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Cesàro Means of Weighted Orthogonal Expansions on Regular Domains

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Abstract: In this paper, we investigate Cesàro means for the weighted orthogonal polynomial expansions on spheres with weights being invariant under a general finite reflection group on \mathbb{R}^d . Our theorems extend previous results only for specific reflection groups. Precisely, we consider the weight function $h_\kappa(x) := \prod_{\nu \in R_+} |\langle x, \nu \rangle|^{\kappa_\nu}$, $x \in \mathbb{R}^d$ on the unit sphere; the upper estimates of the Cesàro kernels and Cesàro means are obtained and used to prove the convergence of the Cesàro (C, δ) means in the weighted L^p space for δ above the corresponding index. We also establish similar results for the corresponding estimates on the unit ball and the simplex.

Keywords: spherical h -harmonics; Cesàro means; Christoffel functions

MSC: 33C50; 33C52; 42B08; 42C10



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1. Introduction

Cesàro summation (also known as the Cesàro mean) is a collection of methods for the infinite summation of a series of numbers or functions. It was stated by E. Cesàro [1] in 1890 and ever since has been extensively studied in mathematical analysis. In the orthogonal expansion theory, generally for $N \in \mathbb{N}$ and $\delta \geq -1$, a Cesàro operator for a function f with a series of orthogonal projections $\{\text{proj}_j f\}_{j=0}^\infty$ is defined as

$$S_N^\delta(f) := \frac{1}{A_N^\delta} \sum_{j=0}^N A_{N-j}^\delta \text{proj}_j f,$$

where $A_N^\delta := \frac{\Gamma(N+\delta+1)}{\Gamma(\delta+1)\Gamma(N+1)}$ and Γ is the Gamma function. It coincides with the partial sum for $\delta = 0$ and the Fejèr sum for $\delta = 1$.

Classical results about the Cesàro operator were established for Fourier expansion of periodic functions [2]. In order to recover a periodic function f from its Fourier coefficients, it would be more convenient to use the Cesàro mean method than taking the limit of the partial sums of its Fourier series since this approach does not always work well. As straight extension but far beyond trivial, these classical results have been developed for spherical harmonic expansions on unit spheres. In the 1980s, C.D. Sogge [3,4] proved the boundedness and convergence of Cesàro operators. Furthermore, the critical index of δ for convergence was obtained under certain restrictions. The approach of Sogge is based on some delicate global estimates of the orthogonal projection operators, which, however, significantly relied on the translation invariance of the Lebesgue measure on the sphere. In recent decades, the theory of spherical h -harmonics was developed and attracted much attention. This theory was initially studied by Dunkl in [5–7] and has been applied in physics (see, for instance, [8,9] (pp. 360–370)). The weight functions in Dunkl theory are invariant under a finite reflection group G on \mathbb{R}^d . Specifically, for the case of group $G = \mathbb{Z}_2^d$ and $G = S_3$ (see [10,11]), Dai and Xu obtained a pointwise estimate for Cesàro

kernel and proved the convergence of Cesàro means. Their analysis relied on an explicit “closed” integral representation of Cesàro kernels. It should be pointed out that the explicit integral representation, which is only known for several special groups $G = \mathbb{Z}_2^d$ and S_3 , is essential in the works of [10–15].

In this paper, our goal is to extend the previous results about the convergence of Cesàro means for the weighted orthogonal polynomials expansions (WOPEs) with general finite reflection groups and to give a condition for the convergence of the Cesàro means with respect to the weights. To overcome the difficulty of the results of the integral representation, we shall apply the weighted Christoffel function to establish a delicate pointwise estimate of Cesàro kernels.

The paper is organized as follows: in Section 2, we describe some necessary notations and preliminary results on Jacobi polynomials. We also discuss the Dunkl operators in detail and the theory of spherical h -harmonics, which was developed by Dunkl ([5,6]). An important tool for our proofs of the main theorems is the weighted Christoffel function on the sphere. We will present some sharp asymptotic estimates of the weighted Christoffel functions. In Sections 3 and 4, we shall state and prove our main theorems. We apply the weighted Christoffel function to establish a highly localized pointwise estimate of a Cesàro kernel in spherical h -harmonic expansions. This pointwise kernel estimate plays a crucial role in proving an integral estimate and a convergence result. Finally, Section 5 is devoted to results on the unit ball \mathbb{B}^d and the simplex \mathbb{T}^d : necessary notations and results on WOPEs on \mathbb{B}^d and \mathbb{T}^d are described briefly in this section, while we establish similar results on the ball and simplex.

Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm in \mathbb{R}^d . We denote by c a generic constant that may depend on fixed parameters such as κ and d , whose value may change from line to line. Furthermore, we write $A \sim B$ if there exists a constant $c > 0$ such that $A \geq cB$ and $B \geq cA$.

2. Preliminaries

In this section, we describe the material necessary for the spherical h -harmonic analysis of the sphere.

2.1. Dunkl Theory and Spherical h -Harmonic Expansions

Let $G \subset O(d)$ be a finite reflection group on \mathbb{R}^d . Let ν be a nonzero vector in \mathbb{R}^d . The reflection σ_ν along ν is defined by

$$\sigma_\nu x = x - \frac{2\langle x, \nu \rangle}{\|\nu\|^2} \nu, \quad x \in \mathbb{R}^d,$$

that is the reflection with respect to the hyperplane perpendicular to ν . Let R be the root system of G , normalized so that $\langle x, \nu \rangle = 2$ for all $\nu \in R$, and fix a positive subsystem R_+ of R . It is known that (see, for instance, [16]) the set of reflections in G coincides with the set $\{\sigma_\nu : \nu \in R_+\}$, which also generates the group G . Let $\kappa : R \rightarrow [0, \infty), \nu \rightarrow \kappa_\nu = \kappa(\nu)$ be a nonnegative multiplicity function on R (i.e., a nonnegative G -invariant function on R). Define weight function,

$$h_\kappa(x) := \prod_{\nu \in R_+} |\langle x, \nu \rangle|^{\kappa_\nu}, \quad x \in \mathbb{R}^d, \tag{1}$$

as a homogeneous function of degree $\gamma_\kappa = \sum_{\nu \in R_+} \kappa_\nu$ and invariant under G .

The sphere $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ is a metric space equipped with the geodesic metric $\rho(x, y) := \arccos \langle x, y \rangle, x, y \in \mathbb{S}^{d-1}$ and the usual rotation-invariant surface Lebesgue measure $d\sigma(x)$. We denote by $B_r(x)$ the spherical cap $\{y \in \mathbb{S}^{d-1} : \rho(x, y) < r\}$ centered at $x \in \mathbb{S}^{d-1}$ and having radius $r > 0$. Given a positive constant c and a spherical cap $B = B_r(x) \subset \mathbb{S}^{d-1}$, we use the notation cB to denote the spherical cap $B_{cr}(x)$ with the same center as that of B but c times the radius of B . The weight function we shall consider on

the sphere \mathbb{S}^{d-1} is $h_\kappa^2(x)$, which can also be written as $h_\kappa^2(x) := \prod_{\nu \in R} |\langle x, \nu \rangle|^{2\kappa_\nu}$. Given a measurable set $E \subset \mathbb{S}^{d-1}$, we write $\text{meas}_\kappa(E) := \omega_d^\kappa \int_E h_\kappa^2(x) d\sigma(x)$, where

$$\omega_d^\kappa := \left(\int_{\mathbb{S}^{d-1}} h_\kappa^2(x) d\sigma(x) \right)^{-1}. \tag{2}$$

It is easily seen that for $0 < r \leq \pi$, (see [17]),

$$\text{meas}_\kappa(B_r(x)) \sim r^{d-1} \prod_{\nu \in R_+} (|\langle x, \nu \rangle| + r)^{2\kappa_\nu}, \quad x \in \mathbb{S}^{d-1}, \tag{3}$$

where the constants of equivalence depend only on d and κ . This in particular implies that meas_κ is a doubling measure on \mathbb{S}^{d-1} satisfying that for any $x \in \mathbb{S}^{d-1}$ and $r \in (0, \pi)$,

$$\text{meas}_\kappa(B_{2^m r}(x)) \leq C 2^{ms_\kappa} \text{meas}_\kappa(B_r(x)), \quad m = 1, 2, \dots,$$

where $C > 0$ is a constant depending only on κ and d , and s_κ is the smallest positive number s for which

$$\sup_B \sup_{m \in \mathbb{N}} \frac{\text{meas}_\kappa(2^m B)}{2^{ms} \text{meas}_\kappa(B)} \leq C < \infty,$$

with the first supremum being taken over all spherical caps $B \subset \mathbb{S}^{d-1}$ with radius $\leq 2^{-m}$.

Given $0 < p \leq \infty$, we denote by $L^p(h_\kappa^2) \equiv L^p(h_\kappa^2; \mathbb{S}^{d-1})$ the Lebesgue L^p -space defined with respect to the measure $h_\kappa^2(x) d\sigma(x)$ on \mathbb{S}^{d-1} , and $\|\cdot\|_{\kappa,p}$ the L^p -norm of the space $L^p(h_\kappa^2; \mathbb{S}^{d-1})$. Denote Π_n^d be the space of all spherical polynomials of degree at most n on the sphere \mathbb{S}^{d-1} . Set $\Pi_{-1}^d = \{0\}$, and let $\mathcal{H}_n^d(h_\kappa^2)$ denote the orthogonal complement of the space Π_{n-1}^d in the Hilbert space $\Pi_n^d \subset L^2(h_\kappa^2)$ (relative to the norm of $L^2(h_\kappa^2)$). Then the $\mathcal{H}_n^d(h_\kappa^2)$, $n = 0, 1, \dots$ are mutually orthogonal, finite-dimensional linear subspaces of $L^2(h_\kappa^2)$. Denote by $P_n(h_\kappa^2)$ the reproducing kernel of the space $\mathcal{H}_n^d(h_\kappa^2)$; that is,

$$P_n(h_\kappa^2; x, y) := \sum_{j=1}^{a_n^d} Y_{n,j}^\kappa(x) \overline{Y_{n,j}^\kappa(y)}, \quad x, y \in \mathbb{S}^{d-1},$$

where $a_n^d = \dim \mathcal{H}_n^d(h_\kappa^2)$ and $\{Y_{n,j}^\kappa : j = 1, 2, \dots, a_n^d\}$ is an orthonormal basis of the space $\mathcal{H}_n^d(h_\kappa^2) \subset L^2(h_\kappa^2)$. Then the standard Hilbert space theory shows that each $f \in L^2(h_\kappa^2)$ can be represented as an orthogonal series converging in the norm of $L^2(h_\kappa^2)$:

$$f = \sum_{n=0}^\infty \text{proj}_n(h_\kappa^2; f), \tag{4}$$

where $\text{proj}_n(h_\kappa^2) : L^2(h_\kappa^2; \mathbb{S}^{d-1}) \mapsto \mathcal{H}_n^d(h_\kappa^2)$ is the orthogonal projection operator, which can be expressed as an integral operator,

$$\text{proj}_n(h_\kappa^2; f, x) = \omega_d^\kappa \int_{\mathbb{S}^{d-1}} f(y) P_n(h_\kappa^2; x, y) h_\kappa^2(y) d\sigma(y), \quad x \in \mathbb{S}^{d-1}. \tag{5}$$

Clearly, in the case of $h_\kappa(x) \equiv 1$, the orthogonal expansion in (4) coincides with the ordinary spherical harmonic expansion on \mathbb{S}^{d-1} .

Let $\Pi^d := \Pi(\mathbb{R}^d)$ be the linear space of algebraic polynomials on \mathbb{R}^d , and \mathbb{P}_n^d be the space of homogeneous polynomials of degree n on \mathbb{R}^d . One of the most important results in the Dunkl theory states that, associated with a reflection group G and multiplicity κ , there exists a unique linear operator $V_\kappa : \Pi^d \rightarrow \Pi^d$ called the Dunkl intertwining operator, such that:

$$V_\kappa(\mathbb{P}_n^d) = \mathbb{P}_n^d, \quad V_\kappa(1) = 1.$$

The intertwining operator V_κ commutes with the G -action; that is, $g^{-1} \circ V_\kappa \circ g = V_\kappa$ for all $g \in G$. Here and throughout, we use the notation $g \circ f(x) := f(gx)$ for $g \in G$, $f \in C(\mathbb{S}^{d-1})$, and $x \in \mathbb{S}^{d-1}$. An explicit “closed” form for the intertwining operator is known so far only in the case of $G = \mathbb{Z}_2^d$ (see [6,18]) and the case of $G = S_3$ (see [11,19]). At the moment, little information is known about the intertwining operator for general finite reflection groups other than \mathbb{Z}_2^d and S_3 , except the following important result of Rösler (see [8]).

