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Total 2-Rainbow Domination in Graphs

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Abstract: A total k-rainbow dominating function on a graph $G = (V, E)$ is a function $f : V(G) \to 2^{\{1,2,\ldots,k\}}$ such that (i) $\cup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ for every vertex v with $f(v) = \emptyset$, (ii) $\cup_{u \in N(v)} f(u) \neq \emptyset$ for $f(v) \neq \emptyset$. The weight of a total 2-rainbow dominating function is denoted by $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The total k-rainbow domination number of G is the minimum weight of a total k-rainbow dominating function of G. The minimum total 2-rainbow domination problem (MT2RDP) is to find the total 2-rainbow domination number of the input graph. In this paper, we study the total 2-rainbow domination number of graphs. We prove that the MT2RDP is NP-complete for planar bipartite graphs, chordal bipartite graphs, undirected path graphs and split graphs. Then, a linear-time algorithm is proposed for computing the total k-rainbow domination number of trees. Finally, we study the difference in complexity between MT2RDP and the minimum 2-rainbow domination problem.

Keywords: total 2-rainbow domination; total 2-rainbow domination number; NP-complete; linear-time algorithm

MSC: 05C69

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1. Introduction

In this paper, only graphs without multiple edges or loops are considered. Let $G = (V, E)$ be an undirected graph with $|V(G)| = n$ and $|E(G)| = m$. The open neighborhood and closed neighborhood of a vertex v in G are denoted by $N(v) = \{u | uv \in E(G)\}$ and $N[v] = \{v\} \cup N(v)$, respectively. The degree of a vertex v is denoted by $d(v) = |N(v)|$. A graph is called k-regular if $d(v) = k$ for $v \in V(G)$. For a positive integer n, we write $[n] = \{0, 1, 2, \cdots, n-1\}$.

Let $G = (V, E)$, $\mathcal{F}$ be a finite family of subsets of a non-empty set, if there is a correspondence between $V(G)$ and $\mathcal{F}$ such that $f_i \cap f_j \neq \emptyset$ if and only if $uv \in E(G)$, where $f_i, f_j \in \mathcal{F}$ and f_i, f_j are the corresponding sets of vertices u and v, respectively, then G is an intersection graph. A graph G is chordal if every cycle C_k with $k \geq 4$ in G has a chord, i.e., an edge joining two non-consecutive vertices of the cycle, and a chordal graph is also an intersection graph with $\mathcal{F}$ which is a finite family of subtrees of a tree, see [1]. If G is a chordal graph and $\mathcal{F}$ is a finite family of paths of a tree, then G is an undirected path graph [2]. A chordal bipartite graph is a bipartite graph and every cycle in G of length ≥ 4 has a chord [3].

In a graph $G = (V, E)$, a subset $D \subset V(G)$ is called a dominating set if $N(u) \cap D \neq \emptyset$ for $u \notin D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G [4]. The problems of finding a minimum cardinality dominating set in graphs [5], split graphs and bipartite graphs [6], chordal bipartite graphs [7], planar bipartite graphs [8] and undirected path graphs [9] are NP-complete. For relevant papers on dominating set in graphs, see [10–14].

One of the most famous open problem involving domination in graphs is that $\gamma(G)\gamma(H) \leq \gamma(G \Box H)$ for any graphs G and H, called Vizing's conjecture [15], where

$G \square H$ denotes the Cartesian product of graphs G and H [16]. To investigate a similar problem for paired domination, Brešar et al. proposed k-rainbow domination [17]. A k-rainbow dominating function on a graph G is a function $f : V(G) \rightarrow 2^{\{1,2,\ldots,k\}}$ such that every vertex u for which $f(u) = \emptyset$, then $\cup_{u\in N(v)} f(u) = \{1, 2, \ldots, k\}$. The weight $\omega(f)$ of a k-rainbow dominating function f is denoted by $\omega(f) = \sum_{u\in V(G)} |f(u)|$. The k-rainbow domination number $\gamma_{rk}(G)$ of G is the minimum weight over all k-rainbow dominating functions on the graph G. A k-rainbow dominating function f of G with $\omega(f) = \gamma_{rk}(G)$ is called a γ_{rk}-function of G.

The minimum k-rainbow domination problem (MkRDP) is to finda minimum weight of a k-rainbow dominating function in graphs. *Brešar* et al. proved that M2RDP is NP-complete for chordal graphs and bipartite graphs [18]. Later, Chang et al. showed that the MkRDP for chordal graphs and bipartite graphs are NP-complete [19]. The linear-time algorithms for computing 2-rainbow domination number [20] and k-rainbow domination number [19] of trees are proposed. For a more detailed discussion of k-rainbow domination number in graphs, see [21–27].

Ahangar et al. [28] proposed a new dominating function named total k-rainbow dominating function for protecting the Empire under a more complex situation where the Empire is guarded by different types of guards, and where every location without guards needs all types of guards in its neighborhood and every location with guards needs at least one guard in its neighborhood. A total k-rainbow dominating function on a graph $G = (V, E)$ is a k-rainbow dominating function f such that $\cup_{u\in N(v)} f(u) \neq \emptyset$ for $f(v) \neq \emptyset$. The weight of a total k-rainbow dominating function is denoted by $\omega(f) = \sum_{v\in V(G)} |f(v)|$. The total k-rainbow domination number of G is the minimum weight of a total k-rainbow dominating function on the graph G, denoted by $\gamma_{trk}(G)$. A total k-rainbow dominating function f of G with $\omega(f) = \gamma_{trk}(G)$ is called a γ_{trk}-function of G.

The properties of total k-rainbow dominating function in graphs was studied [28,29]. Ahangar et al. characterized all graphs G, where $\gamma_{tr2}(G) = |V(G)| - 1$ [30]. The problems of finding a minimum weight of a total 2-rainbow dominating function (MT2RDP) in chordal graphs and bipartite graphs are NP-complete [29].

In this paper, we study the total 2-rainbow domination number of graphs. Then, we prove that MT2RDP is NP-complete even when restricted to planar bipartite graphs, chordal bipartite graphs, undirected path graphs and split graphs. Next, a linear-time algorithm is proposed for computing the total k-rainbow domination number of trees. Finally, we study the difference in complexity between MT2RDP and the minimum 2-rainbow domination problem.

2. Results

Lemma 1. *If the graph G is k-regular with order n, then $\gamma_{tr2}(G) \geq \frac{2n}{k+1}$.*

Proof. Let f be a γ_{tr2}-function of G. To prove the lower bound, we define an initial charge function s corresponding to f, such that $s(v) = |f(v)|$. Then, we apply the following two discharging rules to lead to the final charge function s' corresponding to s of G, such that $\sum_{v\in V(G)} s'(v) = \sum_{v\in V(G)} s(v)$.

Rule 1: For the vertex $s(v) = 1$, $N(v) = \{v_1, v_2, v_3, \ldots, v_k\}$, suppose that $f(v_1) \neq \emptyset$. Then, $s(v)$ transmits $\frac{1}{1+k}$ charge to $v_2, v_3, \ldots, v_k$.

Rule 2: For each $s(v) = 2$, $N(v) = \{v_1, v_2, v_3, \ldots, v_k\}$, suppose that $f(v_1) \neq \emptyset$. Then, $s(v)$ transmits $\frac{2}{1+k}$ charge to $v_2, v_3, \ldots, v_k$.

Thus, for $s(v) = 1$, we have $s'(v) \geq s(v) - \frac{k-1}{k+1} = \frac{2}{k+1}$ by Rule 1.

For $s(v) = 2$, then $s'(v) \geq s(v) - \frac{2(k-1)}{k+1} > \frac{2}{k+1}$ by Rule 2.

For $s(v) = 0$, since v is adjacent to at least one vertex u such that $s(u) = 2$ or two vertices $s(x) \geq 1$ and $s(y) \geq 1$, then v will receive $\frac{2}{k+1}$ charge form u or x, y by Rules 2 and 1, then $s'(v) \geq s(v) + \frac{2}{k+1} = \frac{2}{k+1}$.

Therefore, $\omega(f) = \sum_{v\in V(G)} s'(v) = \sum_{v\in V(G)} s(v) \geq \frac{2}{k+1} \times n = \frac{2n}{k+1}$. □

To show that $\gamma_{tr2}(G) \geq \frac{2n}{k+1}$ is sharp for k-regular graphs with order $n = 4t$ when $k = 3$, $t \geq 2$, we investigate the total 2-rainbow domination number of double generalized Petersen graphs.

