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On the Geometry in the Large of Einstein-like Manifolds

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Abstract: Gray has presented the invariant orthogonal irreducible decomposition of the space of all covariant tensors of rank 3, obeying only the identities of the gradient of the Ricci tensor. This decomposition introduced the seven classes of Einstein-like manifolds, the Ricci tensors of which fulfill the defining condition of each subspace. The large-scale geometry of such manifolds has been studied by many geometers using the classical Bochner technique. However, the scope of this method is limited to compact Riemannian manifolds. In the present paper, we prove several Liouville-type theorems for certain classes of Einstein-like complete manifolds. This represents an illustration of the new possibilities of geometric analysis.

Keywords: Einstein-like manifold; Bochner method; Sampson Laplacian; Bourguignon Laplacian; vanishing theorem

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1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold with the Levi–Civita connection ∇. If H is the vector bundle over (M, g) of all covariant tensors of rank 3 satisfying only the identities of the covariant derivative ∇Ric of the Ricci tensor Ric, then H decomposes into the pointwise orthogonal sum $H = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{S}$ of its three subbundles, $\mathcal{A}, \mathcal{B}$, and $\mathcal{S}$ (see [1,2] (pp. 432–433). Moreover $\mathcal{A}, \mathcal{B}$, and $\mathcal{S}$ are pointwise irreducible under the action of the orthogonal group $O(n)$. Therefore, the Riemannian manifold is called an *an Einstein-like manifold of type* $\mathcal{A}$ (respectively, $\mathcal{B}$ or $\mathcal{S}$) if $\nabla Ric \in C^\infty \mathcal{A}$ (respectively, $\nabla Ric \in C^\infty \mathcal{B}$ or $\nabla Ric \in C^\infty \mathcal{S}$). In the first case Ric is called the *Killing–Ricci tensor* (see [3]), while in the second case Ric is called the *Codazzi–Ricci tensor* (see [4]) and in the third case Ric is called the *Sinyukov–Ricci tensor* (see [5–7]).

A discussion of the geometry of the above and other types of Riemannian Einstein-like manifolds can be found in the paper [1] and the monograph [2] (pp. 432–455). In turn, the application of such manifolds in general relativity can be found in [7,8]. For example, it is well known that the scalar curvature $s = \mathrm{trace}_g Ric$ is constant for an arbitrary Einstein-like manifold (M, g) belonging to either class $\mathcal{A}$ or $\mathcal{B}$. Moreover, an arbitrary manifold belonging to $\mathcal{A} \cap \mathcal{B}$ must have a parallel Ricci tensor. An example of this type of Einstein-like manifold is a Riemannian *locally symmetric space* (see [9], p. 369).

In turn, we use the *Bourguignon Laplacian* (see [10]) and the *Sampson Laplacian* (see [11–13]) to study the global geometry of the above three classes of Einstein-like manifolds. Both of these Laplacians admit *Weitzenböck decompositions* (see [2], p. 53). We recall here that a Laplace operator D permits a *Weitzenböck decomposition* if $D = \bar{\Delta} + \Re$, where $\Re$ is the Weitzenböck curvature operator, which depends linearly on the curvature R and the Ricci tensors of (M, g) and $\bar{\Delta} = \nabla^* \nabla$ is the *Bochner Laplacian* (see [2], pp. 52–53).

Weitzenböck decompositions are important because (see [2], p. 53) there is a method, per Bochner [14], of proving vanishing theorems for null space of a Laplace operator that allows a Weizenböck decomposition.

For example, several fundamental results formulated in various theorems are based on the Bochner technique, which usually shows that the assumption of positive or negative sectional curvatures of compact Riemannian manifolds yields the vanishing of certain geometric and topological invariants (such as the Betti numbers), geometrically interesting tensor fields (such as the Killing–Yano tensors), and mappings (such as harmonic mappings); see for example [9] (pp. 333–364), [15–17].

However, we have already entered the era of geometric analysis and its applications in studying geometric and topological properties of complete Riemannian manifolds (see for example [18]). Therefore, in this article we discuss the global geometry of Einstein-like complete manifolds using a generalized version of the Bochner technique (see for example [19]). Furthermore, in the three sections below we demonstrate the application of various methods of the generalized Bochner technique (see for example [20]) to the study of the above-mentioned three classes of complete Einstein-like manifolds. In particular, the results obtained in our paper generalize well-known results on Einstein-like compact manifolds to the case of Einstein-like complete non-compact manifolds.