Proposition 1 ([8] (Theorem 1.2 and Corollary 5.3)). *For every $x \in \mathbb{R}^d$ there exists a unique probability measure μ_x^κ on the Borel σ -algebra of \mathbb{R}^d such that:*

$$V_\kappa P(x) = \int_{\mathbb{R}^d} P(\xi) d\mu_x^\kappa(\xi), \quad P \in \Pi_n^d.$$

Furthermore, the representing measures μ_x^κ are compactly supported in the convex hull $\widehat{G}_x := \text{co}\{gx : g \in G\}$ of the orbit of x under G , and satisfy:

$$\mu_{rx}^\kappa(E) = \mu_x^\kappa(r^{-1}E), \quad \text{and} \quad \mu_{gx}^\kappa(E) = \mu_x^\kappa(g^{-1}E)$$

for all $r > 0, g \in G$ and each Borel subset E of \mathbb{R}^d .

In the theory of spherical h -harmonics, a crucial fact is that the reproducing kernel $P_n(h_\kappa^2; x, y)$ can be expressed in terms of the intertwining operator V_κ as (see [18] (Theorems 3.1 and 3.2)):

$$P_n(h_\kappa^2; x, y) = \frac{n + \lambda_\kappa}{\lambda_\kappa} V_\kappa \left[C_n^{\lambda_\kappa}(\langle x, \cdot \rangle) \right] (y), \quad x, y \in \mathbb{S}^{d-1} \tag{6}$$

with $\lambda_\kappa := \frac{d-2}{2} + \sum_{\nu \in R_+} \kappa_\nu$. By means of (5) and (6), the projection $\text{proj}_n(h_\kappa^2; f)$ can be extended to all $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$.

2.2. Jacobi Polynomials

We denote by $P_n^{(\alpha, \beta)}$ the usual Jacobi polynomial of degree n with indices α and β . According to [20] (4.21.2), we have:

$$P_n^{(\alpha, \beta)}(x) = \frac{1}{n!} \sum_{v=0}^n \binom{n}{v} (n + \alpha + \beta + 1) \cdots (n + \alpha + \beta + v) \times \\ \times (\alpha + v + 1) \cdots (\alpha + n) \left(\frac{x-1}{2} \right)^v,$$

where $x \in [-1, 1], n = 0, 1, \dots$, and the general coefficient,

$$\binom{n}{v} (n + \alpha + \beta + 1) \cdots (n + \alpha + \beta + v) (\alpha + v + 1) \cdots (\alpha + n),$$

has to be replaced by $(\alpha + 1)(\alpha + 2) \cdots (\alpha + n)$ for $v = 0$, and by $(n + \alpha + \beta + 1)(n + \alpha + \beta + 2) \cdots (2n + \alpha + \beta)$ for $v = n$. They are mutually orthogonal with respect to the weight function $\omega_{\alpha, \beta}(x) = (1 - x)^\alpha (1 + x)^\beta$ on $[-1, 1]$. In the case when $\alpha, \beta > -1$, we have the following well known estimate on the Jacobi polynomials [20] ((7.32.5) and (4.1.3)):

Lemma 1. *For an arbitrary real number α and $t \in [0, 1]$,*

$$|P_n^{(\alpha, \beta)}(t)| \leq cn^{-1/2} (1 - t + n^{-2})^{-(\alpha+1/2)/2}. \tag{7}$$

The estimate on $[-1, 0]$ follows from the fact that $P_n^{(\alpha, \beta)}(t) = (-1)^n P_n^{(\beta, \alpha)}(-t)$.

Next, we denote by C_n^λ the usual Gegenbauer polynomial of degree n with parameter $\lambda > -\frac{1}{2}$. As is well known, for $\alpha > -1$:

$$C_n^{\alpha+\frac{1}{2}}(x) = \frac{\Gamma(\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(n+\alpha+1)} P_n^{(\alpha,\alpha)}(x).$$

2.3. Doubling Weights on the Sphere

Given a weight function w on \mathbb{S}^{d-1} , we write $w(E) := \int_E w(x)d\sigma(x)$ for $E \subset \mathbb{S}^{d-1}$. A weight function ω on \mathbb{S}^{d-1} is called a doubling weight if there exists a constant $L > 0$ such that

$$w(2B) \leq Lw(B), \text{ for all spherical caps } B \subset \mathbb{S}^{d-1},$$

where the least constant L is called the doubling constant of w . The following lemma collects some useful properties on doubling weights [21]:

Proposition 2. *Let w be a doubling weight on \mathbb{S}^{d-1} with the doubling constant L . Then the following statements hold:*

1. *There exists a positive number s such that:*

$$w(2^m B) \leq C2^{ms}w(B), \forall m \in \mathbb{N}, \forall \text{ spherical caps } B \subset \mathbb{S}^{d-1}, \tag{8}$$

where the constant C is independent of m and B .

2. *For $0 < r < t$, and $x \in \mathbb{S}^{d-1}$,*

$$w(B_t(x)) \leq C\left(\frac{t}{r}\right)^s w(B_r(x)), \tag{9}$$

where s is a positive number satisfying (8).

3. *For $0 < r < \pi$, and $x, y \in \mathbb{S}^{d-1}$,*

$$w(B_r(x)) \leq C(1+r^{-1}\rho(x,y))^s w(B_r(y)), \tag{10}$$

where s is a positive number satisfying (8).

As we stated for Equations (2) and (3), the weight function

$$h_\kappa(x) := \prod_{\nu \in R_+} |\langle x, \nu \rangle|^{\kappa_\nu}$$

satisfies the doubling condition. Indeed, a slight modification of the proof in [22] (5.3) shows that for $r \in (0, \pi)$ and $x \in \mathbb{S}^{d-1}$,

$$\int_{B_r(x)} h_\kappa^2(x) d\sigma(x) \sim r^{d-1} \prod_{\nu \in R_+} (|\langle x, \nu \rangle| + r)^{2\kappa_\nu}.$$

Thus, for a spherical cap $B := B_\theta(x)$ with $x \in \mathbb{S}^{d-1}$ and $\theta \in (0, \pi)$,

$$\text{meas}_\kappa(B) \sim \theta^{d-1} \prod_{\nu \in R_+} (|\langle x, \nu \rangle| + \theta)^{\kappa_\nu}, \tag{11}$$

which, in particular, implies that h_κ^2 is a doubling weight on \mathbb{S}^{d-1} .

2.4. Weighted Christoffel Functions

The main tool in our study is the weighted Christoffel function defined for a weight function w on \mathbb{S}^{d-1} by

$$\lambda_n(w, x) := \inf_{P_n(x)=1} \int_{\mathbb{S}^{d-1}} |P_n(x)|^2 w(x) d\sigma(x), \quad n = 0, 1, 2, \dots,$$

where the infimum is taken over all spherical polynomials of degree n on \mathbb{S}^{d-1} that take the value 1 at the point $x \in \mathbb{S}^{d-1}$. The connection between weighted Christoffel functions and weighted orthogonal polynomial expansions can be seen in the following lemma.

Lemma 2. Let $P_{n,1}, \dots, P_{n,a_n}$ be an orthonormal basis of the space Π_n^d with respect to the inner product $\langle f, g \rangle_w := \int_{\mathbb{S}^{d-1}} f(x)g(x)w(x)d\sigma(x)$. Then,

$$\lambda_n(w, x) = \left(\sum_{j=1}^{a_n} |P_{n,j}(x)|^2 \right)^{-1}, \quad x \in \mathbb{S}^{d-1}.$$

Lemma 3. If w is a doubling weight on \mathbb{S}^{d-1} , then for $x \in \mathbb{S}^{d-1}$ and $n \in \mathbb{N}$,

$$\lambda_n(w, x) \sim \int_{B_{n^{-1}}(x)} w(y) d\sigma(y),$$

where the constant of equivalence is independent of x and n .

We will then deduce the following lemma and proposition, which generalize the results of [23] (Lemma 3.5 and Theorem 3.1).

Lemma 4. For each positive integer n , there exists a non-negative algebraic polynomial of degree n of the form:

$$P_n(t) = \sum_{j=0}^n c_{n,j} \frac{j + \lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa}(t), \quad t \in [-1, 1], \tag{12}$$

which satisfies that:

$$P_n(\cos \theta) \sim n^\beta (1 + n\theta)^{-\alpha}, \quad \theta \in (0, \pi), \tag{13}$$

and

$$\sup_j |c_{n,j}| = \begin{cases} n^{\beta-2\lambda_\kappa-1} & \text{if } \alpha > 2\lambda_\kappa + 1 \\ n^{\beta-2\lambda_\kappa-1} \log n & \text{if } \alpha = 2\lambda_\kappa + 1 \\ n^{\beta-\alpha} & \text{if } \alpha < 2\lambda_\kappa + 1. \end{cases} \tag{14}$$

Proof. We first prove (13). By Lemma 4.6 of [22], we can set $f(x) = n^\beta (1 + nd(x, e))^{-\alpha}$ where $e \in \mathbb{S}^{d-1}$ be a fixed point, and prove that such f satisfies [22] (Lemma 4.6, (4.10)). That is, there exists α' , such that:

$$(1 + nd(x, e))^{-\alpha} (1 + nd(y, e))^\alpha \leq (1 + nd(x, y))^{\alpha'}.$$

In fact, if $d(x, e) = d(y, e)$, then

$$\text{LHS} = (1 + nd(x, e))^{-\alpha} (1 + nd(y, e))^\alpha = 1 \leq (1 + nd(x, e))^{\alpha'}$$

for some $\alpha' \geq 0$. If $d(x, e) > d(y, e)$, we have

$$\text{LHS} = (1 + nd(x, e))^{-\alpha} (1 + nd(y, e))^\alpha \leq (1 + nd(x, e))^{-\alpha} (1 + nd(x, e))^\alpha = 1 \leq (1 + nd(x, e))^{\alpha'}$$

for some $\alpha' \geq 0$. Lastly, if $d(x, e) < d(y, e)$, we have

$$\begin{aligned} \text{LHS} &= \left(\frac{1 + nd(y, e)}{1 + nd(x, e)} \right)^\alpha = \left(\frac{1 + nd(x, e) + nd(y, e) - nd(x, e)}{1 + nd(x, e)} \right)^\alpha \\ &\leq \left(1 + \frac{nd(x, y)}{1 + nd(x, e)} \right)^\alpha \leq (1 + nd(x, y))^{\alpha'} \end{aligned}$$

for some $\alpha' \geq 0$. Then, by Lemma 4.6 of [22] and setting $p = 1, \theta = d(x, e)$, we can confirm there exists a non-negative algebraic polynomial $g = P_n(\cos \theta)$ such that $P_n(\cos \theta) \sim n^\beta(1 + n\theta)^{-\alpha}$. Let the ultraspherical polynomial expansion of P_n be given by (12). It remains to show (14). We apply the same argument in [23] (Lemma 3.5). Recall that

$$\|C_j^{\lambda_\kappa}\|_{2, \lambda_\kappa}^2 = \frac{c}{j + \lambda_\kappa} C_j^{\lambda_\kappa}(1) = \frac{c'}{j + \lambda_\kappa} \frac{\Gamma(2\lambda_\kappa + j)}{\Gamma(j + 1)} \sim j^{2\lambda_\kappa - 2},$$

and

$$\max_{\theta \in [0, \pi]} |C_j^{\lambda_\kappa}(\cos \theta)| = |C_j^{\lambda_\kappa}(1)| \sim j^{2\lambda_\kappa - 1},$$

here $\|\cdot\|_{2, \lambda_\kappa}$ denotes the L^2 -norm computed with respect to the measure $(1 - t^2)^{\lambda_\kappa - \frac{1}{2}}$ on $[-1, 1]$. By orthogonality of the ultraspherical polynomials, we have:

$$\begin{aligned} j^{2\lambda_\kappa - 1} |c_{n,j}| &\sim |c_{n,j}| \frac{j + \lambda_\kappa}{\lambda_\kappa} \|C_j^{\lambda_\kappa}\|_{2, \lambda_\kappa}^2 = c \left| \int_0^\pi P_n(\cos \theta) C_j^{\lambda_\kappa}(\cos \theta) (\sin \theta)^{2\lambda_\kappa} d\theta \right| \\ &\leq cn^\beta j^{2\lambda_\kappa - 1} \int_0^\pi (1 + n\theta)^{-\alpha} (\sin \theta)^{2\lambda_\kappa} d\theta. \end{aligned}$$

