mathematics

MDPI

Article

Asymptotic Hyperstability and Input–Output Energy Positivity of a Single-Input Single-Output System Which Incorporates a Memoryless Non-Linear Device in the Feed-Forward Loop

Manuel De la Sen

Institute of Research and Development of Processes, Department of Electricity and Electronics, Faculty of Science and Technology, University of the Basque Country (UPV/EHU), 48940 Leioa, Bizkaia, Spain; manuel.delasen@ehu.eus

Abstract: This paper visualizes the role of hyperstable controllers in the closed-loop asymptotic stability of a single-input single-output system subject to any nonlinear and eventually time-varying controller within the hyperstable class. The feed-forward controlled loop (or controlled plant) contains a strongly strictly positive real transfer function in parallel with a non-linear and memory-free device. The properties of positivity and boundedness of the input–output energy are examined based on the "ad hoc" use of the Rayleigh energy theorem on the truncated relevant signals for finite time intervals. The cases of minimal and non-minimal state-space realizations of the linear part are characterized from a global asymptotic stability (asymptotic hyperstability) point of view. Some related extended results are obtained for the case when the linear part is both positive real and externally positive and for the case of incorporation of other linear components which are stable but not necessarily positive real.

Keywords: positive realness; hyperstability; asymptotic hyperstability; passivity; sample and hold devices

MSC: 93D10; 93D15; 93D20; 34D05

check for updates

Citation: De la Sen, M. Asymptotic Hyperstability and Input–Output Energy Positivity of a Single-Input Single-Output System Which Incorporates a Memoryless Non-Linear Device in the Feed-Forward Loop. *Mathematics* **2022**, *10*, 2051. https://doi.org/10.3390/math10122051

Academic Editors: Lijun Pei and Youming Lei

Received: 24 April 2022
Accepted: 12 June 2022
Published: 13 June 2022

1. Introduction

Positive realness is a very relevant property of linear systems. A positive real transfer function has non-negative real part on the closed complex right half-plane. It has a relative degree (that is, a pole-zero excess) of 0, +1, or −1 [1–8]. Several ways and methods of designing such transfer functions in circuitry synthesis problems are given in [3–7]. Their design in the context of recursive parameter adaptation is focused on in [8]. In [9], the global asymptotic stability property is studied for a composite system with an asymptotically hyperstable subsystem. A consequence of positive realness of transfer functions is that the frequency response hodograph is confined within the first and four complex quadrants so that the maximum absolute phase of the frequency response is not larger than $\pi/2$. On the other hand, if the system is state-space realizable, then its transfer function is proper, that is, with not less than zero poles, so that its relative degree is either 0 (i.e., the transfer function is biproper, that is, it is proper with a proper inverse) or 1. Another property of such transfer functions is that they are stable, including the critical case, and so they are non-necessarily strictly stable, but eventual critical poles, if any, have to be single and with non-negative associated residuals. Furthermore, the inverses of positive real transfer functions are also positive real. On the other hand, the input–output energy of the systems described by positive real transfer functions is non-negative for all times. In this way, the positive realness of a transfer is associated with input–output energy dissipation for all times of the corresponding dynamic system. It has to be pointed out that positive realness does not directly imply the joint non-negativity of the input and the output through time, which is the so-called external positivity property, which also implies that the non-negativity for all times of both the input–output power and the input–output energy. A particular subclass of that set of positive real transfer functions is that of the so-called strictly positive real transfer functions which are strictly stable, that is, without poles at the imaginary and whose real parts are strictly positive at the open right half-plane. Positive real transfer functions are very common in the description of classical circuitry involving tandems of resistors, capacitors and inductances.

On the other hand, the class of non-linear and eventually time-varying hyperstable controllers is defined by the set of controllers which satisfy a so-called Popov's type input–output integral inequality (referred to as Popov's hyperstability condition of the whole class of controllers) [9–20]. In particular, the use of theory in different adaptive control problems is widely developed in [14,15] and some of the references therein. Its usefulness in qualitative behaviours of dynamic systems and in neural networks are focused on in [16,17], while the hyperstability in the discrete-time context is addressed in [18] for linear time-varying systems. The case of impulsive controls in hyperstability problems is focused on in [19]. On the other hand, an important stability property of the obtained closed-loop system is that a positive real transfer function under any controller belonging to the hyperstable class of controllers is "hyperstable". What this means is that it is globally stable in the large (that is, for any given finite initial condition) in the Lyapunov's sense. If the feed-forward transfer function is strictly positive real, then the closed-loop system is "globally hyperstable", that is, globally asymptotically stable in the large. It can be pointed out that Popov's hyperstability condition on the controller is also satisfied for more elementary static non-linear controllers invoked in the context of absolute stability (like the well-known Lur'e absolute stability problem within a Lur'e's sector, Popov's absolute stability criterion within a Popov's sector, etc.) [20–22]. The concept of hyperstability is closely related to the more general one of dissipativity, or its particular version of passivity, through the above-mentioned positivity/boundedness properties of the input–output energy [23–27]. On the other hand, a variety of applications in different designs in the fields of mechanics, electric machinery, circuit synthesis, model reference adaptive control, and delayed systems has been performed. See, for instance, refs. [28–35] and some references therein for more details.

Positive realness has been widely applied and linked to the hyperstability concept in adaptive control designs by taking advantage of the large universe of useful controllers, which allows a large flexibility in the design of the adaptive laws, the input/output filters to be used, and the family of free-design parameters of the estimation algorithm being compatible with the closed-loop stabilization. For similar reasons, they have been very popular for the synthesis of a wide set of regulators in electrical machinery problems. Basically, the hyperstability condition of the feedback part obtained under appropriate transformations and equivalence manipulations of the involved equations is used to get the adaptive law, which ensures the global stability of the whole scheme [14,15,33,34]. In [36], a double-convection system exhibiting chaotic behaviour with three nonlinearities is discussed, and its stability and dissipativity properties and their equilibria are investigated. On the other hand, in [37], a chaotic dissipative attractor with two quadratic nonlinearities, which possesses three unstable equilibrium points, is investigated. Its realization through an electronic circuit is also described. On the other hand, it is well-known that discrete-time models are widely invoked in practical applications because of their flexibility for the design of appropriate controllers which do not need to pick up information for all times, but only at certain sampling instants, even if the controlled system is totally or, part of, a continuous-time nature. This fact allows for simplification of the whole design, and, in general, closed-loop stabilization is achievable anyway, as are the basic needed design performances [38–42]. In this context, this paper also gives some further ideas about hyperstability designs when the continuous-time control input to the continuous-time controlled plant is generated by sample and hold devices, which pick up input-registered values at previous sampling instants, which are used to generate the continuous-time input.

The main objective of this paper is to visualize the role of the class of hyperstable controllers in the closed-loop asymptotic stability of a single-input single-output system subject to negative feedback generated by, in general, a nonlinear and eventually time-varying controller. The controller is any element within a class which satisfies a Popov's type time-integral inequality. The feed-forward loop consists of a strongly strictly positive real transfer function operating in parallel with, in general, a non-linear, memory-free device. The incorporation of such a device in the whole configuration, while guaranteeing the asymptotic hyperstability of the closed-loop system, is the main contribution of this work. Because of the intrinsic nature of the hyperstability concept, the global asymptotic stability in the large of the obtained closed-loop system is characterized for the whole class of controllers satisfying a Popov's type integral inequality. Special attention is also paid to the properties of positivity and the uniform boundedness of the input–output energy of the feed-forward-loop for all times so that the controlled system has a dissipative nature. In particular, the minimum upper-bound of such an input–output energy is given for all times by the largest negative parameter, which bounds from below the time–integral Popov's constraint that defines the class of hyperstable controllers. The main results are derived for the case that the state-space realization of the transfer function is minimal, that is, controllable and observable. There are also some further extensions of the main above results that dealt with the case of non-minimal realizations, which are stable, and for the case when the transfer function is weakly positive real, or, simply, (non-strictly) positive real. In this last case, the input–output energy is guaranteed to be non-negative and uniformly bounded for all times, and the closed-loop hyperstability is not asymptotic. Further related results are also obtained for the case when the linear part is both positive real and externally positive. In this case, the non-negative and boundedness properties of the input–output energy for all times are also fulfilled by the instantaneous input–output power, or passivity supply rate.

The results are obtained for the mentioned devices being saturated and linear, while non-necessarily being proportional to the input, and nonlinear, including the constant, linear, and quadratic terms of the input. Some further results are also obtained for when the linear part of the system is a parallel connection of a strictly positive real transfer function with a strictly stable one, which has a sufficiently small resonance peak compared to the minimum (positive) value of the real part of its counterpart integrated with the mentioned linear tandem and connected in parallel. Further, some applications are developed for the case when the continuous-time input is generated from a very general sampling and hold device, which generates the current inter-sample input value, in general, from its two last previous sampled values.

The paper is organized as follows: Section 2 states the main results for a closed-loop system whose feed-forward loop is a linear system described by a strictly positive real transfer function operating in series with a bounded nonlinear operator on the input, and the feedback loop is any controller belonging to an hyperstable class defined by an integral-type, Popov's-type hyperstability constraint. The closed-loop system is proven to be asymptotically hyperstable if the transfer function of the feed-forward loop is strongly strictly positive real. Other proven results are the integrability of the squared input and the squared output on the whole interval of time and the non-negativity and boundedness of the input–output integral energy. Some extensions are given in Section 3 for: (a) weaker constraints related to weak strict positive realness on the transfer function; (b) the tandems of the strictly positive real transfer function with another strictly stable one which does not have, in general, positive realness properties; and (c) other alternative constraints on the cascaded nonlinear operator on the input combined with the above variants. In addition, in the case where the transfer function is only positive real but not strictly positive real, some further parallel conditions are obtained for the input and output to those in Section 2. In particular, some hyperstability conditions are proven if the transfer function is both externally positive and positive real. However, the asymptotic hyperstability property is not concluded, in general. Section 4 develops some applications of the former theoretical results to the case where the input is generated from a general sampling and hold device of speed correction, which generates the current inter-sample input value from the two last sampled values according to a correcting design coefficient, and, at the same time, it satisfies an "ad hoc" Popov's-type hyperstability integral constraint. Finally, conclusions end the paper.

