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Markovian Demands on Two Commodity Inventory System with Queue-Dependent Services and an Optional Retrial Facility

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Abstract: The use of a Markovian inventory system is a critical part of inventory management. The purpose of this study is to examine the demand for two commodities in a Markovian inventory system, one of which is designated as a major item (Commodity-I) and the other as a complimentary item (Commodity-II). Demand arrives according to a Poisson process, and service time is exponential at a queue-dependent rate. We investigate a strategy of (s, Q) type control for commodity-I with a random lead time but instantaneous replenishment for commodity-II. If the waiting hall reaches its maximum capacity of N, any arriving primary client may enter an infinite capacity orbit with a specified ratio. For orbiting consumers, the classical retrial policy is used. In a steady-state setting, the joint probability distributions for commodities and the number of demands in the queue and the orbit, are derived. From this, we derive a waiting time analysis and a variety of system performance metrics in the steady-state. Additionally, the physical properties of various performance measures are evaluated using various numerical assumptions associated with diverse stochastic behaviours.

Keywords: classical retrial policy; queue dependent service rate; waiting time analysis; infinite orbit

MSC: 60K20; 60K25

1. Introduction

The function or usage of one product may be dependent on another product in general. The first product is the major commodity and the second one is the complimentary of the first product, such as mobile phone with memory card, bike with helmet, printer with ink cartridge, computer with software, torch with battery, etc. From the production point of view, both commodities are correlated with each other. According to the demand of both commodities, a firm will sell them abundantly to the targeted population who are economically benefited with the purchase.

Elaborately if the cost of one product increases, the customer demand for the corresponding complimentary product decreases. So the customers' interest towards the product will be changed. Furthermore, it will spoil the existing quantity of the product and hence, the company may encounter lose in their sales. In order to maintain the goodwill of business, the company should sell their product along with its complimentary product. Furthermore any company must plan to introduce a new product as a compliment of another product that will gain customers interests towards the new product with positive feedback. It can be applied in any industry right from software to dairy products. These impacts make us analyze the economic strategy of multi inventory system which sells both products with an affordable cost.

In reality, many servers can adjust the speed of service according to customer satisfaction which impacts brand reputation. In a queuing system, some authors considered queue length-dependent service times (see Abolnikov [1], Dshalalow [2], Fakinos [3], Harris [4] and Ivnitskiy et al. [5]). Furthermore, if an arriving customer finds the waiting room full, the customer decides to reattempt to get the entry with certain proportion. Under such real conditions, the proposed model deals with two commodity perishable inventory system that is the novelty of this paper. The next section elaborates the Review Literature of this model. System description is presented in Section 3. Under stability conditions, the mathematical and waiting time of the model is studied in Sections 4 and 5, respectively. System characteristics and numerical analysis are presented in Sections 6 and 7, respectively. Conclusion of the deal is presented in the final section.

Related Works

The relevant features of the present study are discussed in various modeling of queuing inventory system. Before making procurement of some products in an inventory system, a customer should need some demonstration about the product. It will take some positive time for a server showing its demonstration. In that duration, any arriving customer may wait for his/her turn. An inventory system with arbitrary service time was first studied by Sigman and Simchi-levi [6]. Schwarz et al. [7] first developed an inventory queuing system for which lead time, service time and the time between each arrival into the system are all considered to be independent exponential distributions. This was discussed by Berman et al. [8]. Recently, regarding service facilities one can refer [9].

Gebhard [10] considered a M/M/1 queuing system with two rate of service policy. Sangeetha and Sivakumar [11] studied a perishable inventory system for MAP arrival and Phase type service distribution environment and found an optimal policies of service rates in order to minimize the expected total cost. Under the steady state approximation, the probability of empty state and mean queue length are also obtained. Furthermore Doo Il Choi et al. [12] individually studied the queue dependent service time with finite capacity and infinite capacity queue. In these models, one rate of service is fixed up to a certain level in the queue and another rate of service is provided after finding the queue size beyond that level. Jeganathan et al. [13] considered an inventory queuing system which provides k independent phases of services with each service rate depending upon two discrete states of queue length.

Keerthana et al. [14] considered the postponed and renewal demands on their inventory system in which they follow arbitrary demand distribution for the inter-arrival times. The economical advantages of an inventory system are analyzed with three kinds of retrial mode by Krishnamoorthy and Jose [15]. Amirthakodi and Sivakumar [16] investigated the feedback mechanism on their inventory system with an orbital search policy. Dhanya Shajin and Krishnamoorthy [17] explored the advanced reservation, cancellation, overbooking with an impatient customers. Furthermore, more queries regarding inventory and production inventory discussion, can be referred through [18–21].

Paul Manuel et al. [22] studied a perishable finite capacity retrial inventory system with service facilities. The inter-retrial time and life time of a stored item for each model discussed here are exponentially distributed. Furthermore, Paul Manuel et al. [23] counted a constant rate of retrial from infinite space orbit is dealt with a finite queuing perishable inventory. Kathiresan et al. [24] considered an inventory system with finite capacity waiting hall. In this model,any arriving customer can also use a finite capacity orbit whenever there is no vacant in waiting hall and there is a vacant in the orbit with constant rate of retrial. Most of the times, the nature of the retrial policy is dependent on the size of the customers in the orbit which is discussed as a classical retrial policy (CRP). Artaljeo et al. [25] and Ushakumari [26] both elaborately discussed CRP in an inventory system.

The merits of multi item service facilities are pointed out in some publications. The optimum ordering policy of multi-item inventory system under some different constraints of total cost function was studied by Veinott and Wagner [27] and Wagner et al. [28]. Alscher and Schneider [29] determined a multi-item inventory control model undertaking with varying costs of multi items. Kalpakam and Arivarignan [30] considered a joint (s,S) reordering policy of a multi-item inventory. In this model, S denotes the aggregate of maximum stock level of each commodity and whenever the aggregate stock level reaches to s, the quantity of each commodity are ordered up to its maximum stock level and the replenishment time of any joint new order is zero. Anbazhagan and Arivarignan [31–33] analyzed the different ordering policies of two commodities inventory system under the assumption of various random conditions. Sivakumar et al. [34] studied two commodity inventory queuing system where any one of two item is replaceable when a demanded item is not available. Instead of a finite waiting hall, Sivakumar [35] studied the same with retrial demand.

Yadavalli et al. [36] studied a two commodity coordinated inventory system in which demand of each commodity is sold with another commodity under a distinct Bernoulli schedule and arrival pattern of a customer is poisson. Furthermore, Yadavalli et al. [36] thoroughly investigated two perishable commodity inventory system with three types of customers where the demand of one commodity is always bulk. Anbazhagan et al. [37] introduced the gift item for a customer whenever the quantity of the demand of the main product is beyond certain level. Under a base stock ordering policy, a two commodity inventory system with a finite capacity waiting hall was analyzed by Gomathi et al. [38]. Anbazhagan and Jeganathan [39] independently studied the ordering policies of primary product and gift item, but compliment item may also be sold when a customer do not make any demand of primary product.

2. System Description

A single server two commodity inventory system with a limited queue of size N and an optional retrial facility of indefinite size is considered in this model, in which one commodity is designated as a significant item and the other as a complimentary item is taken into consideration. When a customer first enters the system, they purchase a large item and depart with a complementary item after the service is completed. The interval between the arrival times of any two customers is exponentially distributed with rate λ between them. The single server delivers queue-dependent service at a rate of $x\mu$, where x denotes the number of consumers currently waiting in line at the time of the request. If there is no available space in the queue, any new customers who join in an endless retrial orbit with a rate of $q\lambda$. After some exponential time has passed, the customer can be reintroduced into the system to satisfy his or her demand, with the rate of reintroduction being $u\lambda_r$, where u represents the number of orbital clients present at the time.

For each commodity, a continuous review ordering policy is considered: (s, Q) and $(0, S_2)$(instantaneous) are the ordering policies for the major item and complimentary item, respectively; further, anytime the level of stock for a major item falls below s, the system immediately places an order for quantity Q of the major item, and the time it takes for the ordered quantity of major item to arrive is exponential with rate β, as shown in the following diagram. However, whenever the stock of the complimentary item reaches zero, the order for quantity S_2 is placed immediately. The deteriorating times of both items are random and exponential in nature, with intensities of γ_1 and γ_2 for the major item and γ_2 for the complimentary item, respectively.

