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# Extension of the $q$-Pfaff-Saalschütz Theorem by Two Integer Parameters

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**Abstract:** We investigate a class of terminating ${}_3\phi_2$-series that comes from the balanced series perturbed by two extra integer parameters. By making use of the linearization method, a general summation formula is established that extends the well-known $q$-Pfaff-Saalschütz theorem. Five closed formulae are exemplified as applications.

**Keywords:** basic hypergeometric series; balanced series; linearization method

**MSC:** 33D15; 05A30

## 1. Introduction and Outline

Denote by $\mathbb{N}$ the set of natural numbers with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For an indeterminate $x$, the rising and falling $q$-shifted factorials of order $n \in \mathbb{N}_0$ are defined by

$$\left.\begin{aligned}(x;q)_0 = 1 \ \text{ and } \ (x;q)_n &= (1-x)(1-xq)\cdots(1-q^{n-1}x)\\ \langle x;q\rangle_0 = 1 \ \text{ and } \ \langle x;q\rangle_n &= (1-x)(1-x/q)\cdots(1-q^{1-n}x)\end{aligned}\right\} \text{ for } n \in \mathbb{N}$$

with the quotient form for the former being abbreviated compactly to

$$\left[\begin{matrix}a,\ b,\ \ldots,\ c\\ \alpha,\ \beta,\ \ldots,\ \gamma\end{matrix}\middle|q\right]_n = \frac{(a;q)_n(b;q)_n\cdots(c;q)_n}{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}.$$

Following Gasper and Rahman ([1], §1.2), the basic hypergeometric series reads as

$${}_{1+p}\phi_p\left[\begin{matrix}a_0, a_1, \ldots, a_p\\ b_1, \ldots, b_p\end{matrix}\middle|q;z\right] = \sum_{n=0}^{\infty}\left[\begin{matrix}a_0, a_1, \cdots, a_p\\ q,\ b_1, \cdots, b_p\end{matrix}\middle|q\right]_n z^n.$$

This series is terminating when one of the numerator parameters has the form $q^{-m}$ with $m \in \mathbb{N}_0$. It is said to be balanced when the quotient between the product of its denominator parameters and the product of its numerator parameters is equal to $q$. If that quotient results in $q^m$, where $m$ is a small positive integer greater than 1, then the series will be said to be "quasi-balanced".

There are numerous $q$-series identities. Among them, the $q$-Saalschütz summation theorem (cf. [1], Equation (II.12)) is fundamental:

$${}_3\phi_2\left[\begin{matrix}q^{-n},\ a,\ b\\ c, q^{1-n}ab/c\end{matrix}\middle|q;q\right] = \left[\begin{matrix}c/a, c/b\\ c, c/(ab)\end{matrix}\middle|q\right]_n. \tag{1}$$

For $\lambda, \mu \in \mathbb{N}_0$ subject to $\lambda \geq \mu$, we shall investigate by means of the linearization method (cf. [2–4]), the following quasi-balanced series

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$$\Omega_n(\lambda,\mu|a,b,c) = {}_3\phi_2\left[\begin{matrix} q^{-n},\ a,\ b \\ c, q^{1-n+\lambda}ab/c \end{matrix}\middle| q; q^{1+\mu}\right]. \tag{2}$$

The main theorem will be proved in the next section which expresses the series $\Omega_n(\lambda,\mu|a,b,c)$ explicitly as a linear combination of $\Omega_n(0,0|a,b,c)$ with the number of terms being independent of $n$. Then the paper will end in Section 3, where five explicit formulae are presented as examples.

## 2. Main Theorem and Proof

Recently, the linearization method has successfully been employed in [2–4], to extend several important known theorems of hypergeometric series and $q$-series by integer parameters. In this section, we are going first to establish the linearization lemma below and then apply it to reduce the series $\Omega_n(\lambda,\mu|a,b,c)$ in terms of $\Omega_n(0,0|a,b,c)$, which has the closed expression as in (1).

