbb{N} \setminus \{1,2\})$$

is presumably still an open problem. Recently, there has been interest in studying bi-univalent function class Σ and obtained non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1). This subject was extensively discussed in the pioneering work by Srivastava et al. [26], who revived the study of analytic and bi-univalent functions in recent years. It was followed by many sequels to Srivastava et al. [26] (see, for example, [27–36]), certain subclasses of the bi-univalent analytic functions class Σ were introduced, and nonsharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ were found. The study of operators plays an significant role in geometric function theory. Many differential and integral operators can be written in terms of the convolution of certain analytic functions. This formalism brings ease in further mathematical exploration, and helps in better understanding the symmetric and geometric properties of such operators. Inspired by the aforementioned works on bi-univalent functions, and by using Sălăgean-Erdély–Kober operator in the present paper, we define two new subclasses as in Definitions 3 and 4 of function class Σ, determine the estimates on coefficients $|a_2|$, $|a_3|$, and attempted to find $|a_4|$ for the functions in these new subclasses of function class Σ. We also discussed the Fekete-Szegö inequalities results [37] for $f \in \mathcal{YN}^{a,c,\varrho}_{\vartheta,m,\ell}(\xi,t)$, and $f \in \mathcal{YM}^{a,c,\varrho}_{\vartheta,m,\ell}(\zeta,t)$. Further we discussed the results several consequences of the results for the new subclasses of Σ that are not studied in association with telephone numbers based on Sălăgean-Erdély–Kober operator as illustrated in Definitions 5–8 .

Definition 3. *For $0 < \xi \leq 1, t \geq 1$, and $f \in \Sigma$ be assumed by (1) is supposed to be in class $\mathcal{YN}^{a,c,\varrho}_{\vartheta,m,\ell}(\xi,t)$ if the following conditions are satisfied:*

$$\left| arg\left((1-t)\frac{\Xi^{a,c,\varrho}_{\vartheta,m,\ell}f(z)}{z} + t\left(\Xi^{a,c,\varrho}_{\vartheta,m,\ell}f(z)\right)'\right)\right| < \frac{\xi\pi}{2}, \tag{13}$$

and

$$\left| arg\left((1-t)\frac{\Xi^{a,c,\varrho}_{\vartheta,m,\ell}g(w)}{w} + t\left(\Xi^{a,c,\varrho}_{\vartheta,m,\ell}g(w)\right)'\right)\right| < \frac{\xi\pi}{2}, \tag{14}$$

where $z, \in \Delta; w \in \Delta_\rho$ and g are the inverse of f given by (10).

Definition 4. *For $0 < \zeta \leq 1, t \geq 1$; and $f \in \Sigma$ are given by (1); then, $f \in \mathcal{YM}^{a,c,\varrho}_{\vartheta,m,\ell}(\zeta,t)$ if the following conditions are satisfied:*

$$\Re\left((1-t)\frac{\Xi^{a,c,\varrho}_{\vartheta,m,\ell}f(z)}{z} + t\left(\Xi^{a,c,\varrho}_{\vartheta,m,\ell}f(z)\right)'\right) > \zeta \tag{15}$$

and

$$\Re\left((1-t)\frac{\Xi^{a,c,\varrho}_{\vartheta,m,\ell}g(w)}{w} + t\left(\Xi^{a,c,\varrho}_{\vartheta,m,\ell}g(w)\right)'\right) > \zeta, \tag{16}$$

where $z, \in \Delta; w \in \Delta_\rho$ and g is the inverse of f given by (10).

By fixing $t = 1$, we define a new subclass of Σ due to Noshiro [38].

Definition 5. *For $0 < \xi \leq 1$ and $f \in \Sigma$ be given by (1) then $f \in \mathcal{N}^{a,c,\varrho}_{\vartheta,m,\ell}(\xi)$ if it holds the following conditions :*

$$\left| arg \left(\left(\Xi^{a,c,\varrho}_{\vartheta,m,\ell} f(z) \right)' \right) \right| < \frac{\xi\pi}{2} \quad and \quad \left| arg \left(\left(\Xi^{a,c,\varrho}_{\vartheta,m,\ell} g(w) \right)' \right) \right| < \frac{\xi\pi}{2}, \tag{17}$$

where $z, \in \Delta; w \in \Delta_\rho$ and g is the inverse of f given by (10).

Definition 6. *A function $f \in \Sigma$ given by (1) then $f \in \mathcal{M}^{a,c,\varrho}_{\vartheta,m,\ell}(\zeta)$ if it satisfy the following conditions:*

$$\Re\left(\left[\Xi^{a,c,\varrho}_{\vartheta,m,\ell} f(z) \right]' \right) > \zeta \quad and \quad \Re\left(\left[\Xi^{a,c,\varrho}_{\vartheta,m,\ell} g(w) \right]' \right) > \zeta, \tag{18}$$

where $z, \in \Delta; w \in \Delta_\rho$ and g is the inverse of f given by (10).

By fixing $t = 0$, we define a new subclass of Σ due to Yamaguchi [39].

