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Research on Some Problems for Nonlinear Operators in the Z-Z-B Space

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Abstract: In this paper, we first propose a new concept of Z-Z-B spaces, which is a generalization of Z-C-X spaces. Meanwhile, the new concept of the superior cone is introduced. Secondly, we study some new problems for semi-closed 1-set-contractive operators in the Z-Z-B space and obtain some new results. These new theorems are proven by combining partial order theory with fixed point index theory. Regarding these theorems, in the latter part of the paper, the proofs are omitted since the methods of proving these theorems are similar. Moreover, two important inequality lemmas are proven.

Keywords: Z-Z-B space; semi-closed 1-set-contractive operator; operator equation; excellent cone; fixed point; homotopy invariance property

MSC: 47H10

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1. Introduction and Preliminaries

The theory of fixed points and equations of nonlinear operators is an important component of nonlinear functional analysis, and nonlinear functional analysis is one of the most important research threads of modern mathematics. The effective theoretical tools for studying nonlinear problems mainly include partial order theory, topological degree theory and variational theory [1]. Among them, using partial order theory to deal with equations is a very useful method, and the combination with topological degree theory has effectively promoted the development of modern nonlinear functional analysis. In this respect, the school represented by Dajun Guo has shown brilliant achievements in the fixed point theory of cone expansion–compression [1]. In addition, the new concept of the fixed point index of different operators was also obtained, and some theorems in the theory of the fixed point index were proven [1,2]. In 1988, Guozhen Li proposed a new concept of the fixed point index of 1-set-contractive operators and a new concept of the semi-closed 1-set-contractive operator, and proved the fixed point theorems of 1-set-contractive operators in Banach spaces [3].

Furthermore, Guozhen Li proposed a new concept of the random fixed point index of random 1-set-contractive operators and a new concept of the random semi-closed 1-set-contractive operator, and proved the random fixed point theorems of random 1-set-contractive operators in separable real Banach spaces [4]. For random fixed point theorems for 1-set-contractive random operators, readers can also read article [5]. In [6], the new concepts of the Z-C-X space and excellent cone were introduced. Meanwhile, some problems of random semi-closed 1-set-contractive operators were investigated in the Z-C-X space. In Article [7], Chuanxi Zhu and Zongben Xu investigated some problems for semi-closed 1-set-contractive operators in real Banach spaces, and proved some fixed point theorems with different boundary conditions. Chuanxi Zhu also studied some problems of nonlinear operators in Menger PN-spaces, and proved some fixed point theorems in Menger PN-spaces [8].

The fixed points of nonlinear operators and their equation theory play an important role in solving the existence and uniqueness of solutions of various differential equations and integral equations. In this regard, readers can read articles [9,10]. Meanwhile, fixed point theory also plays a great role in solving problems in the real world. Fixed point theorems have proven to be a useful tool in many application fields, such as physics, chemistry, biology, medicine, cybernetics, aerospace technology, mathematical economics, non-cooperative game theory, dynamic optimization and stochastic games, functional analysis, variational calculus, etc. For example, in the research of [11], the researchers utilized the generalized integral transform and the Adomian decomposition method to derive a fascinating explicit pattern for outcomes of the biological population model (BPM); in [12], the researchers proved the existence of solutions to a new family of fractional boundary value problems (FBVPs) on the methylpropane graph by means of Krasnoselskii's and Scheafer's fixed point theorems; in addition, by using Banach fixed point theorem, researchers have studied the existence and uniqueness of a solution of the proposed traumatic avoidance learning model in [13].

Let R be the set of all real numbers and Q be the set of all rational numbers. Let E be a real Banach space and P be a cone in E; thus, the partial order relation $(\preceq)$ in E is determined as follows [1]:

$$\forall x, y \in E, x \preceq y \Leftrightarrow y - x \in P$$

If $x \preceq y$, $x \neq y$, then write it as $x \prec y$.

Let U be an excellent cone in E, V be a bounded open subset of U, $\overline{V}$ be the closure of V and ∂V be the boundary of V. A continuous operator T: $\overline{V} \to U$ is said to be a semi-closed 1-set-contractive operator if T is a 1-set-contraction operator and $I - T$ is a closed operator [3]. We know that $\alpha(TV) \leq \alpha(V)$, where α denotes the non-compact measure.

Definition 1 (See [6])**.** *Suppose that a separable real Banach space E satisfies the following conditions:*

(H_1) *E is an algebra over the real number field R, namely:*

(1°) *E is closed under multiplication; that is,* $\forall x, y \in E$, $x \cdot y \in E$.

(2°) $\forall a \in R$, $\forall x, y \in E$, $(ax)y = x(ay) = a(xy)$.

(H_2) $\forall x \in E$ *and* $x \neq \theta \Rightarrow x^n \neq \theta$, $n \in N$.

Then, E is called the $Z - C - X$ *space.*

By (H_1), we obtain the following formula:

(3°) $\forall a, \lambda \in R$, $ax \cdot (\lambda y) = (a\lambda)(x \cdot y)$.