The double generalized Petersen graphs $DP(n,k)$ was introduced by Kutnar and Petecki [31], $n \geq 3$ and $1 \leq k \leq n-1$, with vertex set:

$$V(DP(n,k)) = U \cup V \cup X \cup Y$$

where $U = \{u_0, u_1, \ldots, u_{n-1}\}$, $V = \{v_0, v_1, \ldots, v_{n-1}\}$, $X = \{x_0, x_1, \ldots, x_{n-1}\}$, $Y = \{y_0, y_1, \ldots, y_{n-1}\}$, and its edge set is the union:

$$E(DP(n,k)) = E_1 \cup E_2 \cup E_3$$

where $E_1 = \{u_i u_{i+1}, y_i y_{i+1} | i \in [n]\}$, $E_2 = \{u_i v_i, x_i y_i | i \in [n]\}$, $E_3 = \{v_i x_{i+k}, x_i v_{i+k} | i \in [n]\}$, and the subscripts are reduced modulo n (see, e.g., $(n,k) \in \{(6,1),(9,4),(n,1)\}$ in Figures 1 and 2).

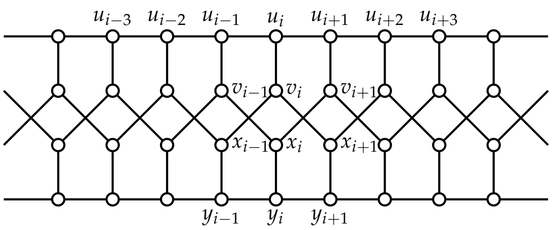
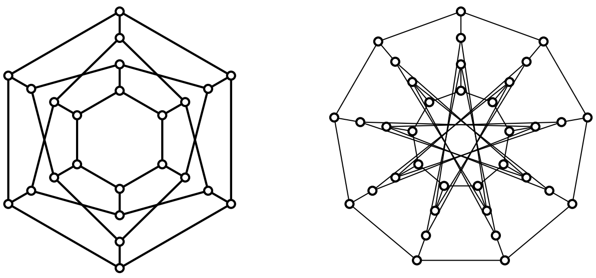


Figure 1. The graph $DP(n,1)$.



(**a**) (**b**)

Figure 2. (**a**) The graph $DP(6,1)$; (**b**) The graph $DP(9,4)$.

Let

$$A_{4\times 4} = \begin{bmatrix} a_{1,1} & a_{12} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{2,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{2,2} & a_{4,3} & a_{4,4} \end{bmatrix}, \quad B_{4\times t} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \ldots & b_{1,t} \\ b_{2,1} & b_{2,2} & b_{2,3} & \ldots & b_{2,t} \\ b_{3,1} & b_{2,2} & b_{3,3} & \ldots & b_{3,t} \\ b_{4,1} & b_{2,2} & b_{4,3} & \ldots & b_{4,t} \end{bmatrix}.$$

Let $f_{A_{4\times 4}, B_{4\times t}}$ be a function of $DP(n,1)$ with $n = 4s + t$, such that $f(u_p) = f(u_{p+4k}) = a_{1,p+1}$, $f(v_p) = f(v_{p+4k}) = a_{2,p+1}$, $f(x_p) = f(x_{p+4k}) = a_{3,p+1}$, $f(y_p) = f(y_{p+4k}) = a_{4,p+1}$, $f(u_{n-1-q}) = b_{1,t-q}$, $f(v_{n-1-q}) = b_{2,t-q}$, $f(x_{n-1-q}) = b_{3,t-q}$, $f(y_{n-1-q}) = b_{4,t-q}$, where $p \in \{0,1,2,3\}$, $k \in \{1,2,\ldots,s\}$, $q \in \{0,1,2,\ldots,t-1\}$.

Lemma 2. $\gamma_{tr2}(DP(n,1)) \geq 2n+1$, *where* $n \geq 9$ *and* $n \equiv 1,2,3$ *(mod 4)*.

Proof. Let f be aγ_{tr2}-function of $DP(n,1)$, $V_0 = \{v | f(v) = \emptyset\}$, $V_1 = \{v | f(v) = \{1\}\}$, $V_2 = \{v | f(v) = \{2\}\}$, $V_3 = \{v | f(v) = \{1,2\}\}$. Suppose that $\gamma_{tr2}(DP(n,1)) < 2n+1$, that is, $\omega(f) = \gamma_{tr2}(DP(n,1)) = 2n$.

Then, $V_3 = \emptyset$. Otherwise, $s'(v) = 2 - \frac{1}{2} - \frac{1}{2} = 1, \omega(f) = \sum_{v \in V(G)} s(v) = \sum_{v \in V(G)} s'(v)$ $= \sum_{v \in V_0} s'(v) + \sum_{v \in V_1} s'(v) + \sum_{v \in V_2} s'(v) + \sum_{v \in V_3} s'(v) \geq \frac{1}{2} \times 4n + \frac{1}{2}$ according to the proof of Lemma 1, contradicting with $\omega(f) = \gamma_{tr2}(DP(n,1)) = 2n$.

Similarly, $|N(v) \cap V_1| = |N(v) \cap V_2| = |N(v) \cap V_0| = 1$ for every vertex $v \in V_0$, and $|N(v) \cap V_1| + |N(v) \cap V_2| = 1$, $|N(v) \cap V_0| = 2$ for every vertex $v \in V_1 \cup V_2$.

Let $s \cup \bar{s} = \{1,2\}$, $|s| = |\bar{s}| = 1$, $t \cup \bar{t} = \{1,2\}$, $|t| = |\bar{t}| = 1$, $p \cup \overline{p} = \{1,2\}$, $|p| = |\overline{p}| = 1$, $q \cup \bar{q} = \{1,2\}$, $|q| = |\bar{q}| = 1$, $I_i = \{v_i, u, x_i, y_i\}$, where $i \in [n]$.

Case 1: $\sum_{v \in I_i} |f(v)| = 4$ for some $i \in [n]$.

In this case, $f(u_i) = s$, $f(v_i) = t$, $f(x_i) = p$, $f(y_i) = q$. Then, $f(u_{i+1}) = f(v_{i+1}) = f(x_{i+1}) = f(y_{i+1}) = f(u_{i-1}) = f(v_{i-1}) = f(x_{i-1}) = f(y_{i-1}) = \emptyset$. To dominate $u_{i+1}, x_{i+1}, v_{i+1}, y_{i+1}$, we have $f(u_{i+2}) = \bar{s}$, $f(v_{i+2}) = \bar{t}$, $f(x_{i+2}) = \overline{p}$, $f(y_{i+2}) = \bar{q}$. Since $|N(v) \cap V_0| = 2$ for every vertex $v \in V_1 \cup V_2$, thus $f(u_{i+3}) = f(v_{i+3}) = f(x_{i+3}) = f(y_{i+3}) = \emptyset$. To dominate $u_{i+3}, x_{i+3}, v_{i+3}, y_{i+3}$, then $f(u_{i+4}) = f(u_i) = s$, $f(v_{i+4}) = f(v_i) = t$, $f(x_{i+4}) = f(x_i) = p$, $f(y_{i+4}) = f(y_i) = q$. Therefore, $f(u_{i+4k}) = f(u_i) = s$, $f(v_{i+4k}) = f(v_i) = t$, $f(x_{i+4k}) = f(x_i) = p$, $f(y_{i+4k}) = f(y_i) = q$, $f(u_{i+4k-2}) = f(u_{i+2}) = \bar{s}$, $f(v_{i+4k-2}) = f(v_{i+2}) = \bar{t}$, $f(x_{i+4k-2}) = f(x_{i+2}) = \overline{p}$, $f(y_{i+4k-2}) = f(y_{i+2}) = \bar{q}$, $f(u_{i+4k-3}) = f(u_{i+1}) = f(u_{i+4k-1}) = f(u_{i+3}) = f(v_{i+4k-3}) = f(v_{i+1}) = f(v_{i+4k-1}) = f(v_{i+3}) = f(x_{i+4k-3}) = f(x_{i+1}) = f(x_{i+4k-1}) = f(x_{i+3}) = f(y_{i+4k-3}) = f(y_{i+1}) = f(y_{i+4k-1}) = f(y_{i+3}) = \emptyset$, where $k \geq 1$ and the subscripts are reduced modulo n. Thus, $n \equiv 0 \pmod 4$, a contradiction.