**Lemma 1.** *For a variable $y$, non-negative integers $\{\lambda,\mu\}$ with $\lambda \geq \mu$ and two indeterminates $\{A,B\}$, the following linear relation holds:*

$$y^\mu = \sum_{i=0}^{\lambda} (Ay;q)_{\lambda-i}\langle By;q\rangle_i W_i(\lambda,\mu). \tag{3}$$

*The connection coefficients $W_i(\lambda,\mu)$ are independent of $y$ and given explicitly by the expressions*

$$W_i(\lambda,\mu) = \frac{(-1)^i B^{-\mu}}{(A/B;q)_\lambda}\begin{bmatrix}\lambda\\ i\end{bmatrix}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}^{-1}\sum_{j=0}^{\mu} q^{\binom{j}{2}+(i-j)(\mu-j)}\begin{bmatrix}i\\ j\end{bmatrix}\begin{bmatrix}\lambda-i\\ \mu-j\end{bmatrix}\left(\frac{A}{B}\right)^{i-j} \tag{4}$$

$$= (-1)^i\begin{bmatrix}\lambda\\ i\end{bmatrix}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}^{-1}\frac{A^{i-\mu}q^{\binom{i+\mu}{2}-\mu^2}}{B^{\mu+i}(A/B;q)_\lambda}[x^\mu](-qAx;q)_{\lambda-i}\langle -Bx;q\rangle_i, \tag{5}$$

*where $[x^n]P(x)$ stands for the coefficient of $x^n$ in the polynomial $P(x)$ and the Gaussian binomial coefficient is defined by*

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = \frac{\langle q^\lambda;q\rangle_\mu}{(q;q)_\mu} = \frac{(q;q)_\lambda}{(q;q)_\mu(q;q)_{\lambda-\mu}}.$$

**Proof.** It suffices to substitute (4) or (5) into (3) and then verify that the resulting sum equals $y^\mu$ for $\mu \leq \lambda$. First, denote by S the resulting double sum after having replaced $W_i(\lambda,\mu)$ in (3) by (4). In view of the $q$-binomial product

$$\begin{bmatrix}\lambda\\ i\end{bmatrix}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}^{-1}\begin{bmatrix}i\\ j\end{bmatrix}\begin{bmatrix}\lambda-i\\ \mu-j\end{bmatrix} = \begin{bmatrix}\mu\\ j\end{bmatrix}\begin{bmatrix}\lambda-\mu\\ i-j\end{bmatrix},$$

we can manipulate, by interchanging the summation order, S as follows:

$$\begin{aligned}\mathrm{S} &= \sum_{i=0}^{\lambda}\frac{(-1)^i B^{-\mu}}{(A/B;q)_\lambda}\begin{bmatrix}\lambda\\ i\end{bmatrix}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}^{-1}(Ay;q)_{\lambda-i}\langle By;q\rangle_i \\ &\quad\times\sum_{j=0}^{\mu}\left(\frac{A}{B}\right)^{i-j}\begin{bmatrix}i\\ j\end{bmatrix}\begin{bmatrix}\lambda-i\\ \mu-j\end{bmatrix}q^{\binom{j}{2}+(i-j)(\mu-j)} \\ &= \sum_{j=0}^{\mu}\left(\frac{B}{A}\right)^j\begin{bmatrix}\mu\\ j\end{bmatrix}\frac{B^{-\mu}}{(A/B;q)_\lambda}\,\mathrm{S}(j),\end{aligned} \tag{6}$$

where $\mathrm{S}(j)$ is given by

$$\mathrm{S}(j)=\sum_{i=0}^{\lambda}\Big(-\frac{A}{B}\Big)^{i}\begin{bmatrix}\lambda-\mu\\ i-j\end{bmatrix}(Ay;q)_{\lambda-i}\langle By;q\rangle_{i}q^{\binom{i}{2}+(i-j)(\mu-j)}.$$

Observe that for the above sum $\mathrm{S}(j)$, the summation index $i$ runs, in fact, from $j$ to $\lambda$ since $\begin{bmatrix}\lambda-\mu\\ i-j\end{bmatrix}$ vanishes when $i<j$. We can rewrite it, by letting $i=k+j$, as

$$\begin{aligned}\mathrm{S}(j)&=\sum_{k=0}^{\lambda-j}\Big(-\frac{A}{B}\Big)^{k+j}\begin{bmatrix}\lambda-\mu\\ k\end{bmatrix}(Ay;q)_{\lambda-j-k}\langle By;q\rangle_{j+k}q^{\binom{k+j}{2}+k(\mu-j)}\\&=q^{\binom{j}{2}}\Big(-\frac{A}{B}\Big)^{j}\langle By;q\rangle_{j}(Ay;q)_{\lambda-j}\sum_{k=0}^{\lambda-j}\frac{(q^{\mu-\lambda};q)_{k}(q^{j}/(By);q)_{k}}{(q;q)_{k}(q^{1+j-\lambda}/(Ay);q)_{k}}q^{k},\end{aligned}$$

where the three equalities below have been utilized:

$$\begin{aligned}\begin{bmatrix}\lambda-\mu\\ k\end{bmatrix}&=(-1)^{k}\frac{(q^{\mu-\lambda};q)_{k}}{(q;q)_{k}}q^{k(\lambda-\mu)-\binom{k}{2}},\\ \langle By;q\rangle_{j+k}&=(-By)^{k}\langle By;q\rangle_{j}(q^{j}/(By);q)_{k}q^{-kj-\binom{k}{2}},\\ (Ay;q)_{\lambda-j-k}&=(-Ay)^{-k}\frac{(Ay;q)_{\lambda-j}}{(q^{1+j-\lambda}/(Ay);q)_{k}}q^{k(1+j-\lambda)+\binom{k}{2}}.\end{aligned}$$