In the $Z - C - X$ space, let $\underbrace{x \cdot x \cdots x}_{n} = x^n$, where $x \in E$, n denotes a natural number, which is $n \in N$.

Definition 2. *Suppose that a separable real Banach space E satisfies the following conditions:*

(H_1) *E is an algebra over the real number field R, namely:*

(1°) *E is closed under multiplication; that is,* $\forall x, y \in E$, $x \cdot y \in E$.

(2°) $\forall a \in R$, $\forall x, y \in E$, $(ax)y = x(ay) = a(xy)$.

(H_2) $\forall x, y \in E$, $\forall m, n \in N^+$, *by* $x^m = y^n \Rightarrow y = x^{\frac{m}{n}}$, *let* $\beta = \frac{m}{n}$, *then* $y = x^\beta$, *where* $\beta > 0, \beta \in Q$.

(H_3) $\forall x \in E$, *by* $x \neq \theta \Rightarrow x^\beta \neq \theta$, *where* $\beta > 0, \beta \in Q$.

Then, E is called the $Z - Z - B$ *space.*

Therefore, Z-C-X space is a subspace of Z-Z-B space.

Example 1. *A real number space is a* $Z - Z - B$ *space.*

Definition 3 (See [6])**.** *Let X be a cone in the E, and suppose that X satisfies the following conditions:*

(1)$\forall x, y \in X$, *by* $\theta \prec x \prec y \Rightarrow 0 <\| x \|<\| y \|$.
(2)X *is closed under multiplication, and* X *is the subspace of* E.
Then, X *is called the excellent cone.*

$(E, \preceq)$ is assumed to be a partial order derived from excellent cone X.
By Definition 3, we also obtain: $\forall x, y \in X$, by $\theta \preceq x \preceq y \Rightarrow 0 \leq\| x \|\leq\| y \|$.
This is because $\theta = x = y$, and we have $\| x \|=\| y \|= 0$.

Example 2. *Let* E *be an n-dimensional Euclidean space,* $X = x \mid x = (x_1, x_2, \cdots, x_n), x_i \geq 0$, $(i = 1, 2, \cdots, n)$, *and then* X *is a cone in* E. *By* $\| x \|= \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2}$, *then* X *is obviously an excellent cone in* E.

Definition 4. *Let* Y *be a cone in the* E, *and suppose that* Y *satisfies the following conditions:*
(1)$\forall x, y \in Y, n \in N^+$, *by* $\theta \prec x \prec y \Rightarrow 0 <\| x \|^n<\| y \|^n$.
(2)Y *is closed under multiplication, and* Y *is the subspace of* E.
Then, Y *is called the superior cone.*

$(E, \preceq)$ is assumed to be a partial order derived from superior cone Y.
By Definition 4, we also obtain: $\forall x, y \in Y, n \in N^+$, by $\theta \preceq x \preceq y \Rightarrow 0 \leq\| x \|^n\leq\| y \|^n$.
This is because $\theta = x = y$, and we have $\| x \|=\| y \|= 0$, so $\| x \|^n=\| y \|^n= 0$.

Example 3. *Let* E *be an n-dimensional Euclidean space,* $X = x \mid x = (x_1, x_2, \cdots, x_n), x_i \geq 0$, $(i = 1, 2, \cdots, n)$, *and then* X *is a cone in* E. *By* $\| x \|= \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2}$, *then* X *is also a superior cone in* E.

Some new concepts in this article are taken from [1–6].
We obtain the following results.

2. Main Results

Theorem 1. *Let* E *be a* $Z - Z - B$ *space,* U *be an excellent cone of* E, V *be a bounded open subset of* U, *and* $\theta \in V$. *Let* Q *be the set of all rational numbers,* $\overline{V}$ *be the closure of* V, *and* ∂V *be the boundary of* V. *Suppose that* T: $\overline{V} \to U$ *is a semi-closed 1-set-contractive operator and satisfies the following condition:*

$$(Tx + \mu x)^{\beta+1} \preceq (Tx)^\beta \cdot \mu x, \quad for\, every\ x \in \partial V, where\ \beta \geq 1, \beta \in Q, \mu \geq 1. \tag{1}$$

Then, the operator equation $Tx = \mu x$ *has a solution in* $\overline{V}$.

Proof. We can assume that $Tx = \mu x$ has no solution on ∂V (otherwise, the result holds). That is, $\frac{1}{\mu}Tx \neq x$, for every $x \in \partial V$, $\mu \geq 1$.

Let $l_s(x) = x - \frac{s}{\mu}Tx$, $L_s(x) = \frac{s}{\mu}(Tx)$, $s \in [0, 1]$, for every $x \in \overline{V}$, $\mu \geq 1$; then, L_s: $[0, 1] \times \overline{V} \to U$ is a semi-closed 1-set-contractive operator.