Notation

The following notation will be used through the manuscript:

$$\boldsymbol{R}_{0+} = \boldsymbol{R}_+ \cup \{0\};\ \boldsymbol{R}_+ = \{r \in \boldsymbol{R} : r > 0\},$$

$$\boldsymbol{Z}_{0+} = \boldsymbol{Z}_+ \cup \{0\};\ \boldsymbol{Z}_+ = \{z \in \boldsymbol{Z} : z > 0\}, \text{ and}$$

$$\boldsymbol{C}_{0+} = \boldsymbol{C}_+ \cup \{\boldsymbol{iR}\};\ \boldsymbol{C}_+ = \{w \in \boldsymbol{C} : Re\, w > 0\},$$

where $\boldsymbol{R}$, $\boldsymbol{Z}$, and $\boldsymbol{C}$ are the sets of real, integer, and complex numbers, respectively, the real set $\boldsymbol{R}$ can be extended, including the infinity points, to $\overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{\pm\infty\}$. In the same way, the extended $\overline{\boldsymbol{R}}_{0+} = \boldsymbol{R}_{0+} \cup \{+\infty\}$, $\overline{\boldsymbol{R}}_+ = \boldsymbol{R}_+ \cup \{+\infty\}$, and $\boldsymbol{iR} = \{\boldsymbol{i}\omega : \omega \in \boldsymbol{R}\}$ are defined as the set of pure imaginary complex numbers, $\boldsymbol{i} = \sqrt{-1}$ is the complex unit, u_t is the truncation in the $[0, t]$ of $u : \boldsymbol{R} \to \boldsymbol{R}$, that is, $u_t(\tau) = u(\tau)$ if $\tau \in [0, t]$ and $u_t(\tau) = 0$ if $\tau \in (-\infty, 0) \cup (t, +\infty)$ and $(f * h)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$, and $\forall t \in \boldsymbol{R}_{0+}$ is the convolution of $f, h : \boldsymbol{R} \to \boldsymbol{R}$. If $f, h : \boldsymbol{R} \to \boldsymbol{R}$, then

$$(f*h)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f_t(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(\tau)g_t(t-\tau)d\tau = \int_{-\infty}^{\infty} f_t(\tau)g_t(t-\tau)d\tau$$

$$= (f_t * h)(t) = (f * h_t)(t) = (f_t * h_t)(t)$$

$$= (f_t * h)_t(t) = (f * h_t)_t(t) = (f_t * h_t)_t(t) = (f * h)_t(t);\ \forall t \in \boldsymbol{R}_{0+},$$

where $\hat{g}(s)$ and $\hat{g}(\boldsymbol{i}\omega)$ are the Laplace and Fourier transforms of $g : \boldsymbol{R}_{0+} \to \boldsymbol{R}$, if they exist.

The strictly positive real transfer functions (in the set ***SPR***), and, respectively, positive real transfer functions (in the set ***PR***) $\hat{g}(s)$ and $s \in \boldsymbol{C}$ are analytic in $Re\, s \geq 0$ (respectively, in $Re\, s > 0$) and, if they are state-space realizable, then they have a relative degree (i.e., a pole-zero excess) of either unity or zero. The set ***SPR*** of the strictly positive real transfer functions is included in the set ***PR*** of (non-strict) positive real transfer functions, the first ones being strictly stable while those ones in the second set are required to be only stable. The set ***SSPR***, a subset of ***SPR***, is a set of strongly strictly positive real transfer functions of interest though the manuscript such that $\hat{g} \in$ ***SSPR*** if, and only if, $Re\, \hat{g}(s) > 0$ for all $Re\ s \geq 0$ and also for $Re\, s \to +\infty$. In addition, the set of the so-called weakly strictly positive real transfer functions, ***WSPR*** [1], does not necessarily maintain the strict positive realness of $Re\, s \to +\infty$ and can be proper (that is, with the number of zeros not exceeding the number of poles), while not necessarily bi-proper (i.e., those being proper with a proper inverse, so with an identical number of poles and zeros). We note that the above sets possess the set inclusion properties ***SSPR*** $\subset$ ***SPR*** $\subset$ ***PR*** and ***WSPR*** $\subset$ ***SPR*** $\subset$ ***PR*** from more restrictive to less restrictive conditions. On the other hand, strictly positive real transfer functions are strictly stable, while positive real transfer functions can have single poles at the imaginary complex axis.

The main of the above-mentioned sets of transfer functions for our central purposes in this paper is that of the strongly strictly positive real transfer functions ***SSPR***, whose members have a strictly positive real part of the transfer function on the open right-half-plane. Such transfer functions are also bi-proper (that is, they have the same number of poles and zeros) and strictly stable (all poles are in Res < 0).

We will refer to a strictly stable linear system as being one with all the poles of its transfer function in the open left-hand-side complex plane $\boldsymbol{C}_{0-} = \{s \in \boldsymbol{C} : Re\, s < 0\}$, and we refer to a stable linear system as being one with poles in the closed left-hand-side complex plane. In the first case, the matrix of dynamics is a stability matrix whose eigenvalues are such poles. In the second case, some eigenvalues can be allocated at the imaginary complex axis. We refer indistinctly to both of the above system stability concepts, as well to the respective transfer functions, as a strictly stable, or respectively stable, systems or transfer functions.

2. Problem Statement and Main Results

It is well known that the Fourier transform $\hat{g}(\boldsymbol{i}\omega) = F(g(t)) = \int_{-\infty}^{\infty} g(t)e^{-\boldsymbol{i}\omega t}dt$ with $\hat{g} : \boldsymbol{i}\boldsymbol{R} \to \boldsymbol{C}$ of $g : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ exists if g is absolutely integrable on $\boldsymbol{R}$ and the Laplace transform $\hat{g}(s) = L(g(t)) = \int_{0}^{\infty} g(t)e^{-(\sigma+\boldsymbol{i}\omega)t}dt$ is defined for a real $\sigma \geq \sigma_0$ and some $\sigma_0 \in \boldsymbol{R}$.

For a given stable linear dynamic single-input $(u(t))$ single-output $(y(t))$ system of impulse response $g(t)$: $\hat{g}(s) = \hat{y}(s)/\hat{u}(s)$ is the so-called transfer function, which is the Laplace transform of $g(t)$, which equalizes the quotient of the Laplace transform of the output to the Laplace transform of the input under null initial conditions, and $\hat{g}(\boldsymbol{i}\omega)$ is its so-called frequency response, which is the Fourier transform of $g(t)$.

Note that unstable linear systems can still be analysed though a Laplace transforms context, but not under a Fourier transform one.

Remember also that $\hat{g}(s) \in \boldsymbol{SSPR}$ if, and only if, $Re\ \hat{g}(s) > 0$ for $Re\ s \geq 0$. This property also implies that $Re\ \hat{g}(\boldsymbol{i}\omega) > 0$ and $\forall \omega \in \boldsymbol{R}$ (thus, $Re\ \hat{g}(\boldsymbol{i}\omega) > 0$, $\forall \omega \in \boldsymbol{R}$, and $\lim_{\omega \to \pm\infty} Re\ \hat{g}(\boldsymbol{i}\omega) > 0$), and that $\hat{g}(s)$ is bi-proper (i.e., it has the same number of zeros and poles) and strictly stable, i.e., all its poles are in $Re\ s < 0$.

L_∞ is the set of essentially bounded real functions on $\boldsymbol{R}$ and L_2 is the set of square-integrable functions on $\boldsymbol{R}$. The functions considered in this paper are identically zero on the negative real semi-axis. Therefore, if essential boundedness and square-integrability, respectively, are proven to hold on $\boldsymbol{R}_{0+}$, then they are in L_∞, respectively, in L_2.

For any given control $u : \boldsymbol{R}_{0+} \to \boldsymbol{R}$, the output of the controlled dynamic system $\boldsymbol{P}$ (or plant), under zero initial conditions, is:

$$y(t) = (g * u)(t) + W(u_t);\ \forall t \in \boldsymbol{R}_{0+} \tag{1}$$

where $g : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ is the impulse response of the linear part, which is the Laplace inverse transform of the transfer function $\hat{g}(s)$, $W : \boldsymbol{R} \to \boldsymbol{R}$ is, in general, nonlinear, and

$$u(t) = -v(t);\ \forall t \in \boldsymbol{R}_{0+} \tag{2}$$

gives the control action under negative feedback of the hyperstable feedback controller $K \in \boldsymbol{K}$ (the class of hyperstable controllers) of input $y(t)$ (that is, is, the output of the feed-forward controlled system) and output $v : \boldsymbol{R}_{0+} \to \boldsymbol{R}$, which is assumed to satisfy the subsequent Popov's hyperstability input–output integral condition for some nonzero finite real constant γ_0:

$$\int_0^t v(\tau)y(\tau)d\tau \geq -\gamma_0^2 > -\infty;\ \forall t \in \boldsymbol{R}_{0+} \tag{3}$$

The definitions of hyperstability and asymptotic hyperstability in Popov's sense follow below. See, for instance, refs. [9–15].

Definition 1. *The controlled dynamic system $\boldsymbol{P}$ of an input–output relation defined by (1) is hyperstable if, for any control input satisfying the integral inequality (3), the zero-state solution $x(t)$ for all $t \in \boldsymbol{R}_{0+}$ of any minimal state-space realization of n-th order of the linear part of (1) is globally stable in the large (that is, for any given finite initial condition $x_0 \in \boldsymbol{R}^n$) in the sense that the subsequent relation holds for some positive real constants δ and K:*

$$\|x(t)\| \leq K(\|x(0)\| + \delta);\ \forall t \in \boldsymbol{R}_{0+}$$

Definition 2. *The controlled dynamic system* $\boldsymbol{P}$ *of an input–output relation defined by (1) is asymptotically hyperstable if it is hyperstable in the sense of Definition 1 and, in addition,* $x(t) \to 0$ *as* $t \to \infty$.

The following features can be emphasized concerning the above hyperstability concepts:

1. The hyperstability (asymptotic hyperstability) property is a global Lyapunov stability (global Lyapunov asymptotic stability) property in the large (i.e., on the whole state space) for any control which satisfies (3), which defines a whole class of admissible controllers. Thus, it is not a global stability property for a particular control law, but it holds inherently for a whole class of controllers. The whole class of controllers can include linear and nonlinear static members, as well as time-varying ones, subject to the constraints of (3). If the feed-forward controlled plant is linear and time-invariant, it is well known that it has to be defined by a positive real (strictly positive real) transfer function in order to achieve the hypesrtability (asymptotic hyperstability) of the closed-loop system for the whole class of controllers satisfying the integral inequality of (3).
2. It has been common in the classical background literature to use the terminology that a closed-loop configuration (1)–(3) is hyperstable if both the feed-forward block (or controlled plant (1)) is hyperstable and the feedback loop (or the class of stabilizing controller (2) and (3)) is hyperstable as well. See, for instance [13–15] and some references therein. However, it can be pointed out that the class of stabilizing feedback controllers (in the hyperstability context) can include static members so that it can be preferable to refer to the hyperstability as a property of the plant under all controller members belonging to the hyperstable class of controllers.
3. Since hyperstability and asymptotic hyperstability are very wide classes of global Lyapunov's stability, those properties can be characterized via Lyapunov function candidates. Exhaustive discussion on their associated Lyapunov function can be found in [13–15]. It can be pointed out as well that the hyperstability approach is not more general for stability characterization than the standard Lyapunov theory, but it allows for characterization of the stability for a whole class of controllers which satisfy and input–output integral constraint. This class contains eventually linear controllers, classes of static non-linear ones under a sector-type (Lur'e or Popov type) constraint, or eventually time-varying controllers.
4. In Definition 1, it is assumed that the state-space realization is minimal, that is, of minimum order of the state for the given transfer function, which implies that the transfer function has no zero-pole cancellation and the state-space realization is jointly controllable and observable. This constraint is not strictly necessary and some extensions under its removal will be given in Section 3. However, that minimality constraint helps to create an easy understanding of the property at a first glance since it becomes obvious that non-minimal realizations with eventual zero-pole unstable cancellations in the transfer function are not stable, and so they could never be hyperstable, either.
5. Some basic properties associated with hyperstability of a closed-loop configuration rely on the fact that the input–output energy of the feed-forward block is both non-negative and uniformly bounded for all times. This is the main mathematical tool addressed in this research to obtain the given results.
6. The main objective of this study is to extend the asymptotic hyperstability property to the presence of certain nonlinear devices in the feed-forward loop, which are allocated in a series tandem with the linear time-invariant part, and to characterize the strong type of strict positive realness of the linear time-invariant part, leading to the asymptotic hyperstability of the closed-loop configuration.