Case 2: $\sum_{v \in I_i} |f(v)| = 3$ for some $i \in [n]$.

Suppose that $f(u_i) = s$, $f(v_i) = t$, $f(x_i) = p$, $f(y_i) = \emptyset$. Then $f(u_{i+1}) = f(x_{i+1}) = f(u_{i-1}) = f(x_{i-1}) = \emptyset$. To dominate y_i, without loss of gravity, we assume that $f(y_{i+1}) = \overline{p}$, $f(y_{i-1}) = \emptyset$. Then $f(y_{i-2}) = \{1,2\}$ for dominating y_{i-1}, contradicting with $V_3 = \emptyset$.

Now we consider that $f(u_i) = s$, $f(v_i) = t$, $f(x_i) = \emptyset$, $f(y_i) = q$. Then $f(u_{i+1}) = f(x_{i+1}) = f(u_{i-1}) = f(x_{i-1}) = \emptyset$. To dominate x_i, wl.o.g we assume that $f(v_{i+1}) = \bar{q}$, $f(v_{i-1}) = \emptyset$. Thus, $f(x_{i-2}) = \{1,2\}$ for dominating v_{i-1}, contradicting with $V_3 = \emptyset$.

Case 3: $\sum_{v \in I_i} |f(v)| = 1$ for some $i \in [n]$.

Suppose that $f(u_i) = s$, $f(v_i) = f(x_i) = f(y_i) = \emptyset$. To dominate x_i, y_i, v_i we have $|f(v_{i+1})| = |f(v_{i-1})| = 1$, $|f(y_{i+1})| = |f(y_{i-1})| = 1$, $|f(x_{i+1})| + |f(y_{i-1})| = 1$. Therefore, $\sum_{v \in I_{i+1}} f(v) \geq 3$ or $\sum_{v \in I_{i-1}} |f(v)| \geq 3$. The result is entirely consistent with Case 2, then contradicting with $V_3 = \emptyset$.

Now we consider that $f(v_i) = t$, $f(u_i) = f(x_i) = f(y_i) = \emptyset$. To dominate x_i, y_i, u_i, we have $|f(v_{i+1})| = |f(v_{i-1})| = 1$, $|f(y_{i+1})| = |f(y_{i-1})| = 1$, $|f(u_{i+1})| + |f(u_{i-1})| = 1$. Therefore, $\sum_{v \in I_{i+1}} f(v) \geq 3$ or $\sum_{v \in I_{i-1}} |f(v)| \geq 3$. The result is entirely consistent with Case 2, then contradicting with $V_3 = \emptyset$.

Case 4: $\sum_{v \in I_i} |f(v)| = 2$ for some $i \in [n]$.

In this case, it is sufficient to consider the following three subcases.

Subcase 4.1: $f(u_i) = s$, $f(v_i) = t$, $f(x_i) = f(y_i) = \emptyset$.

In this case, $f(u_{i+1}) = f(u_{i-1}) = f(x_{i+1}) = f(x_{i-1}) = \emptyset$. To dominate x_i, we have $f(v_{i-1}) = q$, $f(v_{i-1}) = \bar{q}$. To dominate u_{i+1}, then $s = p$. However, $s = \overline{p}$ for dominating v_{i-1}, a contradiction.

Subcase 4.2: $f(u_i) = s$, $f(x_i) = p$, $f(v_i) = f(y_i) = \emptyset$.

To dominate y_i, we may assume $f(y_{i+1}) = \overline{p}$ and $f(y_{i-1}) = \emptyset$. Since $|N(v) \cap V_0| = 2$ for every vertex $v \in V_1 \cup V_2$, assume that $f(u_{i+1}) = \emptyset$. Then, $f(v_{i+1}) = \bar{s}$, $f(v_{i-1}) = \emptyset$, $f(u_{i-1}) = \overline{p}$. To dominate v_i, then $f(x_{i-1}) = \bar{s}$. Therefore, $f(u_{i+4k}) = f(u_i) = s$, $f(v_{i+4k}) = f(v_i) = \emptyset$, $f(x_{i+4k}) = f(x_i) = p$, $f(y_{i+4k}) = f(y_i) = \emptyset$, $f(u_{i+4k-3}) = f(u_{i+1}) = \emptyset$, $f(v_{i+4k-3}) = f(v_{i+1}) = \bar{s}$, $f(x_{i+4k-3}) = f(x_{i+1}) = \emptyset$, $f(y_{i+4k-3}) = f(y_{i+1}) = \overline{p}$, $f(u_{i+4k-2}) = f(u_{i+2}) = \emptyset$, $f(v_{i+4k-2}) = f(v_{i+2}) = p$, $f(x_{i+4k-2}) = f(x_{i+2}) = \emptyset$, $f(y_{i+4k-2}) = f(y_{i+2}) = s$, $f(u_{i+4k-1}) = f(u_{i+3}) = \overline{p}$, $f(v_{i+4k-1}) = f(v_{i+3}) = \emptyset$, $f(x_{i+4k-1}) = f(x_{i+3}) = \bar{s}$, $f(y_{i+4k-1}) = f(y_{i+3}) = \emptyset$, where $k \geq 1$ and the subscripts are reduced modulo n. Thus, $n \equiv 0 \pmod 4$, a contradiction.

Now we consider $f(u_{i-1}) = \emptyset$, then $f(v_{i-1}) = \bar{s}$, $f(v_{i+1}) = \emptyset$, $f(u_{i+1}) = \overline{p}$. To dominate v_i, then $f(x_{i-1}) = \bar{s}$. Therefore, $f(u_{i+4k}) = f(u_i) = s$, $f(v_{i+4k}) = f(v_i) = \emptyset$,

$f(x_{i+4k}) = f(x_i) = p$, $f(y_{i+4k}) = f(y_i) = \emptyset$, $f(u_{i+4k-3}) = f(u_{i+1}) = \overline{p}$, $f(v_{i+4k-3}) = f(v_{i+1}) = \emptyset$, $f(x_{i+4k-3}) = f(x_{i+1}) = \emptyset$, $f(y_{i+4k-3}) = f(y_{i+1}) = \overline{p}$, $f(u_{i+4k-2}) = f(u_{i+2}) = \emptyset$, $f(v_{i+4k-2}) = f(v_{i+2}) = p$, $f(x_{i+4k-2}) = f(x_{i+2}) = \emptyset$, $f(y_{i+4k-2}) = f(y_{i+2}) = s$, $f(u_{i+4k-1}) = f(u_{i+3}) = \emptyset$, $f(v_{i+4k-1}) = f(v_{i+3}) = \overline{s}$, $f(x_{i+4k-1}) = f(x_{i+3}) = \overline{s}$, $f(y_{i+4k-1}) = f(y_{i+3}) = \emptyset$, where $k \geq 1$ and the subscripts are reduced modulo n. Thus, $n \equiv 0$ (mod 4), a contradiction.

Subcase 4.3: $f(u_i) = s$, $f(y_i) = q$, $f(v_i) = f(x_i) = \emptyset$ (or $f(v_i) = t$, $f(x_i) = p$, $f(u_i) = f(y_i) = \emptyset$).

In this case, we have $f(u_{i+4k}) = f(u_i) = s$, $f(v_{i+4k}) = f(v_i) = \emptyset$, $f(x_{i+4k}) = f(x_i) = \emptyset$, $f(y_{i+4k}) = f(y_i) = q$, $f(u_{i+4k-3}) = f(u_{i+1}) = \emptyset$, $f(v_{i+4k-3}) = f(v_{i+1}) = \overline{p}$, $f(x_{i+4k-3}) = f(x_{i+1}) = \overline{s}$, $f(y_{i+4k-3}) = f(y_{i+1}) = \emptyset$, $f(u_{i+4k-2}) = f(u_{i+2}) = \emptyset$, $f(v_{i+4k-2}) = f(v_{i+2}) = \overline{s}$, $f(x_{i+4k-2}) = f(x_{i+2}) = \overline{p}$, $f(y_{i+4k-2}) = f(y_{i+2}) = \emptyset$, $f(u_{i+4k-1}) = f(u_{i+3}) = s$, $f(v_{i+4k-1}) = f(v_{i+3}) = \emptyset$, $f(x_{i+4k-1}) = f(x_{i+3}) = \emptyset$, $f(y_{i+4k-1}) = f(y_{i+3}) = p$, where $k \geq 1$ and the subscripts are reduced modulo n. Thus, $n \equiv 0$ (mod 4), a contradiction. □

Theorem 1. $\gamma_{tr2}(DP(n,1)) = \begin{cases} 2n, & n \equiv 0 (mod\ 4), \\ 2n+1, & n \equiv 1,2,3 (mod\ 4), \end{cases}$ *where* $n \geq 8$.