3. Analysis of the Model

Let $X_1(t), X_2(t), X_3(t), W(t)$ described number of demands in the orbit, current inventory level of first commodity (CIL1), current inventory level of second commodity (CIL2), number of demands in the waiting hall (queue), respectively. Assumption developed on the birth death process make a stochastic process $Y(t) = \{(X_1(t), X_2(t), X_3, W(t)), t \geq 0\}$ and it is also said to be a continuous-time stochastic process (CTSP) having the state space E such that $E = \{(\wp_1, \wp_2, \wp_3, \wp_4) : \wp_1 = 0, 1, \cdots ; \wp_2 = 0, 1, \cdots , S_1; \wp_3 = 1, \cdots , S_2;$ $\wp_4 = 0, 1, \cdots , N\}$.

3.1. Construction of Infinitesimal Generator Matrix

Indicating to the discrete state space and continuous time Markov chain, the transition matrix of $Y(t)$ having the structure as follows:

$$B = \begin{pmatrix} \mathbb{B}_{00} & \mathbb{B}_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbb{B}_{10} & \mathbb{B}_{11} & \mathbb{B}_{01} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbb{B}_{20} & \mathbb{B}_{21} & \mathbb{B}_{01} & \mathbf{0} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$\mathbb{B}_{01} = \begin{cases} q\lambda & \wp_1' = \wp_1, \quad \wp_1 \in \{0, 1, 2, \cdots\} \\ & \wp_2' = \wp_2, \quad \wp_2 \in \{0, 1, 2, \cdots S_1\} \\ & \wp_3' = \wp_3, \quad \wp_3 \in \{1, 2, \cdots S_2\} \\ & \wp_4' = \wp_4, \quad \wp_4 = N \\ 0, & \text{otherwise.} \end{cases},$$

$$\mathbb{B}_{\wp_1 0} = \begin{cases} \wp_1\lambda_r, & \wp_1' = \wp_1 - 1, \quad \wp_1 \in \{1, 2, \cdots\} \\ & \wp_2' = \wp_2, \quad \wp_2 \in \{0, 1, 2, \cdots , S_1\} \\ & \wp_3' = \wp_3, \quad \wp_3 \in \{1, 2, \cdots , S_2\} \\ & \wp_4' = \wp_4, \quad \wp_4 \in \{0, 1, 2, \cdots , N-1\} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$
\mathbb{B}_{\wp_1,\wp_1'} = \begin{cases}
\beta & \begin{array}{ll} \wp_1' = \wp_1, & \wp_1 \in \{0,1,2,\ldots\} \\ \wp_2' = \wp_2 + Q, & \wp_2 \in \{0,1,2,\ldots,s\} \\ \wp_3' = \wp_3, & \wp_3 \in \{1,2,\ldots,S_2\} \\ \wp_4' = \wp_4, & \wp_4 \in \{0,1,\ldots,N\} \end{array} \\[2ex]
\lambda & \begin{array}{ll} \wp_1' = \wp_1, & \wp_1 \in \{0,1,2,\ldots\} \\ \wp_2' = \wp_2, & \wp_2 \in \{0,1,2,\ldots,S_1\} \\ \wp_3' = \wp_3, & \wp_3 \in \{1,2,\ldots,S_2\} \\ \wp_4' = \wp_4, & \wp_4 \in \{0,1,\ldots,N-1\} \end{array} \\[2ex]
\wp_2\gamma_1 & \begin{array}{ll} \wp_1' = \wp_1, & \wp \in \{0,1,2,\ldots\} \\ \wp_2' = \wp_2 - 1, & \wp_2 \in \{1,2,\ldots,S_1\} \\ \wp_3' = \wp_3, & \wp_3 \in \{1,2,\ldots,S_2\} \\ \wp_4' = \wp_4, & \wp_4 \in \{0,1,\ldots,N\} \end{array} \\[2ex]
\wp_3\gamma_2 & \begin{array}{ll} \wp_1' = \wp_1, & \wp_1 \in \{0,1,2,\ldots\} \\ \wp_2' = \wp_2, & \wp_2 \in \{0,1,2,\ldots,S_1 - 1\} \\ \wp_3' = S_2, & \wp_3 = 1 \\ \wp_3' = \wp_3 - 1, & \wp_3 \in \{2,3,\ldots,S_2\} \\ \wp_4' = \wp_4, & \wp_4 \in \{0,1,\ldots,N\} \end{array} \\[2ex]
\wp_4\mu & \begin{array}{ll} \wp_1' = \wp_1, & \wp_1 \in \{0,1,2,\ldots\} \\ \wp_2' = \wp_2 - 1, & \wp_2 \in \{1,2,\ldots,S_1\} \\ \wp_3' = S_2, & \wp_3 = 1 \\ \wp_3' = \wp_3 - 1, & \wp_3 \in \{2,3,\ldots,S_2\} \\ \wp_4' = \wp_4 - 1, & \wp_4 \in \{1,2,\ldots,N\} \end{array} \\[2ex]
\begin{array}{l} -[H(s-\wp_2)\beta + \wp_2\gamma_1 \\ +\wp_3\gamma_2 + v_2\bar{\delta}_{0\wp_2}\wp_4\mu \\ \delta_{N\wp_4}q\lambda + \bar{\delta}_{N\wp_4}\wp_1\lambda_r] \end{array} & \begin{array}{ll} \wp_1' = \wp_1, & \wp_1 \in \{0,1,2,\ldots\} \\ \wp_2' = \wp_2, & \wp_2 \in \{0,1,2,\ldots,S_1\} \\ \wp_3' = \wp_3, & \wp_3 \in \{1,2,\ldots,S_2\} \\ \wp_4' = \wp_4, & \wp_4 \in \{0,1,\ldots,N\} \end{array} \\[2ex]
0, & \text{otherwise.}
\end{cases}
$$

The infinitesimal generator matrix, B, is to be obtained by the transitions as follows:

1. $(\wp_1, \wp_2, \wp_3, N) \xrightarrow{q\lambda} (\wp_1 + 1, \wp_2, \wp_3, N)\wp_1 = 0,1,2,\cdots; \wp_2 = 0,1,2,\cdots S_1;$ $\wp_3 = 1,2,\cdots S_2.$
2. $(\wp_1, \wp_2, \wp_3, \wp_4) \xrightarrow{\wp_1\lambda_r} (\wp_1 - 1, \wp_2, \wp_3, \wp_4 + 1)\wp_1 = 0,1,2,\cdots; \wp_2 = 0,1,2,\cdots S_1;$ $\wp_3 = 1,2,\cdots S_2; \wp_4 = 0,1,2,\cdots N-1.$
3. $(\wp_1, \wp_2, \wp_3, \wp_4) \xrightarrow{\beta} (\wp_1, \wp_2 + Q, \wp_3, \wp_4)\wp_1 = 0,1,2,\cdots; \wp_2 = 0,1,2,\cdots s;$ $\wp_3 = 1,2,\cdots S_2; \wp_4 = 0,1,2,\cdots N.$
4. $(\wp_1, \wp_2, \wp_3, \wp_4) \xrightarrow{\lambda} (\wp_1, \wp_2, \wp_3, \wp_4 + 1)\wp_1 = 0,1,2,\cdots; \wp_2 = 0,1,2,\cdots S_1;$ $\wp_3 = 1,2,\cdots S_2; \wp_4 = 0,1,2,\cdots N-1.$
5. $(\wp_1, \wp_2, \wp_3, \wp_4) \xrightarrow{\wp_2\gamma_1} (\wp_1, \wp_2 - 1, \wp_3, \wp_4)\wp_1 = 0,1,2,\cdots; \wp_2 = 1,2,\cdots S_1;$ $\wp_3 = 1,2,\cdots S_2; \wp_4 = 0,1,2,\cdots N.$
6. $(\wp_1, \wp_2, \wp_3, \wp_4) \xrightarrow{\wp_3\gamma_1} (\wp_1, \wp_2, \wp_3 - 1, \wp_4)\wp_1 = 0,1,2,\cdots; \wp_2 = 0,1,2,\cdots S_1;$ $\wp_3 = 1,2,\cdots S_2; \wp_4 = 0,1,2,\cdots N.$
7. $(\wp_1, \wp_2, \wp_3, \wp_4) \xrightarrow{\wp_4\mu} (\wp_1, \wp_2 - 1, \wp_3 - 1, \wp_4 - 1)\wp_1 = 0,1,2,\cdots; \wp_2 = 1,2,\cdots S_1;$ $\wp_3 = 1,2,\cdots S_2; \wp_4 = 1,2,\cdots N.$

3.2. Matrix Geometric Approximation

Steady State Analysis

Consider the point at which this truncation procedure stops for the matrix-geometric approximation to be K. In order to identify the steady state of the considered system using Neut's Rao truncation approach, we make the assumptions that $B_{i0} = B_{K0}$ and $B_{i1} = B_{K1}$ for all $i \geq K$. In addition, the updated generator matrix for the truncated system with the following structure is created.