Assumption 1. *The control* $u : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ *is admitted to have "a priori", any number of finite bounded discontinuities, and a finite number of impulsive discontinuities.*

The following result holds for a feed-forward controlled dynamic system under the class of hyperstable controllers $\boldsymbol{K}$. It is concerned with sufficiency-type conditions for the positivity and boundedness of the input–output energy and asymptotic vanishing conditions of the input and output of $\boldsymbol{P}$ under certain stipulations on $g : \boldsymbol{R}_{0+} \to \boldsymbol{R}$.

Theorem 1. *The following properties hold for any controller $K \in \boldsymbol{K}$:*

(i) *The input–output energy is bounded for all time, that is,*

$$E(t) = \int_0^t y(\tau)u(\tau)d\tau \leq \gamma_0^2 < \infty;\ \forall t \in \boldsymbol{R}_{0+}$$

(ii) *Assume that $\hat{g}(s) \in \boldsymbol{SSPR}$ and that $W(u)$ is bounded. Then, $\underset{t \in \boldsymbol{R}_{0+}}{ess\,sup}\,|u(t)| < \infty$ and $\underset{t \to \infty}{lim\,inf}|u(t)| \leq \frac{\underset{u \in \boldsymbol{R}}{sup}|W(u)|}{\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(i\omega)}$.*

(iii) *If, in addition to the conditions of Property (ii), $W(u_t(\tau)) \geq \lambda(t, u(t))u(\tau)$, $\forall \tau \in [0, t]$, and $\forall t \in \boldsymbol{R}_{0+}$ for some $\lambda : \boldsymbol{R}_{0+} \times \boldsymbol{R} \to \boldsymbol{R}$, subject to $\underset{t \in \boldsymbol{R}_{0+}}{inf}\ \lambda(t, u(t)) > -\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega)$, then $u \in L_\infty \cap L_2$ and $|u| \in L_1$ as well, so that $u(t) \to 0$ as $t \to \infty$, except, eventually, on an interval of zero measure, and $E(t) \in \left[0, \gamma_0^2\right]$ and $\forall t \in \boldsymbol{R}_{0+}$, that is, the input–output energy is non-negative bounded for all time. If, furthermore, $u(t)$ has support on some real interval of nonzero measure S, then $E(t) \in \left(0, \gamma_0^2\right]$ and $\forall t \geq t_1 > 0$, where $(0, t_1)$ is the first connected component of S, that is, the input–output energy is jointly positive and bounded on $[t_1, \infty)$.*

(iv) *If, in addition to the conditions of Properties (ii)–(iii), $W(0) = 0$, then $y(t)$ is bounded, $\forall t \in \boldsymbol{R}_{0+}$ for any given finite initial conditions, and $y(t) \to 0$ as $t \to \infty$. Furthermore, $y \in L_2$ so that both $|u|$, $|y| \in L_\infty \cap L_1 \cap L_2$.*

Proof. Note that (3) combined with (2) proves Property (i).

Now, note that by using Rayleigh (or Parseval's) energy theorem [13,19,43], it follows that $\int_0^\infty y(\tau)u_t(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^\infty \hat{y}(\boldsymbol{i}\omega)\hat{u}_t(-\boldsymbol{i}\omega)d\omega$, so that, from the symmetry property of the Fourier transform, one gets:

$$\begin{aligned} +\infty > \gamma_0^2 \geq E(t) &= \textstyle\int_0^\infty y(\tau)u_t(\tau)d\tau = \int_0^\infty\ [(g * u)(\tau) + W(u_\tau)]u_t(\tau)d\tau \\ &= \tfrac{1}{2\pi}\textstyle\int_{-\infty}^\infty \hat{y}(\boldsymbol{i}\omega)\hat{u}_t(-\boldsymbol{i}\omega)d\omega = \tfrac{1}{2\pi}\int_{-\infty}^\infty \hat{g}(\boldsymbol{i}\omega)\hat{u}_t(\boldsymbol{i}\omega)\hat{u}_t(-\boldsymbol{i}\omega)d\omega \\ &= \tfrac{1}{2\pi}\textstyle\int_{-\infty}^\infty \hat{g}(\boldsymbol{i}\omega)|\hat{u}_t(\boldsymbol{i}\omega)|^2 d\omega + \int_0^\infty W(u_t(\tau))(u_t(\tau))d\tau \\ &\geq \tfrac{1}{2\pi}\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega)\textstyle\int_{-\infty}^\infty |\hat{u}_t(\boldsymbol{i}\omega)|^2 d\omega + \int_0^\infty W(u_t(\tau))(u_t(\tau))d\tau;\ \forall t \in \boldsymbol{R}_{0+}, \end{aligned} \tag{4}$$

since $\hat{g} \in SPR$ implies that $Re\,\hat{g}(\boldsymbol{i}\omega) \geq d = \underset{\omega \in \boldsymbol{R}}{\underline{inf}}\,Re\,\hat{g}(\boldsymbol{i}\omega) > 0$ and since the hodograph $\hat{g}(i\omega)$ again has the symmetry property $Re\ \hat{g}(\boldsymbol{i}\omega) = Re\ \hat{g}(-\boldsymbol{i}\omega)$, and $Im\hat{g}(\boldsymbol{i}\omega) = -Im\hat{g}(-\boldsymbol{i}\omega)$ and $\forall \omega \in \boldsymbol{R}$ (due to the symmetry of the Fourier transform), one has to infer from the above inequality and the Rayleigh energy theorem that:

$$\begin{aligned}\infty > \gamma_0^2 \geq E(t) &\geq \underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega)\int_0^\infty u_t^2(\tau)d\tau + \int_0^\infty W(u_t(\tau))(u_t(\tau))d\tau \\ &= \underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega)\int_0^t u^2(\tau)d\tau + \int_0^t W(u_t(\tau))(u(\tau))d\tau \\ &\geq \underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(i\omega)\int_0^t u^2(\tau)d\tau - \underset{u\in\boldsymbol{R}}{sup}|W(u)|\int_0^t|u(\tau)|d\tau;\ \forall t\in\boldsymbol{R}_{0+}\end{aligned} \tag{5}$$

It is proven by contradiction that $u(t)$ is essentially bounded. If we assume that it is not essentially bounded, then there is a strictly increasing sequence $\{t_i\}_{i=0}^\infty\,(\subset \boldsymbol{R}_{0+})$ such that

$$\left(\int_0^{t_i} u^2(\tau)d\tau\right)/\left(\int_0^{t_i}|u(\tau)|d\tau\right) \geq M_i$$

for some strictly increasing sequence $\{M_i\}_0^\infty(\subset \boldsymbol{R}_{0+}) \to \infty$ as $i \to \infty$. Then,

$$\frac{\gamma_0^2}{\int_0^{t_i}|u(\tau)|d\tau} \geq \frac{\underset{\omega\in\boldsymbol{R}_{0+}}{inf}\ Re\,\hat{g}(\boldsymbol{i}\omega)\int_0^t u^2(\tau)d\tau}{\int_0^{t_i}|u(\tau)|d\tau} - \underset{u\in\boldsymbol{R}}{sup}|W(u)| \geq \underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(i\omega)M_i - \underset{u\in\boldsymbol{R}}{sup}|W(u)|$$

so that, since $W(u)$ is bounded,

$$\infty > \underset{t\to\infty}{lim\,inf}\left(\underset{u\in\boldsymbol{R}}{sup}|W(u)| + \frac{\gamma_0^2}{\int_0^{t_i}|u(\tau)|d\tau} - M_i\ \underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega)\right) \geq 0, \tag{6}$$

a contradiction since $\underset{\omega\in\boldsymbol{R}}{\underline{inf}}Re\ \hat{g}(\boldsymbol{i}\omega) > 0$ and $\{M_i\}_0^\infty(\subset \boldsymbol{R}_{0+}) \to \infty$ as $i \to \infty$. Thus, $\underset{t\in\boldsymbol{R}_{0+}}{ess\,sup}\,|u(t)| < \infty$.

It is now proven that $\underset{t\to\infty}{lim\,inf}|u(t)| \leq \frac{\underset{u\in\boldsymbol{R}}{sup}|W(u)|}{\underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega)}$. Assume, on the contrary, that there is some $|\underline{u}| > \frac{\underset{u\in\boldsymbol{R}}{sup}|W(u)|}{\underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(i\omega)} > 0$ such that $\underset{t\to\infty}{lim\,inf}|u(t)| = |\underline{u}|$. Then,

$$\gamma_0^2 \geq \underset{t\to\infty}{lim\,inf}\left(\underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega)|\underline{u}| - \underset{u\in\boldsymbol{R}}{sup}|W(u)|\right)\left(\int_t^{t+\theta}|u(\tau)|d\tau\right) \text{ and } \forall\theta\in\boldsymbol{R}_{0+} \tag{7}$$

so that one gets the subsequent contradiction:

$$\infty > \gamma_0^2 \geq \underset{\theta\to\infty}{lim}\left[\underset{t\to\infty}{lim\,inf}\left(\underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega)|\underline{u}| - \underset{u\in\boldsymbol{R}}{sup}|W(u)|\right)\left(\int_t^{t+\theta}|u(\tau)|d\tau\right)\right] = \infty.$$

Then, either $\underset{t\to\infty}{lim\,inf}|u(t)| \leq \frac{\underset{u\in\boldsymbol{R}}{sup}|W(u)|}{\underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(i\omega)}$ or $\underset{t\to\infty}{lim\,inf}|u(t)| > \frac{\underset{u\in\boldsymbol{R}}{sup}|W(u)|}{\underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(i\omega)}$ and $u(t) \to 0$ as $t \to \infty$, except eventually on a time interval of zero measure, the second condition being a contradiction itself. Therefore, $\underset{t\to\infty}{lim\,inf}|u(t)| \leq \frac{\underset{u\in\boldsymbol{R}}{sup}|W(u)|}{\underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\; Re\,\hat{g}(\boldsymbol{i}\omega)}$. Property (ii) has been proven.

Now, also assume that $W(u_t(\tau)) \geq \lambda(t\,,u(t))u(\tau), \forall\tau \in [0\,,\,t]$, and $\forall t \in \boldsymbol{R}_{0+}$, so that one obtains from (5) $\infty > \gamma_0^2 \geq E(t) \geq \underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\; Re\,\hat{g}(\boldsymbol{i}\omega)\int_0^\infty u_t^2(\tau)d\tau + \int_0^\infty \lambda(t\,,u(t))u(\tau)u_t(\tau)d\tau$

$$\geq \left(\underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\; Re\,\hat{g}(\boldsymbol{i}\omega) + \underset{t\in\boldsymbol{R}_{0+}}{inf}\;\lambda(t\,,\,u(t))\right)\int_0^\infty u_t^2(\tau)d\tau$$

$$= \left(\underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\; Re\,\hat{g}(\boldsymbol{i}\omega) + \underset{t\in\boldsymbol{R}_{0+}}{inf}\;\lambda(t\,,\,u(t))\right)\int_0^t u^2(\tau)d\tau;\ \forall t \in \boldsymbol{R}_{0+}, \tag{8}$$

and, if, furthermore, $\underset{t\in\boldsymbol{R}_{0+}}{inf}\;\lambda(t\,,\,u(t)) > -\underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\; Re\,\hat{g}(\boldsymbol{i}\omega)$, then, in addition to the previously proved properties $\underset{t\in\boldsymbol{R}_{0+}}{ess\,sup}\,|u(t)| < \infty$ and $u \in L_2$, then $u \in L_\infty \cap L_2$, so that $u(t) \to 0$ as $t \to \infty$, except eventually on a time interval of zero measure, and it follows from (8) that $E(t) \in \left[0\,,\gamma_0^2\right]$ and $\forall t \in \boldsymbol{R}_{0+}$; that is the, input–output energy is non-negative bounded for all times. If, in addition, $u(t)$ has support on some real interval of nonzero measure S, then $E(t) \in \left(0, \gamma_0^2\right]$ and $\forall t \geq t_1$, where $(0, t_1)$ is the first connected component of S. Property (iii) has been proven.