By the known fact that:

$$\int_0^\pi (1 + n\theta)^{-a} \theta^b d\theta \sim \begin{cases} n^{-a} & \text{if } a < b + 1 \\ n^{-b-1} & \text{if } a > b + 1 \\ n^{-a} \log n & \text{if } a = b + 1, \end{cases}$$

we then have

$$\sup_j |c_{n,j}| = \begin{cases} n^{\beta - 2\lambda_\kappa - 1} & \text{if } \alpha > 2\lambda_\kappa + 1 \\ n^{\beta - 2\lambda_\kappa - 1} \log n & \text{if } \alpha = 2\lambda_\kappa + 1 \\ n^{\beta - \alpha} & \text{if } \alpha < 2\lambda_\kappa + 1. \end{cases} \tag{15}$$

□

Proposition 3. Let $\Phi_n, n = 1, 2, \dots$, be a sequence of continuous functions on $[-1, 1]$ satisfying that:

$$|\Phi_n(\cos \theta)| \leq cn^\beta(1 + n\theta)^{-\alpha}. \tag{16}$$

Then we have for any $x, y \in \mathbb{S}^{d-1}$,

$$V_\kappa[\Phi_n(\langle y, \cdot \rangle)](x) \leq c \begin{cases} n^{\beta - 2\lambda_\kappa - 1} \frac{(1 + n\tilde{\rho}(x, y))^{-\alpha + \tau + \frac{s_\kappa}{2}}}{\text{meas}_\kappa(B_{n-1}(x))} & \text{if } \alpha > 2\lambda_\kappa + 1, \\ n^{\beta - 2\lambda_\kappa - 1} (\log n) \cdot \frac{(1 + n\tilde{\rho}(x, y))^{\frac{s_\kappa}{2}}}{\text{meas}_\kappa(B_{n-1}(x))} & \text{if } \alpha = 2\lambda_\kappa + 1 \\ n^{\beta - \alpha} \frac{(1 + n\tilde{\rho}(x, y))^{s_\kappa/2}}{\text{meas}_\kappa(B_{n-1}(x))} & \text{if } \alpha < 2\lambda_\kappa + 1, \end{cases}$$

where τ is a positive number satisfying $2\lambda_\kappa + 1 < \tau \leq \alpha$, and $\tilde{\rho}(x, y) = \min_{g \in G} \rho(gx, y)$ for $x, y \in \mathbb{S}^{d-1}$.

Proof. This is the analogue of [23] (Theorem 3.1). Using Proposition 1, we have

$$V_\kappa[\Phi_n(\langle y, \cdot \rangle)](x) = \int_{\widehat{G}_x} \Phi_n(\langle y, z \rangle) d\mu_x(z),$$

where \widehat{G}_x denotes the convex hull of the orbit $G_x := \{gx : g \in G\}$ of x under the group G . Since the group G has finite order, it follows that every element $z \in \widehat{G}_x$ can be written in the form $z = \sum_{g \in G} t_{g,z} gx$ for some $t_{g,z} \in [0, 1]$ satisfying $\sum_{g \in G} t_{g,z} = 1$. This implies that

$$\langle z, y \rangle = \sum_{g \in G} t_{g,z} \langle gx, y \rangle \leq \max_{g \in G} \langle gx, y \rangle, \quad \forall z \in \widehat{G}_x,$$

and

$$\rho(z, y) \geq \min_{g \in G} \rho(gx, y) =: \widetilde{\rho}(x, y), \quad \forall z \in \widehat{G}_x.$$

Thus, using (16), we deduce that:

$$V_\kappa[\Phi_n(\langle y, \cdot \rangle)](x) \leq V_\kappa \left[n^\beta (1 + n\rho(\cdot, y))^{-\alpha} \right](x) = \int_{\widehat{G}_x} n^\beta (1 + n\rho(z, y))^{-\alpha} d\mu_x(z). \quad (17)$$

(i) If $\alpha > 2\lambda_\kappa + 1$, we have

$$\int_{\widehat{G}_x} n^\beta (1 + n\rho(z, y))^{-\alpha} d\mu_x(z) \leq (1 + n\widetilde{\rho}(x, y))^{-\alpha + \tau} n^{\beta - 2\lambda_\kappa - 1} \int_{\widehat{G}_x} n^{2\lambda_\kappa + 1} (1 + n\rho(z, y))^{-\tau} d\mu_x(z).$$

Since $\tau > 2\lambda_\kappa + 1$, we can use the Lemma 4 and follow the same argument as in the proof of [23] (Theorem 3.1, (3.12)) to get

$$\int_{\widehat{G}_x} n^{2\lambda_\kappa + 1} (1 + n\rho(z, y))^{-\tau} d\mu_x(z) \leq c \frac{(1 + n\widetilde{\rho}(x, y))^{\frac{s_\kappa}{2}}}{\text{meas}_\kappa(B_{n^{-1}}(x))},$$

and thus

$$V_\kappa[\Phi_n(\langle y, \cdot \rangle)](x) \leq n^{\beta - 2\lambda_\kappa - 1} \frac{(1 + n\widetilde{\rho}(x, y))^{-\alpha + \tau + \frac{s_\kappa}{2}}}{\text{meas}_\kappa(B_{n^{-1}}(x))};$$

(ii) If $\alpha < 2\lambda_\kappa + 1$, we use the linearity of V_κ , Lemma 4, and the fact that

$$V_\kappa \left[\frac{\lambda_\kappa + j}{\lambda_\kappa} C_j^\lambda(\langle \cdot, y \rangle) \right](x) = \sum_{k=1}^{a_j} p_{j,k}(x) p_{j,k}(y), \quad x, y \in \mathbb{S}^{d-1},$$

where $\{p_{j,k}\}_{k=1}^{a_j}$ be an orthonormal basis of the space $\mathcal{H}_j^d(h_\kappa^2)$, to get:

$$\begin{aligned} \int_{\widehat{G}_x} n^\beta (1 + n\rho(z, y))^{-\alpha} d\mu_x(z) &= \int_{\widehat{G}_x} P_n(\langle y, z \rangle) d\mu_x(z) = V_\kappa \left[\sum_{j=0}^n c_{n,j} \frac{j + \lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa}(\langle y, \cdot \rangle) \right](x) \\ &= \sum_{j=0}^n c_{n,j} \frac{j + \lambda_\kappa}{\lambda_\kappa} V_\kappa \left[C_j^{\lambda_\kappa}(\langle y, \cdot \rangle) \right](x) = \sum_{j=0}^n \sum_{k=1}^{a_j} c_{n,j} p_{j,k}(x) p_{j,k}(y). \end{aligned}$$

Then by (17), Lemma 4, and Hölder’s inequality, we have:

$$\begin{aligned}
 V_\kappa[\Phi_n(\langle y, \cdot \rangle)](x) &\leq cn^{\beta-\alpha} \sum_{j=0}^n \sum_{k=1}^{a_j} |p_{j,k}(x)p_{j,k}(y)| \\
 &\leq cn^{\beta-\alpha} \left(\sum_{j=0}^n \sum_{k=1}^{a_j} p_{j,k}(x)^2 \right)^{1/2} \left(\sum_{j=0}^n \sum_{k=1}^{a_j} p_{j,k}(y)^2 \right)^{1/2} \\
 &\leq cn^{\beta-\alpha} \left(\int_{B_{n-1}(x)} h_\kappa^2(z) d\sigma(z) \right)^{-\frac{1}{2}} \left(\int_{B_{n-1}(y)} h_\kappa^2(z) d\sigma(z) \right)^{-\frac{1}{2}} \\
 &\leq cn^{\beta-\alpha} \frac{(1 + n\tilde{\rho}(x, y))^{s_\kappa/2}}{\text{meas}_\kappa(B_{n-1}(x))},
 \end{aligned}$$

where the last second inequality is followed by Lemmas 2 and 3. The last inequality followed by the fact the weight h_κ^2 is invariant under the group G , and by using (10), we have

$$\int_{B_{n-1}(x)} h_\kappa^2(z) d\sigma(z) = \int_{B_{n-1}(g_0x)} h_\kappa^2(z) d\sigma(z) \leq c(1 + n\rho(g_0x, y))^{s_\kappa} \int_{B_{n-1}(y)} h_\kappa^2(z) d\sigma(z),$$

where $g_0 \in G$ is such that $\rho(g_0x, y) = \tilde{\rho}(x, y) = \min_{g \in G} \rho(gx, y)$ for $x, y \in \mathbb{S}^{d-1}$;

(iii) If $\alpha = 2\lambda_\kappa + 1$, following the same idea as (ii), and using the Lemma 4, we have:

$$\begin{aligned}
 &\int_{\widehat{G}_x} n^\beta (1 + n\rho(z, y))^{-\alpha} d\mu_x(z) \\
 &= \int_{\widehat{G}_x} P_n(\langle y, z \rangle) d\mu_x(z) = V_\kappa \left[\sum_{j=0}^n c_{n,j} \frac{j + \lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa}(\langle y, \cdot \rangle) \right](x) \\
 &= \sum_{j=0}^n c_{n,j} \frac{j + \lambda_\kappa}{\lambda_\kappa} V_\kappa [C_j^{\lambda_\kappa}(\langle y, \cdot \rangle)](x) \\
 &\leq cn^{\beta-2\lambda_\kappa-1} (\log n) \sum_{j=0}^n \sum_{k=1}^{a_j} |p_{j,k}(x)p_{j,k}(y)| \\
 &\leq cn^{\beta-2\lambda_\kappa-1} (\log n) \left(\sum_{j=0}^n \sum_{k=1}^{a_j} p_{j,k}(x)^2 \right)^{1/2} \left(\sum_{j=0}^n \sum_{k=1}^{a_j} p_{j,k}(y)^2 \right)^{1/2} \\
 &\leq cn^{\beta-2\lambda_\kappa-1} (\log n) \left(\int_{B_{n-1}(x)} h_\kappa^2(z) d\sigma(z) \right)^{-\frac{1}{2}} \left(\int_{B_{n-1}(y)} h_\kappa^2(z) d\sigma(z) \right)^{-\frac{1}{2}} \\
 &\leq cn^{\beta-2\lambda_\kappa-1} (\log n) \frac{(1 + n\tilde{\rho}(x, y))^{s_\kappa/2}}{\text{meas}_\kappa(B_{n-1}(x))}.
 \end{aligned}$$

□

3. Main Results

We define the n -th Cesàro mean of order $\delta > 0$ of the WOPE (4) of f by:

$$S_n^\delta(h_\kappa^2; f, x) := \frac{1}{A_n^\delta} \sum_{j=0}^n A_{n-j}^\delta \text{proj}_j(h_\kappa^2; f, x), \quad x \in \mathbb{S}^{d-1},$$

where $A_j^\delta = \frac{\Gamma(j+\delta+1)}{\Gamma(j+1)\Gamma(\delta+1)}$ for $j = 0, 1, \dots$. According to (5), the Cesàro (C, δ) operators $S_n^\delta(h_\kappa^2)$ can be represented as:

$$S_n^\delta(h_\kappa^2; f, x) = \omega_d^\kappa \int_{\mathbb{S}^{d-1}} f(y) K_n^\delta(h_\kappa^2; x, y) h_\kappa^2(y) d\sigma(y), \quad x \in \mathbb{S}^{d-1},$$

where

$$K_n^\delta(h_\kappa^2; x, y) = \sum_{j=0}^n \frac{A_{n-j}^\delta}{A_n^\delta} P_j(h_\kappa^2; x, y), \quad x, y \in \mathbb{S}^{d-1}.$$

The main point-wise estimate of the Cesàro kernel is as follows:

Theorem 1. For $\delta > 0$ and $\ell \geq \tau > 2\lambda_\kappa + 1$, we have:

$$\begin{aligned} |K_n^\delta(h_\kappa^2; x, y)| \leq & c_\kappa \cdot \left[n^{-1} \sum_{j=1}^n (1+j)^{d-1} \cdot \frac{\prod_{v \in R_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}(x, y) + j^{-1})^{-2\kappa_v}}{(1 + j\tilde{\rho}(x, y))^{\lambda_\kappa + \ell - \tau - \frac{3}{2}s_\kappa + d - 1}} \right. \\ & + \sum_{i=2}^{\lfloor \log_2 n \rfloor + 2} 2^{-i - i\delta + i\ell} \frac{n^{d-1} (1 + n\tilde{\rho}(x, y))^{-\ell + \tau + \frac{3}{2}s_\kappa - d + 1}}{\prod_{v \in R_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}(x, y) + n^{-1})^{2\kappa_v}} \\ & \left. + \frac{n^{d-\delta-1} (1 + n\tilde{\rho}(x, y))^{\frac{3}{2}s_\kappa - d + 1}}{\prod_{v \in R_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + n^{-1} + \tilde{\rho}(x, y))^{2\kappa_v}} \right], \end{aligned}$$

where $g_0 \in G$ is such that $\rho(g_0 x, y) = \tilde{\rho}(x, y) = \min_{g \in G} \rho(gx, y)$ for $x, y \in \mathbb{S}^{d-1}$.