In fact, T: $\overline{V} \to U$ is a continuous operator, L_s: $[0, 1] \times \overline{V} \to U$ is also a continuous operator. T: $\overline{V} \to U$ is a semi-closed 1-set-contractive operator; let $w = \frac{s}{\mu}$, because $\forall s \in [0, 1]$, $\mu \geq 1$, so $w \in [0, 1]$, and we have $\alpha(L_s(V)) = \alpha(\frac{s}{\mu}T(V)) = \alpha(wT(V)) = w\alpha(T(V)) \leq \alpha(T(V)) \leq \alpha(V)$, where α denotes a noncompact measure. Thus, L_s: $[0, 1] \times \overline{V} \to U$ is a 1-set-contractive operator.

At the same time, $I - L_s = I - \frac{s}{\mu}T = I - wT = I - wT + wI - wI = (1 - w)I + w(I - T)$, and we know that L_s: $[0, 1] \times \overline{V} \to U$ is a semi-closed operator. Thus, L_s: $[0, 1] \times \overline{V} \to U$ is a semi-closed 1-set-contractive operator.

Moreover, $L_s(\cdot, x)$: $[0, 1] \to U$ is uniformly continuous with respect to $x \in \overline{V}$.

Below, we prove that

$$\theta \overline{\in} l_s(\partial V), s \in [0, 1]. \tag{2}$$

In fact, suppose that (2) is not true; then, there exist a $s_0 \in [0,1]$ and an $x_0 \in \partial V$, such that $\theta = l_{s_0}(x_0)$, that is,

$$x_0 - \frac{s_0}{\mu} T x_0 = \theta. \tag{3}$$

That is, $\frac{s_0}{\mu} T x_0 = x_0$; then, $s_0 \neq 0$(otherwise $s_0 = 0$, and we have $x_0 = \theta \in \partial V$; this is in contradiction to $\theta \in V$) and $s_0 \neq 1$ (otherwise, $s_0 = 1$, and we have $T x_0 = \mu x_0$; this is in contradiction to the given condition $Tx \neq \mu x$, for every $x \in \partial V$); hence, $s_0 \in (0,1)$. By (3), we obtain $T x_0 = \frac{\mu}{s_0} x_0$, where $x_0 \in \partial V$, $s_0 \in (0,1)$, $\mu \geq 1$. Inserting $T x_0 = \frac{\mu}{s_0} x_0$ into (1), we have

$$(\frac{\mu}{s_0} x_0 + \mu x_0)^{\beta+1} \preceq (\frac{\mu}{s_0} x_0)^{\beta} \cdot \mu x_0.$$

Since E is a $Z - Z - B$ space,

$$\mu^{\beta+1} (\frac{1}{s_0} + 1)^{\beta+1} (x_0)^{\beta+1} \preceq \mu^{\beta+1} (\frac{1}{s_0})^{\beta} (x_0)^{\beta+1}.$$

Because U is an excellent cone in E, we can write

$$\| \mu^{\beta+1} (\frac{1}{s_0} + 1)^{\beta+1} (x_0)^{\beta+1} \| \leq \| \mu^{\beta+1} (\frac{1}{s_0})^{\beta} (x_0)^{\beta+1} \| .$$

It follows that

$$\mu^{\beta+1} (\frac{1}{s_0} + 1)^{\beta+1} \| (x_0)^{\beta+1} \| \leq \mu^{\beta+1} (\frac{1}{s_0})^{\beta} \| (x_0)^{\beta+1} \| . \tag{4}$$

Owing to the fact that E is a $Z - Z - B$ space and $x_0 \neq \theta$, we thus have $(x_0)^{\beta+1} \neq \theta$. By (4), we have

$$(\frac{1}{s_0} + 1)^{\beta+1} \leq (\frac{1}{s_0})^{\beta}.$$

Since $s_0 \in (0,1)$, thus $\frac{1}{s_0} > 1$, $\beta \geq 1$, $\beta \in Q$, that is,

$$(\frac{1}{s_0} + 1)^{\beta+1} > (\frac{1}{s_0})^{\beta}.$$

Therefore, $\theta \overline{\in} l_s(\partial V)$; that is, $\theta \overline{\in} (I - L_s)(\partial V)$. We obtain that $x \neq L_s(s,x)$, for every $x \in \partial V$,$s \in [0,1]$.

According to the homotopy invariance and normalization in [3], we have

$$i(\frac{1}{\mu} T, V, U) = i(\theta, V, U) = 1.$$

Moreover, according to the solution property in [3], we know that $Tx = \mu x$ has a solution in V.