$$\hat{B} = \begin{pmatrix} \mathbb{B}_{00} & \mathbb{B}_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbb{B}_{10} & \mathbb{B}_{11} & \mathbb{B}_{01} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbb{B}_{20} & \mathbb{B}_{21} & \mathbb{B}_{01} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbb{B}_{K0} & \mathbb{B}_{K1} & \mathbb{B}_{01} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbb{B}_{K0} & \mathbb{B}_{K1} & \mathbb{B}_{01} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

Theorem 1. *The steady-state probability vector, χ, where*

$$\chi = (\chi^{(0)}, \chi^{(1)}, \cdots, \chi^{(S_1)}),$$
$$\chi^{(\wp_2)} = \chi^{(\wp_2,0)}, \chi^{(\wp_2,1)}, \cdots, \chi^{(\wp_2,S_2)} \wp_2 = 0,1,\cdots,S_1$$
$$\chi^{(\wp_2,\wp_3)} = \chi^{(\wp_2,\wp_3,0)}, \chi^{(\wp_2,\wp_3,1)}, \cdots, \chi^{(\wp_2,\wp_3,N)} \wp_2 = 0,1,\cdots,S_1 and \wp_3 = 0,1,\cdots,S_2$$

that corresponds to the generator matrix is denoted by $\mathbb{B}_K$ where $\mathbb{B}_K = \mathbb{B}_{K0} + \mathbb{B}_{K1} + \mathbb{B}_{01}$ is given by

$$\chi^{(i)} = \chi^{(Q)} e_i, \quad i = 0,1,\ldots,S_1.$$

where

$$e_i = \begin{cases} (-1)^{Q-i} F_Q E_{Q-1}^{-1} F_{Q-1} \cdots F_{i+1} E_i^{-1}, & i = 0,1,\cdots,Q-1 \\ I, & i = Q \\ (-1)^{2Q-i+1} \sum\limits_{j=0}^{S_1-i} [(F_Q E_{Q-1}^{-1} F_{Q-1} \cdots F_{s+1-j} E_{s-j}^{-1}) \times & \\ G E_{S_1-j}^{-1} (F_{S_1-j} E_{S_1-j-1}^{-1} F_{S_1-j-1} \cdots F_{i+1} E_i^{-1})], & i = Q+1, Q+2, \cdots, S_1 \end{cases}$$

and $\chi^{(Q)}$ is obtained by solving

$$\chi^{(Q)}[(-1)^Q \sum_{j=0}^{s-1} [(F_Q E_{Q-1}^{-1} F_{Q-1} \cdots F_{s+1-j} E_{s-j}^{-1}) G E_{S_1-j}^{-1} (F_{S_1-j} E_{S_1-j-1}^{-1} F_{S_1-j-1} \cdots F_{i+1} E_i^{-1})]$$
$$F_{Q+1} + E_Q + (-1)^Q F_Q E_{Q-1}^{-1} F_{Q-1} \cdots F_1 E_0^{-1} G] = \mathbf{0}$$

and $\sum\limits_{j=1}^{S_1} \chi^{(i)} \mathbf{e} = 1.$

Proof. We have

$$\chi \mathbb{B}_K = \mathbf{0} \text{ and } \chi \mathbf{e} = 1 \tag{1}$$

where

$$[\mathbb{B}_K]_{ij} = \begin{cases} E_i & j = i, \quad i = 0,1,2,\cdots,S_1; \\ F_i & j = i-1, \quad i = 1,2,\cdots,S_1; \\ G & j = i+Q, \quad i = 0,1,2,\cdots,s; \\ \mathbf{0} & \text{otherwise} \end{cases}$$

The first equation of the above framework yields the following set of equations:

$$\begin{array}{lcl} \chi^{(i+1)}F_{i+1}+\chi^{(i)}E_i & = & \mathbf{0}, i=0,1,\cdots,Q-1 \\ \chi^{(i+1)}F_{i+1}+\chi^{(i)}E_i+\chi^{(i-Q)}G & = & \mathbf{0}, i=Q,Q+1,\cdots,S_1-1 \\ \chi^{(i)}E_i+\chi^{(i-Q)}G & = & \mathbf{0}, i=S_1 \end{array}$$

Solving the system of equations, we will get the stated result. □

Theorem 2. *The stability condition of the system at the truncation point K is given by*

$$r_1 q\lambda < r_2 K\lambda_r$$

where $r_1=\sum_{\wp_2=0}^{S_1}\sum_{\wp_3=1}^{S_2}\chi^{(\wp_2,\wp_3,N)}$ *and* $r_2=\sum_{\wp_2=0}^{S_1}\sum_{\wp_3=1}^{S_2}\sum_{\wp_4=1}^{N}\chi^{(\wp_2,\wp_3,\wp_3)}$.

Proof. From the well known result of Neuts on the positive recurrence of $\mathbb{B}_K$ we have

$$\chi^{(K)}\mathbb{B}_{01}\mathbf{e}<\chi^{(K)}\mathbb{B}_{K0}\mathbf{e}$$

and by exploiting the structure of the matrices $\mathbb{B}_{01}$ and $\mathbb{B}_{K0}$, we get, for $\wp_2=0,1,2,\cdots,S_1$, $\wp_3=1,2,\cdots,S_2$ and $\wp_4=0,1,2,\cdots,N$.

$$\chi^{(\wp_2,\wp_3,\wp_4)}\mathbb{B}_{01}\mathbf{e}<\chi^{(\wp_2,\wp_3,\wp_4)}\mathbb{B}_{K0}\mathbf{e}.$$

First, $[\chi^{(0)},\chi^{(1)},\cdots,\chi^{(S_1)}]\mathbb{B}_{01}\mathbf{e}<[\chi^{(0)},\chi^{(1)},\cdots,\chi^{(S_1)}]\mathbb{B}_{K0}\mathbf{e}$
Due to the structure of B_0, L.H.S becomes

$$\chi^{(\wp_2)}B_0=\chi^{(\wp_2,\wp_3,N)}q\lambda.$$

On the other hand, due to structure of B_1 R.H.S becomes

$$\chi^{(\wp_2)}B_1=[\chi^{(\wp_2,\wp_3,1)},\chi^{(\wp_2,\wp_3,2)},\cdots,\chi^{(\wp_2,\wp_3,N)}]K\lambda_r$$

Therefore, the last inequality becomes

$$\sum_{\wp_2=0}^{S_1}\sum_{\wp_3=1}^{S_2}\chi^{(\wp_2,\wp_3,N)}q\lambda<\sum_{\wp_2=0}^{S_1}\sum_{\wp_3=1}^{S_2}\sum_{\wp_4=1}^{N}\chi^{(\wp_2,\wp_3,\wp_4)}K\lambda_r$$

Hence,

$$r_1 q\lambda < r_2 K\lambda_r$$

where $r_1=\sum_{\wp_2=0}^{S_1}\sum_{\wp_3=1}^{S_2}\chi^{(\wp_2,\wp_3,N)}$ and $r_2=\sum_{\wp_2=0}^{S_1}\sum_{\wp_3=1}^{S_2}\sum_{\wp_4=1}^{N}\chi^{(\wp_2,\wp_3,\wp_4)}$ as desired. □

Remark 1. *Using* $r_1q\lambda<r_2K\lambda_r$, *we get,*

$$\frac{r_1 q\lambda}{r_2 K\lambda_r}<1$$

If $\frac{r_1}{r_2K}=m(say)$, *then*

$$\frac{\lambda}{\lambda_r}<\frac{1}{m}.$$

3.3. Limiting Probability Distribution

It can be seen from the structure of the rate matrix B and from the Theorem 3, that the Markov process $\{X_1(t),X_2(t),X_3(t),W(t),t\geq 0\}$ with the state space E is regular. Henceforth, the limiting probability distribution

$$\Upsilon^{(\wp_1,\wp_2,\wp_3,\wp_4)} = \lim_{t\to\infty} Pr[X_1(t) = \wp_1, X_2(t) = \wp_2, X_3(t) = \wp_3, W(t) = \wp_4 | X_1(0), X_2(0), X_3(0), W(0)],$$

exists and is independent of the initial state.