Our second main result can be stated as follows:

Theorem 2. Let $\delta > 0$ and $\tau > 2\lambda_\kappa + 1$. Then

$$\int_{\mathbb{S}^{d-1}} |K_n^\delta(h_\kappa^2; x, y)| h_\kappa^2(y) d\sigma(y) \leq \begin{cases} 1 & \text{if } \delta > \frac{3}{2}s_\kappa + \tau - 1 \\ \log n & \text{if } \delta = \frac{3}{2}s_\kappa + \tau - 1 \\ n^{-\delta + \frac{3}{2}s_\kappa - \tau + 1} & \text{if } \delta < \frac{3}{2}s_\kappa + \tau - 1. \end{cases}$$

As a consequence of the main kernel estimate, we can immediately obtain the following convergence result:

Corollary 1. Let $\tau > 2\lambda_\kappa + 1$ and $\sigma_\kappa := \frac{3}{2}s_\kappa + \tau - 1$. Then if $\delta > \sigma_\kappa$, $S_n^\delta(h_\kappa^2; f)$ converges in $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ for all $1 \leq p \leq \infty$.

In Section 5, we will also establish similar results for WOPEs on unit ball \mathbb{B}^d and the simplex \mathbb{T}^d , with weights being given by:

$$W_{\kappa, \mu}^B(x) := h_\kappa^2(x) (1 - \|x\|^2)^{\mu-1/2}, \quad \mu \geq 0, x \in \mathbb{B}^d,$$

and

$$W_{\kappa, \mu}^T(x) := \frac{h_\kappa^2(\sqrt{x_1}, \dots, \sqrt{x_d})}{\sqrt{x_1 \cdots x_d}} (1 - |x|)^{\mu-1/2}, \quad \mu \geq 0, x \in \mathbb{T}^d,$$

respectively.

4. Proofs of Main Theorems

In this section, we will give the proofs of Theorems 1 and 2. Our main references are [12,14,23].

4.1. Proof of Theorem 1

Let $\varphi_0 \in C^\infty[0, \infty)$ be such that $\chi_{[0,1]} \leq \varphi_0 \leq \chi_{[0,2]}$, where χ_E denotes the characteristic function of the interval E , and let $\varphi(t) := \varphi_0(t) - \varphi_0(2t)$. Clearly, φ is a C^∞ -function supported in $(\frac{1}{2}, 2)$ and satisfying $\sum_{i=0}^\infty \varphi(2^i t) = \varphi_0(t)$ for all $t > 0$. Thus, let $t = \frac{n-j}{n}$, we have $\sum_{i=0}^{\lfloor \log_2 n \rfloor + 2} \varphi\left(\frac{2^i(n-j)}{n}\right) = 1$ for $0 \leq j \leq n - 1$.

We decompose the Cesàro kernel as follows:

$$\begin{aligned}
 K_n^\delta(h_{\kappa}^2; x, y) &= \sum_{j=0}^n \frac{A_{n-j}^\delta}{A_n^\delta} \frac{\lambda_\kappa + j}{\lambda_\kappa} V_\kappa \left[C_j^{\lambda_\kappa}(\langle x, \cdot \rangle) \right] (y) \\
 &= \sum_{i=0}^{\lfloor \log_2 n \rfloor + 2} \sum_{j=0}^n \varphi \left(\frac{2^i(n-j)}{n} \right) \frac{A_{n-j}^\delta}{A_n^\delta} \frac{\lambda_\kappa + j}{\lambda_\kappa} V_\kappa \left[C_j^{\lambda_\kappa}(\langle x, \cdot \rangle) \right] (y) \\
 &\quad + \frac{1}{A_n^\delta} \frac{\lambda_\kappa + n}{\lambda_\kappa} V_\kappa \left[C_n^{\lambda_\kappa}(\langle x, \cdot \rangle) \right] (y) \tag{18}
 \end{aligned}$$

$$= \sum_{i=0}^{\lfloor \log_2 n \rfloor + 2} V_\kappa \left[\Psi_{\varphi_i}(\langle x, \cdot \rangle) \right] (y) + \frac{1}{A_n^\delta} \frac{\lambda_\kappa + n}{\lambda_\kappa} V_\kappa \left[C_n^{\lambda_\kappa}(\langle x, \cdot \rangle) \right] (y), \tag{19}$$

where

$$\Psi_{\varphi_i}(\cos \theta) = \sum_{j=0}^n \varphi_i \left(\frac{j}{2^i} \right) \frac{\lambda_\kappa + j}{\lambda_\kappa} C_j^{\lambda_\kappa}(\cos \theta), \text{ and } \varphi_i(t) = \varphi \left(\frac{2^i(n-2^i t)}{n} \right) \frac{A_{n-2^i t}^\delta}{A_n^\delta}.$$

To show the Theorem 1, we will consider three parts to estimate Equation (18):

Part 1: When $2 \leq i \leq \lfloor \log_2 n \rfloor + 2$;

Part 2: When $i = 0, 1$;

Part 3: The last term, i.e., the reproducing kernel.

We follow the proof of Lemma 3.3 of [24] (pp. 413–414). We shall use the following formula for Jacobi polynomials (see [20] (4.5.3)):

$$\begin{aligned}
 \sum_{n=0}^k \frac{(2n + \alpha + \beta + j + 1)\Gamma(n + \alpha + \beta + j + 1)}{\Gamma(n + \beta + 1)} P_n^{(\alpha+j, \beta)}(t) \\
 = \frac{\Gamma(k + \alpha + \beta + j + 2)}{\Gamma(k + \beta + 1)} P_k^{(\alpha+j+1, \beta)}(t), \tag{20}
 \end{aligned}$$

where $j = 0, 1, \dots$.

Define a sequence of functions $\{a_{n,v,\ell}(\cdot)\}_{\ell=0}^\infty$ recursively by:

$$\begin{aligned}
 a_{n,i,0}(j) &= 2(j + \lambda_\kappa) \varphi_i \left(\frac{j}{2^i} \right) = 2(j + \lambda_\kappa) \varphi \left(\frac{2^i(n-j)}{n} \right) \frac{A_{n-j}^\delta}{A_n^\delta}, \\
 a_{n,i,\ell+1}(j) &= \frac{a_{n,i,\ell}(j)}{2j + 2\lambda_\kappa + \ell} - \frac{a_{n,i,\ell}(j+1)}{2j + 2\lambda_\kappa + \ell + 2} \quad \ell \geq 0.
 \end{aligned}$$

Using (20) and summation by parts finite times, we have for any integer $\ell \geq 0$,

$$\Psi_{\varphi_i}(t) = \sum_{j=0}^n \varphi_i \left(\frac{j}{2^i} \right) \frac{\lambda_\kappa + j}{\lambda_\kappa} C_j^{\lambda_\kappa}(t) = c_\kappa \sum_{j=0}^\infty a_{n,i,\ell}(j) \frac{\Gamma(j + 2\lambda_\kappa + \ell)}{\Gamma(j + \lambda_\kappa + \frac{1}{2})} P_j^{(\lambda_\kappa + \ell - \frac{1}{2}, \lambda_\kappa - \frac{1}{2})}(t), \tag{21}$$

where $\lambda_\kappa = \frac{d-2}{2} + \sum_{v \in R_+} \kappa_v$.

Part 1: When $2 \leq i \leq \lfloor \log_2 n \rfloor + 2$

Note that $a_{n,i,\ell}(j) = 0$ if $j + \ell \leq (1 - \frac{1}{2^{i-1}})n$ or $j \geq (1 - \frac{1}{2^{i+1}})n$, so that the sum is over $j \sim n$. Furthermore, it follows from the definition, and Leibniz rule that

$$\left| \Delta^\ell \left(\varphi_i \left(\frac{j}{2^i} \right) \right) \right| \leq c 2^{-i\delta} \left(\frac{2^i}{n} \right)^\ell, \quad \forall \ell \in \mathbb{N}, 0 \leq j \leq n,$$

and

$$|\Delta^s a_{n,i,\ell}(j)| \leq c 2^{-i\delta} n^{-\ell+1} \left(\frac{2^i}{n}\right)^{s+\ell}, \quad s, \ell = 0, 1, \dots \tag{22}$$

Consequently, using (21) and (22) with $s = 0$, and the following well-known estimates on Jacobi polynomials [20] (7.32.6) for $k \geq 1$ and $\alpha, \beta > -\frac{1}{2}$,

$$P_k^{(\alpha,\beta)}(\cos \theta) \leq C_{\alpha,\beta} \begin{cases} \min\{k^\alpha, k^{-\frac{1}{2}}\theta^{-\alpha-\frac{1}{2}}\} & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\ \min\{k^\beta, k^{-\frac{1}{2}}(\pi - \theta)^{-\beta-\frac{1}{2}}\} & \text{if } \frac{\pi}{2} \leq \theta \leq \pi, \end{cases}$$

we obtain for $\theta \in [0, n^{-1}]$, and $\ell \geq 1$,

$$\begin{aligned} |\Psi_{\varphi_i}(\cos \theta)| &= |c_\kappa \sum_{j=0}^n a_{n,i,\ell}(j) \frac{\Gamma(j + 2\lambda_\kappa + \ell)}{\Gamma(j + \lambda_\kappa + \frac{1}{2})} P_j^{(\lambda_\kappa + \ell - \frac{1}{2}, \lambda_\kappa - \frac{1}{2})}(\cos \theta)| \\ &\leq c_\kappa \sum_{\substack{j \sim n \\ n-j \sim \frac{n}{2^i}}} 2^{-i\delta} n^{-\ell+1} \left(\frac{2^i}{n}\right)^\ell \cdot (j + \lambda_\kappa + \frac{1}{2})^{\lambda_\kappa + \ell - \frac{1}{2}} \cdot j^{\lambda_\kappa + \ell - \frac{1}{2}} \\ &\leq c_\kappa \cdot 2^{-i-i\delta+i\ell} \cdot n^{2\lambda_\kappa+1}. \end{aligned}$$

For $\theta \in [n^{-1}, \frac{\pi}{2}]$ and $\ell \geq 1$,

$$\begin{aligned} |\Psi_{\varphi_i}(\cos \theta)| &= |c_\kappa \sum_{j=0}^n a_{n,i,\ell}(j) \frac{\Gamma(j + 2\lambda_\kappa + \ell)}{\Gamma(j + \lambda_\kappa + \frac{1}{2})} P_j^{(\lambda_\kappa + \ell - \frac{1}{2}, \lambda_\kappa - \frac{1}{2})}(\cos \theta)| \\ &\leq c_\kappa \sum_{\substack{j \sim n \\ n-j \sim \frac{n}{2^i}}} 2^{-i\delta} n^{-\ell+1} \left(\frac{2^i}{n}\right)^\ell \cdot (j + \lambda_\kappa + \frac{1}{2})^{\lambda_\kappa + \ell - \frac{1}{2}} \cdot j^{-\frac{1}{2}} \theta^{-(\lambda_\kappa + \ell)} \\ &\leq c_\kappa \cdot 2^{-i-i\delta+i\ell} \cdot n^{2\lambda_\kappa+1} (n\theta)^{-(\lambda_\kappa + \ell)}. \end{aligned}$$

For $\theta \in [\frac{\pi}{2}, \pi]$ and $\ell \geq 1$,

$$\begin{aligned} |\Psi_{\varphi_i}(\cos \theta)| &= c_\kappa \sum_{j=0}^\infty a_{n,i,\ell}(j) \frac{\Gamma(j + 2\lambda_\kappa + \ell)}{\Gamma(j + \lambda_\kappa + \frac{1}{2})} P_j^{(\lambda_\kappa + \ell - \frac{1}{2}, \lambda_\kappa - \frac{1}{2})}(\cos \theta) \\ &\leq c_\kappa \sum_{\substack{j \sim n \\ n-j \sim \frac{n}{2^i}}} |a_{n,i,\ell}(j)| \cdot (j + \lambda_\kappa + \frac{1}{2})^{\lambda_\kappa + \ell - \frac{1}{2}} \cdot j^{\lambda_\kappa - \frac{1}{2}} \\ &\sim c_\kappa \cdot n^{-\ell+2\lambda_\kappa+1} \cdot 2^{-i-i\delta+i\ell} \\ &\sim c_\kappa \cdot 2^{-i-i\delta+i\ell} \cdot n^{2\lambda_\kappa+1} (n\theta)^{-\ell}. \end{aligned}$$