2. $\mathcal{A}$-Spaces and the Sampson Laplacian

A Riemannian manifold (M, g) is said to be Einstein-like of type $\mathcal{A}$ if its Ricci tensor Ric is cyclic parallel, that is, if $(\nabla_X Ric)(X, X) = 0$ for all $X \in TM$ (see [21]). In particular, from [2] (p. 451) it is known that if (M, g) is a compact (without boundary) Einstein-like manifolds of type $\mathcal{A}$ with nonpositive sectional curvature, then $\nabla Ric = 0$. If, in addition, there exists a point in M where the sectional curvature of every two-plane is strictly negative, then (M, g) is Einsteinian, i.e., its Ricci tensor satisfies $Ric = \rho\, g$ for some constant ρ (see [2], p. 451).

On the other hand, Deng (see [22]) studied the rigidity of complete $\mathcal{A}$-manifolds and showed that (M, g) is an n-dimensional complete Einstein-like manifold of type $\mathcal{A}$ with a *Yamabe constant* $Q(M, g) > 0$ and nonpositive scalar curvature, and is an Einstein manifold if there exists a small number C depending on the dimension n and $Q(M, g)$ such that $\int_M (\|Ric - s/n \cdot g\|^{n/2} + \|W\|^{n/2}) d\mathrm{vol}_g \leq C$ for the Weyl curvature tensor, W. In turn, Chu modernized this result in his article [23]. We remark here that the above results were obtained using the methods of the classical Bochner technique.

This section studies complete Einstein-like manifolds of type $\mathcal{A}$ with nonpositive sectional curvature. As the starting point in the study of Riemannian manifolds of nonpositive curvature, we first recall the following well known Cartan–Hadamard theorem: *Let (M, g) be an n-dimensional simply connected complete Riemannian manifold of nonpositive curvature; then, (M, g) is diffeomorphic to the n-dimensional Euclidean space $\mathbb{R}^n$.* Therefore, a simply connected complete Riemannian manifold of nonpositive curvature is called a *Hadamard manifold* or a *Cartan–Hadamard manifold*, after the Cartan–Hadamard theorem (see for example [9], p. 241; [18], pp. 391–381).

Remark 1. *From the Cartan–Hadamard theorem, one can conclude, in particular, that no compact simply connected manifold admits a metric of nonpositive curvature (see [9], p. 162). Therefore, compact Hadamard manifolds do not exist.*

Here, we recall that the function $f \in C^2M$ is subharmonic if $\Delta f \geq 0$, where $\Delta = \mathrm{div} \circ \mathrm{grad}$ is the Beltrami Laplacian on functions. Then, we can formulate the following proposition.

Lemma 1. *On a Hadamard manifold (M, g) any non-negative subharmonic function $f \in C^2M$ such that $f \in L^q(M)$ for $q \in (0, \infty)$ is equal to zero.*

Proof. The following theorem holds (see [24]): on a complete simply connected Riemannian manifold (M, g) of nonpositive sectional curvature, every nonnegative subharmonic function $f \in L^q(M)$ for $q \in (0, \infty)$ is a constant C. In this case, we have $C \cdot \int_M d\mathrm{vol}_g < +\infty$. We observe that Hadamard manifolds have infinite volume (see [25]); hence, from the last inequality, we obtain $C = 0$. This finishes the proof of our Lemma 1. $\square$

In turn, we introduce the Sampson Laplacian into consideration and consider several of its properties. In order to do this, we define the differential operator δ^* as follows: $C^\infty S^p M \to C^\infty S^{p+1} M$ of degree 1 such that $\delta^* = (p+1)\,\mathrm{Sym} \circ \nabla$, where Sym: $\otimes^p T^*M \to S^p M$ is the linear algebraic operator of symmetrization. This means that δ^* is a symmetrized covariant derivative defined by the following equation (see [2], pp. 355–356):

$$(\delta^* \varphi)(X_1, X_2, \dots, X_p, X_{p+1}) = (\nabla_{X_1} \varphi)(X_2, \dots, X_p, X_{p+1}) + \dots + (\nabla_{X_{p+1}} \varphi)(X_1, X_2, \dots, X_p)$$

for any $\varphi \in C^\infty S^p M$, and $X_1, X_2, X_3, \dots, X_p, X_{p+1} \in TM$. Then, there exists its formal adjoint operator δ: $C^\infty S^{p+1} M \to C^\infty S^p M$, which is called the *divergence operator* (see [2], p. 356). Notice that δ is nothing other than the $\otimes^{p+1} T^*M$ restriction of ∇^* to $S^{p+1}M$ (see [2], p. 35).