Evaluating the last sum with respect to $k$ by the $q$-Chu-Vandermonde theorem (cf. [1], Equation (II.6))

$${}_2\phi_1\left[\begin{matrix}q^{\mu-\lambda},q^{j}/(By)\\ q^{1+j-\lambda}/(Ay)\end{matrix}\Big|q;q\right]=\frac{(q^{1-\lambda}B/A;q)_{\lambda-\mu}}{(q^{1+j-\lambda}/(Ay);q)_{\lambda-\mu}}(q^{j}/(By))^{\lambda-\mu}=\frac{(q^{\mu}A/B;q)_{\lambda-\mu}}{(q^{\mu-j}Ay;q)_{\lambda-\mu}},$$

we find that

$$\begin{aligned}\mathrm{S}(j)&=q^{\binom{j}{2}}\Big(-\frac{A}{B}\Big)^{j}\langle By;q\rangle_{j}(Ay;q)_{\lambda-j}\frac{(q^{\mu}A/B;q)_{\lambda-\mu}}{(q^{\mu-j}Ay;q)_{\lambda-\mu}}\\&=q^{\binom{j}{2}}\Big(-\frac{A}{B}\Big)^{j}(q^{\mu}A/B;q)_{\lambda-\mu}\langle By;q\rangle_{j}(Ay;q)_{\mu-j}.\end{aligned}$$

Substituting this into (6) leads us to the expression

$$\mathrm{S}=\frac{(q^{\mu}A/B;q)_{\lambda-\mu}}{B^{\mu}(A/B;q)_{\lambda}}\sum_{j=0}^{\mu}(-1)^{j}\begin{bmatrix}\mu\\ j\end{bmatrix}q^{\binom{j}{2}}\langle By;q\rangle_{j}(Ay;q)_{\mu-j}.$$

Taking into account that

$$\begin{aligned}\begin{bmatrix}\mu\\ j\end{bmatrix}&=(-1)^{j}\frac{(q^{-\mu};q)_{j}}{(q;q)_{j}}q^{j\mu-\binom{j}{2}},\\ \langle By;q\rangle_{j}&=(-By)^{j}(1/(By);q)_{j}q^{-\binom{j}{2}},\\ (Ay;q)_{\mu-j}&=(-Ay)^{-j}\frac{(Ay;q)_{\mu}}{(q^{1-\mu}/(Ay);q)_{j}}q^{\binom{j}{2}-j\mu+j};\end{aligned}$$

we can reduce further

$$\mathrm{S}=\frac{(Ay;q)_{\mu}}{B^{\mu}(A/B;q)_{\mu}}\sum_{j=0}^{\mu}\Big(\frac{qB}{A}\Big)^{j}\frac{(q^{-\mu};q)_{j}(1/(By);q)_{j}}{(q;q)_{j}(q^{1-\mu}/(Ay);q)_{j}}.$$

According to another $q$-Chu-Vandermonde theorem (cf. [1], Equation (II.7)), the last sum admits the closed form

$$ {}_2\phi_1\left[\begin{matrix} q^{-\mu}, 1/(By) \\ q^{1-\mu}/(Ay) \end{matrix}\Big| q; \frac{qB}{A}\right] = \frac{(q^{1-\mu}B/A;q)_\mu}{(q^{1-\mu}/(Ay);q)_\mu} = \frac{(A/B;q)_\mu}{(Ay;q)_\mu}(By)^\mu. $$

This confirms finally that

$$ \mathrm{S} = y^\mu \quad \text{for} \quad \mu \le \lambda. $$

In order to show the equivalent expression (5), consider the $q$-binomial expansions ([1], Equation (II.4))

$$ \langle -Bx;q\rangle_i = \sum_{j=0}^{i}(Bx)^j \begin{bmatrix} i \\ j \end{bmatrix} q^{\binom{j+1}{2}-ij}, $$

$$ (-qAx;q)_{\lambda-i} = \sum_{k=0}^{\lambda}(Ax)^k \begin{bmatrix} \lambda-i \\ k \end{bmatrix} q^{\binom{k+1}{2}}. $$