The proof of Theorem 1 is completed. □

Theorem 2. *Let E be a $Z - Z - B$ space, U be an excellent cone in E, V be a bounded open subset of U, and $\theta \in V$. Let Q be the set of all rational numbers, $\overline{V}$ be the closure of V, and ∂V be the boundary of V. Suppose that T: $\overline{V} \to U$ is a semi-closed 1-set-contractive operator and satisfies the following condition:*

$$(Tx + \mu x)^{\beta} + (Tx - \mu x)^{\beta} \succeq (2Tx + \mu x)^{\beta}, \quad for\ every\ x \in \partial V, where\ \beta \geq 1, \beta \in Q, \mu \geq 1. \tag{5}$$

Then, the operator equation $Tx = \mu x$ has a solution in $\overline{V}$.

Proof. We can assume that $Tx = \mu x$ has no solution on ∂V (otherwise, the result holds). That is, $Tx \neq \mu x$, for every $x \in \partial V$, $\mu \geq 1$.

Let $l_s(x) = x - \frac{s}{\mu}Tx$, $L_s(x) = \frac{s}{\mu}(Tx)$, $s \in [0,1]$, for every $x \in \overline{V}$, $\mu \geq 1$; then, L_s: $[0,1] \times \overline{V} \to U$ is a semi-closed 1-set-contractive operator.

In fact, T: $\overline{V} \to U$ is a continuous operator, and L_s: $[0,1] \times \overline{V} \to U$ is also a continuous operator. T: $\overline{V} \to U$ is a semi-closed 1-set-contractive operator; let $w = \frac{s}{\mu}$, $\forall s \in [0,1]$, $\mu \geq 1$; hence, $w \in [0,1]$, and we have $\alpha(L_s(V)) = \alpha(\frac{s}{\mu}T(V)) = \alpha(wT(V)) = w\alpha(T(V)) \leq \alpha(T(V)) \leq \alpha(V)$, where α denotes a noncompact measure. Thus, L_s: $[0,1] \times \overline{V} \to U$ is a 1-set-contractive operator.

Meanwhile, $I - L_s = I - \frac{s}{\mu}T = I - wT = I - wT + wI - wI = (1-w)I + w(I-T)$, and we know that L_s: $[0,1] \times \overline{V} \to U$ is a semi-closed operator. Thus, L_s: $[0,1] \times \overline{V} \to U$ is a semi-closed 1-set-contractive operator.

Moreover, $L_s(\cdot, x)$: $[0,1] \to U$ is uniformly continuous with respect to $x \in \overline{V}$.

Below, we prove that

$$\theta \overline{\in} l_s(\partial V), s \in [0,1]. \tag{6}$$

In fact, suppose that (1) is not true; then, there exist a $s_0 \in [0,1]$ and an $x_0 \in \partial V$, such that $\theta = l_{s_0}(x_0)$, that is,

$$x_0 - \frac{s_0}{\mu}Tx_0 = \theta. \tag{7}$$

That is, $\frac{s_0}{\mu}Tx_0 = x_0$; then, $s_0 \neq 0$ (otherwise, $s_0 = 0$, and we have $x_0 = \theta \in \partial V$; this is in contradiction to $\theta \in V$) and $s_0 \neq 1$ (otherwise, $s_0 = 1$, and we have $Tx_0 = \mu x_0$; this is in contradiction to the given condition $Tx \neq \mu x$, for every $x \in \partial V$); hence, $s_0 \in (0,1)$. By (7), we obtain $Tx_0 = \frac{\mu}{s_0}x_0$, where $x_0 \in \partial V$, $s_0 \in (0,1)$, $\mu \geq 1$. Inserting $Tx_0 = \frac{\mu}{s_0}x_0$ into (5), we have

$$(\frac{\mu}{s_0}x_0 + \mu x_0)^\beta + (\frac{\mu}{s_0}x_0 - \mu x_0)^\beta \succeq (2\frac{\mu}{s_0}x_0 + \mu x_0)^\beta.$$

Since E is a $Z-Z-B$ space,

$$\mu^\beta(\frac{1}{s_0}+1)^\beta(x_0)^\beta + \mu^\beta(\frac{1}{s_0}-1)^\beta(x_0)^\beta \succeq \mu^\beta(\frac{2}{s_0}+1)^\beta(x_0)^\beta.$$

It follows that

$$\mu^\beta[(\frac{1}{s_0}+1)^\beta + (\frac{1}{s_0}-1)^\beta](x_0)^\beta \succeq \mu^\beta(\frac{2}{s_0}+1)^\beta(s_0)^\beta.$$

Because U is an excellent cone in E, that is,

$$\| \mu^\beta[(\frac{1}{s_0}+1)^\beta + (\frac{1}{s_0}-1)^\beta](x_0)^\beta \| \geq \| \mu^\beta(\frac{2}{s_0}+1)^\beta(x_0)^\beta \| . \tag{8}$$

Owing to the fact that E is a $Z-Z-B$ space and $x_0 \neq \theta$, we thus have $(x_0)^\beta \neq \theta$. By (6), we have

$$(\frac{1}{s_0}+1)^\beta + (\frac{1}{s_0}-1)^\beta \geq (\frac{2}{s_0}+1)^\beta.$$

Since $s_0 \in (0,1)$, thus $\frac{1}{s_0} > 1$, $\beta \geq 1$, $\beta \in Q$, that is,

$$(\frac{1}{s_0}+1)^\beta + (\frac{1}{s_0}-1)^\beta < (\frac{1}{s_0}+1+\frac{1}{s_0}-1+1)^\beta.$$

Therefore, $\theta \overline{\in} l_s(\partial V)$, that is, $\theta \overline{\in} (I - L_s)(\partial V)$. We obtain that $x \neq L_s(s,x)$, for every $x \in \partial V$,$s \in [0,1]$.