Using the operators δ^* and δ, we can define the second order differential operator Δ_S, $C^\infty S^p M \to C^\infty S^p M$, by the formula $\Delta_S = \delta\,\delta^* - \delta^* \delta$. At the same time, if $\Delta_S \varphi = 0$, then the tensor field φ is called a *Δ_S-harmonic symmetric tensor*, as it is an analog of the harmonic forms of the Hodge–de Rham theory (see [9], p. 335).

The Weitzenböck decomposition formula for the Sampson Laplacian Δ_S: $C^\infty S^p M \to C^\infty S^p M$ has the form

$$\Delta_S\, \varphi = \bar{\Delta}\, \varphi - \Re\,(\varphi). \tag{1}$$

The second component of the right-hand side of (1) is called the Weitzenböck curvature operator of the Sampson Laplacian Δ_S.

Next, from (1) by direct calculation, we obtain the *Bochner–Weitzenböck formula*,

$$\frac{1}{2}\,\Delta\,\|\varphi\|^2 = -\,g(\Delta_S \varphi, \varphi) + \|\nabla \varphi\|^2 - g\,(\Re(\varphi), \varphi), \tag{2}$$

for $\|\nabla\varphi\|^2 = g(\nabla\varphi, \nabla\varphi)$. If $p = 2$, then for any point $x \in M$ there exists an orthonormal eigenframe $e_1, \dots, e_n$ of $T_x M$ such that $\varphi_x(e_i, e_j) = \mu_i\, \delta_{ij}$ for the Kronecker delta, δ_{ij}. In this case (see [2], p. 436, [26], p. 388), we have

$$g(\,\Re\,(\varphi_x),\ \varphi_x) = 2 \cdot \sum_{i<j} \sec\,(e_i \wedge e_j)\,(\,\mu_i - \mu_j)^{\,2} \tag{3}$$

where $\sec\,(e_i \wedge e_j)$ is the *sectional curvature* in the direction of the two-plane $\sigma_x = \mathrm{span}\,\{e_i, e_j\}$ of $T_x M$ at an arbitrary point $x \in M$. In this case, the Formula (2) can be rewritten in the form

$$\frac{1}{2}\,\Delta\,\|\varphi\|^2 = -\,g(\Delta_S \varphi, \varphi) + \|\nabla\varphi\|^2 -\ 2 \cdot \sum_{i<j} \sec\,(e_i \wedge e_j)\ (\mu_i - \mu_j)^2. \tag{4}$$

We can now prove the following statement.

Theorem 1. *Let (M, g) be a Hadamard manifold and φ a Δ_S-harmonic symmetric 2-tensor on (M, g) such that $\int_M \|\varphi\|^q\, d\mathrm{vol}_g < +\infty$ for at least one $q > 1$; then, $\varphi \equiv 0$.*

Proof. Let φ be a non-zero Δ_S-harmonic symmetric 2-tensor on a Riemannian manifold with non-positive sectional curvature; then, from Formula (4), we obtain

$$\frac{1}{2}\,\Delta\|\varphi\|^2 = \|\nabla\varphi\|^2 - 2\,\cdot\,\sum_{i<j} \sec\,(e_i \wedge e_j)(\mu_i - \mu_j)^2. \tag{5}$$

On the other hand, we have $\frac{1}{2}\,\Delta\|\varphi\|^2 = \|\varphi\| \cdot \Delta\|\varphi\| + \|\nabla\|\varphi\|\|^2$. Then, from the last formula and (5), we obtain

$$\|\varphi\| \cdot \Delta \|\varphi\| = \|\nabla\varphi\|^2 - 2 \cdot \sum_{i<j} \sec(e_i \wedge e_j)\,(\mu_i - \mu_j)^2 - \|\nabla\|\varphi\|\,\|^2 \tag{6}$$

where $\|\nabla\varphi\|^2 \geq \|\nabla\|\varphi\|\,\|^2$ due to the *Kato inequality* $\|\nabla\varphi\| \geq \|\nabla\|\varphi\|\,\|$.