Definition 7. *For $0 < \xi \leq 1$ and $f \in \Sigma$, as assumed as (1); then, $f \in \mathcal{Y}^{a,c,\varrho}_{\vartheta,m,\ell}(\xi)$ if the following conditions hold:*

$$\left| arg \left(\frac{\Xi^{a,c,\varrho}_{\vartheta,m,\ell} f(z)}{z} \right) \right| < \frac{\xi\pi}{2} \quad and \quad \left| arg \left(\frac{\Xi^{a,c,\varrho}_{\vartheta,m,\ell} g(w)}{w} \right) \right| < \frac{\xi\pi}{2}, \tag{19}$$

where $z, \in \Delta; w \in \Delta_\rho$ and g are the inverse of f given by (10).

Definition 8. *For $0 \leq \zeta < 1$ and a function $f \in \Sigma$ given by (1); then, $f \in \mathcal{X}^{a,c,\varrho}_{\vartheta,m,\ell}(\zeta)$ if the following conditions are satisfied:*

$$\Re\left(\frac{\Xi^{a,c,\varrho}_{\vartheta,m,\ell} f(z)}{z} \right) > \zeta \quad and \quad \Re\left(\frac{\Xi^{a,c,\varrho}_{\vartheta,m,\ell} g(w)}{w} \right) > \zeta, \tag{20}$$

where $z, \in \Delta; w \in \Delta_\rho$ and g are the inverse of f given by (10).

2. Coefficient Bounds

In order to find the initial coefficient bounds, namely, $|a_2|$, $|a_3|$ and $|a_4|$ for $f \in \mathcal{YN}^{a,c,\varrho}_{\vartheta,m,\ell}(\xi, t)$ and $f \in \mathcal{YM}^{a,c,\varrho}_{\vartheta,m,\ell}(\zeta, t)$ of Σ, we recall the following lemma:

Lemma 1. *(see [1], p. 41) Let $\phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be in $\mathcal{P}$, the class of all analytic functions with $\Re(\phi(z)) > 0$ $(z \in \Delta)$ and $\phi(0) = 1$. Then,*

$$|c_n| \leq 2 \;\; (n = 1, 2, 3, \ldots).$$

This inequality is sharp $\forall n$. In particular, for $\phi(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n$. equality holds $\forall n$.

Theorem 1. *Let $f(z)$ given by (1) be in the class $\mathcal{YN}^{a,c,\varrho}_{\vartheta,m,\ell}(\xi, t)$. Then*

$$|a_2| \leq \frac{2\xi}{\sqrt{2\xi(1+2t)\Upsilon_3 + (1-\xi)(1+t)^2\Upsilon_2^2}}, \tag{21}$$

$$|a_3| \leq \frac{2\xi}{(1+2t)\Upsilon_3}, \tag{22}$$

and

$$|a_4| \leq \frac{2\xi}{(1+3t)\Upsilon_4}\left[1+\frac{2(1-\xi)(1+t)\Upsilon_2\{6\xi(1+2t)\Upsilon_3+(1-2\xi)(1+t)^2\Upsilon_2^2\}}{3\{2\xi(1+2t)\Upsilon_3+(1-\xi)(1+t)^2\Upsilon_2^2\}^{\frac{3}{2}}}\right]. \tag{23}$$

Proof. Let $f \in \mathcal{YN}^{a,c,\varrho}_{\vartheta,m,\ell}(\xi,t)$. Hence, by Definition 3, there exists $\phi(z)$ and $\psi(w) \in \mathcal{P}$ such that

$$(1-t)\frac{\Xi^{a,c,\varrho}_{\vartheta,m,\ell}f(z)}{z}+t\left(\Xi^{a,c,\varrho}_{\vartheta,m,\ell}f(z)\right)' = [\phi(z)]^{\xi} \tag{24}$$

and

$$(1-t)\frac{\Xi^{a,c,\varrho}_{\vartheta,m,\ell}g(w)}{w}+t\left(\Xi^{a,c,\varrho}_{\vartheta,m,\ell}g(w)\right)' = [\psi(w)]^{\xi}. \tag{25}$$

Write

$$\phi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \tag{26}$$

and

$$\psi(w) = 1 + d_1 w + d_2 w^2 + d_3 w^3 + \ldots, \tag{27}$$

Now, equating the coefficients in (24) and (25), we obtain

$$(1+t)\Upsilon_2 a_2 = \xi c_1 \tag{28}$$

$$(1+2t)\Upsilon_3 a_3 = \xi c_2 + \frac{\xi(\xi-1)}{2}c_1^2 \tag{29}$$

$$(1+3t)\Upsilon_4 a_4 = \xi c_3 + \xi(\xi-1)c_1c_2 + \frac{\xi(\xi-1)(\xi-2)}{6}c_1^3, \tag{30}$$

and

$$-(1+t)\Upsilon_2 a_2 = \xi d_1 \tag{31}$$

$$(1+2t)\Upsilon_3(2a_2^2-a_3) = \xi d_2 + \frac{\xi(\xi-1)}{2}d_1^2 \tag{32}$$

$$-(1+3t)\Upsilon_4(5a_2^3-5a_2a_3+a_4) = \xi d_3 + \xi(\xi-1)d_1d_2 + \frac{\xi(\xi-1)(\xi-2)}{6}d_1^3. \tag{33}$$