Proof. Case 1: $n \equiv 0$ (mod 4).

Let:

$$A_{4\times4} = \begin{bmatrix} \emptyset & \emptyset & \{2\} & \{2\} \\ \{1\} & \{1\} & \emptyset & \emptyset \\ \emptyset & \{1\} & \{1\} & \emptyset \\ \{2\} & \emptyset & \emptyset & \{2\} \end{bmatrix}.$$

Then $f_{A_{4\times4},A_{4\times4}}$ is a total 2-rainbow dominating function of $DP(n,1)$ with $\omega(f) = 2n$, and $\gamma_{tr2}(DP(n,1)) \leq 2n$ for $n \equiv 0$ (mod 4).

Case 2: $n \equiv 1$ (mod 4).

Let:

$$A_{4\times4} = \begin{bmatrix} \{2\} & \emptyset & \{1\} & \emptyset \\ \{1\} & \emptyset & \{2\} & \emptyset \\ \{1\} & \emptyset & \{2\} & \emptyset \\ \{2\} & \emptyset & \{1\} & \emptyset \end{bmatrix}, \ B_{4\times5} = \begin{bmatrix} \{2\} & \emptyset & \{1\} & \{1\} & \emptyset \\ \{1\} & \emptyset & \emptyset & \{2\} & \emptyset \\ \{1\} & \{1\} & \{2\} & \{2\} & \emptyset \\ \{2\} & \emptyset & \emptyset & \{1\} & \emptyset \end{bmatrix},$$

Then $f_{A_{4\times4},B_{4\times5}}$ is a total 2-rainbow dominating function of $DP(n,1)$ with $\omega(f) = 2n+1$, and $\gamma_{tr2}(DP(n,1)) \leq 2n+1$ for $n \equiv 1$ (mod 4).

Case 3: $n \equiv 2$ (mod 4).

Let:

$$A_{4\times4} = \begin{bmatrix} \emptyset & \emptyset & \{1\} & \{2\} \\ \{1\} & \emptyset & \emptyset & \{2\} \\ \emptyset & \{2\} & \{1\} & \emptyset \\ \{1\} & \emptyset & \emptyset & \{2\} \end{bmatrix}, \ B_{4\times6} = \begin{bmatrix} \emptyset & \{2\} & \{1\} & \emptyset & \{2\} & \{1\} \\ \{1\} & \emptyset & \emptyset & \{1\} & \emptyset & \{2\} \\ \emptyset & \{2\} & \{1\} & \{1\} & \emptyset & \emptyset \\ \{1\} & \emptyset & \emptyset & \{2\} & \emptyset & \{1\} \end{bmatrix}.$$

Then $f_{A_{4\times4},B_{4\times6}}$ is a total 2-rainbow dominating function of $DP(n,1)$ with $\omega(f) = 2n+1$, and $\gamma_{tr2}(DP(n,1)) \leq 2n+1$ for $n \equiv 2$ (mod 4).

Case 4: $n \equiv 3$ (mod 4).

Let:

$$A_{4\times4} = \begin{bmatrix} \{2\} & \emptyset & \emptyset & \{1\} \\ \emptyset & \{1\} & \{2\} & \emptyset \\ \{2\} & \{1\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \{2\} & \{1\} \end{bmatrix}, \ B_{4\times7} = \begin{bmatrix} \{2\} & \emptyset & \{1\} & \{1\} & \emptyset & \emptyset & \{1\} \\ \{2\} & \{2\} & \emptyset & \emptyset & \{2\} & \{2\} & \emptyset \\ \{2\} & \emptyset & \emptyset & \{2\} & \{2\} & \emptyset & \emptyset \\ \emptyset & \{1\} & \{1\} & \emptyset & \emptyset & \{1\} & \{1\} \end{bmatrix}.$$

Then $f_{A_{4\times4},B_{4\times7}}$ is a total 2-rainbow dominating function of $DP(n,1)$ with $\omega(f) = 2n+1$, and $\gamma_{tr2}(DP(n,1)) \leq 2n+1$ for $n \equiv 3 \pmod 4$.

Furthermore, by Lemmas 1 and 2 , this theorem holds. □

Therefore, $\gamma_{tr2}(DP(n,1)) \geq 2n$ is sharp when $n \equiv 0 \pmod 4$.

2.1. Complexity

In this section, we show that the problems of finding a minimum weight of a total 2-rainbow dominating function in planar bipartite graphs, chordal bipartite graphs, undirected path graphs and split graphs are NP-complete, by giving two polynomial time reductions from two NP-complete problems, MINIMUM DOMINATION PROBLEM and 3-SAT, which are defined as follows.

MINIMUM DOMINATION PROBLEM(MDP)
INSTANCE: A simple and undirected graph $G = (V, E)$ and a positive integer $k \leq |V(G)|$.
QUESTION: Does G have a dominating set with cardinality at most k?

3-SAT
INSTANCE: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set U of variables such that $|C_j| = 3$ for $j = 1, 2, \ldots, m$.
QUESTION: Is there a truth assignment for U that satisfies all the clauses in $\mathcal{C}$?

MINIMUM TOTAL 2-RAINBOW DOMINATION PROBLEM(MT2RDP)
INSTANCE: A simple and undirected graph $G = (V, E)$ and a positive integer $k \leq |V(G)|$.
QUESTION: Does G have a total 2-rainbow dominating function of weight at most k?

Theorem 2. *The MT2RDP is NP-complete for planar bipartite graphs, chordal bipartite graphs and undirected path graphs.*

Proof. Given a graph $G = (V, E)$, then let each vertex $v \in V(G)$ be the tree T_v, where $V(T_v) = \{v, v_1, v_2, v_3, v_4, v_5\}$, $E(T_v) = \{vv_1, v_1v_2, v_2v_3, v_2v_4, v_2v_5\}$. Let $A = \{T_{v^i} | i \in \{1, 2, \ldots, n\}\}$ be the set of disjoint trees corresponding to the graph G. If $v^iv^j \in E(G)$, then add an edges v^iv^j between the trees $T_{v^i} \in A$ and $T_{v^j} \in A$. Therefore, we obtain a graph G', see Figure 3.

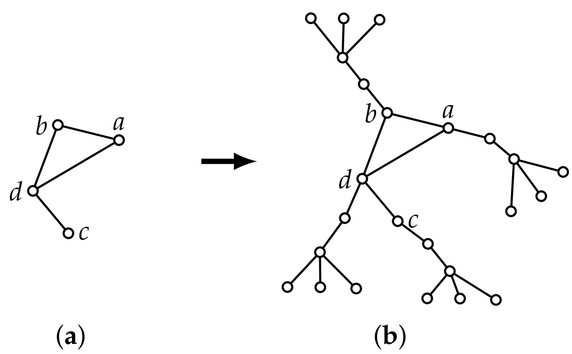


Figure 3. (**a**) The graph G. (**b**) The graph G' obtained from G.

Claim 1. *The graph G has a dominating set with cardinality at most k if and only if there is a total 2-rainbow dominating function f of the graph G' such that $\omega(f) \leq k + 3n$.*

Proof. Suppose G has a dominating set D and $|D| \leq k$. We define a function $f : V(G') \to \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ such that $f(v_1) = \{2\}$,$f(v_2) = \{1, 2\}$,$f(v_3) = f(v_4) = f(v_5) = \emptyset$ for every tree T_v, and if $v \in V(G) \cap D$, $f(v) = \{1\}$, if $v \in V(G) \setminus D$, $f(v) = \emptyset$.

Thus, f is a total 2-rainbow dominating function of G and $\omega(f) \leq k + 3n$.

Conversely, suppose the graph G' has a total 2-rainbow dominating function f such that $\omega(f) \leq k + 3n$. It is immediate that $|f(v_1)| + |f(v_2)| + |f(v_3)| + |f(v_4)| + |f(v_5)| \geq 3$ with equality if and only if $f(v_2) = \{1,2\}$. If $f(v) \neq \emptyset$, let $v \in D$, then v is dominated. If $f(v) = \emptyset$ and $|f(v_1)| \leq 1$, v is total 2-rainbow dominated by $u \in N_G(v)$ and v_1, that is, $f(u) \neq \emptyset$. Let $v \notin D$, then v is dominated by u. If $f(v) = \emptyset$ and $f(v_1) = \{1,2\}$, let $v \in D$, then v is dominated by u.