Moreover, if we suppose that (M,g) is a manifold with nonpositive sectional curvature, then from (6) we obtain the inequality $\|\varphi\|\,\Delta\|\varphi\| \geq 0$. It is known from [27] that if the inequality $(q-1)\,f \cdot \Delta f \geq 0$ holds for a nonnegative function $f \in C^2(M)$ defined on a complete Riemannian manifold, then either $\int_M f^q d\mathrm{vol}_g = +\infty$ for all $q \neq 1$ or f = const. In particular, q may even be less than one here (see [27]). In this case, the inequality $\int_M f^q d\mathrm{vol}_g < \infty$ becomes $C^q \cdot \int_M d\mathrm{vol}_g < \infty$. At the same time, we know that a Hadamard manifold (M,g) has an infinite volume; hence, from the last inequality we obtain $C = 0$. This completes our proof. □

On the other hand, let $S_0^p M$ be the bundle of traceless symmetric p-tensors on (M,g). Then, the fact that $\sec \leq 0$ implies the negative semi-definiteness of the quadratic $g(\Re(\varphi),\varphi)$ for any $p \geq 2$, while $\varphi \in C^\infty S_0^p M$ is proven in [28]. For this case, the following proposition holds:

Theorem 2. *Let (M,g) be a Hadamard manifold and φ a Δ_S-harmonic traceless symmetric p-tensor $(p \geq 2)$ on (M,g) such that $\int_M \|\varphi\|^q\, d\mathrm{vol}_g < +\infty$ for at least one $q > 0$; then, $\varphi \equiv 0$.*

Let (M,g) be a Riemannian Einstein-like manifold (M,g) of type $\mathcal{A}$. Then, its Ricci tensor, Ric, satisfies the equations $\delta^* Ric = 0$ and has a constant trace, i.e., the scalar curvature $s = \mathrm{trace}_g Ric$ is a constant function. This means that $\delta Ric = 0$. Therefore, Ric is a Δ_S-harmonic symmetric 2-tensor. Then, we can formulate the following lemma.

Lemma 2. *A Killing–Ricci tensor is a Δ_S-harmonic symmetric 2-tensor.*

In this case, the following proposition is an immediate consequence of Lemma 2 and Theorem 1.

Corollary 1. *Let an n-dimensional Riemannian Einstein-like manifold (M,g) of type $\mathcal{A}$ be a Hadamard manifold. If $\int_M \|Ric\|^q\, d\mathrm{vol}_g < +\infty$ for at least one $q > 1$, then (M,g) is isometric to $\mathbb{R}^n$.*

Proof. We know that a Killing–Ricci tensor of (M,g) is a Δ_S-harmonic symmetric 2-tensor. Moreover, if (M,g) is a Hadamard manifold and $\|Ric\| \in L^q(M)$ for at least one $q \in (1,+\infty)$, then $Ric \equiv 0$ per Theorem 2. Next, we need to prove one obvious statement. If the sectional curvature is non-positive and the Ricci curvature is zero, then the Riemannian manifold is flat; that is, let $X \in T_x M$ be a unit vector. We can complete it on an orthonormal basis, $\{X,\ e_2,\ldots,e_n\}$, for $T_x M$ at an arbitrary point $x \in M$; then, (see [9], p. 86):

$$Ric(X,X) = \sum_{a=2}^{n} \sec(X \wedge e_a).$$

In this case, from the conditions $Ric \equiv 0$ and $\sec \leq 0$ we obtain $\sec \equiv 0$, i.e., the sectional curvature vanishes identically. In this case, (M,g) is a flat Riemannian manifold. If (M,g) is simply connected, it follows that (M,g) is isometric to the Euclidean space $\mathbb{R}^n$. □

We now consider a three-dimensional Riemannian Einstein-like manifold (M,g) of type $\mathcal{A}$. In this case, the following corollary holds.

Corollary 2. *Let (M,g) be a three-dimensional simply connected Riemannian Einstein-like manifold (M,g) of type $\mathcal{A}$. If its Ricci tensor Ric satisfies the conditions $Ric \geq \frac{1}{2} sg$ and $\int_M \|Ric\|^q\, d\mathrm{vol}_g < +\infty$ for at least one $q > 1$, then (M,g) is isometric to $\mathbb{R}^3$.*

Proof. In dimensions up to three, Einstein-like manifolds of type $\mathcal{A}$ have been classified in [29]; in particular, they are homogeneous (see [30]). Therefore, a three-dimensional Einstein-like manifold of type $\mathcal{A}$ is complete, because any homogeneous Riemannian manifold is complete (see [2], p. 181). Moreover, it is well known (see [9], p. 86 and [31]) that