Property (iv) follows since $\hat{g}(s) \in \boldsymbol{SSPR}$, thus:

(1) the bi-proper and strictly stable system of the form $\hat{g}(s) = \hat{g}_1(s) + d$ with $d = \underset{\omega\in\boldsymbol{R}_{0+}}{\underline{inf}}\; Re\,\hat{g}(\boldsymbol{i}\omega) > 0$, since $\hat{g}_1(s) \in \boldsymbol{PR}$ is the proper of unity relative order with $Re\,\hat{g}_1(s) \geq 0$ for $Re\,s \geq 0$ and $\underset{|s|\to\infty}{lim}\; Re\,\hat{g}_1(s) = 0$;

(2) $W(u_t(\tau)) \geq \lambda(t\,,u(t))u(\tau)$, $\tau \in [0\,,\,t]$; $\forall t \in \boldsymbol{R}_{0+}$; and

(3) $\underset{t\in\boldsymbol{R}_{0+}}{inf}\;\lambda(t\,,\,u(t)) > -d$ then $\underset{t\in\boldsymbol{R}_{0+}}{sup}\;|y(t)| < \infty$ and $y(t) \to 0$ as $t \to \infty$ for any given finite initial conditions since:

(a) It has already been proven that $\underset{t\in\boldsymbol{R}_{0+}}{ess\,sup}\,|u(t)| < \infty$ and $u(t) \to 0$ as $t \to \infty$ (except eventually on a set of zero measure), and furthermore,

(b) for any eventually non-zero initial conditions $x(0) = x_0$, the solution of (1) has an extra additive function $y_0 : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$, which is bounded and exponentially vanishing, since $\hat{g}(s)$ is strictly stable and $g(t)$ also asymptotically vanishes, so that

$$|y(t)| = |(g * u)(t)| + |W(u_t)| + |y_0(t)| \leq \overline{y} < \infty;\ \forall t \in \boldsymbol{R}_{0+} \tag{9}$$

$$\underset{t\to\infty}{lim}\,y(t) = \underset{t\to\infty}{lim}\,(g * u)(t) + W(0) + \underset{t\to\infty}{lim}\,y_0(t) = 0;\ \forall t \in \boldsymbol{R}_{0+} \tag{10}$$

since $W(0) = 0$. Since $|u| \in L_\infty \cap L_1 \cap L_2$, then $|y| \in L_\infty \cap L_1 \cap L_2$. □

The following result is a direct extension of Theorem 1 if there are two linear strictly stable systems in parallel connection in the feed-forward loop, with one of them having strict positive realness properties.

Corollary 1. *Assume that the output of the system is*

$$y(t) = ((g + g_a) * u)(t) + W(u_t);\ \forall t \in \boldsymbol{R}_{0+} \tag{11}$$

Instead of (1), while (2) remains identical, with $g : \mathbf{R}_{0+} \to \mathbf{R}$ and $W : \mathbf{R} \to \mathbf{R}$, satisfy the conditions of Theorem 1 and $\hat{g}(s)$ and $g_a : \mathbf{R}_{0+} \to \mathbf{R}$ is the impulse response of a strictly stable linear system.

The following properties hold for any controller $K \in \mathbf{K}$:

(i) *Property (i) of Theorem 1 holds.*

(ii) *Assume that $\hat{g}(s) \in$ **SSPR**, $\hat{g}_a(s)$ is strictly stable, and $W(u)$ is bounded with $\underset{\omega\in\mathbf{R}_{0+}}{\overline{sup}}\,|\hat{g}_a(\boldsymbol{i}\omega)| < \underset{\omega\in\mathbf{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega)$.*

Then, $\underset{t\in\mathbf{R}_{0+}}{ess\,sup}\,|u(t)| < \infty$ and $\underset{t\to\infty}{lim\,inf}\,|u(t)| \le \dfrac{\underset{u\in\mathbf{R}}{sup}|W(u)|}{\underset{\omega\in\mathbf{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega) - \underset{\omega\in\mathbf{R}_{0+}}{\overline{sup}}\ |\hat{g}_a(\boldsymbol{i}\omega)|}$.

(iii) *If, in addition to the conditions of Property (ii), $W(u_t(\tau)) \ge \lambda(t, u(t))u(\tau)$, $\forall\tau \in [0, t]$, and $\forall t \in \mathbf{R}_{0+}$ for some $\lambda : \mathbf{R}_{0+} \times \mathbf{R} \to \mathbf{R}$, subject to $\underset{t\in\mathbf{R}_{0+}}{inf}\ \lambda(t, u(t)) > \underset{\omega\in\mathbf{R}_{0+}}{\overline{sup}}\ |\hat{g}_a(\boldsymbol{i}\omega)| - \underset{\omega\in\mathbf{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega)$, then $u \in L_\infty \cap L_2$, thus $u(t) \to 0$ as $t \to \infty$, except eventually on an interval of zero measure, and $E(t) \in [0, \gamma_0^2]$ and $\forall t \in \mathbf{R}_{0+}$, that is the, input–output energy is non-negative bounded for all times. If, furthermore, $u(t)$ has support on some real interval of nonzero measure S, then $E(t) \in (0, \gamma_0^2]$ and $\forall t \ge t_1 > 0$, where $(0, t_1)$ is the first connected component of S, that is, the input–output energy is jointly positive and bounded on $[t_1, \infty)$.*

(iv) *If, in addition to the conditions of Properties (ii) and (iii), $W(0) = 0$, then $y(t)$ is bounded, $\forall t \in \mathbf{R}_{0+}$ for any given finite initial conditions, and $y(t) \to 0$ as $t \to \infty$ and $y \in L_\infty$.*

Sketch of Proof. Equations (7)–(10) are modified as follows by taking into account the symmetry of $\hat{g}_a(\boldsymbol{i}\omega)$, which allows for taking its absolute maximum for $\omega \in \mathbf{R}_{0+}$:

$$\gamma_0^2 \ge \underset{t\to\infty}{lim\,inf}\left(\left(\underset{\omega\in\mathbf{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega) - \underset{\omega\in\mathbf{R}_{0+}}{\overline{sup}}\ |\hat{g}_a(\boldsymbol{i}\omega)|\right)|\underline{u}| - \underset{u\in\mathbf{R}}{sup}|W(u)|\right)\left(\int_t^{t+\theta}|u(\tau)|d\tau\right);\ \forall\theta \in \mathbf{R}_{0+} \tag{12}$$

$$\gamma_0^2 \ge \left(\underset{\omega\in\mathbf{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega) - \underset{\omega\in\mathbf{R}_{0+}}{\overline{sup}}\ |\hat{g}_a(\boldsymbol{i}\omega)| + \underset{t\in\mathbf{R}_{0+}}{inf}\ \lambda(t, u(t))\right)\int_0^t u^2(\tau)d\tau;\ \forall t \in \mathbf{R}_{0+} \tag{13}$$

$$|y(t)| = |(g * u)(t)| + |(g_a * u)(t)| + |W(u_t)| + |y_0(t)| \le \overline{y} < \infty;\ \forall t \in \mathbf{R}_{0+} \tag{14}$$

$$\underset{t\to\infty}{lim}\,y(t) = \underset{t\to\infty}{lim}(g * u)(t) + \underset{t\to\infty}{lim}(g_a * u)(t) + W(0) + \underset{t\to\infty}{lim}\,y_0(t) = 0;\ \forall t \in \mathbf{R}_{0+} \tag{15}$$

since $W(0) = 0$, and $g(t)$, $g_a(t)$, and $y_0(t)$ vanish exponentially. □

Remark 1. *Note that $\overline{\hat{g}}_a = \underset{\omega\in\mathbf{R}_{0+}}{\overline{sup}}\ |\hat{g}_a(\boldsymbol{i}\omega)|$ in Corollary 1 is the resonance peak of the impulse response $\hat{g}(\boldsymbol{i}\omega)$, which, if it fulfils $\underset{\omega\in\mathbf{R}_{0+}}{\overline{sup}}\ |\hat{g}_a(\boldsymbol{i}\omega)| < \underset{\omega\in\mathbf{R}_{0+}}{\underline{inf}}\ Re\,\hat{g}(\boldsymbol{i}\omega)$, then Properties (ii) to (iv) are guaranteed. Thus, we can observe the following interesting engineering viewpoint on the problem: the positivity and asymptotically stability properties of the whole closed-loop system basically rely on the fact that the resonance peak associated with the strictly stable transfer function $\hat{g}_a(s)$ is smaller than the minimum value of the real part of $\hat{g}(\boldsymbol{i}\omega)$ in the case where such a $\hat{g}_a(s)$ is not also strictly positive real. In this last case, we can omit this discussion on this added transfer function and replace $Re\,\hat{g}(\boldsymbol{i}\omega) \to Re\hat{g}(\boldsymbol{i}\omega) + Re\hat{g}_a(\boldsymbol{i}\omega)$ in Theorem 1.*

Remark 2. *Note that in Theorem 1 (iii) and Corollary 1 (iii), $u(t) \to 0$ as $t \to \infty$ if $u : \mathbf{R}_{0+} \to \mathbf{R}$ is constrained to be piecewise continuous on its definition domain, except with eventual bounded isolated discontinuities on a time subinterval of finite measure. Such a restriction on Assumption 1* *ensures that all the either finite or impulsive isolated discontinuities, if any, of the control input take place at finite times so that its square-integrability property guarantees its asymptotic vanishing property as time tends to infinity.*

In the case that the control possesses are infinitely many isolated bounded discontinuities, it can still be guaranteed under Theorem 1 or Corollary 1 that $u(t) \to 0$ as $t \to \infty$, except at its finite discontinuities at infinite times.

3. Some Direct Extensions for More General Nonlinear Operators in the Feed-Forward Loop and Further Internal Stability Considerations

Concerning the more general nonlinear operators of the form $W : \boldsymbol{R} \to \boldsymbol{R}$ [44], it is possible to obtain some further conclusions by properly extending Theorem 1 and Corollary 1 based on the related appropriate modifications of (5), (7), and (12). In this way, the following results follow:

Corollary 2. *Assume that $W(u)$ is linear, including, eventually, a constant term, that is, it is of the form $W(u) = w_0 + w_1 u$. Equation (5) becomes modified as follows:*

$$+\infty > \gamma_0^2 \geq \left(\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}} \; Re\hat{g}(\boldsymbol{i}\omega) + w_1 \right) \int_0^t u^2(\tau) d\tau - |w_0| \left(\int_0^t |u(\tau)| d\tau \right); \forall t \in \boldsymbol{R}_{0+} \tag{16}$$

Then, Theorem 1 holds with the subsequent modifications:
(a) Property (i) holds, and
If $\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}} \; Re\hat{g}(\boldsymbol{i}\omega) > -w_1$, then
(b) Property (ii) holds with $\underset{t \to \infty}{lim\,inf} |u(t)| \leq \frac{|w_0|}{\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}} \; Re\hat{g}(\boldsymbol{i}\omega) + w_1}$, and Property (iii) and Property (iv) hold.