Let $\Upsilon = \left(\Upsilon^{(0)}, \Upsilon^{(1)}, \ldots,\right)$ satisfies

$$\Upsilon B = \mathbf{0}, \quad \Upsilon \mathbf{e} = 1.$$

We can partition the vector $\Upsilon^{(\wp_1)}$, as

$$\Upsilon^{(\wp_1)} = \left(\Upsilon^{(\wp_1,0)}, \Upsilon^{(\wp_1,1)}, \ldots, \Upsilon^{(\wp_1,S_1)}\right), \wp_1 \geq 0$$

and

$$\Upsilon^{(\wp_1,\wp_2)} = \left(\Upsilon^{(\wp_1,\wp_2,1)}, \Upsilon^{(\wp_1,\wp_2,2)}, \ldots, \Upsilon^{(\wp_1,\wp_2,S_2)}\right), \wp_1 \geq 0, 0 \leq \wp_2 \leq S_1$$

$$\Upsilon^{(\wp_1,\wp_2,\wp_3)} = \left(\Upsilon^{(\wp_1,\wp_2,\wp_3,0)}, \Upsilon^{(\wp_1,\wp_2,\wp_3,1)}, \ldots, \Upsilon^{(\wp_1,\wp_2,\wp_3,N)}\right), \wp_1 \geq 0, 0 \leq \wp_2 \leq S_1, 0 \leq \wp_3 \leq N$$

Theorem 3. *Utilizing the vector $\Upsilon = (\Upsilon^{(0)}, \Upsilon^{(1)}, \ldots,)$ and the specific structure of B, R can be determined by*

$$R^2\mathbb{B}_{K0} + R\mathbb{B}_{K1} + \mathbb{B}_{01} = \mathbf{0}$$

where R is the minimal non-negative of the matrix quadratic equation.

Proof. Assume

$$R = \begin{pmatrix} D_{00} & D_{01} & \cdots & D_{0S_1} \\ D_{10} & D_{11} & \cdots & D_{1S_1} \\ D_{20} & D_{21} & \cdots & D_{2S_1} \\ \vdots & \vdots & \ddots & \vdots \\ D_{S_10} & D_{S_11} & \cdots & D_{S_1S_1} \end{pmatrix}$$

where

$$D_{ij} = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1S_2} \\ R_{21} & R_{22} & \cdots & R_{2S_2} \\ R_{31} & R_{32} & \cdots & R_{3S_2} \\ \vdots & \vdots & \ddots & \vdots \\ R_{S_21} & R_{S_22} & \cdots & R_{S_2S_2} \end{pmatrix}, i, j \in 1, 2, \cdots, S_2$$

Since the block matrix $\mathbb{B}_{01}$ has $(S_1+1)(S_2)$ number of nonzero rows, the assumed R matrix also has the same number of nonzero rows. Now, due to the specified structure of $\mathbb{B}_{01}$, the structure of the block matrix R_{mn} is of the form

$$R_{mn} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ l^0_{mn} & l^1_{mn} & l^2_{mn} & l^3_{mn} & l^4_{mn} & \cdots & l^N_{mn} \end{pmatrix}, m, n \in 1, 2, \cdots, S_2$$

is also square matrix of dimension $N+1$.

Now, exploiting the coefficient matrices $\mathbb{B}_{K0}, \mathbb{B}_{K1}, \mathbb{B}_{01}$ with R^2 and R equating with $\mathbf{0}$, we obtain a system of $(N+1)$-dimensional vector as follows:

For $i = 0,1,\cdots,S_1, j = 0, m = 1,2,\cdots,S_2, n = 1,2,\cdots,S_2-1$, $C_h^{(j)}$ be the diagonal elements of the corresponding matrix E_j where $j = 0,1,2,\cdots,S_1$ and $h = 1,2,\cdots,S_2(N+1)$.

$$(l^0_{mn}C^{(j)}_{(n-1)(N+1)+1} + l^0_{m(n+1)}(n+1)\gamma_2 + l^0_{mn}(j+1)\gamma_1 + l^1_{m(n+1)}1\mu, l^0_{mn}\lambda + l^1_{mn}C^{(j)}_{(n-1)(N+1)+2} + l^1_{m(n+1)}(n+1)\gamma_2 + l^1_{mn}(j+1)\gamma_1 + l^2_{m(n+1)}2\mu,\cdots, l^{N-2}_{mn}\lambda + l^{N-1}_{mn}C^{(j)}_{(n-1)(N+1)+N} + l^{N-1}_{m(n+1)}(n+1)\gamma_2 + l^{N-1}_{mn}(j+1)\gamma_1 + l^N_{m(n+1)}N\mu, l^{N-1}_{mn}\lambda + l^N_{mn}C^{(j)}_{n(N+1)} + l^N_{m(n+1)}(n+1)\gamma_2 + l^N_{mn}(j+1)\gamma_1 + \delta_{i0}q\lambda) = \mathbf{0}$$

For $j = 0, m = 1,2,\cdots,S_2, n = S_2$

$$(l^0_{mn}C_{(n-1)(N+1)+1} + l^0_{m1}1\gamma_2 + l^0_{mn}(j+1)\gamma_1 + l^1_{m1}1\mu, l^0_{mn}\lambda + l^1_{mn}C_{(n-1)(N+1)+2} + l^1_{m1}1\gamma_2 + l^1_{mn}(j+1)\gamma_1 + l^2_{m1}2\mu,\cdots, l^{N-2}_{mn}\lambda + l^{N-1}_{mn}C_{(n-1)(N+1)+N} + l^{N-1}_{m1}1\gamma_2 + l^{N-1}_{mn}(j+1)\gamma_1 + l^N_{m1}N\mu, l^{N-1}_{mn}\lambda + l^N_{mn}C_{n(N+1)} + l^N_{m1}1\gamma_2 + l^N_{mn}(j+1)\gamma_1 + \delta_{i0}q\lambda) = \mathbf{0}$$

For $j = 1,2,\cdots,S_1-1, m = 1,2,\cdots,S_2, n = 1,2,\cdots,S_2-1$

$$(l^0_{mn}C^{(j)}_{(n-1)(N+1)+1} + l^0_{m(n+1)}(n+1)\gamma_2 + l^0_{mn}(j+1)\gamma_1 + l^1_{m(n+1)}1\mu + H(s-j)l^0_{mn}\beta \sum_{d=1}^{S_2} l^0_{md}l^0_{dn}N\lambda_r + l^0_{mn}\lambda + l^1_{mn}C^{(j)}_{(n-1)(N+1)+2} + l^1_{m(n+1)}(n+1)\gamma_2 + l^1_{mn}(j+1)\gamma_1 + l^2_{m(n+1)}2\mu + H(s-j)l^1_{mn}\beta,\cdots, \sum_{d=1}^{S_2} l^{N-2}_{md}l_d n^{N-2}N\lambda_r + l^{N-2}_{mn}\lambda + l^{N-1}_{mn}C^{(j)}_{(n-1)(N+1)+N} + l^{N-1}_{m(n+1)}(n+1)\gamma_2 + l^{N-1}_{mn}(j+1)\gamma_1 + l^N_{m(n+1)}N\mu + H(s-j)l^{N-1}_{mn}\beta, \sum_{d=1}^{S_2} l^{N-1}_{md}l^{N-1}_{dn}N\lambda_r + l^{N-1}_{mn}\lambda + l^N_{mn}C^{(j)}_{n(N+1)} + l^N_{m(n+1)}(n+1)\gamma_2 + l^N_{mn}(j+1)\gamma_1 + \delta_{ij}q\lambda + H(s-j)l^N_{mn}\beta) = \mathbf{0}$$

For $j = 1,2,\cdots,S_1-1, m = 1,2,\cdots,S_2, n = S_2$

$$(l^0_{mn}C_{(n-1)(N+1)+1} + l^0_{m1}1\gamma_2 + l^0_{mn}(j+1)\gamma_1 + l^1_{m1}1\mu + H(s-j)l^0_{mn}\beta, \sum_{d=1}^{S_2} l^0_{md}l^0_{dn}N\lambda_r + l^0_{mn}\lambda + l^1_{mn}C^{(j)}_{(n-1)(N+1)+2} + l^1_{m1}1\gamma_2 + l^1_{mn}(j+1)\gamma_1 + l^2_{m1}2\mu + H(s-j)l^1_{mn}\beta,\cdots, \sum_{d=1}^{S_2} l^{N-2}_{md}l^{N-2}_{dn}N\lambda_r + l^{N-2}_{mn}\lambda + l^{N-1}_{mn}C_{(n-1)(N+1)+N} + l^{N-1}_{m1}1\gamma_2 + l^{N-1}_{mn}(j+1)\gamma_1 + l^N_{m1}N\mu + H(s-j)l^{N-1}_{mn}\beta, \sum_{d=1}^{S_2} l^{N-1}_{md}l^{N-1}_{dn} N\lambda_r + l^{N-1}_{mn}\lambda + l^N_{mn}C_{n(N+1)} + l^N_{m1}1\gamma_2 + l^N_{mn}(j+1)\gamma_1 + \delta_{ij}q\lambda + H(s-j)l^N_{mn}\beta) = \mathbf{0}$$