$$\sec\left(X_x \wedge Y_x\right) = \frac{1}{2}\, s(x) - Ric(Z_x, Z_x) \tag{7}$$

for unit orthogonal vectors X_x, Y_x, $Z_x \in T_xM$ at any point $x \in M$ such that Z_x is orthogonal to the plane $\sigma = \operatorname{span}\{X_x, Y_x\}$. In this case, the condition $\sec \leq 0$ can be rewritten in the form $Ric \geq \frac{1}{2}\, s\, g$. Then, from Theorem 1 and Corollary 2, we can conclude that if the Ricci tensor Ric satisfies the two conditions $Ric \geq \frac{1}{2}\, s\, g$ and $\int_M \|\, Ric\, \|^q\, d\mathrm{vol}_g < +\infty$ for at least one $q > 1$, then (M, g) is a Ricci-flat manifold, and hence is a flat manifold due to equality (7). In this case, (M, g) is isometric to the Euclidean space $\mathbb{R}^3$. $\square$

3. $\mathcal{B}$-Spaces and the Bourguignon Laplacian

A Riemannian manifold (M, g) is said to have a *harmonic curvature tensor* if $\delta R = 0$ (see [9], p. 362). This happens if and only if the Ricci tensor Ric is a Codazzi–Ricci tensor, i.e., $(\nabla_X Ric)(Y, Z) = (\nabla_Y Ric)(X, Z)$ for any $X, Y, Z \in TM$. This means that (M, g) is an Einstein-like manifold belonging to class $\mathcal{B}$. There exist numerous examples of compact Riemannian manifolds with this property (see [2], pp. 443–447; [4]). On the other hand, the following classical *Berger–Ebin theorem* is well known. If (M, g) is a compact (without boundary) Einstein-like manifold of type $\mathcal{B}$ with non-negative sectional curvature, then $\nabla\, Ric = 0$. If, in addition, there exists a point in M where the sectional curvature of every two-plane is strictly positive, then (M, g) is Einsteinian (see [2], p. 445). Based on the results obtained above, we can supplement this theorem as follows: a three-dimensional compact Einstein-like manifold of type $\mathcal{B}$ with $Ric < \frac{1}{2} s \cdot g$ has constant positive sectional curvature.

In this section, we generalize this result to the case of a complete Riemannian manifold. In order to do this, we use the *Bourguignon Laplacian* (see [10]).

Here, we consider a symmetric tensor $\varphi \in C^\infty S^2M$ as a one-form with values in the cotangent bundle T^*M on M. This bundle is equipped with the Levi–Civita covariant derivative ∇; thus, there is an induced exterior differential $d^\nabla \colon C^\infty S^2M \to C^\infty(\Lambda^2 M \otimes T^*M)$ on T^*M-valued differential one-forms such as

$$\mathrm{d}^\nabla \varphi(X, Y, Z) := \left(\nabla_X \varphi\right)(Y, Z) - \left(\nabla_Y \varphi\right)(X, Z)$$

for any tangent vector fields X, Y, Z on M and an arbitrary $\varphi \in C^\infty S^2M$ (see [2], pp. 133–134, 355; [9], pp. 349–350; and [32]). In this case, $\varphi \in C^\infty S^2M$ is a *Codazzi tensor* if and only if $d^\nabla \varphi = 0$ (see [9], p. 350). The formal adjoint of d^∇ is denoted by δ^∇(see [2], p. 134). Moreover, Bourguignon proved in [33] (see p. 271) that

$$\delta^\nabla \varphi = -\, d(\operatorname{trace}_g \varphi) \tag{8}$$

for an arbitrary Codazzi tensor $\varphi \in C^\infty S^2M$. At the same time, he defined a *harmonic symmetric 2-tensor* as a tensor $\varphi \in C^\infty S^2M$ such that $\varphi \in \operatorname{Ker} \mathrm{d}^\nabla \bigcap Ker\, \delta^\nabla$ (see [9], p. 350 and [33] p. 270). Next, Bourguignon defined the Laplacian $\Delta_B \colon C^\infty S^2M \to C^\infty S^2M$ by the formula $\Delta_\mathrm{B} := \delta^\nabla \mathrm{d}^\nabla + d^\nabla \delta^\nabla$ (see [33], p. 273). Then, the symmetric harmonic 2-tensors belong to the kernel of the Bourguignon Laplacian Δ_B. The converse is true in the compact case as well; namely, if (M, g) is a compact manifold (without boundary), then $L^2(M)$ denotes the usual Hilbert space of functions or tensors with the global product (with respect to the global norm)

$$\langle u, w \rangle = \int_M g(u, w)\, d\mathrm{vol}_g.$$

Then, by direct computation, we obtain the following integral formula (see [10]):

$$\langle \Delta_B \varphi, \varphi \rangle = \langle d^{\nabla} \varphi, d^{\nabla} \varphi \rangle + \langle \delta^{\nabla} \varphi, \delta^{\nabla} \varphi \rangle \tag{9}$$

Next, an easy computation yields the *Weitzenböck decomposition formula* (see [2], p. 355 and [33], p. 273)

$$\Delta_B\, \varphi = \bar{\Delta}\, \varphi + \Re(\varphi). \tag{10}$$

The second component of the right-hand side of (10) is called the Weitzenböck curvature operator of the Bourguignon Laplacian Δ_B (see [2], p. 356).