According to the homotopy invariance and normalization in [3], we have

$$i(\frac{1}{\mu}T, V, U) = i(\theta, V, U) = 1.$$

Moreover, according to the solution property in [3], we know that $Tx = \mu x$ has a solution in V.

The proof of Theorem 2 is completed. □

Lemma 1. *When $\eta \in (0, \frac{1}{2})$, $\lambda \in (0,1)$, $\lambda > \eta$, $\eta + \lambda = 1$, $t > 1$, and $\gamma \geq 1$, the following inequality holds:*

$$\lambda t^\gamma - \eta > (\lambda - \eta)(t-1)^\gamma.$$

Proof. Let

$$f(t) = \lambda t^\gamma - \eta - (\lambda - \eta)(t-1)^\gamma,$$

where $\eta \in (0, \frac{1}{2})$, $\lambda \in (0,1)$, $\lambda > \eta$, $\eta + \lambda = 1$, $t > 1$, and $\gamma \geq 1$.

Then,

$$\begin{aligned} f'(t) &= \lambda\gamma t^{\gamma-1} - (\lambda-\eta)\gamma(t-1)^{\gamma-1} \\ &= \lambda\gamma t^{\gamma-1} - \lambda\gamma(t-1)^{\gamma-1} + \eta\gamma(t-1)^{\gamma-1} \\ &= \lambda\gamma(t^{\gamma-1} - (t-1)^{\gamma-1}) + \eta\gamma(t-1)^{\gamma-1} \end{aligned}$$

when $\gamma \geq 1$, $t^{\gamma-1} - (t-1)^{\gamma-1} \geq 0$, $\eta\gamma(t-1)^{\gamma-1} > 0$.

Thus, $f'(t) > 0$; therefore, $f(t)$ is a monotonous increasing function.

When $t > 1$, we have $f(t) > f(1)$ and $f(1) = \lambda - \eta > 0$.

Hence $f(t) > 0$, that is,

$$\lambda t^\gamma - \eta > (\lambda - \eta)(t-1)^\gamma.$$

The proof of Lemma 1 is completed. □

Theorem 3. *Let E be a $Z - Z - B$ space, U be an excellent cone in E, V be a bounded open subset of U, and $\theta \in V$. Let Q be the set of all rational numbers, $\overline{V}$ be the closure of V, and ∂V be the boundary of V. Suppose that T:$\overline{V} \to U$ is a semi-closed 1-set-contractive operator and satisfies the following condition:*

$$\lambda(Tx)^\beta - \eta(\mu x)^\beta \preceq (\lambda - \eta)(Tx - \mu x)^\beta, \quad for\ every\ x \in \partial V, \tag{9}$$

where $\eta \in (0, \frac{1}{2})$, $\lambda \in (0,1)$, $\lambda > \eta$, $\eta + \lambda = 1$, $\beta \geq 1$, $\beta \in Q$, $\mu \geq 1$.

Then, the operator equation $Tx = \mu x$ has a solution in $\overline{V}$.

Proof. We can assume that $Tx = \mu x$ has no solution on ∂V (otherwise, the result holds). That is, $\frac{1}{\mu}Tx \neq x$, for every $x \in \partial V$, $\mu \geq 1$.

Let $l_s(x) = x - \frac{s}{\mu}Tx$, $L_s(x) = \frac{s}{\mu}(Tx)$, $s \in [0,1]$, for every $x \in \overline{V}$, $\mu \geq 1$; then, L_s: $[0,1] \times \overline{V} \to U$ is a semi-closed 1-set-contractive operator.

In fact, T: $\overline{V} \to U$ is a continuous operator, L_s: $[0,1] \times \overline{V} \to U$ is also a continuous operator. T: $\overline{V} \to U$ is a semi-closed 1-set-contractive operator; let $w = \frac{s}{\mu}$, by $\forall s \in [0,1]$, $\mu \geq 1$, so $w \in [0,1]$, and we have $\alpha(L_s(V)) = \alpha(\frac{s}{\mu}T(V)) = \alpha(wT(V)) = w\alpha(T(V)) \leq \alpha(T(V)) \leq \alpha(V)$, where α denotes a noncompact measure. Thus, L_s: $[0,1] \times \overline{V} \to U$ is a 1-set-contractive operator.