Since every vertex $v \in V(G)$ is dominated, D is a dominating set of G with cardinality at most k. $\square$

If the graph G is a chordal bipartite graph or planar bipartite graph, so is G'. Recall that the MDP is NP-complete for chordal bipartite graphs [7], planar bipartite graphs [8] and undirected path graphs [9]; thus, it can be immediately concluded that the MT2RDP is NP-complete for chordal bipartite graphs and planar bipartite graphs.

Now we show that if the graph G is an undirected path graph, so is G'. Suppose G is an undirected path graph. Then, there exists a finite family $\mathcal{F}$ of paths $\{P_{v_i} | i \in \{1,2,\ldots,n\}\}$ of a tree T. Let x_i be the one of the end points of path P_{v_i}, T_{v_i} be a tree with $V(T_{v_i}) = \{a_i, b_i, s_i, t_i, r_i\}$, $E(T_v) = \{a_ib_i, b_is_i, b_it_i, b_ir_i\}$, where $i \in \{1,2,\ldots,n\}$. Construct T' from T by adding edges x_ia_i between $a_i \in V(T_{v_i})$ and $x_i \in V(P_{v_i})$, where $i \in \{1,2,\ldots,n\}$. Now let $P'_{v_i} = \{x_ia_i, a_ib_i, b_is_i, b_it_i, b_ir_i\}$, where $i \in \{1,2,\ldots,n\}$, $\mathcal{F}' = \mathcal{F} \cup \{P'_{v_i} | \in \{1,2,\ldots,n\}\}$. Thus, there is a 1-1 correspondence f between $V(G')$ and $\mathcal{F}'$ such that $f(a_i) = x_ia_i$, $f(b_i) = a_ib_i$, $f(s_i) = b_is_i$, $f(t_i) = b_it_i$, $f(r_i) = b_ir_i$, and $uv \in E(G')$ if and only if $f(u) \cap f(v) \neq \emptyset$. Therefore, G' is an undirected path graph.

The proof is completed. $\square$

Theorem 3. *The MT2RDP is NP-complete for split graphs.*

Proof. Let $U = \{x_1, x_2, \ldots, x_n\}$, $\mathcal{C}$ =$\{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance F of 3-SAT .

Let $G_F = (V, E)$ be a graph, $V(G_F) = V^1 \cup V^2 \cup V^3$, $E(G_F) = E^1 \cup E^2 \cup E^3$, where $V^1 = \{x_i, \overline{x}_i | i \in \{1,2,\ldots,n\}\}$, $V^2 = \{c_j | j \in \{1,2,\ldots,m\}\}$, $V^3 = \{a_i, b_i | i \in \{1,2,\ldots,n\}\}$, $E^1 = \{uv | u, v \in \{V^1\}\}$, $E^2 = \{c_jx_i \text{ (or } c_j\overline{x}_i) | x_i \in C_j \text{ (or } \overline{x}_i \in C_j)\}$, $E^3 = \{x_ia_i, x_ib_i, \overline{x}_ia_i, \overline{x}_ib_i | x_i \in C_j\}$, $i \in \{1,2,\ldots,n\}$, $j \in \{1,2,\ldots,m\}$, see Figure 4.

It is immediate that the graph G_F is a split graph with a partitioning $V(G_F)$ into a clique v^1 and a stable set $V^2 \cup V^3$.

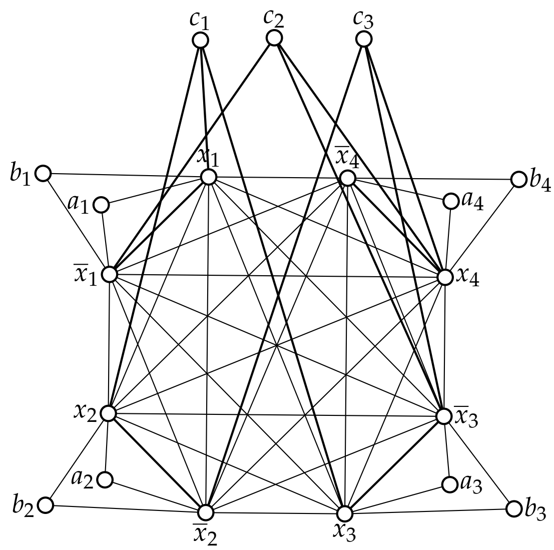


Figure 4. The graph G_F constructed from the instance F, where $C_1 = x_1 \vee x_2 \vee x_3$, $C_2 = \overline{x}_1 \vee \overline{x}_3 \vee x_4$, $C_3 = \overline{x}_2 \vee \overline{x}_3 \vee x_4$.

If $\mathcal{C}$ is satisfiable, then we define a function $f : V(G') \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, such that $f(a_i) = f(b_i) = f(c_j) = \emptyset$. If x_i is true, then $f(x_i) = \{1,2\}$, $f(\overline{x}_i) = \emptyset$; otherwise, $f(\overline{x}_i) = \{1,2\}$, $f(x_i) = \emptyset$. Thus, f is a total 2-rainbow dominating function of G_F and $\omega(f) \leq 2\text{n}$.

Conversely, suppose the graph G_F has a total 2-rainbow dominating function f such that $\omega(f) \leq 2n$. Let $V_1 = \{v|f(v) = \{1\}\}$, $V_2 = \{v|f(v) = \{2\}\}$, $V_3 = \{v|f(v) = \{1,2\}\}$. To dominate a_i, b_i for $i \in \{1,2,..,n\}$, if $|f(a_i)| \geq 1$ (or $|f(b_i)| \geq 1$), then $|f(x_i)| + |f(\overline{x}_i)| \geq 1$ with equality if and only if $|f(b_i)| \geq 1$ (or $|f(a_i)| \geq 1$). Thus, $|f(a_i)| + |f(b_i)| + |f(x_i)| + |f(\overline{x}_i)| \geq 2$ with equality if and only if $|f(a_i)| = |f(b_i)| = 0$. Note that $\omega(f) \leq 2n$, then $|f(x_i)| + |f(\overline{x}_i)|$ =2, $|f(a_i)| = |f(b_i)| = 0$, $|f(c_j)| = 0$, where $i \in \{1,2,..,n\}$, $j \in \{1,2,..,m\}$.

Since c_j is dominated by one vertex $v \in V_3$ or two vertices $u \in V_1, v \in V_2$, where $j \in \{1,2,..,m\}$ then let x be true for $x \in V_1 \cup V_3$. Therefore, the clause C_j is satisfied for $j \in \{1,2,..,m\}$. Note that $V_1 \cup V_3 \subseteq V^1$, so $V_1 \cup V_3$ is the true assignment for U that satisfies all the clauses in $\mathcal{C}$. □

2.2. A Linear-Time Algorithm for Trees

In this section, we propose a linear-time algorithm for computing the total k-rainbow domination number of trees. Let f be a total k-rainbow dominating function of G. If $S \subset$ V(G), $f(S) = \bigcup_{v \in S} f(v)$.

If $u \in V(G)$, an H-trk function of (G,u) is a function $g : V(G) \to 2^{\{1,2,\ldots,k\}}$, such that g is a total k-rainbow dominating function of $G - u$, that is, every vertex $v \in V(G) \setminus \{u\}$ is dominated by the vertices in $V(G)$, the weight of g is denoted by $\omega(g) = \sum_{v \in V(G)} |g(v)|$.