For $j = S_1, m = 1,2,\cdots,S_2, n = 1,2,\cdots,S_2-1$

$$(l^0_{mn}C^{(j)}_{(n-1)(N+1)+1} + l^0_{m(n+1)}(n+1)\gamma_2 + H(s-j)l^0_{mn}\beta, \sum_{d=1}^{S_2} l^0_{md}l^0_{dn}N\lambda_r + l^0_{mn}\lambda + l^1_{mn}C^{(j)}_{(n-1)(N+1)+2} + l^1_{m(n+1)}(n+1)\gamma_2 + H(s-j)l^1_{mn}\beta,\cdots, \sum_{d=1}^{S_2} l^{N-2}_{md}l^{N-2}_{dn}N\lambda_r + l^{N-2}_{mn}\lambda + l^{N-1}_{mn}C^{(j)}_{(n-1)(N+1)+N} + l^{N-1}_{m(n+1)}(n+1)\gamma_2 + H(s-j)l^{N-1}_{mn}\beta, \sum_{d=1}^{S_2} l^{N-1}_{md}l^{N-1}_{dn}N\lambda_r + l^{N-1}_{mn}\lambda + l^N_{mn}C^{(j)}_{n(N+1)} + l^N_{m(n+1)}(n+1)\gamma_2 + \delta_{ij}q\lambda + H(s-j)l^N_{mn}\beta) = \mathbf{0}$$

For $j = S_1, m = 1,2,\cdots,S_2, n = S_2$

$$(l^0_{mn}C_{(n-1)(N+1)+1} + l^0_{m1}1\gamma_2 + H(s-j)l^0_{mn}\beta, \sum_{d=1}^{S_2} l^0_{md}l^0_{dn}N\lambda_r + l^0_{mn}\lambda + l^1_{mn}C^{(j)}_{(n-1)(N+1)+2}+$$
$$l^1_{m1}1\gamma_2 + H(s-j)l^1_{mn}\beta, \cdots, \sum_{d=1}^{S_2} l^{N-2}_{md}l^{N-2}_{dn}N\lambda_r + l^{N-2}_{mn}\lambda + l^{N-1}_{mn}C_{(n-1)(N+1)+N} + l^{N-1}_{m1}1\gamma_2+$$
$$H(s-j)l^{N-1}_{mn}\beta, \sum_{d=1}^{S_2} l^{N-1}_{md}l^{N-1}_{dn}N\lambda_r + l^{N-1}_{mn}\lambda + l^N_{mn}C_{n(N+1)} + l^N_{m1}1\gamma_2 + \delta_{ij}q\lambda + H(s-j)l^N_{mn}\beta) = \mathbf{0}$$

Equating the $(N+1)$-dimensional vector to zero vector we obtain a set of equations, after solving such equations, one can obtain the elements of R matrix. □

Theorem 4. *Due to the specific structure of B the vector Y can be determined by*

$$Y^{(i+K-1)} = Y^{(K-1)}R^i; i \geq 0 \tag{2}$$

where R is the solution of

$$R^2\mathbb{B}_{K0} + R\mathbb{B}_{K1} + \mathbb{B}_{01} = \mathbf{0} \tag{3}$$

and the vector $Y^{(i)}, i \geq 0$

$$Y^{(i)} = \begin{cases} \sigma X^{(0)} \prod_{j=i+1}^{K} \mathbb{B}_{j0}(-\mathbb{B}_{j-1}), 0 \leq i \leq K-1 \\ \sigma X^{(0)} R^{(i-K)}, i \geq K \end{cases} \tag{4}$$

where

$$\sigma = [1 + X^{(0)} \sum_{i=0}^{K-1} \prod_{j=i+1}^{K} \mathbb{B}_{j0}(-\mathbb{B}_{j-1})\mathbf{e}]^{-1}. \tag{5}$$

Proof. The sub vector $(Y^{(0)}, Y^{(1)}, \ldots, Y^{(K-1)})$ and the block partitioned matrix of B satisfies the following set of equations

$$\begin{aligned} Y^{(0)}\mathbb{B}_{00} + Y^{(1)}\mathbb{B}_{10} &= \mathbf{0} \\ Y^{(i-1)}\mathbb{B}_{01} + Y^{(i)}\mathbb{B}_{i1} + Y^{(i+1)}\mathbb{B}_{(i+1)0} &= \mathbf{0}; 1 \leq i \leq K-1 \\ Y^{(K-2)}\mathbb{B}_{01} + Y^{(K-1)}(\mathbb{B}_{(K-1)1} + R\mathbb{B}_2) &= \mathbf{0}; \end{aligned} \tag{6}$$

using Equation (6), we get,

$$Y^{(i)} = Y^{(i+1)}\mathbb{B}_{(i+1)0}(-B_i)^{-1}, 0 \leq i \leq K-1 \tag{7}$$

where

$$B_i = \begin{cases} \mathbb{B}_{i0}, & i = 0 \\ (\mathbb{B}_{i1} - \mathbb{B}_{i0}(-B_{i-1})^{-1}\mathbb{B}_{01}), & 1 \leq i \leq K \end{cases}$$

Then

$$(Y^{(K)}, Y^{(K+1)}, Y^{(K+2)} \ldots) \begin{pmatrix} B_K & \mathbb{B}_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbb{B}_{K0} & \mathbb{B}_{K1} & \mathbb{B}_{01} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbb{B}_{K0} & \mathbb{B}_{K1} & \mathbb{B}_{01} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \mathbf{0} \tag{8}$$

Assume,

$$\sigma = \sum_{i=K}^{\infty} Y^{(i)}\mathbf{e} \quad (9)$$

$$X^{(i)} = \sigma^{-1}Y^{(K+i)}, i \geq 0 \quad (10)$$

where X is also a 4-dimensional continuous time Markov chain such as Y. The similar partitions are also applicable to X. From (8) we get

$$\begin{aligned} Y^{(K)}B_N + Y^{(K+1)}\mathbb{B}_{K0} &= \mathbf{0} \\ Y^{(K+i)} &= Y^{(K+i-1)}R, i \geq 1 \end{aligned}$$

This can be written as

$$\begin{aligned} X^{(0)}B_N + X^{(1)}\mathbb{B}_{01} &= \mathbf{0} \\ X^{(i)} &= X^{(i-1)}R, i \geq 1 \end{aligned}$$

Since $\sum_{i=0}^{\infty} X^{(i)}\mathbf{e} = 1$, then

$$X^{(0)}(I-R)^{-1}\mathbf{e} = 1$$

Hence,

$$Y^{(i)} = \sigma X^{(0)}R^{(i-K)}, i \geq K \quad (11)$$

Again by (7) and (10),

$$Y^{(i)} = \sigma X^{(0)} \prod_{j=i+1}^{K} \mathbb{B}_{j0}(-B_{j-1}), 0 \leq i \leq K-1 \quad (12)$$

Therefore, combining (11) and (12), we get (4) and $X^{(0)}$ is the unique solution of the system

$$X^{(0)}(B_K + R\mathbb{B}_{K0}) = \mathbf{0} \quad (13)$$

$$X^{(0)}(I-R)^{-1}\mathbf{e} = 1 \quad (14)$$

Since $\sum_{i=0}^{\infty} Y^{(i)}\mathbf{e} = 1$ and (13),

$$\sigma X^{(0)} \sum_{i=0}^{K-1} \prod_{j=i+1}^{K} \mathbb{B}_{j0}(-B_{j-1})\mathbf{e} + \sigma X^{(0)} \sum_{K}^{\infty} R^{(i-K)}\mathbf{e} = 1$$

which gives (5). □

4. Waiting Time Analysis

The waiting time of a customer is defined as the time interval between the customer enters into the waiting hall and leaves the system after the service completion. Using the Laplace–Stieltjes transform (LST), we look at the waiting time of demand in the queue as well as the orbit independently for each node. Since the state space of the proposed model is infinite, the analytical work on finding the waiting time distribution is a difficult task to do so naturally. Thus, we restrict the orbit size to be finite say $L(L > K)$ to find the waiting time of a customer in the waiting hall and orbit. We denote W_p and W_o are continuous random variables to represent the waiting time distribution of a customer in the queue and orbit, respectively. The objective is to describe the probability that a customer has to wait, the distribution of the waiting time and nth order moments.