From (28) and (31), we obtain

$$a_2 = \frac{\xi c_1}{(1+t)\Upsilon_2} = -\frac{\xi}{(1+t)\Upsilon_2}d_1 \tag{34}$$

which implies

$$c_1 = -d_1.$$

Squaring and adding (28), (31), we obtain

$$a_2^2 = \frac{\xi^2}{(1+t)^2\Upsilon_2^2}(c_1^2+d_1^2). \tag{35}$$

Adding (29) and (32), we obtain

$$2(1+2t)\Upsilon_3 a_2^2 = \xi(c_2+d_2) + \frac{\xi(\xi-1)}{2}(c_1^2+d_1^2).$$

Substitute the value of a_2 from (34) in (35) and noting that $c_1^2 = d_1^2$, we observe that

$$c_1^2 = \frac{(1+t)^2\Upsilon_2^2(c_2+d_2)}{2\xi(1+2t)\Upsilon_3+(1-\xi)(1+t)^2\Upsilon_2^2}. \tag{36}$$

By applications of the triangle inequality and Lemma 1 give

$$|c_1| \leq \frac{2(1+t)\Upsilon_2}{\sqrt{2\xi(1+2t)\Upsilon_3 + (1-\xi)(1+t)^2\Upsilon_2^2}}. \tag{37}$$

Thus, (34) gives

$$|a_2| \leq \frac{2\xi}{\sqrt{2\xi(1+2t)\Upsilon_3 + (1-\xi)(1+t)^2\Upsilon_2^2}}. \tag{38}$$

In order to find the bound on $|a_3|$, subtracting (32) from (29) with $c_1 = -d_1$ gives

$$\begin{aligned} 2(1+2t)\Upsilon_3 a_3 &= 2(1+2t)\Upsilon_3 a_2^2 + \xi(c_2 - d_2) \\ a_3 &= a_2^2 + \frac{\xi(c_2 - d_2)}{2(1+2t)\Upsilon_3}. \end{aligned} \tag{39}$$

Using (34) and (36) in (39) after simplification yields

$$\begin{aligned} 2(1+2t)\Upsilon_3 a_3 &= \frac{2\xi^2\Upsilon_3(1+2t)}{2\xi(1+2t)\Upsilon_3 + (1+t)^2\Upsilon_2^2(1-\xi)}(c_2+d_2) + \xi(c_2-d_2) \\ &= \left(\frac{2\xi^2\Upsilon_3(1+2t)}{2\xi(1+2t)\Upsilon_3 + (1+t)^2\Upsilon_2^2(1-\xi)} + \xi\right)c_2 \\ &+ \left(\frac{2\xi^2\Upsilon_3(1+2t)}{2\xi(1+2t)\Upsilon_3 + (1-\xi)(1+t)^2\Upsilon_2^2} - \xi\right)d_2 \\ &= \frac{\xi\left[\{4\xi(1+2t)\Upsilon_3 + (1-\xi)(1+t)^2\Upsilon_2^2\}c_2 - (1-\xi)(1+t)^2\Upsilon_2^2 d_2\right]}{2\xi(1+2t)\Upsilon_3 + (1-\xi)(1+t)^2\Upsilon_2^2} \end{aligned}$$

By the application of a triangle inequality to the above equation,

$$|a_3| \leq \frac{\xi\left[\{4\xi(1+2t)\Upsilon_3 + (1-\xi)(1+t)^2\Upsilon_2^2\}|c_2| + (1-\xi)(1+t)^2\Upsilon_2^2|d_2|\right]}{2(1+2t)\Upsilon_3\left[2\xi(1+2t)\Upsilon_3 + (1-\xi)(1+t)^2\Upsilon_2^2\right]}.$$

Again, by applying Lemma 1 for the coefficients c_2 and d_2, we obtain

$$|a_3| \leq \frac{2\xi}{(1+2t)\Upsilon_3}.$$

To determine the bound on $|a_4|$, adding (30) and (33) with $c_1 = -d_1$, we have

$$-5(1+3t)\Upsilon_4 a_2^3 + 5(1+3t)\Upsilon_4 a_2 a_3 = \xi(c_3+d_3) + \xi(\xi-1)c_1(c_2-d_2), \tag{40}$$

substitute the values of a_2 and a_3 from (34) and (39) in (40) and simplify, we obtain

$$c_1(c_2-d_2) = \frac{2(1+2t)(1+t)\Upsilon_2\Upsilon_3}{5\xi(1+3t)\Upsilon_4 + 2(1-\xi)(1+2t)(1+t)\Upsilon_2\Upsilon_3}(c_3+d_3). \tag{41}$$