Denote $F = \{g|\ g$ is an H-trk function of $(G,u)\}$:
$\gamma(G,u,t,0) = min\{\omega(f)|f \in F, |f(u)| = t, f(N(u)) = \emptyset\}, 1 \leq t \leq k,$
$\gamma(G,u,t,1) = min\{\omega(f)|f \in F, |f(u)| = t, f(N(u)) \neq \emptyset\}, 1 \leq t \leq k,$
$\gamma(G,u,\emptyset,t) = min\{\omega(f)|f \in F, f(u) = \emptyset, |f(N(u))| \geq t\}, 0 \leq t \leq k.$

Lemma 3. *Let $u \in V(G)$, then: $\gamma_{trk}(G) = \min\{\gamma(G,u,1,1), \gamma(G,u,2,1), \ldots, \gamma(G,u,k,1), \gamma(G,u,\emptyset,k)\}$.*

Lemma 4. *Let G be a graph and $u \in V(G)$. If f is an H-trk function of (G,u) such that $|f(u)| = t$, $1 \leq t \leq k$, then for any $A = \{a_1, a_2, \ldots, a_t\}$, $a_i \in \{1,2,\ldots,k\}$, $i \in \{1,2,\ldots,t\}$, there exists an H-trk function f' of (G,u) with $\sum_{v \in V(G)} |f'(v)| = \sum_{v \in V(G)} |f(v)|$, $f'(u) = A$, $|f'(N(u))| = |f(N(u))|$.*

Proof. Assume that $f(u) = \{x_1, x_2, \ldots, x_t\}$ where $x_i \in \{1,2,\ldots,k\}$, $i \in \{1,2,\ldots,t\}$. Let $B = \{x_1, x_2, \ldots, x_t\} \setminus \{a_1, a_2, \ldots, a_t\}$, $C = \{a_1, a_2, \ldots, a_t\} \setminus \{x_1, x_2, \ldots, x_t\}$, then $|B| = |C|$. Assume that $|B| = p$, $B = \{y_1, y_2, .., y_p\}$, $C = \{z_1, z_2, .., z_p\}$. Let f' be a function of G obtained by changing y_i into z_i, changing z_i into y_i for $y_i, z_i \in f(v)$, $v \in V(G)$, $i \in \{1,2,\ldots,p\}$. Then, $f'(u) = \{a_1, a_2, \ldots, a_t\}$ and f' is an H-trk function of (G,u) and $\sum_{v \in V(G)} |f'(v)| = \sum_{v \in V(G)} |f(v)|$, $|f'(N(u))| = |f(N(u))|$. For example, G is a graph with $V(G) = \{u,v,s\}$, $E(G) = \{uv, us, vs\}$ and f is a total 5-rainbow dominating function of G such that $f(u) = \{1,4,5\}$, $f(v) = \{2,3,4\}$, $f(s) = \{1,2\}$. Then, we try to obtain a total 5-rainbow dominating function f' such that $f'(u) = \{1,2,3\}$. Thus, let $B = \{4,5\}$, $C = \{2,3\}$. Then, for $f(v) = \{2,3,4\}$, change 3 into 5, 4 into 2, 2 into 4, so $f'(v) = \{2,4,5\}$. For $f(s) = \{1,2\}$, change 2 into 4, $f'(s) = \{1,4\}$. For $f(u) = \{1,4,5\}$, change 4 into 2, change 5 into 3, so $f'(u) = \{1,2,3\}$, as desired). □

Lemma 5. *Let P and Q be disjoint graphs and u and v be the vertices of P and Q, respectively. Suppose that $G = (V,E)$ is a new graph with $V(G) = V(P) \cup V(Q)$, $E(G) = E(P) \cup E(Q) \cup \{uv\}$. Then, the following statements hold:*
(a) $\gamma(G,u,t,0) = \gamma(P,u,t,0) + \gamma(Q,v,\emptyset,k-t)$, $1 \leq t \leq k$;
(b) $\gamma(G,u,t,1) = \min\{\gamma(P,u,t,1) + \min\limits_{1 \leq t_1 \leq k}\{\gamma(Q,v,\emptyset,k-t), \gamma(Q,v,t_1,0), \gamma(Q,v,t_1,1)\},$
$\gamma(P,u,t,0) + \min\limits_{1 \leq t_1 \leq k}\{\gamma(Q,v,t_1,0), \gamma(Q,v,t_1,1)\}\}$, $1 \leq t \leq k$;

(c) $\gamma(G,u,\emptyset,t) = \min_{0 \le t_1 < t} \{\gamma(P,u,\emptyset,t_1) + \gamma(Q,v,t-t_1,1), \gamma(P,u,\emptyset,t) + \gamma(Q,v,\emptyset,k)\}, 1 \le t \le k$;
(d) $\gamma(G,u,\emptyset,0) = \gamma(P,u,\emptyset,0) + \min_{1 \le t \le k} \{\gamma(Q,v,\emptyset,k), \gamma(Q,v,t,1),\}$.

Proof. If h_1 is a function of P , h_2 is a function of Q , f is a function of G such that $f(x) = h_1(x)$ for $x \in V(P)$, $f(x) = h_2(x)$ for $x \in V(Q)$, then we write $f = h_1 \cup h_2$.

(a) Let f_1 be an H-trk function of (P,u) with minimum weight such that $|f_1(u)| = t$, $f_1(N_P(u)) = \emptyset$, f_2 be an H-trk function of (Q,v) with minimum weight such that $f_2(v) = \emptyset$, $k \ge q = |f_2(N_Q(v))| \ge k-t$. Then, assume that $f_2(N_Q(v)) = \{s_1, s_2, \dots, s_q\}$, $s_i \in \{1,2,..,k\}$.

If $t+q > k$, there exists a function f_1' such that $f_1'(N_P(u)) = \emptyset$, $\sum_{v \in V(P)} |f_1(v)| = \sum_{v \in V(P)} |f_1'(v)|$ and $f_1'(u) = \{1,2,\dots,k\} \setminus \{s_1, s_2, \dots, s_q\} \cup \{x_0, x_1, .., x_{t-(k-q)}\}$, where $x_i \in \{s_1, s_2, \dots, s_q\}$ for $i \in \{0,1,..,t-(k-q)\}$ by Lemma 4.

If $t+q \le k$, let $A = \{a_1, a_2, \dots, a_t\}$, where $a_i \in \{1,2,\dots,k\} \setminus \{s_1, s_2, \dots, s_q\}$, $i \in \{1,2,\dots,t\}$. Then, there exists a function f_1' such that $f_1'(N_P(u)) = \emptyset$, $\sum_{v \in V(P)} |f_1(v)| = \sum_{v \in V(P)} |f_1'(v)|$ and $f_1'(u) = A$ by Lemma 4. Therefore, $f = f_1' \cup f_2$ is an H-trk function of (G,u) such that $|f(u)| = t$, $f(N_G(u)) = \emptyset$. Thus, $\gamma(G,u,t,0) \le \gamma(P,u,t,0) + \gamma(Q,v,\emptyset,k-t)$.

If f is an H-trk function of (G,u) with minimum weight such that $|f(u)| = t$, $f(N_G(u)) = \emptyset$, then $f = g_1 \cup g_2$, where g_1 is an H-trk function of (P,u) such that $|g_1(u)| = t$, $g_1(N_P(u)) = \emptyset$, g_2 is an H-trk function of (Q,v) such that $g_2(v) = \emptyset$, $|g_2(N_Q(v))| \ge k-t$. Thus, $\gamma(G,u,t,0) \ge \gamma(P,u,t,0) + \gamma(Q,v,\emptyset,k-t)$.

(b) Using similar strategies used in the proof of (a), we obtain the equation from the fact that f is an H-trk function of (G,u) with $|f(u)| = t$ and $f(N_G(u)) \ne \emptyset$ if and only if $f = f_1 \cup f_2$, where f_1 is an H-trk function of (P,u) with $|f_1(u)| = t$ and $f_1(N_P(u)) \ne \emptyset$ and f_2 is a total k-rainbow dominating function (TkRDF) of Q, such that $|f_2(v)| = t_1$, or f_2 is a TkRDF of Q, such that $f_2(v) = \emptyset$, $|f_2(N_Q(v))| \ge k-t$, or f_1 is an H-trk function of (P,u) with $|f_1(u)| = t$ and $f_1(N_P(u)) = \emptyset$ and f_2 is a TkRDF of Q such that $|f_2(v)| = t_1$.

(c) Using similar strategies used in the proof of (a), we obtain the equation from the fact that f is an H-trk function of (G,u) with $f(u) = \emptyset$ and $|f(N_G(u))| \ge t$ if and only if $f = f_1 \cup f_2$, where f_1 is an H-trk function of (P,u) with $f_1(u) = \emptyset$ and $|f_1(N_P(u))| \ge t_1$ and f_2 is a TkRDF of Q, such that $|f_2(v)| = t - t_1$ and $f_2(N_Q(v)) \ne \emptyset$, or f_1 is an H-trk function of (P,u) with $f_1(u) = \emptyset$ and $|f_1(N_P(u))| \ge t$ and f_2 is a TkRDF of Q such that $f_2(v) = \emptyset$ and $|f_2(N_Q(v))| = k$.