Waiting Time of a Demand in Queue

To enable waiting time distribution of W_p, we shall define some complementary variables. Suppose that the QIS is at state $(\wp_1, \wp_2, \wp_3, \wp_4)$, $\wp_4 > 0$ at an arbitrary time t:

1. $W_p(\wp_1, \wp_2, \wp_3, \wp_4)$ is the time until chosen demand become satisfied.
2. LST of $W_p(\wp_1, \wp_2, \wp_3, \wp_4)$ is ${}^*W_p(\wp_1, \wp_2, \wp_3, \wp_4)(y)$, and we denote W_p by ${}^*W_p(y)$.
3. ${}^*W_p(y) = E\left[e^{yW_p}\right]$ LST of unconditional waiting time (UWT).
4. ${}^*W_p(\wp_1, \wp_2, \wp_3, \wp_4)(y) = E\left[e^{yW_p(\wp_1,\wp_2,\wp_3,\wp_4)}\right]$ LST of conditional waiting time (CWT).

Theorem 5. *The expected waiting time of a demand in the queue is defined by*

$$E[W_p] = \sum_{\wp_1=0}^{L} \sum_{\wp_2=0}^{S_1} \sum_{\wp_3=0}^{S_2} \sum_{\wp_4=0}^{N-1} \Upsilon^{(\wp_1,\wp_2,\wp_3,\wp_4)} E[W_p(\wp_1, \wp_2, \wp_3, \wp_4 + 1)] \tag{15}$$

Proof. To obtain the CWT, we apply first step analysis as follows:

For $0 \le \wp_1 \le L, \wp_2 = 0, 1 \le \wp_3 \le S_2, 1 \le \wp_4 \le N$

$$\begin{aligned}
{}^*W_p(\wp_1, 0, \wp_3, \wp_4)(y) = {} & \frac{\bar{\delta}_{N\wp_4}\lambda}{a}{}^* W_p(\wp_1, 0, \wp_3, \wp_4 + 1)(y) + \\
& \frac{\wp_3\gamma_2}{a}{}^* W_p(\wp_1, 0, \wp_3 - 1, \wp_4)(y) + \frac{\beta}{a}{}^* W_p(\wp_1, Q, \wp_3, \wp_4)(y) + \\
& \frac{\bar{\delta}_{N\wp_4}\wp_1\lambda_r}{a}{}^* W_p(\wp_1 - 1, 0, \wp_3, \wp_4 + 1)(y) + \\
& \frac{\delta_{N\wp_4}\bar{\delta}_{L\wp_1}q\lambda}{a}{}^* W_p(\wp_1 + 1, 0, \wp_3, \wp_4)(y),
\end{aligned} \tag{16}$$

where $a = (y + \bar{\delta}_{N\wp_4}\lambda + \wp_3\gamma_2 + \beta + \bar{\delta}_{N\wp_4}\wp_1\lambda_r + \delta_{N\wp_4}\bar{\delta}_{L\wp_1}q\lambda)$.

For $0 \le \wp_1 \le L, 1 \le \wp_2 \le S_1, 1 \le \wp_3 \le S_2, 1 \le \wp_4 \le N$,

$$\begin{aligned}
{}^*W_p(\wp_1, \wp_2, \wp_3, \wp_4)(y) = {} & \frac{\bar{\delta}_{N\wp_4}\lambda}{b}{}^* W_p(\wp_1, \wp_2, \wp_3, \wp_4 + 1)(y) + \\
& \frac{H(s - \wp_4)\beta}{b}{}^* W_p(\wp_1, \wp_2 + Q, \wp_3, \wp_4)(y) + \\
& \frac{\wp_2\gamma_1}{b}{}^* W_p(\wp_1, \wp_2 - 1, \wp_3, \wp_4)(y) + \frac{w\gamma_2}{b}{}^* W_p(\wp_1, \wp_2, \wp_3 - 1, \wp_4)(y) + \\
& \frac{(\wp_4 - 1)\mu}{b}{}^* W_p(\wp_1, \wp_2 - 1, \wp_3 - 1, \wp_4 - 1)(y) + \\
& \frac{\bar{\delta}_{N\wp_4}\wp_1\lambda_r}{b}{}^* W_p(\wp_1 - 1, \wp_2, \wp_3, \wp_4 + 1)(y) + \frac{\delta_{N\wp_1}\bar{\delta}_{L\wp_1}q\lambda}{b}{}^* W_p(\wp_1 + 1, \wp_2, \wp_3, \wp_4)(y) + \frac{\mu}{b}
\end{aligned} \tag{17}$$

where, $b = (y + \bar{\delta}_{N\wp_4}\lambda + H(s - \wp_2)\beta + \wp_2\gamma_1 + \wp_3\gamma_2 + \wp_4\mu + \bar{\delta}_{N\wp_4}\wp_1\lambda_r + \delta_{N\wp_4}\bar{\delta}_{L\wp_1}q\lambda)$.

Now, we differentiate the Equations (16) and (17) for $(n+1)$ times and computing at $y = 0$, we have,

For $0 \le \wp_1 \le L, \wp_2 = 0, 1 \le \wp_3 \le S_2, 1 \le \wp_4 \le N$,

$$\begin{aligned}
E[W_p^{n+1}(\wp_1, 0, \wp_3, \wp_4)] = {} & \frac{\bar{\delta}_{N\wp_4}\lambda}{a} E[W_p^{n+1}(\wp_1, 0, \wp_3, \wp_4 + 1)] + \\
& \frac{w\gamma_2}{b} E[W_p^{n+1}(\wp_1, \wp_2, \wp_3 - 1, \wp_4)] + \frac{\beta}{a} E[W_p^{n+1}(\wp_1, Q, \wp_3, \wp_4)] + \\
& \frac{\bar{\delta}_{N\wp_4}\wp_1\lambda_r}{a} E[W_p^{n+1}(\wp_1 - 1, 0, \wp_3, \wp_4 + 1)] + \\
& \frac{\delta_{N\wp_4}\bar{\delta}_{L\wp_1}q\lambda}{a} E[W_p^{n+1}(\wp_1 + 1, 0, \wp_3, \wp_4)] + (n+1)E[W_p^{n+1}(\wp_1, \wp_2, \wp_3, \wp_4)],
\end{aligned} \tag{18}$$

where$a = (y + \bar{\delta}_{N\wp_4}\lambda + \wp_3\gamma_2 + \beta + \bar{\delta}_{N\wp_4}\wp_1\lambda_r + \delta_{N\wp_4}\bar{\delta}_{L\wp_1}q\lambda)$.

For $0 \leq \wp_1 \leq L, 1 \leq \wp_2 \leq S_1, 1 \leq \wp_3 \leq S_2, 1 \leq \wp_4 \leq N$,

$$
\begin{aligned}
E[W_p^{n+1}(\wp_1, \wp_2, \wp_3, \wp_4)] &= \frac{\bar{\delta}_{N\wp_4}\lambda}{b}E[W_p^{n+1}(\wp_1, \wp_2, \wp_3, \wp_4 + 1) + \\
&\frac{H(s-v)\beta}{b}E[W_p^{n+1}(u, v+Q, w, x)] + \frac{v\gamma_1}{b}E[W_p^{n+1}(\wp_1, \wp_2 - 1, \wp_3)] + \\
&\frac{w\gamma_2}{b}E[W_p^{n+1}(\wp_1, \wp_2, \wp_3 - 1)] + \frac{(\wp_4 - 1)\mu}{b}E[W_p^{n+1}(\wp_1, \wp_2 - 1, \wp_3 - 1, \wp_4 - 1)] + \\
&\frac{\bar{\delta}_{N\wp_4}\wp_1\lambda_r}{b}E[W_p^{n+1}(\wp_1 - 1, \wp_2, \wp_3, \wp_4 + 1)] + \\
&\frac{\delta_{N\wp_4}\bar{\delta}_{L\wp_1}q\lambda}{b}E[W_p^{n+1}(\wp_1 + 1, \wp_2, \wp_3, \wp_4) + (n+1)E[W_p^{n+1}(\wp_1, \wp_2, \wp_3, \wp_4)]
\end{aligned}
\tag{19}
$$

where, $b = (y + \bar{\delta}_{N\wp_1}\lambda + H(s-\wp_2)\beta + \wp_2\gamma_1 + \wp_3\gamma_2 + \wp_4\mu + \bar{\delta}_{N\wp_4}\wp_1\lambda_r + \delta_{N\wp_4}\bar{\delta}_{L\wp_1}q\lambda)$.