In the meantime, $I - L_s = I - \frac{s}{\mu}T = I - wT = I - wT + wI - wI = (1-w)I + w(I - T)$, we know that L_s: $[0,1] \times \overline{V} \to U$ is a semi-closed operator. Thus, L_s: $[0,1] \times \overline{V} \to U$ is a semi-closed 1-set-contractive operator.

Moreover, $L_s(\cdot, x)$: $[0,1] \to U$ is uniformly continuous with respect to $x \in \overline{V}$.

Below, we prove that

$$\theta \overline{\in} l_s(\partial V), s \in [0,1]. \tag{10}$$

In fact, suppose that (10) is not true; then, there exist a $s_0 \in [0,1]$ and an $x_0 \in \partial V$, such that $\theta = l_{s_0}(x_0)$, that is,

$$x_0 - \frac{s_0}{\mu} T x_0 = \theta. \tag{11}$$

That is, $\frac{s_0}{\mu} T x_0 = x_0$; then, $s_0 \neq 0$ (otherwise, $s_0 = 0$, and we have $x_0 = \theta \in \partial V$; this is in contradiction to $\theta \in V$) and $s_0 \neq 1$ (otherwise, $s_0 = 1$, and we have $Tx_0 = \mu x_0$; this is in contradiction to the given condition $Tx \neq \mu x$, for every $x \in \partial V$); hence, $s_0 \in (0,1)$. By (11), we obtain $Tx_0 = \frac{\mu}{s_0} x_0$, where $x_0 \in \partial V$, $s_0 \in (0,1)$, $\mu \geq 1$. Inserting $Tx_0 = \frac{\mu}{s_0} x_0$ into (9), we have

$$\lambda(\frac{\mu}{s_0} x_0)^\beta - \eta(\mu x_0)^\beta \preceq (\lambda - \eta)(\frac{\mu}{s_0} x_0 - \mu x_0)^\beta.$$

Since E is a $Z - Z - B$ space,

$$\mu^\beta [\lambda(\frac{1}{s_0})^\beta](x_0)^\beta - \mu^\beta \cdot \eta \cdot (x_0)^\beta \preceq \mu^\beta [(\lambda - \eta)(\frac{1}{s_0} - 1)^\beta](x_0)^\beta.$$

It follows that

$$\mu^\beta [\lambda(\frac{1}{s_0})^\beta - \eta](x_0)^\beta \preceq \mu^\beta [(\lambda - \eta)(\frac{1}{s_0} - 1)^\beta](x_0)^\beta.$$

Because U is an excellent cone in E, that is,

$$\| \mu^\beta [\lambda(\frac{1}{s_0})^\beta - \eta](x_0)^\beta \| \leq \| \mu^\beta [(\lambda - \eta)(\frac{1}{s_0} - 1)^\beta](x_0)^\beta \| . \tag{12}$$

Owing to the fact that E is a $Z - Z - B$ space and $x_0 \neq \theta$, we thus have $(x_0)^\beta \neq \theta$. By (6), we have

$$\lambda(\frac{1}{s_0})^\beta - \eta \leq (\lambda - \eta)(\frac{1}{s_0} - 1)^\beta.$$

Since $s_0 \in (0,1)$, thus $\frac{1}{s_0} > 1$, $\beta \geq 1$, $\beta \in Q$, by Lemma 1; that is,

$$\lambda(\frac{1}{s_0})^\beta - \eta > (\lambda - \eta)(\frac{1}{s_0} - 1)^\beta.$$

Therefore, $\theta \overline{\in} l_s(\partial V)$, that is, $\theta \overline{\in} (I - L_s)(\partial V)$. We obtain that $x \neq L_s(s,x)$, for every $x \in \partial V, s \in [0,1]$.

According to the homotopy invariance and normalization in [3], we have

$$i(\frac{1}{\mu} T, V, U) = i(\theta, V, U) = 1.$$

Moreover, according to the solution property in [3], we know that $Tx = \mu x$ has a solution in V.

The proof of Theorem 3 is completed. □

Lemma 2 (See [8]). *When $\delta \in [0,1]$, $\mu \geq 1$, $o \in (0,1)$, the following inequality holds:*

$$\delta + \frac{\mu}{o} > | \mu - \frac{\delta}{o} \mu | .$$

Lemma 3. *When $\delta \in [0,1]$, $\mu \geq 1$, $o \in (0,1)$, $\gamma \geq 1$, the following inequality holds:*

$$(\delta + \frac{\mu}{o})^\gamma > (| \mu - \frac{\delta}{o} \mu |)^\gamma.$$

Proof. Because $\delta \in [0,1]$, $\mu \geq 1$, $o \in (0,1)$, hence $\delta + \frac{\mu}{o} > 1$.