(d) We obtain the equation from the fact that f is an H-trk function of (G,u), with $f(u) = \emptyset$ and $|f(N_G(u))| \ge 0$ if and only if $f = f_1 \cup f_2$, where f_1 is an H-trk function of (P,u), with $f_1(u) = \emptyset$ and $|f_1(N_P(u))| \ge 0$ and f_2 is a TkRDF of Q. □

By Lemmas 3 and 5, we propose the following linear-time algorithm , Algorithm 1, with time-complexity $O(k^2|V(T)|)$ for computing the total k-rainbow domination number of the tree T.

Algorithm 1: $TkRD(T)$.

Input : A tree T with a tree ordering $[v_1, v_2, v_3, \ldots, v_n]$
Output: The total k-rainbow domination number $\gamma_{trk}(T)$ of T

```
for i = 1 to n do
    for t = 1 to k do
        γ(v_i, t, 0) = t;
        γ(v_i, t, 1) = γ(v_i, ∅, t) = ∞;
    end
    γ(v_i, ∅, 0) = 0;
end
for i = 1 to n do
    let v_j be the parent of v_i;
    for t = 1 to k do
        γ(v_j, t, 0) ← γ(v_j, t, 0) + γ(v_i, ∅, k − t);
        γ(v_j, t, 1) ← min{γ(v_j, t, 1) + min_{1≤s≤k} {γ(v_i, ∅, k − t), γ(v_i, s, 0), γ(v_i, s, 1)},
          γ(v_j, t, 0) + min_{1≤s≤k} {γ(v_i, s, 0), γ(v_i, s, 1)}};
        γ(v_j, ∅, t) ← min_{1≤s≤k} {γ(v_j, ∅, s) + γ(v_i, t − s, 1), γ(v_j, ∅, t) + γ(v_i, ∅, k)};
    end
    γ(v_j, ∅, 0) ← γ(v_j, ∅, 0) + min_{1≤s≤k} {γ(v_i, ∅, k), γ(v_i, s, 1)};
end
γ_trk(T) = min{γ(v_n, 1, 1), γ(v_n, 2, 1), ..., γ(v_n, k, 1), γ(v_n, ∅, k)}.
```

2.3. Complexity Difference between Total 2-Rainbow Domination and 2-Rainbow Domination

In this section, we define two classes of graphs for which the complexities of total 2-rainbow domination is different from 2-rainbow domination.

CONSTRUCTION 1: Let $G = (V, E)$ be a graph with $|V(G)| = n$, then let each vertex $v_i \in V(G)$ be the tree T_{v_i}, where $V(T_{v_i}) = \{v_i, a_i, b_i, c_i, d_i, e_i, s_i, t_i, p_i, q_i, r_i\}$, $E(T_{v_i}) = \{v_i a_i, a_i b_i, a_i c_i, c_i e_i, c_i s_i, c_i t_i, d_i p_i, d_i q_i, d_i r_i\}$, $i \in \{1, 2, \ldots, n\}$. Let $B = \{T_{v_i} | i \in \{1, 2, \ldots, n\}\}$ be the set of disjoint trees corresponding to the graph G. If $v_i v_j \in E(G)$, then add an edge $v_i v_j$ between the trees $T_{v_i} \in B$ and $T_{v_j} \in B$. Therefore, we obtain a graph G'. An example is shown in the Figure 5a,b. Let $\mathcal{GT}$ be the set of G' obtained from graphs by CONSTRUCTION 1.

CONSTRUCTION 2: Let $G = (V, E)$ be a graph with $|V(G)| = n$, then let each vertex $v_i \in V(G)$ be the graph G_{v_i}, where $V(G_{v_i}) = \{v_i, a_i, b_i, c_i, d_i, e_i, f_i, g_i, g_i^1, g_i^2, g_i^3, h_i, k_i, m_i, m_i^1, m_i^2, m_i^3, p_i, q_i, r_i, s_i, s_i^1, s_i^2, s_i^3\}$, $E(G_{v_i}) = \{v_i a_i, v_i b_i, a_i c_i, c_i b_i, a_i d_i, d_i f_i, d_i e_i, f_i e_i, f_i g_i, g_i g_i^1, g_i g_i^2, g_i g_i^3, c_i h_i, c_i k_i, h_i k_i, k_i m_i, m_i m_i^1, m_i m_i^2, m_i m_i^3, b_i p_i, p_i q_i, p_i r_i, r_i q_i, r_i s_i, s_i s_i^1, s_i s_i^2, s_i s_i^3\}$, $i \in \{1, 2, \ldots, n\}$. Let $B = \{G_{v_i} | i \in \{1, 2, \ldots, n\}\}$ be the set of disjoint graphs corresponding to the graph G. If $v_i v_j \in E(G)$, then add an edge $v_i v_j$ between the graphs $G_{v_i} \in B$ and $G_{v_j} \in B$. Therefore, we obtain a graph G'. An example is shown in the Figure 5a,c. Let $\mathcal{GG}$ be the set of G' obtained from graphs by CONSTRUCTION 2.

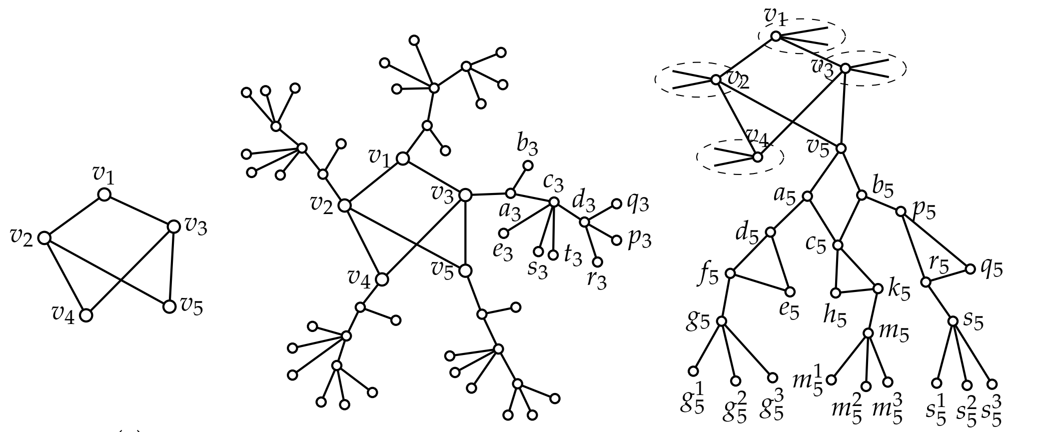


Figure 5. (**a**) The graph G, (**b**) the graph G' obtained from G by CONSTRUCTION 1, (**c**) the graph G' obtained from G by CONSTRUCTION 2.

Lemma 6. *Let $G' = (V', E')$ be a graph constructed from $G = (V, E)$ by CONSTRUCTION 1, then $\gamma_{tr2}(G') = 6n$.*

Proof. First, we define a total 2-rainbow dominating function f of G', $f : V(G') \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, such that $f(a_i) = f(c_i) = f(d_i) = \{1,2\}$, $f(v_i) = f(b_i) = f(e_i) = f(s_i) = f(t_i) = f(p_i) = f(q_i) = f(r_i) = \emptyset$, where $i \in \{1, 2, \ldots, n\}$. Clearly, f is a total 2-rainbow dominating function of G' and $\gamma_{tr2}(G') \leq \omega(f) = 6n$.

Suppose f is a γ_{tr2}-function of G'. To dominate $e_i, s_i, t_i, p_i, q_i, r_i$, it is clear that $f(c_i) = f(d_i) = \{1,2\}$, where $i \in \{1, 2, \ldots, n\}$. Since b_i need to be dominated, we have $|f(a_i)| + |f(b_i)| \geq 2$, $i \in \{1, 2, \ldots, n\}$. Thus, $\gamma_{tr2}(G') = \omega(f) \geq 6n$.