Subtracting (33) from (30) and using (36), (37), (40) and (41) in the result, we obtain

$$\begin{aligned}
2a_4(1+3t)\Upsilon_4 &= -5(1+3t)\Upsilon_4 a_2^3 + 5(1+3t)a_2a_3\Upsilon_4 + \xi(c_3-d_3)\\
&+ \xi(\xi-1)c_1(c_2+d_2) + \tfrac{\xi(\xi-1)(\xi-2)}{3}c_1^3\\
&= \xi(c_3+d_3) + \xi(\xi-1)c_1(c_2-d_2)\\
&+ \xi(c_3-d_3) + \xi(\xi-1)c_1(c_2+d_2) + \tfrac{\xi(\xi-1)(\xi-2)}{3}c_1^3\\
&= 2\xi c_3 + \tfrac{2\xi(\xi-1)(1+2t)(1+t)\Upsilon_2\Upsilon_3}{5\xi(1+3t)\Upsilon_4+2(1-\xi)(1+2t)(1+t)\Upsilon_2\Upsilon_3}(c_3+d_3)\\
&+ \xi(\xi-1)c_1(c_2+d_2) + \tfrac{\xi(\xi-1)(\xi-2)(1+t)^2\Upsilon_2^2}{3\{2\xi(1+2t)\Upsilon_3+(1-\xi)(1+t)^2\Upsilon_2^2\}}\\
&= \tfrac{10\xi^2(1+3t)\Upsilon_4+2\xi(1-\xi)(1+2t)(1+t)\Upsilon_2\Upsilon_3}{5\xi(1+3t)\Upsilon_4+2(1-\xi)(1+2t)(1+t)\Upsilon_2\Upsilon_3}c_3\\
&- \tfrac{2\xi(1-\xi)(1+2t)(1+t)\Upsilon_2\Upsilon_3}{5(1+3t)\xi\Upsilon_4+2(1-\xi)(1+2t)(1+t)\Upsilon_2\Upsilon_3}d_3\\
&- \xi(1-\xi)\left[\tfrac{6\xi(1+2t)\Upsilon_3+(1-2\xi)(1+t)^2\Upsilon_2^2}{6\xi(1+2t)\Upsilon_3+3(1-\xi)(1+t)^2\Upsilon_2^2}\right]c_1(c_2+d_2).
\end{aligned} \tag{42}$$

Applying Lemma 1 with the triangle inequality in (42), we obtain

$$|a_4| \le \frac{2\xi}{(1+3t)\Upsilon_4}\left[1+\frac{2(1-\xi)(1+t)\Upsilon_2\{6\xi(1+2t)\Upsilon_3+(1-2\xi)(1+t)^2\Upsilon_2^2\}}{3\{2\xi(1+2t)\Upsilon_3+(1-\xi)(1+t)^2\Upsilon_2^2\}^{\frac{3}{2}}}\right].$$

This concludes the proof of Theorem 1. □

Theorem 2. *Let $f(z)$ be given by (1) and $f \in \mathcal{YM}_{\vartheta,m,\ell}^{a,c,\varrho}(\zeta,t)$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\zeta)}{(1+2t)\Upsilon_3}}, \tag{43}$$

$$|a_3| \le \frac{2(1-\zeta)}{(1+2t)\Upsilon_3} \tag{44}$$

and

$$|a_4| \le \frac{2(1-\zeta)}{(1+3t)\Upsilon_4}. \tag{45}$$

Proof. Since $f(z) \in \mathcal{YM}_{\vartheta,m,\ell}^{a,c,\varrho}(\zeta,t)$, there exist two functions $\phi(z)$ and $\psi(z) \in \mathcal{P}$ satisfying the conditions of Lemma 1, such that

$$(1-t)\frac{\Xi_{\vartheta,m,\ell}^{a,c,\varrho}f(z)}{z} + t\left(\Xi_{\vartheta,m,\ell}^{a,c,\varrho}f(z)\right)' = \zeta + (1-\zeta)\phi(z) \tag{46}$$

and

$$(1-t)\frac{\Xi_{\vartheta,m,\ell}^{a,c,\varrho}g(w)}{w} + t\left(\Xi_{\vartheta,m,\ell}^{a,c,\varrho}g(w)\right)' = \zeta + (1-\zeta)\psi(w) \tag{47}$$

where $\phi(z)$ and $\psi(w)$ given by (26) and (27) respectively. Equating the coefficients in (46) and (47) gives

$$(1+t)\Upsilon_2 a_2 = (1-\zeta)c_1 \tag{48}$$

$$(1+2t)\Upsilon_3 a_3 = (1-\zeta)c_2 \tag{49}$$

$$(1+3t)\Upsilon_4 a_4 = (1-\zeta)c_3, \tag{50}$$

and

$$-(1+t)\Upsilon_2 a_2 = (1-\zeta)d_1 \tag{51}$$

$$(1+2t)\Upsilon_3(2a_2^2 - a_3) = (1-\zeta)d_2 \tag{52}$$

$$-(1+3t)\Upsilon_4(5a_2^3 - 5a_2a_3 + a_4) = (1-\zeta)d_3, \tag{53}$$

from (48) and (51) gives

$$a_2 = \frac{1-\zeta}{(1+t)\Upsilon_2}c_1 = -\frac{1-\zeta}{(1+t)\Upsilon_2}d_1, \tag{54}$$