So, when $2 \leq i \leq \lfloor \log_2 n \rfloor + 2$, we have

$$|\Psi_{\varphi_i}(\cos \theta)| \leq c_\kappa \cdot 2^{-i-i\delta+i\ell} \cdot n^{2\lambda_\kappa+1} \min\{1, (n\theta)^{-\ell}\} \sim c_\kappa \cdot 2^{-i-i\delta+i\ell} \cdot n^{2\lambda_\kappa+1} (1 + n\theta)^{-\ell}.$$

Then applying the Proposition 3, we can get for $\ell \geq \tau > 2\lambda_\kappa + 1$,

$$V_\kappa[\Psi_{\varphi_i}(\langle x, \cdot \rangle)](y) \leq c_\kappa \cdot 2^{-i-i\delta+i\ell} \frac{(1 + n\tilde{\rho}(x, y))^{-\ell + \tau + s_\kappa/2}}{\text{meas}_\kappa(B_{n^{-1}}(x))}.$$

Following along the same arguments as in [23] (Theorem 3.1, p. 569), we let $m \in \mathbb{N}$ be such that $2^m n^{-1} \sim n^{-1} + \tilde{\rho}(x, y)$ and use the fact (9) to obtain

$$\text{meas}_\kappa(B_{n^{-1} + \tilde{\rho}(x,y)}(x)) \leq c 2^{ms_\kappa} \text{meas}_\kappa(B_{n^{-1}}(x)) \leq c(1 + n\tilde{\rho}(x, y))^{s_\kappa} \text{meas}_\kappa(B_{n^{-1}}(x)), \tag{23}$$

and by the equivalent,

$$\tilde{\rho}(x, y) + n^{-1} + |\langle x, v \rangle| \sim |\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}(x, y) + n^{-1}, \tag{24}$$

we then have

$$V_\kappa [\Psi_{\varphi_i}(\langle x, \cdot \rangle)](y) \leq c_\kappa \cdot 2^{-i-i\delta+il} \frac{n^{d-1}(1+n\tilde{\rho}(x, y))^{-\ell+\tau+\frac{3}{2}s_\kappa-d+1}}{\prod_{v \in R_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}(x, y) + n^{-1})^{2\kappa_v}}. \tag{25}$$

Thus,

$$\sum_{i=2}^{\lfloor \log_2 n \rfloor + 2} V_\kappa [\Psi_{\varphi_i}(\langle x, \cdot \rangle)](y) \leq c_\kappa \sum_{i=2}^{\lfloor \log_2 n \rfloor + 2} 2^{-i-i\delta+il} \frac{n^{d-1}(1+n\tilde{\rho}(x, y))^{-\ell+\tau+\frac{3}{2}s_\kappa-d+1}}{\prod_{v \in R_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}(x, y) + n^{-1})^{2\kappa_v}}. \tag{26}$$

Part 2: When $i = 0, 1$

Next, we deal with the cases of $i = 0, 1$.

The proof is very similar to that of Part 1. The difference here comes from the fact that the coefficients $\varphi_i(\frac{j}{2^i})$ for $i = 0, 1$ are supported in $0 \leq j \leq \frac{3}{4}n$ rather than $\frac{n}{2} \leq j \leq n$. Indeed, for the case of $i = 0, 1$, we have to replace the estimates (22) by:

$$\left| \Delta^k a_{n,i,\ell}(j) \right| \leq \begin{cases} cn^{-k-1}, & \text{if } \ell = 1, \\ cn^{-1}(j+1)^{-k-2\ell+2}, & \text{if } \ell \geq 2. \end{cases} \tag{27}$$

Using (21), we obtain that for $i = 0, 1$ and any $\ell \geq 2$,

$$V_\kappa [\Psi_{\varphi_i}(\langle x, \cdot \rangle)](y) = c_\kappa \sum_{j=0}^{\frac{3}{4}n} a_{n,i,\ell}(j) \frac{\Gamma(j+2\lambda_\kappa+\ell)}{\Gamma(j+\lambda_\kappa+\frac{1}{2})} V_\kappa \left[P_j^{(\lambda_\kappa+\ell-\frac{1}{2}, \lambda_\kappa-\frac{1}{2})}(\langle x, \cdot \rangle) \right](y). \tag{28}$$

By using Equation (7) with $t = \cos \theta$, $\alpha = \lambda_\kappa + \ell - \frac{1}{2}$, and $\beta = \lambda_\kappa - \frac{1}{2}$, we have

$$\left| P_n^{(\lambda_\kappa+\ell-\frac{1}{2}, \lambda_\kappa-\frac{1}{2})}(\cos \theta) \right| \leq cn^{\lambda_\kappa+\ell-\frac{1}{2}}(1+n\theta)^{-(\lambda_\kappa+\ell)}. \tag{29}$$

Then, to estimate $V_\kappa \left[P_j^{(\lambda_\kappa+\ell-\frac{1}{2}, \lambda_\kappa-\frac{1}{2})}(\langle x, \cdot \rangle) \right](y)$ on the right-hand side of (28), by (29) and Proposition 3, we have for $\ell \geq \tau > 2\lambda_\kappa + 1$,

$$V_\kappa \left[P_j^{(\lambda_\kappa+\ell-\frac{1}{2}, \lambda_\kappa-\frac{1}{2})}(\langle x, \cdot \rangle) \right](y) \leq j^{\lambda_\kappa+\ell-\frac{1}{2}-2\lambda_\kappa-1} \frac{(1+j\tilde{\rho}(x, y))^{-(\lambda_\kappa+\ell)+\tau+s_\kappa/2}}{\text{meas}_\kappa(B_{j^{-1}}(x))}.$$

We then use a similar argument of the proof in (25) to obtain that:

$$\begin{aligned} V_\kappa \left[P_j^{(\lambda_\kappa+\ell-\frac{1}{2}, \lambda_\kappa-\frac{1}{2})}(\langle x, \cdot \rangle) \right](y) &\leq j^{\ell-\lambda_\kappa-\frac{3}{2}} \frac{j^{d-1}(1+j\tilde{\rho}(x, y))^{-(\lambda_\kappa+\ell)+\tau+\frac{3}{2}s_\kappa-d+1}}{\prod_{v \in R_+} (|\langle x, v \rangle| + \tilde{\rho}(x, y) + j^{-1})^{2\kappa_v}} \\ &\sim j^{\ell-\lambda_\kappa-\frac{3}{2}} \frac{j^{d-1}(1+j\tilde{\rho}(x, y))^{-(\lambda_\kappa+\ell)+\tau+\frac{3}{2}s_\kappa-d+1}}{\prod_{v \in R_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}(x, y) + j^{-1})^{2\kappa_v}}. \end{aligned} \tag{30}$$

Consequently, using (27), (28) and (30), we conclude that for $i = 0, 1$, and a large ℓ ,

$$\begin{aligned}
 V_\kappa[\Psi_{\varphi_i}(\langle x, \cdot \rangle)](y) &= c_\kappa \sum_{j=0}^{\frac{3}{4}n} a_{n,i,\ell}(j) \frac{\Gamma(j + 2\lambda_\kappa + \ell)}{\Gamma(j + \lambda_\kappa + \frac{1}{2})} V_\kappa \left[P_j^{(\lambda_\kappa + \ell - \frac{1}{2}, \lambda_\kappa - \frac{1}{2})}(\langle x, \cdot \rangle) \right](y) \\
 &\leq c_\kappa n^{-1} \sum_{j=1}^n (1+j)^{-2\ell+2} \cdot j^{\lambda_\kappa + \ell - \frac{1}{2}} \cdot j^{-\lambda_\kappa + \ell - \frac{3}{2}} \cdot j^{d-1} \frac{\prod_{\nu \in R_+} (|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + \tilde{\rho}(x, y) + j^{-1})^{-2\kappa_\nu}}{(1 + j\tilde{\rho}(x, y))^{\lambda_\kappa + \ell - \tau - \frac{3}{2}s_\kappa + d - 1}} \\
 &\leq c_\kappa n^{-1} \sum_{j=1}^n (1+j)^{d-1} \cdot \frac{\prod_{\nu \in R_+} (|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + \tilde{\rho}(x, y) + j^{-1})^{-2\kappa_\nu}}{(1 + j\tilde{\rho}(x, y))^{\lambda_\kappa + \ell - \tau - \frac{3}{2}s_\kappa + d - 1}}. \tag{31}
 \end{aligned}$$

Part 3: The last term(reproducing kernel)

Finally, we consider the last term of (18):

$$\frac{1}{A_n^\delta} \frac{\lambda_\kappa + n}{\lambda_\kappa} V_\kappa \left[C_n^{\lambda_\kappa}(\langle x, \cdot \rangle) \right](y),$$

that is, the reproducing kernel $\frac{1}{A_n^\delta} \cdot P_n(h_\kappa^2; x, y)$. Similarly, let $p_{j,1}, \dots, p_{j,a_j}$ be an orthonormal basis of the space $\mathcal{H}_j^d(h_\kappa^2)$ with respect to the inner product of $L^2(h_\kappa^2; \mathbb{S}^{d-1})$. Then, we have

$$\sum_{k=1}^{a_j} p_{j,k}(x)p_{j,k}(y) = \frac{\lambda_\kappa + j}{\lambda_\kappa} V_\kappa \left[C_j^\lambda(\langle \cdot, y \rangle) \right](x), \quad x, y \in \mathbb{S}^{d-1}.$$

Thus,

$$\begin{aligned}
 P_n(h_\kappa^2; x, y) &= \frac{\lambda_\kappa + n}{\lambda_\kappa} V_\kappa \left[C_n^\lambda(\langle \cdot, y \rangle) \right](x) = \sum_{k=1}^{a_n} p_{n,k}(x)p_{n,k}(y) \leq \left(\sum_{m=0}^{n-1} \sum_{k=1}^{a_m} + \sum_{k=1}^{a_n} \right) p_{n,k}(x)p_{n,k}(y) \\
 &= \sum_{m=0}^n \sum_{k=1}^{a_m} p_{n,k}(x)p_{n,k}(y).
 \end{aligned}$$

By the Hölder’s inequality, Lemmas 2 and 3, (11) and (23), and the equivalent (24), we have:

$$\begin{aligned}
 |P_n(h_\kappa^2; x, y)| &\leq \sum_{m=0}^n \sum_{k=1}^{a_m} |p_{n,k}(x)p_{n,k}(y)| \leq \left(\sum_{m=0}^n \sum_{k=1}^{a_m} |p_{n,k}(x)|^2 \right)^{\frac{1}{2}} \left(\sum_{m=0}^n \sum_{k=1}^{a_m} |p_{n,k}(y)|^2 \right)^{\frac{1}{2}} \\
 &\leq c \left(\int_{B_{n-1}(x)} h_\kappa^2(z) d\sigma(z) \right)^{-\frac{1}{2}} \left(\int_{B_{n-1}(y)} h_\kappa^2(z) d\sigma(z) \right)^{-\frac{1}{2}} \\
 &\leq c(1 + n\tilde{\rho}(x, y))^{s_\kappa/2} \left(\int_{B_{n-1}(x)} h_\kappa^2(z) d\sigma(z) \right)^{-1} \\
 &\leq \frac{(1 + n\tilde{\rho}(x, y))^{s_\kappa/2}}{(1 + n\tilde{\rho}(x, y))^{-s_\kappa} \text{meas}_\kappa(B_{n-1+\tilde{\rho}(x,y)}(x))} = \frac{(1 + n\tilde{\rho}(x, y))^{\frac{3}{2}s_\kappa}}{\text{meas}_\kappa(B_{n-1+\tilde{\rho}(x,y)}(x))} \\
 &\sim \frac{(1 + n\tilde{\rho}(x, y))^{\frac{3}{2}s_\kappa}}{(n^{-1} + \tilde{\rho}(x, y))^{d-1} \prod_{\nu \in R_+} (|\langle x, \nu \rangle| + n^{-1} + \tilde{\rho}(x, y))^{2\kappa_\nu}} \\
 &= \frac{n^{d-1} (1 + n\tilde{\rho}(x, y))^{\frac{3}{2}s_\kappa - d + 1}}{\prod_{\nu \in R_+} (|\langle x, \nu \rangle| + n^{-1} + \tilde{\rho}(x, y))^{2\kappa_\nu}} \\
 &\sim \frac{n^{d-1} (1 + n\tilde{\rho}(x, y))^{\frac{3}{2}s_\kappa - d + 1}}{\prod_{\nu \in R_+} (|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + n^{-1} + \tilde{\rho}(x, y))^{2\kappa_\nu}}.
 \end{aligned}$$