Therefore, $\gamma_{tr2}(G') = 6n$. □

Lemma 7. *Let G' be a graph constructed from G by CONSTRUCTION 1, then $\gamma_{r2}(G') = \gamma_{r2}(G) + 5n$.*

Proof. Let f be a 2-rainbow dominating function with minimum weight of G and let g be a function of G', such that $g(b_i) = \{1\}$, $g(c_i) = g(d_i) = \{1,2\}$, $g(e_i) = g(s_i) = g(t_i) = g(p_i) = g(q_i) = g(r_i) = \emptyset$, $g(v_i) = f(v_i)$, where $i \in \{1, 2, \ldots, n\}$. It is clear that g is the 2-rainbow dominating function of G', and $\omega(g) = \omega(f) + 5n = \gamma_{r2}(G) + 5n$. Therefore, $\gamma_{r2}(G') \leq \gamma_{r2}(G) + 5n$.

Conversely, let h be a γ_{r2}-function of G'. To dominate $e_i, s_i, t_i, p_i, q_i, r_i$, $|h(c_i)| + |h(e_i)| + |h(s_i)| + |h(t_i)| \geq 2$, and $|h(d_i)| + |h(p_i)| + |h(q_i)| + |h(r_i)| \geq 2$, where $i \in \{1, 2, \ldots, n\}$. Then, we define a function l of G, $l : V(G') \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ such that $l(v_i) = \emptyset$ if $h(v_i) = h(a_i) = \emptyset$ and $|h(b_i)| = 1$, $l(v_i) = \{1\}$ if $h(v_i) = \emptyset$ and $|h(a_i)| + |h(b_i)| \geq 2$, $l(v_i) = h(v_i)$ if $h(v_i) \neq \emptyset$, where $i \in \{1, 2, \ldots, n\}$. Thence, l is a 2-rainbow dominating function of G with $\omega(l) \leq \gamma_{r2}(G') - 5n$. That is $\gamma_{r2}(G) \leq \gamma_{r2}(G') - 5n$. This completes the proof of the lemma. □

Lemma 8. *Let $G' = (V', E')$ be a graph constructed from $G = (V, E)$ by CONSTRUCTION 2, then $\gamma_{tr2}(G') = \gamma_{tr2}(G) + 12n$.*

Proof. Let f be a total 2-rainbow dominating function with minimum weight of G and let g be a function of G', such that $g(c_i) = g(f_i) = g(r_i) = \{2\}$, $g(d_i) = g(p_i) = g(k_i) = \{1\}$, $g(g_i) = g(m_i) = g(s_i) = \{1,2\}$, $g(a_i) = g(b_i) = g(e_i) = g(h_i) = g(q_i) = g(g_i^1) = g(g_i^2) = g(g_i^3) = g(m_i^1) = g(m_i^2) = g(m_i^3) = g(s_i^1) = g(s_i^2) = g(s_i^3) = \emptyset$, $g(v_i) = f(v_i)$, where $i \in \{1, 2, \ldots, n\}$. It is clear that g is total 2-rainbow dominating function of G', and $\omega(g) = \omega(f) + 11n = \gamma_{tr2}(G) + 12n$. Therefore, $\gamma_{tr2}(G') \leq \gamma_{tr2}(G) + 12n$.

Conversely, let h be a γ_{tr2}-function of G', $V_0 = \{v|h(v) = \emptyset\}$. To dominate g_i^1, g_i^2, g_i^3, we have $|h(g_i^1)| + |h(g_i^2)| + |h(g_i^3)| + |h(g_i)| \geq 2$. To dominate e_i, $|h(e_i)| + |h(f_i)| + |h(d_i)| \geq 2$. Similarly, $|h(m_i^1)| + |h(m_i^2)| + |h(m_i^3)| + |h(m_i)| \geq 2$, $|h(c_i)| + |h(h_i)| + |h(k_i)| \geq 2$, $|h(s_i^1)| + |h(s_i^2)| + |h(s_i^3)| + |h(s_i)| \geq 2$, $|h(p_i)| + |h(r_i)| + |h(q_i)| \geq 2$. Therefore, $\sum_{v \in G_{v_i} \setminus \{v_i\}} |h(v)| \geq 12$ with equality if and only if $h(a_i) = h(b_i) = \emptyset$.

Then we define a function l of G, $l : V(G') \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, such that (1) if $h(a_i) = h(b_i) = \emptyset$, then $l(v_i) = h(v_i)$, (2) if $|h(a_i)| + |h(b_i)| \geq 1$ and $h(v_i) \neq \emptyset$, then $l(v_i) = h(v_i), l(v_j) = \{1\}$ for one vertex $v_j \in N(v_i) \setminus \{a_i, b_i\} \cap V_0$, (3) if $|h(a_i)| + |h(b_i)| = 1$ and $h(v_i) = \emptyset$, then $l(v_i) = \{1\}$, (4) if $|h(a_i)| + |h(b_i)| \geq 2$ and $h(v_i) = \emptyset$, then $l(v_i) = \{1\}$, $l(v_j) = \{1\}$ for one vertex $v_j \in N(v_i) \setminus \{a_i, b_i\} \cap V_0$, where $i \in \{1, 2, \ldots, n\}$.

Hence, l is a total 2-rainbow dominating function of G with $\omega(l) \leq \gamma_{tr2}(G') - 12n$. That is $\gamma_{r2}(G) \leq \gamma_{tr2}(G') - 12n$. This completes the proof of the lemma. □

Lemma 9. *Let G' be a graph constructed from G by CONSTRUCTION 2, then $\gamma_{r2}(G') = 11n$.*

Proof. First, we define a 2-rainbow dominating function f of G', $f : V(G') \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, such that $f(g_i) = f(m_i) = f(s_i) = \{1,2\}$, $f(a_i) = f(h_i) = f(q_i) = \{1\}$, $f(b_i) = f(e_i) = f(k_i) = \{2\}$, $f(v_i) = f(c_i) = f(d_i) = f(f_i) = f(k_i) = f(p_i) = f(r_i) = f(g_i^1) = f(g_i^2) = f(g_i^3) = f(m_i^1) = f(m_i^2) = f(m_i^3) = f(s_i^1) = f(s_i^2) = f(s_i^3) = \emptyset$, where $i \in \{1, 2, \ldots, n\}$. Clearly, f is a 2-rainbow dominating function of G' and $\gamma_{r2}(G') \leq \omega(f) = 11n$.

Suppose h is a γ_{r2}-function of G'. It is immediate that $|h(g_i^1)| + |h(g_i^2)| + |h(g_i^3)| + |h(g_i)| \geq 2$. $|h(m_i^1)| + |h(m_i^2)| + |h(m_i^3)| + |h(m_i)| \geq 2$, $|h(s_i^1)| + |h(s_i^2)| + |h(s_i^3)| + |h(s_i)| \geq 2$, To dominate e_i, d_i, $|h(e_i)| + |h(d_i)| + |h(a_i)| + |h(f_i)| \geq 2$. Similarly, $|h(b_i)| + |h(p_i)| + |h(r_i)| + |h(q_i)| \geq 2$. Since h_i need to be dominated, $|h(h_i)| + |h(c_i)| + |h(k_i)| \geq 1$. Thus, $\gamma_{r2}(G') = \omega(h) \geq 11n$.

Therefore, $\gamma_{r2}(G') = 11n$. □

By Lemmas 6 and 7, Lemmas 8 and 9, and the fact that the M2RDP and MT2RDP are NP-complete, the following results are immediate.

Theorem 4. *For a graph $G \in \mathcal{GT}$, the minimum 2-rainbow domination problem is NP-complete and the minimum total 2-rainbow domination problem is solvable in polynomial time.*

Theorem 5. *For a graph $G \in \mathcal{GG}$, the minimum 2-rainbow domination problem is solvable in polynomial time and the minimum total 2-rainbow domination problem is NP-complete.*

3. Conclusions

In this paper, we study the total 2-rainbow domination numbers of k-regular graphs and prove that the lower bound of total 2-rainbow domination numbers of 3-regular graphs is sharp for the double generalized Petersen graph $DP(n,1)$ when $n = 4t \geq 8$. It will be interesting to characterize the 3-regular graphs with order n, such that $\gamma_{tr2}(G) = \frac{|V(G)|}{2}$. Then, we prove that the decision problem of minimum total 2-rainbow dominating function is NP-complete for planar bipartite graphs, chordal bipartite graphs, undirected path graphs and split graphs, prove the complexity difference between minimum total 2-rainbow domination problem and minimum 2-rainbow domination problem and show a linear-time algorithm for total k-rainbow domination problem on trees. For the algorithm and hardness aspects of the total 2-rainbow domination problem, designing approximation algorithms on general graphs, or polynomial algorithms on some special classes graphs such as interval graphs, deserves further research.

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