which implies

$$c_1 = -d_1.$$

Adding (49) and (52), we obtain

$$2(1+2t)\Upsilon_3 a_2^2 = (1-\zeta)(c_2+d_2) \tag{55}$$

$$a_2^2 = \frac{(1-\zeta)}{2(1+2t)\Upsilon_3}(c_2+d_2), \tag{56}$$

using (54) in (55), we have

$$c_1^2 = \frac{(1+t)^2\Upsilon_2^2}{2(1+2t)(1-\zeta)\Upsilon_3}(c_2+d_2), \tag{57}$$

an applications of the triangle inequality and Lemma 1 in (57) yield

$$|c_1| \le (1+t)\Upsilon_2\sqrt{\frac{2}{(1+2t)(1-\zeta)\Upsilon_3}}, \tag{58}$$

using (58) in (54) gives

$$|a_2| \le \sqrt{\frac{2(1-\zeta)}{(1+2t)\Upsilon_3}}. \tag{59}$$

Now, subtracting (52) from (49) and using (55), we obtain

$$|a_3| \le \frac{2(1-\zeta)}{(1+2t)\Upsilon_3}, \tag{60}$$

which is the direct consequence of (49).

In order to obtain the bounds on $|a_4|$, we proceed as follows:

From (50), it is easy to see that

$$|a_4| = \left|\frac{(1-\zeta)c_3}{(1+3t)\Upsilon_4}\right| \le \frac{2(1-\zeta)}{(1+3t)\Upsilon_4}. \tag{61}$$

On the otherhand, subtracting (53) from (50) and using (54), we obtain

$$a_4 = \frac{1}{2(1+3t)\Upsilon_4}\left[\frac{-5(1+3t)(1-\zeta)^3\Upsilon_4}{(1+t)^3\Upsilon_2^3}c_1^3 + \frac{5(1+3t)(1-\zeta)\Upsilon_4}{(1+t)\Upsilon_2}a_3c_1 + (1-\zeta)(c_3-d_3)\right]. \tag{62}$$

Applying the triangle inequality in (62), we have

$$|a_4| \le \frac{1}{2(1+3t)\Upsilon_4}\left[\frac{5(1+3t)(1-\zeta)^3\Upsilon_4}{(1+t)^3\Upsilon_2^3}|c_1|^3 + \frac{5(1+3t)(1-\zeta)\Upsilon_4}{(1+t)\Upsilon_2}|a_3||c_1| + (1-\zeta)(|c_3|+|d_3|)\right]. \tag{63}$$

Using (58), (60) and Lemma 1 in (63), after simplification yield

$$|a_4| \le \frac{2(1-\zeta)}{(1+3t)\Upsilon_4}\left[1 + \frac{5(1+3t)}{(1+2t)\Upsilon_3}\sqrt{\frac{2(1-\zeta)}{(1+2t)\Upsilon_3}}\right]. \tag{64}$$

From (61) and (64), we observe that

$$
\begin{aligned}
|a_4| &\leq \min\left[\frac{2(1-\zeta)}{(1+3t)\Upsilon_4}, \frac{2(1-\zeta)}{(1+3t)\Upsilon_4}\left\{1+\frac{5(1+3t)}{(1+2t)\Upsilon_3}\sqrt{\frac{2(1-\zeta)}{(1+2t)\Upsilon_3}}\right\}\right] \\
&= \frac{2(1-\zeta)}{(1+3t)\Upsilon_4}.
\end{aligned}
$$

This completes the proof of Theorem 2. □

3. Fekete-Szegö Inequalities

In this section, we obtain Fekete-Szegö inequalities results [37] (see [40]), for $f \in \mathcal{YN}^{a,c,\varrho}_{\vartheta,m,\ell}(\xi,t)$, and $f \in \mathcal{YM}^{a,c,\varrho}_{\vartheta,m,\ell}(\zeta,t)$.

Theorem 3. *For $\nu \in \mathbb{R}$, let f be given by (1) and $f \in \mathcal{YN}^{a,c,\varrho}_{\vartheta,m,\ell}(\xi,t)$, then*

$$
\left|a_3-\nu a_2^2\right| \leq \begin{cases} \frac{2\xi}{(1+2t)\Upsilon_3}, & 0 \leq |h(\nu)| \leq \frac{\xi}{2(1+2t)\Upsilon_3}; \\ 4|h(\nu)|, & |h(\nu)| \geq \frac{\xi}{2(1+2t)\Upsilon_3}, \end{cases}
$$

where

$$
h(\nu) = \frac{2\xi^2(1-\nu)}{4\xi(1+2t)\Upsilon_3-(\xi-1)(1+t)^2\Upsilon_2^2}.
$$

Proof. From (39), we have

$$
\begin{aligned}
a_3-\nu a_2^2 &= \frac{\xi(c_2-d_2)}{2(1+2t)\Upsilon_3}+(1-\nu)a_2^2 &(65)\\
&= \frac{\xi(c_2-d_2)}{2(1+2t)\Upsilon_3}+\frac{2\xi^2(1-\nu)(c_2+d_2)}{4\xi(1+2t)\Upsilon_3-(\xi-1)(1+t)^2\Upsilon_2^2}. &(66)
\end{aligned}
$$