Based on the last two formulas, we conclude that the *Bourguignon Laplacian* Δ_B is a non-negative operator and its kernel is the finite dimensional vector space of harmonic symmetric 2-tensors (or, in other words, Codazzi tensors with constant trace). Therefore, the harmonic symmetric 2-tensor will be called the Δ_B-*harmonic tensor.*

Using (10), the Bochner-Weitzenböck formula (see [10]) can be obtained:

$$\frac{1}{2}\, \Delta \|\varphi\|^2 = -\, g(\Delta_B \varphi, \varphi) + g(\Re(\varphi), \varphi) + \|\nabla \varphi\|^2, \tag{11}$$

where

$$g(\Re(\varphi), \varphi) = \sum_{i \neq j} \sec\,(e_i \wedge e_j)\,(\mu_i - \mu_j)^2$$

for an arbitrary $\varphi \in C^{\infty} S^2 M$ and an orthonormal eigenframe $e_1, \ldots, e_n$ of $T_x M$ such that $\varphi_x(e_i, e_j) = \mu_i \delta_{ij}$ at any point $x \in M$. Let $\varphi \in C^{\infty} S^2 M$ be a Δ_B-harmonic; then, (11) can be rewritten in the following form:

$$\frac{1}{2}\, \Delta \|\varphi\|^2 = \sum_{i \neq j} \sec\,(e_i \wedge e_j)\,(\mu_i - \mu_j)^2 + \|\, \nabla \varphi\|^2. \tag{12}$$

The following theorem supplements the Berger–Ebin theorem for the case of Δ_B-harmonic tensors on a complete noncompact Riemannian manifold.

Theorem 3. *Let (M, g) be a connected complete and noncompact Riemannian manifold with nonnegative sectional curvature. Then, there is no non-zero Δ_B-harmonic tensor $\varphi \in C^{\infty} S^2 M$ such that $\int_M \|\varphi\|^q d\mathrm{vol}_g < +\infty$ for any $q > 1$.*

Proof. Let (M, g) be a connected complete and noncompact Riemannian manifold with nonnegative sectional curvature, and let $\varphi \in C^{\infty} S^2 M$ be a non-zero Δ_B-harmonic symmetric 2-tensor; then, $g(\Re(\varphi), \varphi) \geq 0$. Therefore, from (12), we can obtain the inequality

$$\|\varphi\|\; \Delta \|\varphi\| = \|\nabla \varphi\|^2 + \sum_{i \neq j} \sec\,(e_i \wedge e_j)\,(\mu_i - \mu_j)^2 - \|\nabla \|\varphi\|\,\|^2 \geq 0.$$

where $\|\nabla \varphi\|^2 \geq \|\nabla \|\varphi\|\,\|^2$ due to the *Kato inequality* $\|\nabla \varphi\| \geq \|\nabla \|\varphi\|\,\|$. Then, we can conclude that $\|\varphi\|\; \Delta \|\varphi\| \geq 0$ on a connected complete and noncompact Riemannian manifold with nonnegative sectional curvature. For $q > 1$, then, either $\int_M \|\varphi\|^q\, d\mathrm{vol}_g = \infty$ or $\|\varphi\| = \mathrm{const}$ (see [27]). In a case where (M, g) has infinite volume, all of the constant functions hold while zero is in $L^q(M)$; that is, if the function $f \in L^q(M)$ for some positive number q is a constant function, C, then the inequality $\int_M |f|^q\, d\mathrm{vol}_g < \infty$ becomes $|\, C\,|^q \cdot \int_M d\mathrm{vol}_g < +\infty$. If in addition (M, g) has an infinite volume, then we can obtain $C = 0$ from the last inequality. It must be recalled that a complete Riemannian manifold of non-negative sectional curvature has an infinite volume (see [27,34]). This remark completes the proof. □

We are now able to formulate the following lemma.