Then we can extract the coefficient

$$ [x^{\mu-j}]\langle -Bx;q\rangle_i(-qAx;q)_{\lambda-i} = \sum_{j=0}^{i} A^{\mu-j}B^j \begin{bmatrix} i \\ j \end{bmatrix}\begin{bmatrix} \lambda-i \\ \mu-j \end{bmatrix} q^{\binom{\mu+1}{2}+j(j-i-\mu)}. $$

Substituting this into (5) and then making some simplifications, we find that the resulting expression coincides with (4). This completes the proof of Lemma 1. □

Now, specify the parameters in Lemma 1 by

$$ y \to q^k, \quad A \to a \quad \text{and} \quad B \to q^{\lambda-n}ab/c. $$

Then the linear relation corresponding to (3) becomes

$$ q^{k\mu} = \sum_{i=0}^{\lambda}(q^k a;q)_{\lambda-i}\langle q^{\lambda-n+k}ab/c;q\rangle_i \mathcal{W}_i(\lambda,\mu), \tag{7} $$

where the connection coefficients $\mathcal{W}_i(\lambda,\mu)$ as in (4) are given explicitly by

$$ \begin{aligned} \mathcal{W}_i(\lambda,\mu) = {} & \frac{(-1)^i(c/(ab))^\mu}{(q^{n-\lambda}c/b;q)_\lambda}\begin{bmatrix} \lambda \\ i \end{bmatrix}\begin{bmatrix} \lambda \\ \mu \end{bmatrix}^{-1} \\ & \times \sum_{j=0}^{\mu}\left(\frac{c}{b}\right)^{i-j}\begin{bmatrix} i \\ j \end{bmatrix}\begin{bmatrix} \lambda-i \\ \mu-j \end{bmatrix} q^{\binom{i}{2}+(i-j)(\mu-j)+(\mu+i-j)(n-\lambda)}. \end{aligned} \tag{8} $$

By putting the linear relation (7) inside the series $\Omega_n(\lambda,\mu|a,b,c)$ and then exchanging the summation order, we can proceed with the double series in the following manner:

$$ \begin{aligned} \Omega_n(\lambda,\mu|a,b,c) &= \sum_{k=0}^{n} q^k \left[\begin{matrix} q^{-n},\ a,\ b \\ q,\ c,\ q^{1-n+\lambda}ab/c \end{matrix}\Big| q\right]_k \\ &\quad \times \sum_{i=0}^{\lambda}(q^k a;q)_{\lambda-i}\langle q^{\lambda-n+k}ab/c;q\rangle_i \mathcal{W}_i(\lambda,\mu) \\ &= \sum_{i=0}^{\lambda}\frac{(a;q)_{\lambda-i}\mathcal{W}_i(\lambda,\mu)}{(q^{1-n+\lambda}ab/c;q)_{-i}}\sum_{k=0}^{n} q^k \left[\begin{matrix} q^{-n},\ q^{\lambda-i}a,\ b \\ q,\ c,\ q^{1+\lambda-n-i}ab/c \end{matrix}\Big| q\right]_k. \end{aligned} $$

Evaluating the rightmost sum with respect to $k$ by (1)

$$\Omega_n(0,0|q^{\lambda-i}a,b,c)=\frac{(q^{i-\lambda}c/a,c/b;q)_n}{(c,q^{i-\lambda}c/(ab);q)_n}$$

and then simplifying the result, we establish the following theorem.

**Theorem 2.** *Assuming $\mathcal{W}_i(\lambda,\mu)$ as in (8), we have the following reduction formula*

$$\Omega_n(\lambda,\mu|a,b,c)=\left[\begin{matrix}c/a,c/b\\c,c/(ab)\end{matrix}\middle|q\right]_n\sum_{i=0}^{\lambda}\left[\begin{matrix}a,qa/c,q^{1-n}ab/c\\qab/c,\ q^{1-n}a/c\end{matrix}\middle|q\right]_{\lambda-i}\times\langle q^{\lambda-n}ab/c;q\rangle_i\mathcal{W}_i(\lambda,\mu).$$

Particular cases of this theorem have previously been investigated in the literature.

- Al-Salam and Fields [5] (see also [6]) derived symmetric expressions for the case $\lambda=\mu$.
- Srivastava [7] (see also [6]) considered the case $\mu=0$.
- Verma and Joshi [8] examined both cases $\lambda=\mu$ and $\mu=0$.

Furthermore, an extension of the $q$-Saalschütz summation theorem along a different direction into double series can be found in [9].

## 3. Special Cases as Examples

By assigning $\lambda$ and $\mu$ to small integers in Theorem 2, we can derive concrete formulae. Five examples are highlighted below as applications.