By Lemma 2, $\gamma \geq 1$, we have

$$(\delta + \frac{\mu}{o})^{\gamma} > (\mid \mu - \frac{\delta}{o}\mu \mid)^{\gamma}.$$

The proof of Lemma 3 is completed. □

Theorem 4. *Let E be a $Z - Z - B$ space, U be an excellent cone in E, V be a bounded open subset of U, and $\theta \in V$. Let Q be the set of all rational numbers, $\overline{V}$ be the closure of V, and ∂V be the boundary of V. Suppose that T: $\overline{V} \to U$ is a semi-closed 1-set-contractive operator and satisfies the following condition:*

$$(\delta x + Tx)^{\beta} \preceq (\mu x - \delta Tx)^{\beta}, \quad for\ every\ x \in \partial V, where\ \delta \in [0,1], \beta \geq 1, \beta \in Q, \mu \geq 1. \tag{13}$$

Then, the operator equation $Tx = \mu x$ has a solution in $\overline{V}$.

Proof. We can assume that $Tx = \mu x$ has no solution on ∂V (otherwise, the result holds). That is, $Tx \neq \mu x$, for every $x \in \partial V$, $\mu \geq 1$.

Let $l_s(x) = x - \frac{s}{\mu}Tx$, $L_s(x) = \frac{s}{\mu}(Tx)$, $s \in [0,1]$, for every $x \in \overline{V}$, $\mu \geq 1$; then, L_s: $[0,1] \times \overline{V} \to U$ is a semi-closed 1-set-contractive operator.

In fact, T: $\overline{V} \to U$ is a continuous operator, and L_s: $[0,1] \times \overline{V} \to U$ is also a continuous operator. T: $\overline{V} \to U$ is a semi-closed 1-set-contractive operator; let $w = \frac{s}{\mu}$, by $\forall s \in [0,1]$, $\mu \geq 1$, so $w \in [0,1]$, and we have $\alpha(L_s(V)) = \alpha(\frac{s}{\mu}T(V)) = \alpha(wT(V)) = w\alpha(T(V)) \leq \alpha(T(V)) \leq \alpha(V)$, where α denotes a noncompact measure. Thus, L_s: $[0,1] \times \overline{V} \to U$ is a 1-set-contractive operator.

By $I - L_s = I - \frac{s}{\mu}T = I - wT = I - wT + wI - wI = (1-w)I + w(I-T)$, we know that L_s: $[0,1] \times \overline{V} \to U$ is a semi-closed operator. Thus, L_s: $[0,1] \times \overline{V} \to U$ is a semi-closed 1-set-contractive operator.

Moreover, $L_s(\cdot, x)$: $[0,1] \to U$ is uniformly continuous with respect to $x \in \overline{V}$.

Below, we prove that

$$\theta \overline{\in} l_s(\partial V), s \in [0,1]. \tag{14}$$

In fact, suppose that (14) is not true; then, there exist a $s_0 \in [0,1]$ and an $x_0 \in \partial V$, such that $\theta = l_{s_0}(x_0)$, that is,

$$x_0 - \frac{s_0}{\mu}Tx_0 = \theta. \tag{15}$$

That is, $\frac{s_0}{\mu}Tx_0 = x_0$; then, $s_0 \neq 0$ (otherwise, $s_0 = 0$, and we have $x_0 = \theta \in \partial V$; this is in contradiction to $\theta \in V$) and $s_0 \neq 1$ (otherwise, $s_0 = 1$, and we have $Tx_0 = \mu x_0$; this is in contradiction to the given condition $Tx \neq \mu x$, for every $x \in \partial V$); hence, $s_0 \in (0,1)$. By (15), we obtain $Tx_0 = \frac{\mu}{s_0}x_0$, where $x_0 \in \partial V$, $s_0 \in (0,1)$, $\mu \geq 1$. Inserting $Tx_0 = \frac{\mu}{s_0}x_0$ into (13), we have

$$(\delta x_0 + \frac{\mu}{s_0}x_0)^{\beta} \preceq (\mu x_0 - \delta\frac{\mu}{s_0}x_0)^{\beta}.$$

Since E is a $Z - Z - B$ space,

$$[(\delta + \frac{\mu}{s_0})x_0]^{\beta} \preceq [(\mu - \frac{\delta}{s_0}\mu)x_0]^{\beta}.$$

It follows that

$$(\delta + \frac{\mu}{s_0})^{\beta}(x_0)^{\beta} \preceq (\mu - \frac{\delta}{s_0}\mu)^{\beta}(x_0)^{\beta}.$$

Because U is an excellent cone in E, we can write

$$\| (\delta + \frac{\mu}{s_0})^{\beta}(x_0)^{\beta} \| \leq \| (\mu - \frac{\delta}{s_0}\mu)^{\beta}(x_0)^{\beta} \| .$$

That is,

$$(\delta + \frac{\mu}{s_0})^\beta \parallel (x_0)^\beta \parallel \leq \mid (\mu - \frac{\delta}{s_0}\mu)^\beta \mid\parallel (x_0)^\beta \parallel . \tag{16}$$