Lemma 3. *The Codazzi–Ricci tensor is a Δ_B-harmonic symmetric 2-tensor.*

Proof. Let (M,g) be a Riemannian Einstein-like manifold (M,g) of type $\mathcal{B}$. Then, its Ricci tensor Ric satisfies the equations $d^{\nabla}Ric = 0$ and has a constant trace, i.e., the scalar curvature $s = \mathrm{trace}_g Ric$ is a constant function. This means that $\delta^{\nabla}Ric = 0$; therefore, Ric is a Δ_B-harmonic symmetric 2-tensor. □

In this case, the following proposition is an immediate consequence of the above Lemma 3 and Theorem 3.

Corollary 3. *Let (M,g) be a connected complete and noncompact Einstein-like manifold (M,g) of type $\mathcal{B}$ with non-negative sectional curvature. If $\int_M \|Ric\|^q \, d\mathrm{vol}_g < +\infty$ for some $q > 1$, then (M,g) is flat.*

Proof. If (M,g) is a connected complete and noncompact manifold with non-negative sectional curvature, and $\int_M \|Ric\|^q \, d\mathrm{vol}_g < +\infty$ for some $q > 1$, then (M,g) is Ricci-flat per Lemma 3 and Theorem 3. Next, we need to prove one obvious statement: if the sectional curvature is nonnegative and the Ricci curvature is zero, then the Riemannian manifold is flat. That is, let $X \in T_xM$ be a unit vector which we complete on an orthonormal basis, $\{X, e_2, \ldots, e_n\}$, for T_xM at an arbitrary point $x \in M$; then, (see [9], p. 86)

$$Ric(X,X) = \sum_{i=2}^{n} \sec(X \wedge e_i).$$

In this case, from the conditions $Ric = 0$ and $\sec \geq 0$ we obtain $\sec = 0$, which completes the proof. □

In conclusion, we formulate one more obvious corollary.

Corollary 4. *Let (M,g) be a three-dimensional connected complete and noncompact Einstein-like manifold of type $\mathcal{B}$. If $Ric \leq \frac{1}{2}\, s\, g$ and $\int_M \|Ric\|^q \, d\mathrm{vol}_g < +\infty$ for some $q > 1$, then (M,g) is flat.*

Remark 2. *In Corollaries 3 and 4, we proved that (M,g) is a flat Riemannian manifold. If (M,g) is simply connected, then (M,g) is isometric to the Euclidian space, $\mathbb{R}^n$.*

4. On Compact Einstein-like Manifolds of the Type $\mathcal{S}$

A Riemannian manifold (M,g) is said to be *Einstein-like of type $\mathcal{S}$* if its Ricci tensor Ric satisfies the condition

$$(\nabla_X Ric)(Y,Z) = \sigma(X) g(Y,Z) + v(Y)\, g(X,Z) + v(Z)\, g(Y,Z) \tag{13}$$

where $\sigma(X) = \frac{n}{(n-1)\,(n+2)}\, X(s)$ and $v\,(X) = \frac{n-2}{2(n-1)\,(n+2)}\, X(s)$ for any $X \in TM$. Riemannian manifolds satisfying condition (13) are called *Sinyukov manifolds* in [5]. Besse defined these Equations (13), but did not carry out any research for manifolds of class $\mathcal{S}$. The local properties of such manifolds were studied in [5]. In turn, the purpose of [35] was the local classification of all three-dimensional Riemannian manifolds belonging to class $\mathcal{S}$. The application of such manifolds in general relativity can be found in [7,8].

We now prove a theorem on compact Einstein-like manifolds of type $\mathcal{S}$.

Theorem 4. *If (M,g) is a compact (without boundary) Einstein-like manifold of type $\mathcal{S}$ with non-positive sectional curvature, then $\nabla Ric = 0$. If, in addition, there exists a point in M where the sectional curvature of every two-plane is strictly negative, then (M,g) is Einsteinian.*

Proof. From (13), we can obtain

$$(\nabla_X \varphi)(Y,Z) + (\nabla_Y \varphi)(Z,X) + (\nabla_Z \varphi)(X,Y) = 0 \tag{14}$$

for

$$\varphi(Y,Z) = Ric(Y,Z) - \frac{2}{n+2}\, s \cdot g(Y,Z) \tag{15}$$

for any $X, Y, Z \in TM$. Assume that the manifold M is compact; then, from (2) we can obtain the integral formula

$$\langle \Re(\varphi), \varphi \rangle - \langle \delta\varphi, \delta\varphi \rangle - \langle \nabla\varphi, \nabla\varphi \rangle = 0, \tag{16}$$