So,

$$\frac{1}{A_n^\delta} |P_n(h_{\kappa}^2; x, y)| \leq c \frac{n^{d-\delta-1} (1 + n\tilde{\rho}(x, y))^{\frac{3}{2}s_{\kappa}-d+1}}{\prod_{\nu \in R_+} (|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + n^{-1} + \tilde{\rho}(x, y))^{2\kappa_{\nu}}}. \tag{32}$$

Finally, to estimate the Cesàro kernel, by using the (26), (31) and (32), we have:

$$\begin{aligned} |K_n^\delta(h_{\kappa}^2; x, y)| &= \left| \left(\sum_{i=0,1} + \sum_{i=2}^{[\log_2 n]+2} \right) V_{\kappa} [\Psi_{\varphi_i}(\langle x, \cdot \rangle)](y) + \frac{1}{A_n^\delta} \frac{\lambda_{\kappa} + n}{\lambda_{\kappa}} V_{\kappa} [C_n^{\lambda_{\kappa}}(\langle x, \cdot \rangle)](y) \right| \\ &\leq c_{\kappa} \cdot \left[n^{-1} \sum_{j=1}^n (1+j)^{d-1} \cdot \frac{\prod_{\nu \in R_+} (|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + \tilde{\rho}(x, y) + j^{-1})^{-2\kappa_{\nu}}}{(1 + j\tilde{\rho}(x, y))^{\lambda_{\kappa} + \ell - \tau - \frac{3}{2}s_{\kappa} + d - 1}} \right. \\ &\quad + \sum_{i=2}^{[\log_2 n]+2} 2^{-i-i\delta+il} \frac{n^{d-1} (1 + n\tilde{\rho}(x, y))^{-\ell + \tau + \frac{3}{2}s_{\kappa} - d + 1}}{\prod_{\nu \in R_+} (|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + \tilde{\rho}(x, y) + n^{-1})^{2\kappa_{\nu}}} \\ &\quad \left. + \frac{n^{d-\delta-1} (1 + n\tilde{\rho}(x, y))^{\frac{3}{2}s_{\kappa}-d+1}}{\prod_{\nu \in R_+} (|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + n^{-1} + \tilde{\rho}(x, y))^{2\kappa_{\nu}}} \right]. \end{aligned}$$

4.2. Proof of Theorem 2

To show this Theorem, we will estimate $|K_n^\delta(h_{\kappa}^2; x, y)| h_{\kappa}^2(y)$, where $h_{\kappa}(y) := \prod_{\nu \in R_+} |\langle y, \nu \rangle|^{\kappa_{\nu}}$, and then give the upper estimate of the integral $\int_{\mathbb{S}^{d-1}} |K_n^\delta(h_{\kappa}^2; x, y)| h_{\kappa}^2(y) d\sigma(y)$. By Theorem 1 we proved,

$$\begin{aligned} &|K_n^\delta(h_{\kappa}^2; x, y)| h_{\kappa}^2(y) \\ &\leq c_{\kappa} \cdot \left[n^{-1} \sum_{j=1}^n (1+j)^{d-1} \cdot \frac{\prod_{\nu \in R_+} (|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + \tilde{\rho}(x, y) + j^{-1})^{-2\kappa_{\nu}}}{(1 + j\tilde{\rho}(x, y))^{\lambda_{\kappa} + \ell - \tau - \frac{3}{2}s_{\kappa} + d - 1}} \prod_{\nu \in R_+} |\langle y, \nu \rangle|^{2\kappa_{\nu}} \right. \\ &\quad + \sum_{i=2}^{[\log_2 n]+2} 2^{-i-i\delta+il} \frac{n^{d-1} (1 + n\tilde{\rho}(x, y))^{-\ell + \tau + \frac{3}{2}s_{\kappa} - d + 1}}{\prod_{\nu \in R_+} (|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + \tilde{\rho}(x, y) + n^{-1})^{2\kappa_{\nu}}} \prod_{\nu \in R_+} |\langle y, \nu \rangle|^{2\kappa_{\nu}} \\ &\quad \left. + \frac{n^{d-\delta-1} (1 + n\tilde{\rho}(x, y))^{\frac{3}{2}s_{\kappa}-d+1}}{\prod_{\nu \in R_+} (|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + n^{-1} + \tilde{\rho}(x, y))^{2\kappa_{\nu}}} \prod_{\nu \in R_+} |\langle y, \nu \rangle|^{2\kappa_{\nu}} \right] \\ &=: c_{\kappa} \cdot [A_1 + A_2 + A_3]. \tag{33} \end{aligned}$$

Let

$$\begin{aligned} I_{\nu} &:= \left(|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + \tilde{\rho}(x, y) + n^{-1} \right)^{-2\kappa_{\nu}} |\langle y, \nu \rangle|^{2\kappa_{\nu}} \\ &= \left(|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + \tilde{\rho}(x, y) + n^{-1} \right)^{-2\kappa_{\nu}} |\langle g_0 y, \nu \rangle|^{2\kappa_{\nu}}, \end{aligned}$$

where the second equality is followed by the G-invariant of the weight. Thus, we have

$$\prod_{\nu \in R_+} I_{\nu} \leq \text{Const}. \tag{34}$$

Hence, by (34) and a straightforward calculation, we have:

$$\begin{aligned}
 A_1 &:= n^{-1} \sum_{j=1}^n (1+j)^{d-1} \cdot \frac{\prod_{v \in R_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}(x, y) + j^{-1})^{-2\kappa_v}}{(1 + j\tilde{\rho}(x, y))^{\lambda_\kappa + \ell - \tau - \frac{3}{2}s_\kappa + d - 1}} \prod_{v \in R_+} |\langle y, v \rangle|^{2\kappa_v} \\
 &\leq c_\kappa n^{-1} \sum_{j=1}^n j^{d-1} \cdot \frac{1}{(1 + j\tilde{\rho}(x, y))^{\lambda_\kappa + \ell - \tau - \frac{3}{2}s_\kappa + d - 1}} \\
 &\leq c'_\kappa n^{-1} (\tilde{\rho}(x, y) + n^{-1})^{-d} \\
 &= c_\kappa \frac{n^{d-1}}{(1 + n\tilde{\rho}(x, y))^d}. \tag{35}
 \end{aligned}$$

For A_2 , we will consider two cases below to give the estimate:

(1) If $0 \leq \tilde{\rho}(x, y) \leq n^{-1}$, then by (34),

$$\begin{aligned}
 A_2 &= \sum_{i=2}^{\lfloor \log_2 n \rfloor + 2} 2^{-i - i\delta + i\ell} \frac{n^{d-1} (1 + n\tilde{\rho}(x, y))^{-\ell + \tau + \frac{3}{2}s_\kappa - d + 1}}{\prod_{v \in R_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}(x, y) + n^{-1})^{2\kappa_v}} \prod_{v \in R_+} |\langle y, v \rangle|^{2\kappa_v} \\
 &\leq cn^{d-1} \frac{1}{(1 + n\tilde{\rho}(x, y))^{\ell - \tau - \frac{3}{2}s_\kappa + d - 1}} \sum_{i=2}^{\lfloor \log_2 n \rfloor + 2} 2^{-i - i\delta + i\ell} \\
 &\leq c_{\ell, \kappa, d} \cdot n^{d-1} \frac{1}{(1 + n\tilde{\rho}(x, y))^{\delta - \tau - \frac{3}{2}s_\kappa + d - 1}} \sum_{i=2}^{\lfloor \log_2 n \rfloor + 2} 2^{-i} \quad [\text{by setting } \ell = \delta] \\
 &\leq c_{\ell, \kappa, d} \cdot n^{d-1} \\
 &\sim c'_{\ell, \kappa, d} \cdot n^{d-1} \frac{1}{(1 + n\tilde{\rho}(x, y))^{\delta + \frac{d}{2}}};
 \end{aligned}$$

(2) If $\tilde{\rho}(x, y) \geq n^{-1}$, then we break the sum by $A_2 = \sum_{i=2}^{\lfloor \log_2 n \rfloor + 2} = \sum_{2^i \leq n\tilde{\rho}} + \sum_{2^i > n\tilde{\rho}}$. For large enough ℓ and δ , we apply (34) to obtain:

$$\begin{aligned}
 &\sum_{2^i \leq n\tilde{\rho}} 2^{-i - i\delta + i\ell} \frac{n^{d-1} (1 + n\tilde{\rho}(x, y))^{-\ell + \tau + \frac{3}{2}s_\kappa - d + 1}}{\prod_{v \in R_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}(x, y) + n^{-1})^{2\kappa_v}} \prod_{v \in R_+} |\langle y, v \rangle|^{2\kappa_v} \\
 &\leq cn^{d-1} (n\tilde{\rho}(x, y))^{-\ell + \tau + \frac{3}{2}s_\kappa - d + 1} \sum_{2^i \leq n\tilde{\rho}} 2^{i(\ell - \delta - 1)} \\
 &\leq cn^{d-1} (n\tilde{\rho}(x, y))^{-\ell + \tau + \frac{3}{2}s_\kappa - d + 1} (n\tilde{\rho}(x, y))^{(\ell - \delta - 1)} \quad [\text{if } \ell > \delta + 1] \\
 &\leq cn^{d-1} \frac{1}{(1 + n\tilde{\rho}(x, y))^{\delta + d - \tau - \frac{3}{2}s_\kappa}},
 \end{aligned}$$

and

$$\begin{aligned}
 &\sum_{2^i > n\tilde{\rho}} 2^{-i - i\delta + i\ell} \frac{n^{d-1} (1 + n\tilde{\rho}(x, y))^{-\ell + \tau + \frac{3}{2}s_\kappa - d + 1}}{\prod_{v \in R_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}(x, y) + n^{-1})^{2\kappa_v}} \prod_{v \in R_+} |\langle y, v \rangle|^{2\kappa_v} \\
 &\leq cn^{d-1} (n\tilde{\rho}(x, y))^{-\ell + \tau + \frac{3}{2}s_\kappa - d + 1} \sum_{2^i > n\tilde{\rho}} 2^{i(\ell - \delta - 1)} \\
 &\leq cn^{d-1} (n\tilde{\rho}(x, y))^{-\delta + \tau + \frac{3}{2}s_\kappa - d + 1} (n\tilde{\rho}(x, y))^{-1} \quad [\text{by setting } \ell = \delta] \\
 &\leq cn^{d-1} \frac{1}{(1 + n\tilde{\rho}(x, y))^{\delta + d - \tau - \frac{3}{2}s_\kappa}}.
 \end{aligned}$$

So,

$$A_2 \leq c_\kappa n^{d-1} \frac{1}{(1 + n\tilde{\rho}(x, y))^{\delta + d - \tau - \frac{3}{2}s_\kappa}}. \tag{36}$$

Lastly, for A_3 , by using (34), we have:

$$\begin{aligned}
 A_3 &= \frac{n^{d-\delta-1}(1+n\tilde{\rho}(x,y))^{\frac{3}{2}s_\kappa-d+1}}{\prod_{\nu \in R_+} (|\langle x, \nu \rangle| + |\langle g_0 y, \nu \rangle| + n^{-1} + \tilde{\rho}(x,y))^{2\kappa_\nu}} \prod_{\nu \in R_+} |\langle y, \nu \rangle|^{2\kappa_\nu} \\
 &\leq cn^{d-\delta-1} \frac{1}{(1+n\tilde{\rho}(x,y))^{d-\frac{3}{2}s_\kappa-1}} \\
 &= cn^{d-1}(n^{-1} + \tilde{\rho}(x,y))^\delta \frac{1}{(1+n\tilde{\rho}(x,y))^{\delta+d-\frac{3}{2}s_\kappa-1}} \\
 &\leq cn^{d-1} \frac{1}{(1+n\tilde{\rho}(x,y))^{\delta+d-\frac{3}{2}s_\kappa-1}}. \tag{37}
 \end{aligned}$$

Therefore, by (35)–(37), we have:

$$\begin{aligned}
 |K_n^\delta(h_\kappa^2; x, y)|h_\kappa^2(y) &\leq c_\kappa[A_1 + A_2 + A_3] \\
 &\leq c_\kappa n^{d-1}(1+n\tilde{\rho}(x,y))^{-\beta(\delta)},
 \end{aligned}$$

where

$$\beta(\delta) := \min\{d, \delta + d - \tau - \frac{3}{2}s_\kappa, \delta - \frac{3}{2}s_\kappa + d - 1\} = \min\{d, \delta + d - \tau - \frac{3}{2}s_\kappa\}.$$

Thus, we can get:

$$\begin{aligned}
 \int_{\mathbb{S}^{d-1}} |K_n^\delta(h_\kappa^2; x, y)|h_\kappa^2(y)d\sigma(y) &\leq cn^{d-1} \int_0^{\frac{\pi}{2}} (1+n\theta)^{-\beta(\delta)} (\sin \theta)^{d-2}d\theta \\
 &\sim \begin{cases} 1 & \text{if } \delta > \frac{3}{2}s_\kappa + \tau - 1 \\ \log n & \text{if } \delta = \frac{3}{2}s_\kappa + \tau - 1 \\ n^{-\delta+\frac{3}{2}s_\kappa-\tau+1} & \text{if } \delta < \frac{3}{2}s_\kappa + \tau - 1. \end{cases}
 \end{aligned}$$

5. Weighted Orthogonal Polynomial Expansions (WOPEs) on the Ball and the Simplex

In this section, we shall describe briefly some necessary notations and results for WOPEs on the unit ball \mathbb{B}^d and the simplex \mathbb{T}^d . Unless otherwise stated, most of the results described in this section can be found in the paper [25] and the books [9,21].