Owing to the fact that E is a $Z-Z-B$ space and $x_0 \neq \theta$, we thus have $(x_0)^\beta \neq \theta$, $\beta \geq 1$, $\beta \in Q$. By (16), we have

$$(\delta + \frac{\mu}{s_0})^\beta \leq (\mid \mu - \frac{\delta}{s_0}\mu \mid)^\beta.$$

Since $s_0 \in (0,1)$, thus $\frac{1}{s_0} > 1$, $\beta \geq 1$, $\beta \in Q$, by Lemma 3, that is,

$$(\delta + \frac{\mu}{s_0})^\beta > (\mid \mu - \frac{\delta}{s_0}\mu \mid)^\beta.$$

Therefore, $\theta \overline{\in} l_s(\partial V)$, that is, $\theta \overline{\in} (I - L_s)(\partial V)$. We obtain that $x \neq L_s(s,x)$, for every $x \in \partial V$,$s \in [0,1]$.

According to the homotopy invariance and normalization in [3], we have

$$i(\frac{1}{\mu}T, V, U) = i(\theta, V, U) = 1.$$

Moreover, according to the solution property in [3], we know that $Ax = \mu x$ has a solution in V.

The proof of Theorem 4 is completed. □

Remark 1. *For the proof of the existence of the solution of an operator equation, the above Theorems 1, 2, 4 and 7 may not be applicable. If they are not applicable, it does not mean that the solution of the operator equation does not exist. Moreover, the differences between Theorems 1, 2, 4 and 7 need to be further studied. The examples of Theorems 1, 2, 4 and 7 also need to be further studied.*

The following theorems can also be proven by the method of proving the above theorems.

Theorem 5. *Let E be a $Z-Z-B$ space, U be an excellent cone in E, V be a bounded open subset of U, and $\theta \in V$. Let Q be the set of all rational numbers, $\overline{V}$ be the closure of V, and ∂V be the boundary of V. Suppose that T: $\overline{V} \to U$ is a semi-closed 1-set-contractive operator and satisfies the following condition:*

$$(Tx \cdot \mu x)^\beta \succeq (Tx)^{2\beta} + (\mu x)^{2\beta}, \quad for\ every\ x \in \partial V, where\ \beta \geq 1, \beta \in Q, \mu \geq 1.$$

Then, the operator equation $Tx = \mu x$ has a solution in $\overline{V}$.

Theorem 6. *Let E be a $Z-Z-B$ space, U be an excellent cone in E, V be a bounded open subset of U, and $\theta \in V$. Let Q be the set of all rational numbers, $\overline{V}$ be the closure of V, and ∂V be the boundary of V. Suppose that T: $\overline{V} \to U$ is a semi-closed 1-set-contractive operator and satisfies the following condition:*

$$(Tx - \mu x)^{\beta+1} \succeq (\mu x)^\beta(\mu x - Tx) - (Tx)^{\beta+1}, \quad for\ every\ x \in \partial V, where\ \beta \geq 1, \beta \in Q, \mu \geq 1.$$

Then, the operator equation $Tx = \mu x$ has a solution in $\overline{V}$.

Theorem 7. *Let E be a $Z-Z-B$ space, X be a bounded open convex subset of E, and $\theta \in X$. Let Q be the set of all rational numbers, $\overline{V}$ be the closure of V, and ∂V be the boundary of V. Suppose that T: $\overline{X} \to E$ is a semi-closed 1-set-contractive operator and satisfies the following condition:*

$$\parallel (Tx)^\beta + (\mu x)^\beta \parallel + \parallel x^\beta \parallel \leq \parallel (Tx)^\beta - (\mu x)^\beta \parallel, \quad for\ every\ x \in \partial X, where\ \beta > 0, \beta \in Q, \mu \geq 1.$$

Then, the operator equation $Tx = \mu x$ has a solution in $\overline{X}$.

Theorem 8. *Let E be a $Z - Z - B$ space, X be a bounded open convex subset of E, and $\theta \in X$. Let Q be the set of all rational numbers, $\overline{V}$ be the closure of V, and ∂V be the boundary of V. Suppose that T: $\overline{X} \to E$ is a semi-closed 1-set-contractive operator and satisfies the following condition:*

$$\| (Tx + \mu x)^{\beta} \| + \| x^{\beta} \| \leq \| (Tx)^{\beta} \| + \| (\mu x)^{\beta} \|, \quad for\ every\ x \in \partial X, where\ \beta \geq 1, \beta \in Q, \mu \geq 1.$$

Then, the operator equation $Tx = \mu x$ has a solution in $\overline{X}$.

3. Conclusions

In this paper, we mainly study some nonlinear problems in Z-Z-B space, obtaining the above theorems. The above theorems can be used to prove the existence of solutions of different equations. We can also give those spaces that meet the conditions, and apply the theorem to obtain the desired results under the given conditions of the theorem. This is a future research direction. In addition, applying this article to various fields of reality is also a research direction. It is also an important research direction to further extend the rational numbers β in the theorem to real numbers.

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