The LST of UWT of a demand in the queue is given by

$$
\begin{aligned}
{}^*W_p(y) = 1 - \sum_{\wp_1=0}^{L}\sum_{\wp_2=0}^{S_1}\sum_{\wp_3=1}^{S_2}\sum_{\wp_4=0}^{N-1}\Upsilon^{(\wp_1,\wp_2,\wp_3,\wp_4)} + \\
\sum_{\wp_1=0}^{L}\sum_{\wp_2=0}^{S_1}\sum_{\wp_3=1}^{S_2}\sum_{\wp_4=0}^{N-1}\Upsilon^{(\wp_1,\wp_2,\wp_3,\wp_4)*}W_p(\wp_1, \wp_2, \wp_3, \wp_4 + 1)(y)
\end{aligned}
\tag{20}
$$

The nth moments of UWT, using (20), is given by

$$
E[W_p^n] = \delta_{0n} + (1 - \delta_{0n})\sum_{\wp_1=0}^{L}\sum_{\wp_2=0}^{S_1}\sum_{\wp_3=1}^{S_2}\sum_{\wp_4=0}^{N-1}\Upsilon^{(\wp_1,\wp_2,\wp_3,\wp_4)}E[W_p^n(\wp_1, \wp_2, \wp_3, \wp_4 + 1)] \tag{21}
$$

Using Equation (21) and substitute $n = 1$, we get the desired result as in (15). □

Corollary 1. *The expected waiting time of a orbital demand is defined by*

$$
E[W_o] = \sum_{\wp_1=0}^{L-1}\sum_{\wp_2=0}^{S_1}\sum_{\wp_3=1}^{S_2}\sum_{\wp_4=0}^{N}\Upsilon^{(\wp_1,\wp_2,\wp_3,\wp_4)}E[W_o(\wp_1 + 1, \wp_2, \wp_3, \wp_4)] \tag{22}
$$

5. System Characteristics

This section explores the necessary and sufficient system performance of the proposed model.

5.1. Mean Inventory Level for First Commodity

The mean inventory of the first commodity is defined as

$$
\Delta_1 = \sum_{\wp_1=0}^{\infty}\sum_{\wp_2=1}^{S_1}\sum_{\wp_3=1}^{S_2}\sum_{\wp_4=0}^{N}\wp_2\Upsilon^{(\wp_1,\wp_2,\wp_3,\wp_4)}
$$

5.2. Mean Inventory Level for Second Commodity

The mean inventory of the second commodity is defined as

$$
\Delta_2 = \sum_{\wp_1=0}^{\infty}\sum_{\wp_2=0}^{S_1}\sum_{\wp_3=1}^{S_2}\sum_{\wp_4=0}^{N}\wp_3\Upsilon^{(\wp_1,\wp_2,\wp_3,\wp_4)}
$$

5.3. Mean Reorder Rate for First Commodity

The mean reorder rate of first commodity is defined as

$$\Delta_3 = \sum_{\wp_1=0}^{\infty} \sum_{\wp_3=1}^{S_2} \sum_{\wp_4=1}^{N} \wp_4 \mu \Upsilon^{(\wp_1, s+1, \wp_3, \wp_4)} + \sum_{\wp_1=0}^{\infty} \sum_{\wp_3=1}^{S_2} \sum_{\wp_4=0}^{N} (s+1)\gamma_1 \Upsilon^{(\wp_1, s+1, \wp_3, \wp_4)}$$

5.4. Mean Reorder Rate for Second Commodity

The mean reorder rate of second commodity is defined as

$$\Delta_4 = \sum_{\wp_1=0}^{\infty} \sum_{\wp_2=1}^{S_1} \sum_{\wp_4=1}^{N} \wp_4 \mu \Upsilon^{(\wp_1, \wp_2, 1, \wp_4)} + \sum_{\wp_1=0}^{\infty} \sum_{\wp_2=0}^{S_1} \sum_{\wp_4=0}^{N} \gamma_2 \Upsilon^{(\wp_1, \wp_2, 1, \wp_4)}$$

5.5. Mean Perishable Rate for First Commodity

The mean perishable rate of first commodity is given by

$$\Delta_5 = \sum_{\wp_1=0}^{\infty} \sum_{\wp_2=1}^{S_1} \sum_{\wp_3=1}^{S_2} \sum_{\wp_4=0}^{N} \wp_2 \gamma_1 \Upsilon^{(\wp_1, \wp_2, \wp_3, \wp_4)}$$

5.6. Mean Perishable Rate for Second Commodity

The mean perishable rate of second commodity is given by

$$\Delta_6 = \sum_{\wp_1=0}^{\infty} \sum_{\wp_2=0}^{S_1} \sum_{\wp_3=1}^{S_2} \sum_{\wp_4=0}^{N} w \gamma_2 \Upsilon^{(\wp_1, \wp_2, \wp_3, \wp_4)}$$

5.7. Mean Number of Primary Customer Lost

The mean number of primary customer lost in the system is defined as

$$\Delta_{11} = \sum_{\wp_1=0}^{\infty} \sum_{\wp_2=0}^{S_1} \sum_{\wp_3=1}^{S_2} (1-q)\lambda \Upsilon^{(\wp_1, \wp_2, \wp_3, N)}$$

5.8. Overall Rate of Retrial

The expected overall rate of retrial of the orbit customer in the system is given by

$$\Delta_{12} = \sum_{\wp_1=1}^{\infty} \sum_{\wp_2=0}^{S_1} \sum_{\wp_3=1}^{S_2} \sum_{\wp_4=0}^{N} \wp_1 \lambda_r \Upsilon^{(\wp_1, \wp_2, \wp_3, \wp_4)}$$

5.9. Successful Rate of Retrial

The expected successful rate of retrial of the orbit customer in the system is given by

$$\Delta_{13} = \sum_{\wp_1=1}^{\infty} \sum_{\wp_2=0}^{S_1} \sum_{\wp_3=1}^{S_2} \sum_{\wp_4=0}^{N-1} \wp_1 \lambda_r \Upsilon^{(\wp_1, \wp_2, \wp_3, \wp_4)}$$

5.10. Fraction of Successful Rate of Retrial

The expected fraction of successful rate of retrial of the orbit customer in the system is given by

$$\Delta_{14} = \frac{\Delta_{13}}{\Delta_{12}}$$

6. Cost Analysis and Numerical Illustrations

The mean total cost (MTC) is given by

$MTC = (C_{h1} * \Delta_1) + (C_{h2} * \Delta_2) + (C_{s1} * \Delta_3) + (C_{s2} * \Delta_4) + (C_{p1} * \Delta_5) + (C_{p2} * \Delta_6) + (C_{w1} * E[W_p]) + (C_{w2} * E[W_o]) + (C_b * \Delta_{11})$.

To compute the MTC per unit time, the following costs are considered

C_{h1} = The holding cost of commodity-I per unit item per unit time t.
C_{h2} = The holding cost of commodity-II per unit item per unit time t.
C_{s1} = Set up cost of commodity-I per run.
C_{s2} = Set up cost of commodity-II per run.
C_{p1} = Perishable cost of commodity-I per unit item per unit time.
C_{p2} = Perishable cost of commodity-II per unit item per unit time.
C_{w1} = Waiting cost of a customer in the queue per unit time t.
C_{w2} = Waiting cost of a orbiting customer per unit time t.
C_b = Cost due to loss of customers per unit time.

Numerical Illustrations

Numerical analysis is an applied mathematics technique that allows staggeringly large amount of data to be processed and analyzed for trends, thereby aiding in forming conclusions. This is done nowadays in a computer environment, providing massive increases in speed and usefulness of calculations. This numerical work is carried out by fixing the parameters and cost values which are involved in the proposed mathematical model. Due to the nature of the proposed model, the determination of the parameter values assumed randomly under the satisfaction of the stability condition. The numerical value of stationary probability vector whose sum gives one. After such verification's (stability condition and normalising condition), we proceed the numerical works. To perform these numerical illustrations, we fix the parameter values as $C_{h1} = 0.001, C_{h2} = 0.01, s = 4, C_{s1} = 4.1, C_{s2} = 3.5, C_{p1} = 2, C_{p2} = 1.8, C_{w1} = 1.6, C_{w2} = 1.1, C_b = 3.6, N = 5, \lambda = 3.6, \beta = 0.93, \gamma_1 = 0.21, \gamma_2 = 0.2, \lambda_r = 4.46, q = 0.5, \mu = 25.5$. The proposed study is only valid if the assumed parameters must satisfy the stability condition and normalising condition.