5.1. WOPEs in Several Variables

Let Ω denote a compact domain in \mathbb{R}^d endowed with the usual Lebesgue measure dx . Given a weight function W on Ω , we denote by $L^p(W; \Omega)$ the usual L^p -space defined with respect to the measure Wdx on Ω , and $\mathcal{V}_n^d(W)$ the space of orthogonal polynomials of degree n with respect to the weight function W on Ω . Thus, if we denote by Π_n^d the space of all algebraic polynomials in d variables of total degree at most n , then $\mathcal{V}_n^d(W)$ is the orthogonal complement of Π_{n-1}^d in the space Π_n^d with respect to the inner product of $L^2(W; \Omega)$, where it is agreed that $\Pi_{-1}^d = \{0\}$.

Since Ω is compact, each function $f \in L^2(W; \Omega)$ has a weighted orthogonal polynomial expansion on Ω , $f = \sum_{n=0}^\infty \text{proj}_n(W; f)$, converging in the norm of $L^2(W; \Omega)$, where $\text{proj}_n(W; f)$ denotes the orthogonal projection of f onto the space $\mathcal{V}_n^d(W)$. Let $P_n(W; \cdot, \cdot)$ denote the reproducing kernel of the space $\mathcal{V}_n^d(W)$; that is,

$$P_n(W; x, y) = \sum_{j=1}^{a_n^d} \varphi_{n,j}(x) \overline{\varphi_{n,j}(y)}, \quad x, y \in \Omega$$

for an orthonormal basis $\{\varphi_{n,j} : 1 \leq j \leq a_n^d := \dim \mathcal{V}_n^d(W)\}$ of the space $\mathcal{V}_n^d(W)$. The orthogonal projection operator $\text{proj}_n(W) : L^2(W; \Omega) \mapsto \mathcal{V}_n^d(W)$ can be expressed as an integral operator,

$$\text{proj}_n(W; f, x) = \int_{\Omega} f(y) P_n(W; x, y) W(y) dy, \quad x \in \Omega,$$

which also extends the definition of $\text{proj}_n(W; f)$ to all $f \in L^1(W; \Omega)$ since the kernel $P_n(W; x, y)$ is a polynomial in both x and y .

Let $S_n^\delta(W; f), n = 0, 1, \dots$, denote the Cesàro (C, δ) -means of the WOPEs of $f \in L^1(W; \Omega)$. Each $S_n^\delta(W; f)$ can be expressed as an integral against a kernel, $K_n^\delta(W; x, y)$, called the Cesàro (C, δ) -kernel,

$$S_n^\delta(W; f, x) := \int_{\Omega} f(y) K_n^\delta(W; x, y) W(y) dy, \quad x \in \Omega$$

where

$$K_n^\delta(W; x, y) := \frac{1}{A_n^\delta} \sum_{j=0}^n A_{n-j}^\delta P_j(W; x, y), \quad x, y \in \Omega.$$

5.2. WOPEs on the Unit Ball \mathbb{B}^d

Recall that G is a finite reflection group on \mathbb{R}^d with a root system $R \subset \mathbb{R}^d; \kappa : R \rightarrow [0, \infty)$ is a nonnegative multiplicity function on R ; the weight functions h_κ on \mathbb{S}^{d-1} and $W_{\kappa, \mu}^B$ on \mathbb{B}^d are given by

$$h_\kappa(x) := \prod_{\nu \in R_+} |\langle x, \nu \rangle|^{\kappa_\nu}$$

and

$$W_{\kappa, \mu}^B(x) := h_\kappa^2(x) (1 - \|x\|^2)^{\mu - \frac{1}{2}}, \quad \mu \geq 0, x \in \mathbb{B}^d, \tag{38}$$

respectively. For $1 \leq p \leq \infty$, we denote by $L^p(W_{\kappa, \mu}^B; \mathbb{B}^d)$ the L^p -space defined with respect to the measure $W_{\kappa, \mu}^B(x) dx$ on \mathbb{B}^d , and $\|\cdot\|_{L^p(W_{\kappa, \mu}^B)}$ the norm of $L^p(W_{\kappa, \mu}^B; \mathbb{B}^d)$. Let \tilde{G} be the finite reflection group on \mathbb{R}^{d+1} associated with the root system,

$$\tilde{R} := \left\{ \tilde{\nu} = (\nu, 0) \in \mathbb{R}^{d+1} : \nu \in R \right\} \cup \{ \pm e_{d+1} \},$$

and define $\tilde{\kappa} : \tilde{R} \rightarrow [0, \infty)$ by $\tilde{\kappa}(\tilde{\nu}) = \kappa(\nu)$ for $\nu \in R$ and $\tilde{\kappa}(\pm e_{d+1}) = \mu$. Clearly, $\tilde{\kappa}$ is a \tilde{G} -invariant nonnegative multiplicity function on \tilde{R} . Let $h_{\tilde{\kappa}}$ be the \tilde{G} -invariant weight function on \mathbb{R}^{d+1} associated with the root system \tilde{R} and the multiplicity function $\tilde{\kappa}$ as defined in (1); that is,

$$h_{\tilde{\kappa}}(x, x_{d+1}) = |x_{d+1}|^\mu \prod_{\nu \in R_+} |\langle x, \nu \rangle|^{\kappa_\nu}, \quad x \in \mathbb{R}^d, x_{d+1} \in \mathbb{R}.$$

The weight $h_{\tilde{\kappa}}$ on \mathbb{S}^d is related to the weight function $W_{\kappa, \mu}^B$ on \mathbb{B}^d by

$$h_{\tilde{\kappa}}^2\left(x, \sqrt{1 - \|x\|^2}\right) = W_{\kappa, \mu}^B(x) \sqrt{1 - \|x\|^2}, \quad x \in \mathbb{B}^d.$$

Furthermore, a change of variables $y = \phi(x)$ with:

$$\phi : \mathbb{B}^d \rightarrow \mathbb{S}^d, \quad x \in \mathbb{B}^d \mapsto \left(x, \sqrt{1 - \|x\|^2}\right) \in \mathbb{S}^d$$

shows that

$$\begin{aligned} & \int_{\mathbb{S}^d} f(y)h_{\tilde{\kappa}}^2(y)d\sigma(y) \\ &= \int_{\mathbb{B}^d} \left[f\left(x, \sqrt{1-\|x\|^2}\right) + f\left(x, -\sqrt{1-\|x\|^2}\right) \right] W_{\kappa,\mu}^B(x)dx. \end{aligned} \tag{39}$$

Given a function $f : \mathbb{B}^d \rightarrow \mathbb{R}$, define $\tilde{f} : \mathbb{S}^d \rightarrow \mathbb{R}$ by:

$$\tilde{f}(x, x_{d+1}) = f(x), \quad x \in \mathbb{B}^d, \quad (x, x_{d+1}) \in \mathbb{S}^d.$$

Clearly, $\tilde{f} \circ \phi = f$, and by (39), the mapping $f \rightarrow \tilde{f}$ is an isometry from $L^p(W_{\kappa,\mu}^B; \mathbb{B}^d)$ to $L^p(\mathbb{S}^{d-1}; h_{\tilde{\kappa}}^2/2)$. More importantly, the orthogonal structure on the weighted ball \mathbb{B}^d is preserved under the mapping $\phi : \mathbb{B}^d \rightarrow \mathbb{S}^d$. To be precise, let $v_n^d(W_{\kappa,\mu}^B)$ denote the space of weighted orthogonal polynomials of degree n with respect to the measure $W_{\kappa,\mu}^B(x)dx$ on \mathbb{B}^d , and let $\text{proj}_n(W_{\kappa,\mu}^B; f)$ denote the orthogonal projection of f onto the space $v_n^d(W_{\kappa,\mu}^B)$. Then a function f on \mathbb{B}^d belongs to the space $v_n^d(W_{\kappa,\mu}^B)$ if and only if $\tilde{f} \in \mathcal{H}_n^{d+1}(h_{\tilde{\kappa}}^2)$ and, moreover (see [9,25,26]),

$$\text{proj}_n(W_{\kappa,\mu}^B; f, x) = \text{proj}_n(W_{\kappa,\mu}^B; \tilde{f} \circ \phi, x) = \text{proj}_n(h_{\tilde{\kappa}}^2; \tilde{f}, \phi(x)), \quad x \in \mathbb{B}^d.$$

This relation allows us to deduce results on the convergence of orthogonal expansions with respect to $W_{\kappa,\mu}^B$ on \mathbb{B}^d from those of h -harmonic expansions on \mathbb{S}^d .

5.3. Results on the Ball

For $x \in \mathbb{B}^d$, we set $x_{d+1} := \sqrt{1-\|x\|^2}$. Let $\rho_B : \mathbb{B}^d \times \mathbb{B}^d \rightarrow [0, \pi]$ denote the metric on \mathbb{B}^d given by:

$$\rho_B(x, y) = \arccos(x \cdot y + x_{d+1}y_{d+1}), \quad x, y \in \mathbb{B}^d.$$

For $x \in \mathbb{B}^d$ and $\theta > 0$, define:

$$B^B(x, \theta) := \{y \in \mathbb{B}^d : \rho_B(x, y) \leq \theta\}.$$

We write:

$$\text{meas}_{\tilde{\kappa}}^B(E) := \int_E W_{\kappa,\mu}^B(x)dx, \quad E \subset \mathbb{B}^d,$$

where $W_{\kappa,\mu}^B$ is the weight function on \mathbb{B}^d given in (38). It is easily seen that $\text{meas}_{\tilde{\kappa}}^B$ is a doubling measure on \mathbb{B}^d , satisfying that for any $x \in \mathbb{B}^d$ and $\theta \in (0, \pi]$,

$$\text{meas}_{\tilde{\kappa}}^B\left(B^B(x, 2^j\theta)\right) \leq C2^{js_{\tilde{\kappa}}} \text{meas}_{\tilde{\kappa}}^B\left(B^B(x, \theta)\right), \quad j = 1, 2, \dots, \tag{40}$$

where $C > 0$ is a constant depending only on $\tilde{\kappa}$ and d , and $s_{\tilde{\kappa}}$ is the optimal constant for which (40) holds. Recall that $P(W_{\kappa,\mu}^B; x, y)$ denotes the reproducing kernel of the space $\mathcal{V}_n^d(W_{\kappa,\mu}^B)$ of orthogonal polynomials of degree n with respect to the weight $W_{\kappa,\mu}^B$ on \mathbb{B}^d , $S_n^\delta(W_{\kappa,\mu}^B; f)$ denotes the n -th Cesàro mean of order $\delta \geq 0$ of the WOPE of f with respect to the weight function $W_{\kappa,\mu}^B$ on \mathbb{B}^d , and $K_n^\delta(W_{\kappa,\mu}^B; x, y)$ is the Cesàro kernel of the operator $S_n^\delta(W_{\kappa,\mu}^B)$. The point-wise estimate of the Cesàro kernel $K_n^\delta(W_{\kappa,\mu}^B; x, y)$ is as follows:

Theorem 3. For $\delta > 0$ and $\ell \geq \tau > 2\lambda_{\bar{\kappa}} + 1$, we have:

$$\begin{aligned}
 |K_n^\delta(W_{\kappa,\mu}^B; x, y)| \leq & c_{\bar{\kappa}} \cdot \left[n^{-1} \sum_{j=1}^n (1+j)^d \cdot \frac{\prod_{v \in \tilde{R}_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}_B(x, y) + j^{-1})^{-2\bar{\kappa}_v}}{(1 + j\tilde{\rho}_B(x, y))^{\lambda_{\bar{\kappa}} + \ell - \tau - \frac{3}{2}s_{\bar{\kappa}} + d}}} \right. \\
 & + \sum_{i=2}^{\lfloor \log_2 n \rfloor + 2} 2^{-i-i\delta+i\ell} \frac{n^d (1 + n\tilde{\rho}_B(x, y))^{-\ell + \tau + \frac{3}{2}s_{\bar{\kappa}} - d}}{\prod_{v \in \tilde{R}_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}_B(x, y) + n^{-1})^{2\bar{\kappa}_v}} \\
 & \left. + \frac{n^{d-\delta} (1 + n\tilde{\rho}_B(x, y))^{\frac{3}{2}s_{\bar{\kappa}} - d}}{\prod_{v \in \tilde{R}_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + n^{-1} + \tilde{\rho}_B(x, y))^{2\bar{\kappa}_v}} \right],
 \end{aligned}$$

where $g_0 \in G$ is such that $\rho_B(g_0 x, y) = \tilde{\rho}_B(x, y) = \min_{g \in G} \rho_B(gx, y)$ for $x, y \in \mathbb{B}^d$.

Our next result can be stated as follows:

Theorem 4. Let $\delta > 0, \tau > 2\lambda_{\bar{\kappa}} + 1$,

$$\int_{\mathbb{B}^d} |K_n^\delta(W_{\kappa,\mu}^B; x, y)| W_{\kappa,\mu}^B(y) d\sigma(y) \leq \begin{cases} 1 & \text{if } \delta > \frac{3}{2}s_{\bar{\kappa}} + \tau - 1 \\ \log n & \text{if } \delta = \frac{3}{2}s_{\bar{\kappa}} + \tau - 1 \\ n^{-\delta + \frac{3}{2}s_{\bar{\kappa}} - \tau + 1} & \text{if } \delta < \frac{3}{2}s_{\bar{\kappa}} + \tau - 1. \end{cases}$$

As a consequence of the kernel estimate, we can prove the following:

Corollary 2. Let $\tau > 2\lambda_{\bar{\kappa}} + 1$ and $\sigma_{\bar{\kappa}} := \frac{3}{2}s_{\bar{\kappa}} + \tau - 1$. Then if $\delta > \sigma_{\bar{\kappa}}$, $S_n^\delta(W_{\kappa,\mu}^B; f)$ converges in $L^p(W_{\kappa,\mu}^B; \mathbb{B}^d)$ for all $1 \leq p \leq \infty$.

These results can be deduced directly from the corresponding results on the sphere S^d . Since the proofs are almost identical to those in [10,14], we skip the details here.

5.4. WOPEs on the Simplex \mathbb{T}^d

In this subsection, we will deduce similar results on the simplex \mathbb{T}^d from the already proven results on the ball \mathbb{B}^d . Our argument is based on the connections between WOPEs on \mathbb{B}^d and WOPEs on \mathbb{T}^d , as observed by Y. Xu, see [21,26].

The weight function $W_{\kappa,\mu}^T$ we consider on the simplex \mathbb{T}^d is given by:

$$W_{\kappa,\mu}^T(x) := \frac{h_{\bar{\kappa}}^2(\sqrt{x_1}, \dots, \sqrt{x_d})}{\sqrt{x_1 \cdots x_d}} (1 - |x|)^{\mu - \frac{1}{2}}, \quad \mu \geq 0, x \in \mathbb{T}^d. \tag{41}$$

It is related to the weight $W_{\kappa,\mu}^B$ on \mathbb{B}^d through the mapping,

$$\psi : (x_1, \dots, x_d) \in \mathbb{B}^d \mapsto (x_1^2, \dots, x_d^2) \in \mathbb{T}^d, \tag{42}$$

by

$$W_{\kappa,\mu}^T(\psi(x)) = \frac{W_{\kappa,\mu}^B(x)}{|x_1 \cdots x_d|}, \quad x \in \mathbb{B}^d.$$

Furthermore, a change of variables shows that:

$$\int_{\mathbb{B}^d} g(\psi(x)) W_{\kappa,\mu}^B(x) dx = \int_{\mathbb{T}^d} g(x) W_{\kappa,\mu}^T(x) dx. \tag{43}$$

For $1 \leq p \leq \infty$, we denote by $L^p(W_{\kappa,\mu}^T; \mathbb{T}^d)$ the L^p -space defined with respect to the measure $W_{\kappa,\mu}^T(x)dx$ on \mathbb{T}^d , and by $\|\cdot\|_{L^p(W_{\kappa,\mu}^T)}$ the norm of $L^p(W_{\kappa,\mu}^T; \mathbb{T}^d)$. Note that (43) particularly implies that the mapping,

$$L^p(W_{\kappa,\mu}^T; \mathbb{T}^d) \rightarrow L^p(W_{\kappa,\mu}^B; \mathbb{B}^d), \quad f \mapsto f \circ \psi,$$

is an isometry.

Let $\mathcal{V}_n^d(W_{\kappa,\mu}^T)$ denote the space of weighted orthogonal polynomials of degree n with respect to the weight $W_{\kappa,\mu}^T$ on \mathbb{T}^d . The orthogonal structure is preserved under the mapping (42) in the sense that $R \in \mathcal{V}_n^d(W_{\kappa,\mu}^T)$ if and only if $R \circ \psi \in \mathcal{V}_{2n}^d(W_{\kappa,\mu}^B)$. Furthermore, the orthogonal projection, $\text{proj}_n(W_{\kappa,\mu}^T; f)$, of f onto $\mathcal{V}_n^d(W_{\kappa,\mu}^T)$ can be expressed in terms of the orthogonal projection of $f \circ \psi$ onto $\mathcal{V}_{2n}^d(W_{\kappa,\mu}^B)$ as follows (see [9,13]):

$$\text{proj}_n(W_{\kappa,\mu}^T; f, \psi(x)) = \text{proj}_{2n}(W_{\kappa,\mu}^B; f \circ \psi, x), \quad x \in \mathbb{B}^d.$$

5.5. Results on the Simplex

For $x = (x_1, \dots, x_d) \in \mathbb{T}^d$, let $|x| = x_1 + x_2 + \dots + x_d$ and $x_{d+1} := 1 - |x|$. Let $\rho_T : \mathbb{T}^d \times \mathbb{T}^d \rightarrow [0, \pi]$ be the metric on \mathbb{T}^d given by:

$$\rho_T(x, y) = \arccos\left(\sum_{j=1}^{d+1} \sqrt{x_j y_j}\right), \quad x, y \in \mathbb{T}^d.$$

For $x \in \mathbb{B}^d$ and $\theta > 0$, define:

$$B^T(x, \theta) := \{y \in \mathbb{T}^d : \rho_T(x, y) \leq \theta\}.$$

We write:

$$\text{meas}_{\tilde{\kappa}}^T(E) := \int_E W_{\kappa,\mu}^T(x)dx, \quad E \subset \mathbb{T}^d,$$

where $W_{\kappa,\mu}^T$ is the weight function on \mathbb{T}^d given in (41). It is easily seen that $\text{meas}_{\tilde{\kappa}}^T$ is a doubling measure on \mathbb{T}^d satisfying that for any $x \in \mathbb{T}^d$ and $\theta \in (0, \pi]$,

$$\text{meas}_{\tilde{\kappa}}^T(B^T(x, 2^j\theta)) \leq C2^{js_{\tilde{\kappa}}} \text{meas}_{\tilde{\kappa}}^T(B^T(x, \theta)), \quad j = 1, 2, \dots, \tag{44}$$

where $C > 0$ is a constant depending only on $\tilde{\kappa}$ and d , and $s_{\tilde{\kappa}}$ is the optimal constant for which (44) holds. Recall that $P(W_{\kappa,\mu}^T; x, y)$ denotes the reproducing kernel of the space $\mathcal{V}_n^d(W_{\kappa,\mu}^T)$ of orthogonal polynomials of degree n with respect to the weight $W_{\kappa,\mu}^T$ on \mathbb{T}^d ; $S_n^\delta(W_{\kappa,\mu}^T; f)$ denotes the n -th Cesàro mean of the WOPE of f with respect to the weight function $W_{\kappa,\mu}^T$ on \mathbb{T}^d , and $K_n^\delta(W_{\kappa,\mu}^T; x, y)$ is the Cesàro kernel of the operator $S_n^\delta(W_{\kappa,\mu}^T)$.

The point-wise estimate of the Cesàro kernel $K_n^\delta(W_{\kappa,\mu}^T; x, y)$ is as follows:

Theorem 5. For $\delta > 0$ and $\ell \geq \tau > 2\lambda_{\tilde{\kappa}} + 1$, we have:

$$\begin{aligned}
 |K_n^\delta(W_{\kappa,\mu}^T; x, y)| \leq & c_{\tilde{\kappa}} \cdot \left[n^{-1} \sum_{j=1}^n (1+j)^d \cdot \frac{\prod_{v \in \tilde{R}_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}_T(x, y) + j^{-1})^{-2\tilde{\kappa}_v}}{(1 + j\tilde{\rho}_T(x, y))^{\lambda_{\tilde{\kappa}} + \ell - \tau - \frac{3}{2}s_{\tilde{\kappa}} + d}} \right. \\
 & + \sum_{i=2}^{\lfloor \log_2 n \rfloor + 2} 2^{-i-i\delta+i\ell} \frac{n^d (1 + n\tilde{\rho}_T(x, y))^{-\ell + \tau + \frac{3}{2}s_{\tilde{\kappa}} - d}}{\prod_{v \in \tilde{R}_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + \tilde{\rho}_T(x, y) + n^{-1})^{2\tilde{\kappa}_v}} \\
 & \left. + \frac{n^{d-\delta} (1 + n\tilde{\rho}_T(x, y))^{\frac{3}{2}s_{\tilde{\kappa}} - d}}{\prod_{v \in \tilde{R}_+} (|\langle x, v \rangle| + |\langle g_0 y, v \rangle| + n^{-1} + \tilde{\rho}_T(x, y))^{2\tilde{\kappa}_v}} \right],
 \end{aligned}$$

where $g_0 \in G$ is such that $\rho_T(g_0 x, y) = \tilde{\rho}_T(x, y) = \min_{g \in G} \rho_T(gx, y)$ for $x, y \in \mathbb{T}^d$.

Our next result can be stated as follows:

Theorem 6. Let $\delta > 0, \tau > 2\lambda_{\tilde{\kappa}} + 1$,

$$\int_{\mathbb{T}^d} |K_n^\delta(W_{\kappa,\mu}^T; x, y)| W_{\kappa,\mu}^T(y) d\sigma(y) \leq \begin{cases} 1 & \text{if } \delta > \frac{3}{2}s_{\tilde{\kappa}} + \tau - 1 \\ \log n & \text{if } \delta = \frac{3}{2}s_{\tilde{\kappa}} + \tau - 1 \\ n^{-\delta + \frac{3}{2}s_{\tilde{\kappa}} - \tau + 1} & \text{if } \delta < \frac{3}{2}s_{\tilde{\kappa}} + \tau - 1. \end{cases}$$

As a consequence of the kernel estimate, we can prove the following:

Corollary 3. Let $\tau > 2\lambda_{\tilde{\kappa}} + 1$ and $\sigma_{\tilde{\kappa}} := \frac{3}{2}s_{\tilde{\kappa}} + \tau - 1$. Then, if $\delta > \sigma_{\tilde{\kappa}}$, $S_n^\delta(W_{\kappa,\mu}^T; f)$ converges in $L^p(W_{\kappa,\mu}^T; \mathbb{T}^d)$ for all $1 \leq p \leq \infty$.

These results can be deduced directly from the corresponding results on the sphere \mathbb{B}^d . Since the proofs are almost identical to those in [10,14], we skip the details here.

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