By simple computation , we have

$$
a_3-\nu a_2^2 = \left(h(\nu)+\frac{\xi}{2(1+2t)\Upsilon_3}\right)c_2+\left(h(\nu)-\frac{\xi}{2(1+2t)\Upsilon_3}\right)d_2, \tag{67}
$$

where

$$
h(\nu) = \frac{2\xi^2(1-\nu)}{4\xi(1+2t)\Upsilon_3-(\xi-1)(1+t)^2\Upsilon_2^2}. \tag{68}
$$

Thus by taking modulus of (67), we obtain

$$
\left|a_3-\nu a_2^2\right| \leq \begin{cases} \frac{2\xi}{(1+2t)\Upsilon_3} & ;0 \leq |h(\nu)| \leq \frac{\xi}{2(1+2t)\Upsilon_3} \\ 4|h(\nu)| & ;|h(\nu)| \geq \frac{\xi}{2(1+2t)\Upsilon_3}, \end{cases}
$$

where $h(\nu)$ is given by (68). □

Theorem 4. *For $\nu \in \mathbb{R}$, let f be given by (1) and $f \in \mathcal{YM}^{a,c,\varrho}_{\vartheta,m,\ell}(\zeta,t)$, then*

$$
\left|a_3-\nu a_2^2\right| \leq \frac{(1-\zeta)|2-\nu|}{(1+2t)\Upsilon_3}\left[1+\frac{\nu}{|2-\nu|}\right].
$$

Proof. Subtracting (52) from (49), we have

$$
a_3 = \frac{(1-\zeta)(c_2-d_2)}{2(1+2t)\Upsilon_3}+a_2^2, \tag{69}
$$

and using (56), we obtain

$$a_3 - \nu a_2^2 = \frac{(1-\zeta)(c_2-d_2)}{2(1+2t)\Upsilon_3} + (1-\nu)a_2^2 \quad (70)$$
$$= \frac{(1-\zeta)(c_2-d_2)}{2(1+2t)\Upsilon_3} + \frac{(1-\zeta)(1-\nu)}{2(1+2t)\Upsilon_3}(c_2+d_2). \quad (71)$$

By simple computation, we have

$$\begin{aligned} a_3 - \nu a_2^2 &= \left(\tfrac{(1-\zeta)(1-\nu)}{2(1+2t)\Upsilon_3} + \tfrac{1-\zeta}{2(1+2t)\Upsilon_3}\right)c_2 + \left(\tfrac{(1-\zeta)(1-\nu)}{2(1+2t)\Upsilon_3} - \tfrac{1-\zeta}{2(1+2t)\Upsilon_3}\right)d_2, \\ &= \tfrac{(1-\zeta)}{2(1+2t)\Upsilon_3}((1-\nu)+1)c_2 + \tfrac{(1-\zeta)}{2(1+2t)\Upsilon_3}((1-\nu)-1)d_2, \\ &= \tfrac{(1-\zeta)}{2(1+2t)\Upsilon_3}[(2-\nu)c_2 - \nu d_2], \\ &= \tfrac{(1-\zeta)(2-\nu)}{2(1+2t)\Upsilon_3}\left[c_2 - \tfrac{\nu}{2-\nu}d_2\right]. \end{aligned} \quad (72)$$

Thus, by taking the modulus of (72), we obtain

$$\left|a_3 - \nu a_2^2\right| \leq \frac{(1-\zeta)|2-\nu|}{(1+2t)\Upsilon_3}\left[1 + \frac{\nu}{|2-\nu|}\right].$$

In particular $\nu = 1$, we obtain

$$\left|a_3 - a_2^2\right| \leq \frac{2(1-\zeta)}{(1+2t)\Upsilon_3}.$$

□

4. Conclusions

We defined a unified Yamaguchi–Noshiro type subclass of bi-univalent functions based on Sălăgean-Erdély–Kober operator. We obtained nonsharp bounds for the initial coefficients and the Fekete–Szegö inequalities for the functions in this new class. Some interesting corollaries and applications of the results were also discussed. By suitably fixing the parameters, as illustrated in Remarks 1 and 1, one can easily state the results discussed in this article for the function classes given in Definitions 3–8. One can construct the Yamaguchi–Noshiro class on the basis of the Ma–Minda [41] subordination [42] for a given $f \in \Sigma$ to be given by (1), satisfying the following conditions:

$$\left((1-t)\frac{\Xi^{a,c,\varrho}_{\vartheta,m,\ell}f(z)}{z} + t\left(\Xi^{a,c,\varrho}_{\vartheta,m,\ell}f(z)\right)'\right) \prec \varphi(z) \quad (73)$$

and

$$\left((1-t)\frac{\Xi^{a,c,\varrho}_{\vartheta,m,\ell}g(w)}{w} + t\left(\Xi^{a,c,\varrho}_{\vartheta,m,\ell}g(w)\right)'\right) \prec \varphi(w), \quad (74)$$

where

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, (B_1 > 0)$$

$0 < \zeta \leq 1, t \geq 1$, and $z, \in \Delta; w \in \Delta_\rho$ and g are the inverse of f given by (10).