The proof follows under close derivation steps as in the proof of Theorem 1 by using the appropriate modifications of (5) and (7).

In the same way, we have the following conclusion for the feed-forward parallel combination of the transfer functions $\hat{g}(s) + \hat{g}_a(s)$:

Corollary 3. *The following properties hold:*

(i) *Theorem 1 (i) holds and Corollary 1 [(ii)–(iv)] hold for the output being given by (11), after adding another strictly stable transfer function $\hat{g}_a(s)$ to $\hat{g}(s)$ in a parallel connection and $W(u) = w_0 + w_1 u$, provided that*

$$\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}} \; Re\hat{g}(\boldsymbol{i}\omega) > -w_1 + \underset{\omega \in \boldsymbol{R}_{0+}}{\underline{sup}} \; |\hat{g}_a(\boldsymbol{i}\omega)|$$

with (5) being modified as:

$$+\infty > \gamma_0^2 \geq \left(\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}} \; Re\hat{g}(\boldsymbol{i}\omega) + w_1 - \underset{\omega \in \boldsymbol{R}_{0+}}{\underline{sup}} \; |\hat{g}_a(\boldsymbol{i}\omega)| \right) \int_0^t u^2(\tau) d\tau - |w_0| \left(\int_0^t |u(\tau)| d\tau \right); \; \forall t \in \boldsymbol{R}_{0+} \tag{17}$$

and

$$\underset{t \to \infty}{lim\,inf} |u(t)| \leq \frac{|w_0|}{\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}} \; Re\hat{g}(\boldsymbol{i}\omega) + w_1 - \underset{\omega \in \boldsymbol{R}_{0+}}{\underline{sup}} \; |\hat{g}_a(\boldsymbol{i}\omega)|}$$

in Corollary 1(ii).

(ii) *The above results still hold for* $\hat{g}(s) \in \boldsymbol{WSPR}$, *that is, if* $\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}} Re\hat{g}(\boldsymbol{i}\omega) \geq 0$ *and* $\hat{g}(s)$ *are strictly stable but eventually proper, rather than bi-proper, so that, as can eventually happen in the case where* $\underset{\omega \to \pm\infty}{lim} Re\hat{g}(\boldsymbol{i}\omega) = 0$, *provided that* $w_1 > 0$, *under the condition* $\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}} Re\hat{g}(\boldsymbol{i}\omega) + w_1 - \underset{\omega \in \boldsymbol{R}_{0+}}{\underline{sup}} |\hat{g}_a(\boldsymbol{i}\omega)| \geq w_1 - \underset{\omega \in \boldsymbol{R}_{0+}}{\underline{sup}} |\hat{g}_a(\boldsymbol{i}\omega)| > 0$ *and* $\underset{t \to \infty}{lim\,inf} |u(t)| \leq \frac{|w_0|}{w_1 - \underset{\omega \in \boldsymbol{R}_{0+}}{\underline{sup}} |\hat{g}_a(\boldsymbol{i}\omega)|}$ *in Corollary 1(ii), provided that* $w_1 > \underset{\omega \in \boldsymbol{R}_{0+}}{\underline{sup}} |\hat{g}_a(\boldsymbol{i}\omega)|$, *and* $\underset{t \to \infty}{lim\,inf} |u(t)| \leq \frac{|w_0|}{w_1}$ *in Theorem 1(ii).*

Corollary 4. *Assume that* $W(u)$ *is linear and quadratic, including, eventually, a constant term and a positive saturation level in the quadratic term, that is,* $W(u)$ *is of the form* $W(u) = w_0 + w_1 u + w_2 sat_\Delta(u^2)$ *with* $\Delta > 0$ *and* $sat_\Delta u = u$ *if* $|u(t)| < \Delta$ *and* $sat_\Delta u = sign\ u$ *if* $|u(t)| \geq \Delta$.

Then, the following properties hold:

(i) *Equation (5) becomes modified as follows:*

$$+\infty > \gamma_0^2 \geq \left(\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}} Re\hat{g}(\boldsymbol{i}\omega) + w_1 \right) \int_0^t u^2(\tau) d\tau - (|w_0| + \Delta |w_2|) \left(\int_0^t |u(\tau)| d\tau \right); \ \forall t \in \boldsymbol{R}_{0+} \tag{18}$$

with

$$\underset{t \to \infty}{lim\,inf} |u(t)| \leq \frac{|w_0| + \Delta}{\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}} Re\hat{g}(\boldsymbol{i}\omega) + w_1} \tag{19}$$

in Theorem 1(ii), and

$$\underset{t \to \infty}{lim\,inf} |u(t)| \leq \frac{|w_0| + \Delta |w_2|}{\underset{\omega \in \boldsymbol{R}_{0+}}{\underline{inf}} Re\hat{g}(\boldsymbol{i}\omega) + w_1 - \underset{\omega \in \boldsymbol{R}_{0+}}{\underline{sup}} |\hat{g}_a(\boldsymbol{i}\omega)|} \tag{20}$$

in Corollary 1(ii) for the output being given by (11), after incorporating another strictly stable transfer function $\hat{g}_a(s)$ *to* $\hat{g}(s)$*in a parallel connection.*

(ii) *The above result still holds if* $\hat{g}(s) \in \boldsymbol{WSPR}$ *and* $w_1 > 0$ *with* $\underset{t \to \infty}{lim\,inf} |u(t)| \leq \frac{|w_0| + \Delta}{w_1}$ *in Theorem 1 (ii) and* $\underset{t \to \infty}{lim\,inf} |u(t)| \leq \frac{|w_0| + \Delta |w_2|}{w_1 - \underset{\omega \in \boldsymbol{R}_{0+}}{\underline{sup}} |\hat{g}_a(\boldsymbol{i}\omega)|}$ *in Corollary 1 (ii), provided in this second case that, in addition,* $w_1 > \underset{\omega \in \boldsymbol{R}_{0+}}{\underline{sup}} |\hat{g}_a(\boldsymbol{i}\omega)|$.

On the other hand, note that based on Theorem 1 (iv), respectively, of Corollary 1 (iv), we can conclude the global asymptotic stability of minimal state-space (that is, being both controllable and observable) realizations of $\hat{g}(s)$, respectively, of the transfer function $\hat{g}(s) + \hat{g}_a(s)$. Since such a property holds for any controller in the class $\boldsymbol{K}$, for any controller satisfying the integral inequality (3), we will conclude that the closed-loop system is asymptotically hyperstable. Thus, we can conclude the following result:

Corollary 5. *The following properties hold:*

(i) *Assume that* $\hat{g}(s)$ *has n poles for some integer* $n \geq 1$. *Then, any state-space realization of* $\hat{g}(s)$ *of state* $x : \boldsymbol{R}_{0+} \to \boldsymbol{R}^n$ *(that is, of order n , then being controllable and observable since it has no zero-pole cancellation) is globally asymptotically stable in the large if Theorem 1 (iv) holds, so that, for any given finite initial condition* $x(0) = x_0 \in \boldsymbol{R}^n$, $x : \boldsymbol{R}_{0+} \to \boldsymbol{R}^n$, $y : \boldsymbol{R}_{0+} \to \boldsymbol{R}$, *and*

*$u : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ are bounded and $x(t) \to 0$, $y(t) \to 0$, and $u(t) \to 0$ as $t \to \infty$. As a result, the closed-loop system is asymptotically hyperstable, that is, globally asymptotically stable for any controller K in the class **K** which satisfies the Popov's integral constraint (3).*

(ii) *Assume that $\hat{g}(s) = \hat{n}(s)/\hat{d}(s)$ and $\hat{g}_a(s) = \hat{n}_a(s)/\hat{d}_a(s)$ have, respectively, n and n_a poles (the respective degrees of the denominator polynomials $\hat{d}(s)$ and $\hat{d}_a(s)$ for some integers n, $n_a \geq 1$). Assume also that*

$$\hat{g}(s) + \hat{g}_a(s) = \frac{\hat{n}(s)\hat{d}_a(s) + \hat{n}_a(s)\hat{d}(s)}{\hat{d}(s)\hat{d}_a(s)} \tag{21}$$

*has no zero-pole cancellation, that is, the polynomials $\hat{n}(s)\hat{d}_a(s) + \hat{n}_a(s)\hat{d}(s)$ and $\hat{d}(s)\hat{d}_a(s)$ are coprime, for which a necessary condition is that the polynomials $\hat{n}(s)$ and $\hat{d}(s)$ and the polynomials $\hat{n}_a(s)$ and $\hat{d}_a(s)$ are coprime. Then, any state-space realization of $\hat{g}(s) + \hat{g}_a(s)$ of the state $x : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n+n_a}$ (that is, controllable and observable) is globally asymptotically stable in the large if Theorem 1 (iv) holds, so that, for any given finite initial condition $x(0) = x_0 \in \boldsymbol{R}^{n+n_a}$, $x : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n+n_a}$, $y : \boldsymbol{R}_{0+} \to \boldsymbol{R}$, and $u : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ are bounded and $x(t) \to 0$, $y(t) \to 0$, and $u(t) \to 0$ as $t \to \infty$. As a result, the closed-loop system is asymptotically hyperstable, that is, globally asymptotically stable in the large (that is, for any finite initial condition) for any controller K in the class **K** which satisfies the Popov's integral constraint (3).*

The above result can also be extended for the case of stable cancellations of the transfer and controllable (but then non-observable) realizations. A brief discussion follows for $\hat{g}(s) + \hat{g}_a(s)$ based on Corollary 1. The case for only $g(s)$ directly follows as a particular case by making $\hat{g}_a(s) = 0$. Assume that $\hat{d}_0(s)$ and $\hat{d}_{a0}(s)$ are stable polynomials (i.e., their zeros are in the complex half-plane $|z| < 1$) which contain the respective zero-pole cancellations in $\hat{g}(s)$ and $\hat{g}_a(s)$ so that (21) has a strictly stable zero-pole cancellation polynomial factor $\hat{d}_0(s)\hat{d}_{a0}(s)$ of the degree $\overline{n}_0 = n_0 n_{ao} \geq 1$, such that it is replaced with:

$$\hat{g}(s) + \hat{g}_a(s) = \frac{\hat{n}(s)\hat{d}_a(s) + \hat{n}_a(s)\hat{d}(s)}{\hat{d}(s)\hat{d}_a(s)} . \frac{\hat{d}_0(s)\hat{d}_{a0}(s)}{\hat{d}_0(s)\hat{d}_{a0}(s)} \tag{22}$$

Then, the cancelled modes associated with the zeros of $\hat{d}_0(s)\hat{d}_{a0}(s)$ are strictly stable so that, provided that any eventual cancellation in $\frac{\hat{n}(s)\hat{d}_a(s)+\hat{n}_a(s)\hat{d}(s)}{\hat{d}(s)\hat{d}_a(s)}$ is also strictly stable, it leads to the conclusion that the whole feedback system is globally asymptotically stable under Corollary 1(iv). Thus, Corollary 5(ii) is extendable as follows.