**Example 1** ($\Omega_n(1,0)$)**.**

$${}_3\phi_2\left[\begin{matrix}q^{-n},\ a,\ b\\c,q^{2-n}ab/c\end{matrix}\middle|q;q\right]=\left[\begin{matrix}q^{-1}c/a,q^{-1}c/b\\c,\ q^{-1}c/(ab)\end{matrix}\middle|q\right]_n\left\{1-\frac{(1-q^n)(1-qab/c)}{(1-qa/c)(1-qb/c)}\right\}.$$

**Example 2** ($\Omega_n(1,1)$)**.**

$${}_3\phi_2\left[\begin{matrix}q^{-n},\ a,\ b\\c,q^{2-n}ab/c\end{matrix}\middle|q;q^2\right]=\left[\begin{matrix}q^{-1}c/a,q^{-1}c/b\\c,\ q^{-1}c/(ab)\end{matrix}\middle|q\right]_n\left\{q^n-\frac{q(1-q^n)(1-qab/c)}{c(1-qa/c)(1-qb/c)}\right\}.$$

**Example 3** ($\Omega_n(2,0)$)**.**

$$\begin{aligned}{}_3\phi_2\left[\begin{matrix}q^{-n},\ a,\ b\\c,q^{3-n}ab/c\end{matrix}\middle|q;q\right]&=\left[\begin{matrix}q^{-1}c/a,q^{-2}c/b\\c,\ q^{-2}c/(ab)\end{matrix}\middle|q\right]_n\frac{(1-b)(1-qb)(1-\frac{c}{qab})}{(1-q)(1-q^nb)(1-\frac{c}{qa})}\\&\times\left\{1-\frac{qb^3(1-q)(1-q^n)(1-q^{n-1})(1-\frac{q^2ab}{c})(1-q^{n-1}c)(1-q^{n-2}c)}{ac(1-b)(1-qb)(1-\frac{qb}{c})(1-\frac{q^2b}{c})(1-q^{n-1}b)(1-\frac{q^{n-2}c}{a})}\right.\\&\left.-\frac{(1-\frac{q^2ab}{c})(1-q^nb)(qc+q^2b^2-q^2bc-q^3b-q^nbc+q^{2+n}bc)}{qc(1-qb)(1-\frac{qab}{c})(1-\frac{q^2b}{c})(1-q^{n-1}b)}\right\}.\end{aligned}$$

**Example 4** ($\Omega_n(2,1)$).

$$
{}_3\phi_2\left[\begin{matrix} q^{-n},\ a,\ b \\ c, q^{3-n}ab/c \end{matrix}\Big| q;q^2\right] = \left[\begin{matrix} q^{-1}c/a, q^{-2}c/b \\ c,\ q^{-2}c/(ab) \end{matrix}\Big| q\right]_n \frac{(1-b)(1-q^{-1}/b)}{(1-qa/c)(1-q^nb)}
$$
$$
\times\left\{q^n+\frac{q^2b(1-q^nb)(1-q^2ab/c)(1-q^{n-2}c/b)}{c^2(1-b)(1-qb)(1-qb/c)(1-q^2b/c)}(c-qb-bc+q^nbc)\right.
$$
$$
\left.+\frac{q^2b^2(1-q^n)(1-q^{n-1})(1-q^{n-1}c)(1-q^{n-2}c)(1-q^2ab/c)}{ac(1-b)(1-qb)(1-qb/c)(1-q^2b/c)(1-q^{n-2}c/a)}\right\}.
$$

**Example 5** ($\Omega_n(2,2)$).

$$
{}_3\phi_2\left[\begin{matrix} q^{-n},\ a,\ b \\ c, q^{3-n}ab/c \end{matrix}\Big| q;q^3\right] = \left[\begin{matrix} q^{-1}c/a, c/b \\ c,\ q^{-2}c/(ab) \end{matrix}\Big| q\right]_n
$$
$$
\times\left\{1-\frac{(1-q^n)(1-b)(1-qb)}{q^2b(1-qa/c)(1-q^{n-1}c/b)(1-q^{n-2}c/b)}\right.
$$
$$
\left.-\frac{c(1-q^n)(1-q^{n-1}c)(1-q^{n-2}c)}{q^2ab^2(1-q^{n-2}c/a)(1-q^{n-1}c/b)(1-q^{n-2}c/b)}\right\}.
$$

*Concluding Comments*

In the course of proving Theorem 2, the linear representation formula in Lemma 1 plays a crucial role. The authors believe that this lemma should be useful also for examining other related $q$-series. The interested reader is enthusiastically encouraged to make further exploration.

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