By giving some specific values to φ, as listed below, we define several new subclasses of $\Sigma:$

1. For $\varphi(z) = 1 + \sin z$, we have the function class $\mathcal{S}^*(\varphi)$ of starlike functions associated with the sine functions (see [43]).
2. For $\varphi(z) = 1 + z - \frac{1}{3}z^3$, we have the function class of starlike functions associated with the nephroid (see [44]).
3. For $\varphi(z) = \sqrt{1+z}$, we have the function class of starlike functions associated with the lemniscate of Bernoulli (see [45]).

4. For $\varphi(z) = e^z$, we have the function class of starlike functions associated with the exponintial functions (see [46]).
5. For $\varphi(z) = z + \sqrt{1+z^2}$, we have the function class of starlike functions associated with the crescent shaped region (see [47]).

Author Contributions: Conceptualization, A.A., G.M. and S.M.E.-D.; methodology, A.A., G.M. and S.M.E.-D.; validation, A.A., G.M. and S.M.E.-D.; formal analysis, A.A., G.M. and S.M.E.-D.; investigation, A.A., G.M. and S.M.E.-D.; resources, A.A., G.M. and S.M.E.-D.;writing—original draft preparation, A.A., G.M. and S.M.E.-D.; writing—review and editing, A.A., G.M. and S.M.E.-D.; supervision, A.A., G.M. and S.M.E.-D.; project administration, A.A., G.M. and S.M.E.-D. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The researchers would like to thank the Deanship of Scientific Research, Qassim University for funding the publication of this project. The authors are grateful to the reviewers of this article who gave valuable comments and advice that allowed for us to revise and improve the content of the paper.

Conflicts of Interest: the authors declare no conflict of interest.

References

1. Duren, P.L. Univalent Functions. In *Grundlehren der Mathematischen Wissenschaften*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1983.
2. Samko, G.; Kilbas, A.; Marichev, O. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: Amsterdam, The Netherlands, 1993.
3. Gorenflo, R.; Mainardi, F. Fractional calculus: Integral and differential equations of fractional order. In *Fractals and Fractional Calculus in Continuum Mechanics*; Carpinteri, A., Mainardi, F., Eds.; Springer: New York, NY, USA, 1997; pp. 277–290.
4. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
5. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific Publishing Company: Singapore, 2000.
6. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
7. Bonilla, B.; Rivero, M.; Rodríguez-Germá, L.; Trujillo, J.J. Fractional differential equations as alternative models to nonlinear differential equations. *Appl. Math. Comput.* **2007**, *10*, 79–88. [CrossRef]
8. Sălăgean, G. *Subclasses of Univalent Functions, Complex Analysis: Fifth Romanian—Finnish Seminar, Part I (Bucharest,1981), Lecture Notes in Mathematics 1013*; Springer: New York, NY, USA, 1983.
9. Kiryakova, V. *Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics Series*; John Willey & Sons Inc.: New York, NY, USA, 1994.
10. Erdélyi, A.; Kober, H. Some remarks on Hankel transforms. *Q. J. Math.* **1940**, *11*, 212–221. [CrossRef]
11. Jung, I.B.; Kim, Y.C.; Srivastava, H.M. The Hardy space of analytic functions associated with certain one-parameter families of integral operators. *J. Math. Anal. Appl.* **1993**, *176*, 138–147. [CrossRef]
12. Carlson, B.C.; Shafer, D.B. Starlike and prestarlike Hypergeometric functions. *J. Math. Anal.* **1984**, *15*, 737–745. [CrossRef]
13. Choi, J.H.; Saigo, M.; Srivastava, H.M. Some inclusion properties of a certain family of integral operators. *J. Math. Anal. Appl.* **2002**, *276*, 432–445. [CrossRef]
14. Ruscheweyh, S. New criteria for univalent functions. *Proc. Am. Math. Soc.* **1975**, *49*, 109–115. [CrossRef]
15. Noor, K.I. On new classes of integral operators. *J. Nat. Geom.* **1999**, *16*, 71–80.
16. Noor, K.I.; Noor, M.A. On integral operators. *J. Math. Anal. Appl.* **1999**, *238*, 341–352. [CrossRef]
17. Bernardi, S.D. Convex and starlike univalent functions. *Trans. Am. Math. Soc.* **1969**, *135*, 429–446. [CrossRef]
18. Libera, R.J. Some classes of regular univalent functions. *Proc. Am. Math. Soc.* **1965**, *16*, 755–758. [CrossRef]
19. Livingston, A.E. On the radius of univalence of certain analytic functions. *Proc. Am. Math. Soc.* **1966**, *17*, 352–357. [CrossRef]
20. Al-Oboudi, F.M. On univalent functions defined by a generalized Sălăgean operator. *Inter. J. Math. Math. Sci.* **2004**, *27*, 1429–1436. [CrossRef]
21. Catas, A. On a certain differential sandwich theorem associated with a new generalized derivative operator. *Gen. Math.* **2009**, *17*, 83–95.
22. Komatu, Y. On analytical prolongation of a family of operators. *Mathematica* **1990**, *32*, 141–145.
23. Lewin, M. On a coefficient problem for bi-univalent functions. *Proc. Am. Math. Soc.* **1967**, *18*, 63–68. [CrossRef]