Corollary 6. *Any state-space controllable (but unobservable) realization of $\hat{g}(s) + \hat{g}_a(s)$ of state $x : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n+n_a+\overline{n}_0}$ is globally asymptotically stable in the large if Corollary 1(iv) holds, so that, for any given finite initial condition $x(0) = x_0 \in \boldsymbol{R}^{n+n_a+\overline{n}_0}$, $x : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n+n_a+\overline{n}_0}$, $y : \boldsymbol{R}_{0+} \to \boldsymbol{R}$, and $u : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ are bounded, while $x(t) \to 0$, $y(t) \to 0$, and $u(t) \to 0$ as $t \to \infty$. As a result, the closed-loop system is asymptotically hyperstable, that is, globally asymptotically stable in the large for any controller K in the class **K** which satisfies the Popov's integral constraint (3).*

Corollaries 5 and 6 are easily extendable under the generalizations for the nonlinear operator [44], as discussed in Corollaries 2–4 and for the mentioned generalizations for the case where $\hat{g}(s)$ is weakly (rather than strongly) strictly positive real, combined with the feed-forward operator, and $w_1 > \underset{\omega \in \boldsymbol{R}_{0+}}{\overline{sup}} |\hat{g}_a(\boldsymbol{i}\omega)|$ (see Corollary 1 and Corollary 4(ii)) and $w_1 > 0$ if $\hat{g}_a(s) = 0$ (see Theorem 1 and Corollary 3(ii)). Thus, basically, the global asymptotic stability is also guaranteed if the transfer function $\hat{g}(s)$ is weakly (rather than strongly) strictly positive real and the nonlinear operator W has a positive linear term coefficient sufficiently large enough to strictly compensate the resonance peak of the strictly stable transfer function $\hat{g}_a(s)$, or it is of any positive value if $\hat{g}_a(s) \equiv 0$. On the other hand, it turns out that the former asymptotic stability results do not hold if $\hat{g}(s)$ is only positive real, with single critically stable poles so that it is not strictly positive real.

Remark 3. *Note from Theorem 1(iii) that, since* $0 \leq E(t) \leq \gamma_0^2 < +\infty$ *and* $\forall t \in \boldsymbol{R}_{0+}$*, Popov's inequality(3) also holds for any* $[t_i\,,\,t_2] \subset [0\,,t)$ *and* $\forall t \in \boldsymbol{R}_{0+}$ *under the form* $\int_{t_1}^{t_2} v(\tau)y(\tau)d\tau = -E[t_1\,,t_2] \geq -\gamma_0^2 > -\infty$ *and* $\forall t_2(\geq t_1), t_1 \in \boldsymbol{R}_{0+}$*, namely, the input–output energy of the feed-forward loop along any interval of time under zero initial conditions is non-negative and bounded under the conditions of Theorem 1(iii) and the hyperstability condition of the nonlinear and eventually time-varying class of feedback controllers is the reversed inequality for the upper-bound of the input–output energy. Classically, the above property has been enounced if* $W(u) \equiv 0$ *and* $\hat{g}_a(s) = 0$*, following Theorem 1(iii)–(iv) and Corollary 5(i), as follows:*

If the feed-forward transfer function $\hat{g}(s) \in \boldsymbol{WSPR}$*, then the closed-loop system given by a controllable and observable realization of* $\hat{g}(s)$ *is globally asymptotically stable under any, eventually nonlinear and time-varying, feedback controller satisfying Popov's inequality (3). In brief, the closed-loop system is asymptotically hyperstable, that is, it is globally asymptotically stable in the large for any feedback controller belonging to the class* $\boldsymbol{K}$*.*

Another property related to dynamic systems is that of external positivity which is not directly related to positive realness of its transfer function in the linear case. A system is said to be externally (respectively, internally) positive if, for any non-negative initial conditions and control input, the output (respectively, both the state and the output) is (are) non-negative for all times. In the linear case, such a property holds if, and only if, the impulse response, i.e., the Laplace inverse transform of the transfer function, is non-negative for all times. The following result relies on the fact that if the feed-forward transfer function is positive real and externally positive but not strictly positive real, it can have single poles at the imaginary complex axis. The property of convergence to zero of the output is not guaranteed, in general.

Theorem 2. *Assume that* $\hat{g}(s) \in \boldsymbol{PR} \cap \overline{\boldsymbol{SPR}}$ *and* $W(u)$ *are bounded and* $|W(u(t))|/|u(t)| < g(t)$*, then, implying that* $g : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ *(i.e., the transfer function is externally positive and positive real although not strictly positive real),* $0 \leq E(t) \leq \gamma_0^2 < +\infty$*,* $\forall t \in \boldsymbol{R}_{0+}$*,* $|u| \in L_1 \cap L_\infty$*,* $|u| \in L_2$*,* $u(t) \to 0$ *as* $t \to \infty$*, and* $|y| \in L_\infty$ *for any hyperstable controller satisfying Popov's integral inequality.*

Proof. From the penultimate element of Equation of (4):

$$\gamma_0^2 \geq E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\boldsymbol{i}\omega)|\hat{u}_t(\boldsymbol{i}\omega)|^2 d\omega + \int_0^\infty W(u_t(\tau))(u_t(\tau))d\tau$$

$$\gamma_0^2 \geq E(t) > \frac{1}{2\pi}\int_{-\infty}^{\infty} (\hat{g}(\boldsymbol{i}\omega)|\hat{u}_t(\boldsymbol{i}\omega)| - |W(u_t(\tau))|)\,|\hat{u}_t(\boldsymbol{i}\omega)|d\omega$$

$$= \int_0^t (g(\tau)|\hat{u}_t(\tau)| - |W(u_t(\tau))|)\,|\hat{u}_t(\tau)|d\tau \geq 0;\ \forall t \in \boldsymbol{R}_{0+}$$

Since $\hat{g}(s) \in \boldsymbol{PR}$ and $0 \leq |W(u(t))|/|u(t)| < g(t)$ and $\forall t \in \boldsymbol{R}_{0+}$ so that the linear part is externally positive, since its impulse response $g : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is non-negative for all time. Proceed by contradiction by assuming that $|u(t)|$ is unbounded so that there exists at least one strictly increasing sequence $\{t_i\}_{i=0}^\infty$ such that $\{|u(t_i)|\}_{i=0}^\infty \to \infty$ and $|W(u(t_i))|/|u(t_i)| < g(t_i)$, then

$$\lim_{t\to\infty} sup\left(\lim_{i\to\infty} sup(g(t_i)|u_t((t_i))| - |W(u_t(t_i))|)\right) = 0$$

and $g(t_i) \to 0$ as $i \to \infty$ with $t_i \in \{t_i\}_{i=0}^\infty$ since W is bounded. Now,

(a) either $g(t) \to 0$ as $t \to \infty$ and then $\hat{g}(s)$ is strictly stable so that $\hat{g}(s) \in \boldsymbol{SPR}$. However, this fact contradicts $\hat{g}(s) \in \boldsymbol{PR}$, but $\hat{g}(s) \notin \boldsymbol{SPR}$ (and so it is not strictly stable);

(b) The impulse response $\{g(t_i)\}_{i=0}^{\infty} \to 0$ with $\{|u(t_i)|\}_{i=0}^{\infty} \to \infty$ only at such sequences of time instants $\{t_i\}_{i=0}^{\infty}$. Then, since the impulse response is non-negative for all times since the feed-forward linear part of the system is externally positive, then $\lim\limits_{t\to\infty} \int_0^t g(\tau)|\hat{u}_t(\tau)|d\tau = +\infty$ if $|u(t)|$ is unbounded, then $|W(u_t)| \to \infty$, which contradicts that $W(u)$ is bounded and $\gamma_0^2 \geq \int_0^t (g(\tau)|\hat{u}_t(\tau)| - |W(u_t(\tau))|)\,|\hat{u}_t(\tau)|d\tau \geq 0$.

It has been proved that if $\hat{g}s \in \boldsymbol{PR} \cap \overline{\boldsymbol{SPR}}$ and $g : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$, then $|u(t)| \in L_\infty$ and
$|u(t)| \to 0$
as $t \to \infty$ for any hyperstable controller in the feedback loop. It also holds that $|u| \in L_1$ from
$\gamma_0^2 \geq \int_0^t (g(\tau)|\hat{u}_t(\tau)| - |W(u_t(\tau))|)\,|\hat{u}_t(\tau)|d\tau \geq 0$. Since $|u| \in L_1 \cap L_\infty$, then $|u| \in L_2$. Since $g \in L_\infty$, $W(u)$ is bounded and $|u| \in L_1 \cap L_\infty$, and one can conclude that $|y| \in L_\infty$. $\square$

Therefore, it can also be easily concluded from Theorem 2 that the closed-loop system obtained from any minimal state-space realization of $\hat{g}(s)$, being positive real and externally positive, is hyperstable (Definition 1), i.e., it is globally stable in the large for any controller in the hyperstable class, but not asymptotically hyperstable (Definition 2) unless the positive realness is also strict, i.e., the stability is not asymptotic for any controller belonging to the hyperstable class of controllers according to the integral inequality constraints they satisfy for all times. In general, such a global stability property is not asymptotic. In other words, the closed-loop system is hyperstable, but not asymptotically hyperstable unless the transfer function is rather strictly positive real. It can be pointed out that research in the hyperstability and passivity fields continues to be very active nowadays. In this context, the hyperstability theory is applied to the linearization of a radio-frequency amplifier in [45], while it is applied in [46] for the estimation of the speed of an induction motor which does not have a speed sensor. The technique proved to reduce the motor current distortions. In [47], the adaptive control of a DC motor speed is focused using only measurable signals and considering the boundedness of the control effort and the compensation for the negative effects of the non-stationarity of the load torque and inertia moment. The asymptotic hyperstability of the proposed scheme is also proven. On the other hand, ref. [48] provides a recent review of the research projection of the dissipativity theory as an important input–output energy-based framework for both the analysis and the design of dynamic systems.

In this section, different operators and extra transfer functions in the feed-forward loop have been considered. This global structure might maintain the hyperstability under certain conditions. This fact can be considered as a robustness property for certain perturbations of the plant given by its (nominal) strictly positive real transfer function. Furthermore, there is a freedom in changing the input–output interconnection gain, which is the quotient between the leading coefficients of the numerator and denominator polynomials of the transfer function, while keeping the asymptotic hyperstability, provided that the nominal transfer function keeps its positive realness assumptions. In a whole overview, the various approaches in this section can be considered in a robustness context of guaranteeing the hyperstability of the nominal controlled plant under certain perturbations.

4. An Example Related to the Discretization through Inter-Sample Interpolation Sampling and Hold Devices

Now, the input is generated through a sample and hold device which reconstructs the continuous-time input for sampled valued to yield:

$$u(kT + \tau) = u(kT) + \frac{\lambda_c(kT + \tau)}{T}(u(kT) - u[(k-1)T]); \tag{23}$$

$\forall \tau \in [0\,,\, T]$, $\forall k \in \boldsymbol{Z}_{0+}$, and some function $\lambda_c : \boldsymbol{R}_{0+} \to \boldsymbol{R}$, where $T > 0$ is the sampling period, $\lambda_c(kT) = 0$, and $\forall k \in \boldsymbol{Z}_{0+}$, $u_k = u(kT)$, and $\forall k \in \boldsymbol{Z}_{0+}$ are the sampled input values at the sampling instant $t = kT$ [38–42]. In particular,

(a) if $\lambda_c(t) \equiv 0$ and $\forall t \in \boldsymbol{R}_{0+}$ then the sampling and hold device is a zero-order hold (Z0H) and $u(kT+\tau) = u_k, \forall \tau \in [0, T)$, and $\forall k \in \boldsymbol{Z}_{0+}$;
(b) if $\lambda_c(kT+\tau) = \tau, \forall \tau \in [0, T)$, and $\forall k \in \boldsymbol{Z}_{0+}$, then the sampling and hold device is a first-order-hold (FOH) and $u(kT+\tau) = u(kT) + \frac{\tau}{T}(u(kT) - u[(k-1)T]), \forall \tau \in [0, T)$, and $\forall k \in \boldsymbol{Z}_{0+}$; and
(c) if $\lambda_c : \boldsymbol{R}_{0+} \to (0, T)$, then the sampling and hold device is a speed correction hold (SCH):

$$u(kT+\tau) = u(kT) + \frac{\lambda_c(kT+\tau)}{T}(u(kT) - u[(k-1)T]),\ \forall \tau \in [0, T),\ \text{and}\ \forall k \in \boldsymbol{Z}_{0+}.$$

In the case where $\lambda_c(kT+\tau) = k_c\tau, \forall k \in \boldsymbol{Z}_{0+}$, for some $k_c \in (0, 1), \forall \tau \in [0, T)$, then the SCH is of constant slope k_c, and SCH(k_c). If $k_c \in (0, 1)$, then the device is named as a partial speed correction hold (PSCH).