where, by virtue of (14), we have

$$\langle \Delta_S\, \varphi, \varphi \rangle = \langle \delta^* \varphi, \delta^* \varphi \rangle - \langle \delta\varphi, \delta\varphi \rangle = -\langle \delta\varphi, \delta\varphi \rangle$$

and $g(\Re(\varphi_x), \varphi_x) = 2 \cdot \sum_{i<j} \sec\,(e_i \wedge e_j)\,(\mu_i - \mu_j)^2$ for an orthonormal eigenframe $e_1, \ldots, e_n$ of T_xM such that $\varphi_x(e_i, e_j) = \mu_i\, \delta_{ij}$ at any point $x \in M$. Therefore, if (M, g) is a compact (without boundary) Einstein-like manifolds of type $\mathcal{S}$ with non-positive sectional curvature, then from (16) we obtain $\nabla Ric = 0$. In this case, from (13) we obtain $\nabla\, \varphi = 0$. In addition, if there exists a point in M where the sectional curvature of every two-plane is strictly negative, then from (15) and (16) we can conclude that (M, g) is Einsteinian. □

All three-dimensional Riemannian manifolds belonging to class $\mathcal{S}$ are known from [35]. In turn, we formulate a theorem for a four-dimensional compact Sinyukov manifold.

Theorem 5. *A four-dimensional compact Sinyukov manifold (M, g) with positive sectional curvature is diffeomorphic to the sphere or the real projective space.*

Proof. Our proof is based on three facts. First, if (M, g) is a Sinyukov manifold and dim $M \leq 4$, then (M, g) is locally conformally flat (see [5]). We recall here that a locally conformally flat Riemannian manifold (M, g) is determined by the condition that any point $x \in M$ has a neighborhood $U_x \subset M$ and a C^∞-function f on U_x such that the Riemannian manifold $(U_x, e^{2f} \cdot g\,|_{U_x})$ is flat (see [2], p. 60). Second, we proved in [20] that in the case of a locally conformally flat Riemannian manifold of dimension $n \geq 4$, the conditions $\hat{R} > 0$ and $\sec > 0$ are equivalent for its *curvature operator*, $\hat{R}$ (see [9], p. 83), and its sectional curvature, sec, respectively. Third, it has been proven (see [36]) that the *Ricci flow* deforms g to a metric of constant positive curvature, provided that (M, g) is compact and has the positive curvature operator $\hat{R}$. In this case, (M, g) is diffeomorphic to the sphere $\mathbb{S}^4$ or the real projective space $\mathbb{RP}^4$. □

In conclusion, we formulate a theorem supplementing the previous assertion.

Theorem 6. *Let (M, g) be a four-dimensional complete Sinyukov manifold with $Ric \geq 0$; then, (M, g) belongs to one of the following classes: either flat, or locally isometric to the product of a sphere and a line, which are globally conformally equivalent to either $\mathbb{R}^n$ or a spherical space form.*

Proof. We know that a four-dimensional Sinyukov manifold (M, g) is locally conformally flat (see [5]). Moreover, the main theorem of [37] states that complete locally conformally flat manifolds of dimension $n \geq 3$ with Ricci tensor $Ric \geq 0$ belong to one of the following classes: either flat, or locally isometric to the product of a sphere and a line, and are globally conformally equivalent to either $\mathbb{R}^n$ or a spherical space form (see [38], p. 69). □

5. Conclusions

Gray has presented the $O(n)$-invariant orthogonal irreducible decomposition of the space of all covariant tensors of rank 3 obeying only the identities of the gradient of the Ricci tensor (see above). This decomposition introduced the seven classes of Einstein-like manifolds with Ricci tensors fulfilling the defining condition of each subspace (see [2], pp. 432–455). In the large, the geometry of such manifolds has been studied by many geometers using the classical Bochner technique. This technique's scope is limited to compact Riemannian manifolds (see [9], pp. 333–364; [14,16]; [17], pp. 322–360). However, we have already entered the era of geometric analysis and its applications to the study of relations between the geometric and topological properties of complete Riemannian manifolds (see for example [17] (pp. 361–394) and [18,19]). Yau, Schoen, Hamilton, and others initiated a particularly productive era of geometric analysis in differential geometry in the large, which continues to this day. Most of these results are called Liouville-type theorems, and belong to the generalized Bochner technique (see for example [19,20]). In the present paper, we have proven several Liouville-type theorems for cetain classes of Einstein-like complete manifolds. This paper represents an illustration of the new possibilities in contemporary geometric analysis.

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