24. Brannan, D.A.; Clunie, J.G. (Eds.) *Aspects of Contemporary Complex Analysis*; Academic Press: New York, NY, USA, 1980.
25. Netanyahu, E. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Arch. Ration. Mech. Anal.* **1969**, *32*, 100–112.
26. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.* **2010**, *23*, 1188–1192. [CrossRef]
27. Taha, T.S. Topics in Univalent Function Theory. Ph.D. Thesis, University of London, London, UK, 1981.
28. Brannan, D.A.; Clunie, J.; Kirwan, W.E. Coefficient estimates for a class of star-like functions. *Canad. J. Math.* **1970**, *22*, 476–485. [CrossRef]
29. Frasin, B.A.; Aouf, M.K. New subclasses of bi-univalent functions. *Appl. Math. Lett.* **2011**, *24*, 1569–1973. [CrossRef]
30. Brannan, D.A.; Taha, T.S. On some classes of bi-univalent functions. *Studia Univ. Babes-Bolyai Math.* **1986**, *31*, 70–77.
31. Xu, Q.-H.; Gui, Y.-C.; Srivastava, H.M. Coefficient estimates for a certain subclass of analytic and bi-univalent functions. *Appl. Math. Lett.* **2012**, *25*, 990–994. [CrossRef]
32. Xu, Q.-H.; Xiao, H.-G.; Srivastava, H.M. A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. *Appl. Math. Comput.* **2012**, *218*, 11461–11465. [CrossRef]
33. Lupaş, A.A.; El-Deeb, S.M. Subclasses of bi-univalent functions connected with integral operator based upon Lucas polynomial. *Symmetry* **2022**, *14*, 622. [CrossRef]
34. El-Deeb, S.M. Maclaurin coefficient estimates for new subclasses of bi-univalent functions connected with a $q-$analogue of Bessel function. *Abstr. Appl. Anal.* **2020**, *2020*, 8368951. [CrossRef]
35. El-Deeb, S.M.; Bulboaca, T.; El-Matary, B.M. Maclaurin coefficient estimates of bi-Univalent functions connected with the $q-$derivative. *Mathematics* **2020**, *8*, 418. [CrossRef]
36. El-Deeb, S.M.; El-Matary, B.M. Subclasses of bi-univalent functions associated with q-confluent hypergeometric distribution based upon the Horadam polynomials. *Adv. Theory Nonlinear Anal. Appl.* **2021**, *5*, 82–93. [CrossRef]
37. Fekete, M.; Szegö, G. Eine Bemerkung über ungerade schlichte Functionen. *J. Lond. Math. Soc.* **1933**, *8*, 85–89. [CrossRef]
38. Noshiro, K. On the theory of schlicht functions. *J. Fac. Sci. Hokkaido Univ. Ser.* **1934**, *35*, 129–155. [CrossRef]
39. Yamaguchi, K. On functions satisfying $\Re\{f(z)/z\} > 0$. *Proc. Am. Math. Soc.* **1966**, *17*, 588–591.
40. Zaprawa, P. On the Fekete-Szegö problem for classes of bi-univalent functions. *Bull. Belg. Math. Soc. Simon Stevin.* **2014**, *21*, 169–178. [CrossRef]
41. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*; Li, Z., Ren, F., Yang, L., Zhang, S., Eds.; International Press: Cambridge, MA, USA, 1994; pp. 157–169.
42. Ali, R.M.; Leo, S.K.; Ravichandran, V.; Supramaniam, S. Coefficient estimates for bi-univalent Ma-Minda star-like and convex functions. *Appl. Math. Lett.* **2012**, *25*, 344–351. [CrossRef]
43. Cho, N.E.; Kumar, S.; Kumar, V.; Ravichandran,V. Radius problems for starlike functions associated with the sine function. *Bull. Iran. Math. Soc.* **2019**, *45*, 213–232. [CrossRef]
44. Wani,L. A.; Swaminathan, A. Starlike and convex functions associated with a Nephroid domain. *Bull. Malays. Math. Sci. Soc.* **2021**, *44*, 79–104. [CrossRef]
45. Sokól, J.; Stankiewicz, J. Radius of convexity of some subclasses of strongly starlike function. *Zesz. Nauk. Oficyna Wydawnicza al. Powstancow Warszawy* **1996**, *19*, 101–105.
46. Mendiratta, R.; Nagpal, S.; Ravichandran, V. On a subclass of strongly starlike functions associated exponential function. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 365–386. [CrossRef]
47. Raina, R.K.; Sokól, J. On coefficient estimates for a certain class of starlike functions. *Hacettepe. J. Math. Statist.* **2015**, *44*, 1427–1433. [CrossRef]