We now discuss the hyperstable design for the more general SCH sampling and hold device. Assume that feedback of the hyperstable feedback controller $K \in \boldsymbol{K}_{dr}$ where K_{dr} has an input $y(t)$ (that is, it is the output of the feed-forward controlled system) and output $v(t)(= -u(t)) : \boldsymbol{R}_{0+} \to \boldsymbol{R}$, which is assumed to satisfy the following continuous/discrete Popov's hyperstability input–output integral condition:

$$\begin{gathered}\int_0^t y(\sigma)v(\sigma)d\tau = \int_0^t (-u(\sigma))y(\sigma)d\sigma \\ = -\left(\sum_{j=0}^{k-1}\int_{jT}^{(j+1)T} u(\sigma)y(\sigma)d\sigma + \int_0^\tau u(kT+\sigma)y(kT+\sigma)d\sigma\right) \\ = -\gamma_0^2 + \int_{j=0}^{k-1}\int_{jT}^{(j+1)T} \varepsilon^2(\sigma)d\sigma + \int_0^\tau \varepsilon^2(kT+\sigma)d\sigma \geq -\gamma_0^2 => -\infty \\ t = kT+\tau,\ \forall k \in \boldsymbol{Z}_{0+},\ \tau \in [0,\ T],\end{gathered} \tag{24}$$

for some arbitrary square-integrable γ, $\varepsilon : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ fulfilling $0 \leq \int_0^t \varepsilon^2(\sigma)d\sigma < \int_0^t \gamma^2(\sigma)d\sigma < \gamma_0^2 < +\infty; \forall t \in \boldsymbol{R}_{0+}$, with $(y(kT) = 0) \Rightarrow [(\varepsilon(kT) = 0) \wedge u(kT) = 0]$ and $|\varepsilon(kT)| < |y(kT)|$ if $y(kT) \neq 0, \forall k \in \boldsymbol{Z}_{0+}$, and $\gamma_0^2 = \underset{t\to\infty}{lim} \int_0^t \gamma^2(\tau)d\tau$ for some finite nonzero real constant γ_0. Note that $\gamma(t)$, $\varepsilon(t) \to 0$ as $t \to \infty$. The purpose of the auxiliary functions γ, $\varepsilon : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ is to guarantee (24) under equalities for all $t \in \boldsymbol{R}_{0+}$. Equation (24) holds for all $t \in \boldsymbol{R}_{0+}$, subject to (23), if:

$$\left(\frac{\lambda_c(kT+\tau)}{T}(u(kT) - u[(k-1)T]) + u(kT)\right)y(kT+\tau) = \gamma^2(kT+\tau) - \varepsilon^2(kT+\tau);\ \forall \tau \in [0, T),\ \forall k \in \boldsymbol{Z}_{0+}, \tag{25}$$

which, at the sampling instants, i.e., $\tau = 0$, becomes:

$$u(kT)y(kT) = \gamma^2(kT) - \varepsilon^2(kT);\ \forall k \in \boldsymbol{Z}_{0+}, \tag{26}$$

which, when replaced in (25), yields:

$$\begin{gathered}\left[\left(1 + \frac{\lambda_c(kT+\tau)}{T}\right)\frac{\gamma^2(kT)-\varepsilon^2(kT)}{y(kT)} - \frac{\lambda_c(kT+\tau)}{T}\frac{\gamma^2[(k-1)T]-\varepsilon^2[(k-1)T]}{y[(k-1)T]}\right]y(kT+\tau) \\ = \gamma^2(kT+\tau) - \varepsilon^2(kT+\tau)\ ;\ \forall \tau \in [0, T), \forall k \in \boldsymbol{Z}_{0+},\end{gathered} \tag{27}$$

which ensures the particular needed constraint at the sampling instants:

$$\lambda_c(kT) = \frac{\left(\gamma^2(kT) - \varepsilon^2(kT)\right)y(kT) - \left(\gamma^2(kT) - \varepsilon^2(kT)\right)y(kT)}{(\gamma^2(kT) - \varepsilon^2(kT))y[(k-1)T] + (\varepsilon^2[(k-1)T] - \gamma^2[(k-1)T])y(kT)}\frac{Ty[(k-1)T]}{y(kT)} = 0\ \text{and}\ \forall k \in \boldsymbol{Z}_{0+},$$

which guarantees $u(t) = u(kT) = \frac{\gamma^2(kT)-\varepsilon^2(kT)}{y(kT)}$ if $t = kT$ and $\forall k \in \mathbf{Z}_{0+}$, provided that $y(kT) \neq 0$ and $u(kT) = 0$ if $y(kT) = 0$.

Thus, the hyperstable controller is synthesized as follows:

(a) If $t = kT$ (sampling instants), then $u(kT) = \frac{\gamma^2(kT)-\varepsilon^2(kT)}{y(kT)}$ with $|\varepsilon(kT)| < |y(kT)|$ if $y(kT) \neq 0$ and $u(kT) = 0$, with $|\varepsilon(kT)| = |y(kT)|$ if $y(kT) = 0$ and $\forall k \in \mathbf{Z}_{0+}$.

(b) If $t \neq kT$ (inter-sampling instants):

$$u(kT+\tau) = u(kT) + \frac{\lambda_c(kT+\tau)}{T}(u(kT) - u[(k-1)T]) \text{ and } \forall k \in \mathbf{Z}_{0+},$$

$$\lambda_c(kT+\tau) = \frac{\left(\gamma^2(kT+\tau) - \varepsilon^2(kT+\tau)\right)y(kT) - \left(\gamma^2(kT) - \varepsilon^2(kT)\right)y(kT+\tau)}{\left(\gamma^2(kT) - \varepsilon^2(kT)\right)y[(k-1)T] + \left(\varepsilon^2[(k-1)T] - \gamma^2[(k-1)T]\right)y(kT)} \frac{Ty[(k-1)T]}{y(kT+\tau)},$$

$\forall \tau \in (0\,, T)$, and $\forall k \in \mathbf{Z}_{0+}$

$$|\varepsilon(kT+\tau)| < |y(kT+\tau)|;\ \forall \tau \in (0\,, T), \forall k \in \mathbf{Z}_{0+}.$$

Note from (27) that as $\{u(kT)\}_{k=0}^{\infty} \to 0\,,\ \lim_{k\to\infty}(|y(kT+\tau)| - |y(kT+\tau)|) = 0$; $\forall \tau \in [0\,, T)$ and $\forall k \in \mathbf{Z}_{0+}$.

The control law of the form (23), obtained from an SCH sampling and hold device, which is within the class of hyperstable controllers subject to the integral Popov's-type constraint (24), satisfies Theorem if it is the feedback loop of a transfer function $\hat{g}(s) \in \boldsymbol{SSPR}$ in parallel with a non-linear device $W(u)$ under the given hypotheses in the theorem.

Remark 4. *It can be pointed out that the Popovian hyperstability constraints of (24)* [10–12] *are more general because of the more general inputs generated from their sampled values than the parallel constraints associated with discrete values of the inputs and outputs being of the form* $\sum_{j=0}^{j} y(kT)\, v(kT) \geq -\gamma_0^2 > -\infty$; $\forall k \in \mathbf{Z}_{0+}$. *A parallel result to Theorem 1 and its corollaries could be easily obtained by applying the Rayleigh theorem on the unit complex circle and using the Z-transform of the impulse response of the linear part. Basically, the integrals of (4) would be changed to the sums* $\hat{g}(\boldsymbol{i}\omega) \to \hat{g}_D(z)$ *for* $z = e^{\boldsymbol{i}\theta}$*and* $\theta \in [0\,, 2\pi)$ *(then* $|z| = 1$*), with* $\hat{g}_D(z) = Z\left(\frac{(1-e^{-Ts})\,\hat{g}(s)}{s}\right)$ *where* $\frac{1-e^{-Ts}}{s}$ *is the transfer function of the ZOH. It turns out that* $d = \inf_{\omega \in \overline{\boldsymbol{R}}_{0+}} \hat{g}(\boldsymbol{i}\omega) = \inf_{\theta\in[0\,,2\pi)} \hat{g}_D(e^{\boldsymbol{i}\theta}) > 0$ *since* $\hat{g}(s) \in \boldsymbol{SSPR}$*, which is then also bi-proper and strictly stable, that is,* $\hat{g}_a(z)$ *and* $\hat{g}(s)$ *have an identical input–output interconnection gain, which is the quotient of the leading coefficients of both their numerator and denominator polynomials. However, the discretization approach addressed in this way has only information at the sampling instants, rather than for all time, and it is of interest only for the use of discretization under a ZOH and not for more general sampling and hold devices. Even in this case, note that the output is not fully addressed in the input–output energy formulas of Theorem 1 and the later corollaries since the output is not piece-wise constant.*

5. Conclusions

The paper has investigated the asymptotic hyperstability of a single-input single-output closed-loop control configuration whose feed-forward loop consists of a parallel connection of a strongly strictly positive real transfer function, together with (in general) a non-positive nonlinear operator which has to satisfy some discussed conditions. The feedback loop consists of, in general, a nonlinear and, perhaps, time-varying controller which satisfies a Popov-type integral inequality. The global asymptotic stability is proven to be "in the large", that is, it is guaranteed for any given finite initial condition, and the asymptotic hyperstability property implies that the closed-loop asymptotic stability is guaranteed independently of the particular controller employed within the above class.

The property is addressed by proving, trough Parseval's theorem, that the input–output energy of the feed-forward loop is always positive and bounded for all times. Extra sufficiency-type conditions to keep the asymptotic hyperstability property are obtained under the incorporation of an additional strictly stable linear and time-invariant system. In particular, and in order to keep the hyperstability properties of the whole closed-loop configuration, its frequency response resonance gain, which is sufficiently small, is related to the minimum value of the real part of the impulse response associated with the strongly strictly positive real transfer function. A case study is provided, which is concerned with the use of a fractional sampling and hold device to generate the continuous-time input from their sampled values at a constant sampling rate.

Funding: This research was funded by the Spanish Government and the European Commission, grant number RTI2018-094336-B-I00 (MCIU/AEI/FEDER, UE) and by the Basque Government, grant number IT1207-19. The APC was funded by grant RTI2018-094336-B-I00 (MCIU/AEI/FEDER, UE).

Acknowledgments: The author is grateful to the Spanish Government and the European Commission for its support through grant RTI2018-094336-B-I00 (MCIU/AEI/FEDER, UE) and to the Basque Government for its support through grant IT1207-19. He is also grateful to the Referees by their useful suggestions and comments.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Hakimi-Moghaddam, M. A survey on positive real and strictly positive real scalar transfer functions. *Int. J. Math. Comput. Sci.* **2016**, *10*, 57–63.
2. Fernandez-Anaya, G.; Flores-Godoy, J.-J.; Rodríguez-Palacios, A. Properties of strictly positive real functions: Products and compositions. *Int. J. Syst. Sci.* **2010**, *41*, 457–466. [CrossRef]
3. Marquez, H.J.; Damaren, C.J. On the design of strictly positive real transfer functions. *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.* **1995**, *42*, 214–218. [CrossRef]
4. Marquez, H.J.; Damaren, C.J. Strictly positive real transfer-functions revisited. *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.* **1995**, *40*, 478–479. [CrossRef]
5. De la Sen, M. A method for general design of positive real functions. *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.* **1998**, *45*, 764–769. [CrossRef]
6. Chen, M.Z.Q.; Smith, M.C. A Note on Tests for Positive-Real Functions. *IEEE Trans. Autom. Control.* **2009**, *54*, 390–393. [CrossRef]
7. Chen, M.Z.Q.; Wang, K.; Zou, Y.; Lam, J. Realization of a Special Class of Admittances with One Damper and One Inerter for Mechanical Control. *IEEE Trans. Autom. Control.* **2013**, *58*, 1841–1846. [CrossRef]
8. Xiao, H.; Landau, I.D.; Chen, X. A robust optimal design for strictly positive realness in recursive parameter adaptation. *Int. J. Adapt. Control. Signal. Process.* **2017**, *31*, 1205–1216. [CrossRef]
9. Delasen, M. Stability of composite systems with an asymptotically hyperstable subsystem. *Int. J. Control.* **1986**, *44*, 1769–1775. [CrossRef]
10. Popov, V.M.; Posehn, M.R. Hyperstability of Control Systems. *J. Dyn. Syst. Meas. Control.* **1974**, *96*, 372. [CrossRef]
11. Popov, V.M. The solution of a new hyperstability problem for controlled system. *Autom. Remote Control.* **1965**, *24*, 1–23.
12. Popov, V.M. 1 Problem in theory of absolute stability of controlled systems. *Autom. Remote Control.* **1963**, *25*, 1129.
13. Landau, I.D. *Systèmes Non Linéaires-Notes de Cours, Laboratoire d ´Automatique de Grenoble*; ENSPG: Grenoble, France, 1975.
14. Landau, I. A hyperstability criterion for model reference adaptive control systems. *IEEE Trans. Autom. Control.* **1969**, *14*, 552–555. [CrossRef]
15. Landau, Y.D. Adaptive Control. In *The Model Reference Approach, Control and Systems Theory Series*; Jerry, M., Ed.; Marcel Dekker Inc.: New York, NY, USA, 1979.
16. Rasvan, V. Popov Theories and Qualitative Behavior of Dynamic and Control Systems. *Eur. J. Control.* **2002**, *8*, 190–199. [CrossRef]
17. Warwick, K.; Zhu, Q.M.; Ma, Z. A hyperstable neural network for the modelling and control of nonlinear systems. In *Sadhana*; Springer: Berlin/Heidelberg, Germany, 2000; Volume 25, pp. 169–180.
18. Ionescu, T. Hyperstability of linear time-varying discrete systems. *IEEE Trans. Autom. Control* **1970**, *15*, 645–647. [CrossRef]
19. De la Sen, M.; Ibeas, A.; Alonso-Quesada, S. Asymptotic Hyperstability of a Class of Linear Systems under Impulsive Controls Subject to an Integral Popovian Constraint. *Abstr. Appl. Anal.* **2013**, *2013*, 1–14. [CrossRef]
20. Iqbal, J.; Ullah, M.; Khan, S.G.; Khelifa, B.; Cukovic, S. Nonlinear control systems—A brief overview of historical and recent advances. *Nonlinear Eng.* **2017**, *6*, 301–312. [CrossRef]
21. Narendra, K.S.; Taylor, J.H. *Frequency Domain Criteria for Absolute Stability*; Academic Press Inn: Denver, CO, USA, 1973.
22. Diblík, J.; Khusainov, D.Y.; Shatyrko, A.; Baštinec, J.; Svoboda, Z. Absolute Stability of Neutral Systems with Lurie Type Nonlinearity. *Adv. Nonlinear Anal.* **2021**, *11*, 726–740. [CrossRef]

23. Fradkov, A. Passification of Non-square Linear Systems and Feedback Yakubovich-Kalman-Popov Lemma. *Eur. J. Control.* **2003**, *9*, 577–586. [CrossRef]
24. Chen, L.; Li, T.; Chen, Y.; Wu, R.; Ge, S. Robust passivity and feedback passification of a class of uncertain fractional-order linear systems. *Int. J. Syst. Sci.* **2019**, *50*, 1149–1162. [CrossRef]
25. Lin, Z.; Liu, J.; Niu, Y. Robust Passivity and Feedback Design for Nonlinear Stochastic Systems with Structural Uncertainty. *Math. Probl. Eng.* **2013**, *2013*, 1–9. [CrossRef]
26. Pota, H.R.; Moylan, P.J. Stability of locally dissipative interconnected systems. *IEEE Trans. Autom. Control.* **1993**, *38*, 308–312. [CrossRef]
27. Eremin, E.L.; Nikiforova, L.V.; Shelenok, E.A. Combined Nonlinear Control of Non-Affine MIMO System with Input and State Delays. *J. Phys. Conf. Ser.* **2021**, *1901*, 012035. [CrossRef]
28. Xiang, B.; Wang, X.; Wong, W.O. Process control of charging and discharging of magnetically suspended flywheel energy storage system. *J. Energy Storage* **2021**, *47*, 103629. [CrossRef]
29. Eremin, E.L.; Nikiforova, L.V.; Shelenok, E.A. Nonlinear robust control of large-scale system with input saturation. In Proceedings of the V International Scientific and Technical Conference "Mechanical Science and Technology Update" (MSTU 2021), Omsk, Russia, 16–17 March 2021.
30. Griva, G.; Profumo, F.; Ilas, C.; Magureanu, R.; Vranka, P. A unitary approach to speedy sensorless induction control motor field oriented drives based on various model reference schemes, AS ′96. In Proceedings of the 1996 IEEE Industry Applications Conference Thirty-First IAS Annual Meeting, San Diego, CA, USA, 6–10 October 1996; Volume 3, pp. 1594–1599.
31. Griva, G.; Profumo, F.; Bojoi, I.R.; Bostan, V.; Cuius, M.; Ilas, C. General adaptation law for MRAS high performance sensorless induction motor drives. In Proceedings of the 2001 IEEE 32nd Annual Power Electronics Specialists Conference, Vancouver, BC, Canada, 17–21 June 2001.
32. Chen, F.; Hou, R.; Tao, G. Adaptive Controller Design for Faulty UAVs via Quantum Information Technology. *Int. J. Adv. Robot. Syst.* **2012**, *9*, 256. [CrossRef]
33. Chen, F.; Wu, M.; Huang, X.; Zhang, J. An improved model reference adaptive control for high-speed PMSMs. In Proceedings of the 2021 IEEE 30th International Symposium on Industrial Electronics (ISIE), Kyoto, Japan, 20–23 June 2021; pp. 1–6.
34. Zhao, X.; Guo, G. Model Reference Adaptive Control of Vehicle Slip Ratio Based on Speed Tracking. *Appl. Sci.* **2020**, *10*, 3459. [CrossRef]
35. Glushchenko, A.; Lastochkin, K.; Petrov, V. DC Drive Adaptive Speed Controller Based on Hyperstability Theory. *Computation* **2022**, *10*, 40. [CrossRef]
36. Mamat, M.; Vaidyanathan, S.; Sambas, A.; Mujiarto; Sanjaya, W.S.M.; Subiyanto, S. A novel double-convection chaotic attractor, its adaptive control and circuit simulation. In *IOP Conference Series-Material Science and Engineering, Proceedings of the Indonesian Operations Research Association-International Conference on Operations Research, Tangerang Selatan, Indonesia, 12 October 2017*; IOP Publishing: Bristol, UK, 2018; Volume 332, p. 012033.
37. Vaidyanathan, S.; Sambas, A.; Firman, S.; Mamat, M.; Gundara, G.; Sanjaya, W.S.M.; Subiyanto, S. A new chaotic attractor with two quadratic nonlinearities, its synchronization and circuit implementation. In *IOP Conference Series-Material Science and Engineering, Proceedings of the Indonesian Operations Research Association-International Conference on Operations Research, Tangerang Selatan, Indonesia, 12 October 2017*; IOP Publishing: Bristol, UK, 2018; Volume 332, p. 012048.
38. Bárcena, R.; De la Sen, M.; Sagastabeitia, I. Improving the stability properties of the zeros of sampled systems with fractional order hold. *IEE Proc. Control Theory Appl.* **2000**, *147*, 456–464. [CrossRef]
39. Delasen, M. Multirate hybrid adaptive-control. *IEEE Trans. Autom. Control.* **1986**, *31*, 582–586. [CrossRef]
40. Pelgrom, M.J.M. Sample-and-Hold Circuits. In *Analog-to-Digital Conversion*; Springer: Cham, Switzerland, 2022; pp. 381–423.
41. He, N.; Shi, D.; Chen, T. Self-triggered model predictive control for networked control systems based on first-order hold. *Int. J. Robust Nonlinear Control.* **2017**, *28*, 1303–1318. [CrossRef]
42. Ou, M.; Yang, Z.; Wu, X.; Liang, S.; Ou, M.; Ran, H. Zeros of sampled systems with Backward Triangle Sample and Hold realized by Zero Order Hold. In Proceedings of the 2021 International Conference on Advanced Mechatronic Systems (ICAMechS), Tokyo, Japan, 9–12 December 2021; pp. 151–155. [CrossRef]
43. Naslin, P. *Introduction à la Commande Optimale*; Dunod: Paris, France, 1965.
44. Schaeffer, H.H. On nonlinear positive operators. *Pac. J. Math.* **1959**, *9*, 847–860. [CrossRef]
45. Zozaya, A.; Bertran, E. Passivity Theory Applied to the Design of Power-Amplifier Linearizers. *IEEE Trans. Veh. Technol.* **2004**, *53*, 1126–1137. [CrossRef]
46. Deeksha, R.; Lakshmi, M.; Harshitha, H.M. Performance improvement in sensorless vector control of IM using bus clamped PWM. In Proceedings of the 2021 International Conference on Design Innovations for 3Cs Compute Communicate Control (ICDI3C), Bangalore, India, 10–11 June 2021; pp. 88–92.

47. Glushchenko, A.; Petrov, V.; Lastochkin, K. Hyperstable MRAC System of DC Drive with Reference Model Hedging and Load Torque Compensation. In Proceedings of the 2021 3rd International Conference on Control Systems, Mathematical Modeling, Automation and Energy Efficiency (SUMMA), Lipetsk, Russia, 10–12 November 2021; pp. 1036–1040. [CrossRef]
48. Zakeri, H.; Antsaklis, P.J. Passivity Measures in Cyberphysical Systems Design: An Overview of Recent Results and Applications. *IEEE Control Syst.* **2022**, *42*, 118–130. [CrossRef]