Example 1. *From Table 1, for varying S_1, the economical capacity $S_1(S_1^*)$ is determined at each S_2 and is identified by the expected total cost which is column minimum with underline. Similarly, the economical capacity $S_2(S_2^*)$ is determined at each S_1 and is identified by the expected total cost which is row minimum and is given in bold form. When we increase the number of first commodity, the MTC holds the both decreasing and increasing property. It shows that the MTC is not monotonic regarding the change in S_1. This will produce the minimum MTC at some S_1. Similarly, the same monotonic property holds for the second commodity. Therefore, we also obtain the minimum MTC for S_2. Since both S_1 and S_2 holds the minimum MTC, the proposed model produces the convex point at some (S_1, S_2) locally. Finally the local minimum expected total cost is determined at S_1^* and S_2^* where the row minimum and column minimum matches, so that the cost will be identified with both underlined and in bold script. Using Table 1, $S_1^* = 25, S_2^* = 7, MTC^* = 8.991429$.*

The graphical illustration of the optimum MTC is shown in Figure 1. If the values of S_1 increases, then the value of MTC have both increasing and increasing quality. It means that S_1 will give the optimum MTC. When we work on the S_2 with the similar manner as S_1, this also gives the minimum MTC. Finally, both S_1 and S_2 are increased together, we obtain the convex at some point of this proposed mathematical model. This convexity also helps us to determine the fixation of parameters and cost values of the model. Under the obtained convex result, we perform the further numerical work.

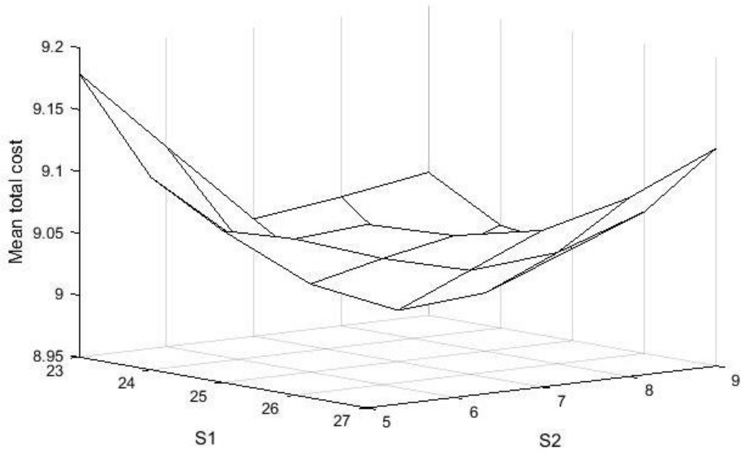

Figure 1. S_1 vs. S_2 on MTC.

Table 1. Mean total cost rate as a function of S_1 and S_2.

S_1/S_2	5	6	7	8	9
23	9.179124	9.105358	**9.072545**	9.076211	9.098163
24	9.111433	9.046548	**9.021119**	9.051946	9.080716
25	9.044719	9.008342	**8.991429**	9.033913	9.076386
26	9.054109	9.008624	**8.996934**	9.039765	9.094405
27	9.065483	9.032995	**9.024460**	9.064498	9.126085

Example 2. *From the exploration of Figures 2–6, the responses of $\Delta_3, \Delta_9, \Delta_{11}, \Delta_{13}$ and MTC under the subject of queue capacity and primary arrival rate are explained below.*

- *When we increase λ, the number of arrival in the system as well as sales of first commodity are increased. So the number of customer and number of sold commodity are directly proportional to each other, the commodity-I reaches the reorder point as soon. Therefore, the mean reorder rate increases according to the increase of λ. Likewise, the queue size N also cause the increase of the mean reorder rate of the commodity-I. It is shown in Figure 2.*
- *As the increase of arrival rate, we observe that the number of customer in the waiting hall will increase from the Figure 3. The expansion of queue size N also cause the increase of number of customer in the waiting hall. Δ_9 monotonically increases as both N and λ increase.*
- *However, as we predicted, the customer lost rate will increase when λ increases due to the restriction of finite waiting hall size. Once the waiting hall starts overflow, the new arrivals at that moment either goes to an orbit or they considered as lost under the Bernoulli's schedule. It is shown in Figure 4. However, this can be controlled when we expand the queue size.*
- *As λ increases, the mean successful rate of retrial decrease. This is because the primary customer occupies the places in the waiting hall. When the orbit customer tries to enter into the waiting hall and finds that there will be very less seats available, the number of customer enters into the waiting hall will decrease. Similarly the same situation will happen when the queue size is expanded. This can be seen from Figure 5.*
- *According to occurrence of the arrival rate, the total cost of the system changed with the direct proportion. Since every customer requires an item, we should maintain the sufficient stock level according to the occurrence of customer in the waiting hall. So that the holding cost of the required items also increased accordingly. Thus the mean total cost value of the system increased as we increase both λ and N. It can be seen in Figure 6.*

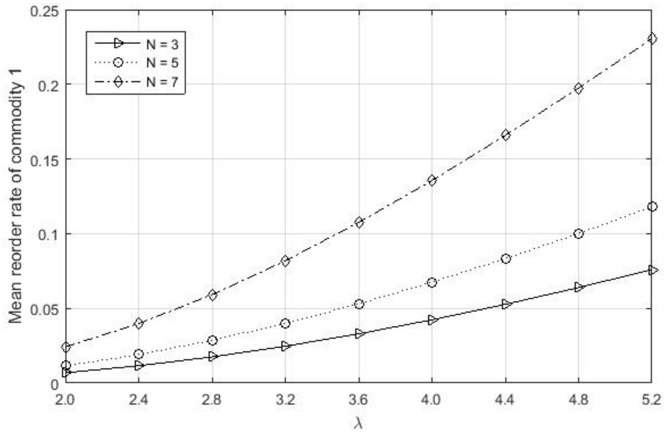

Figure 2. λ vs. N on Δ_3.

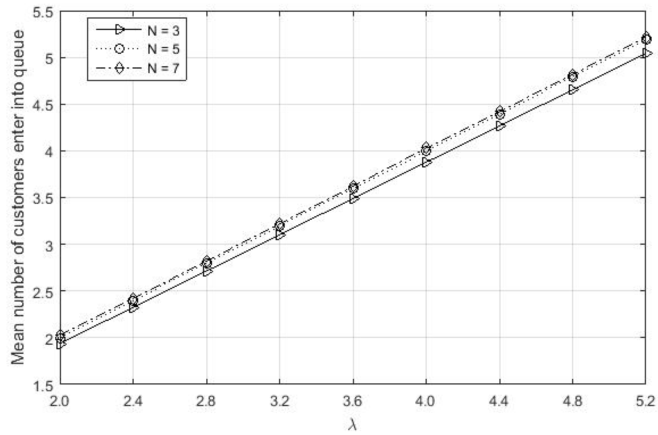

Figure 3. λ vs. N on Δ_9.

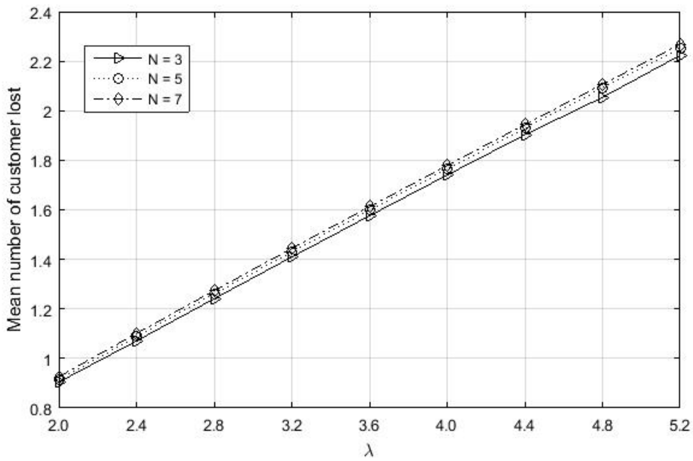

Figure 4. λ vs. N on Δ_{11}.

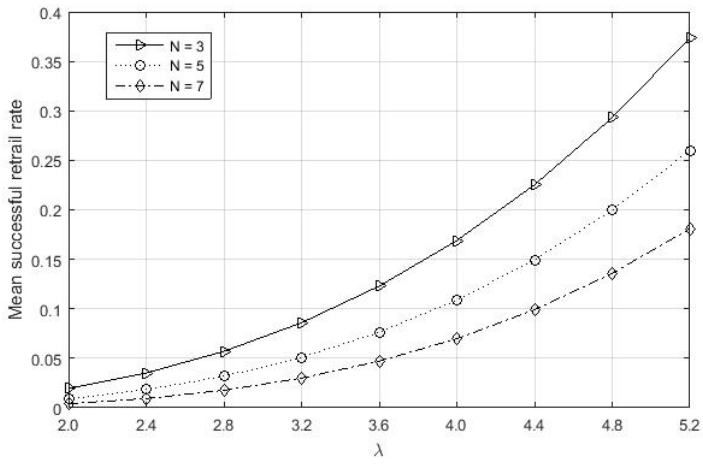

Figure 5. λ vs. N on Δ_{13}.

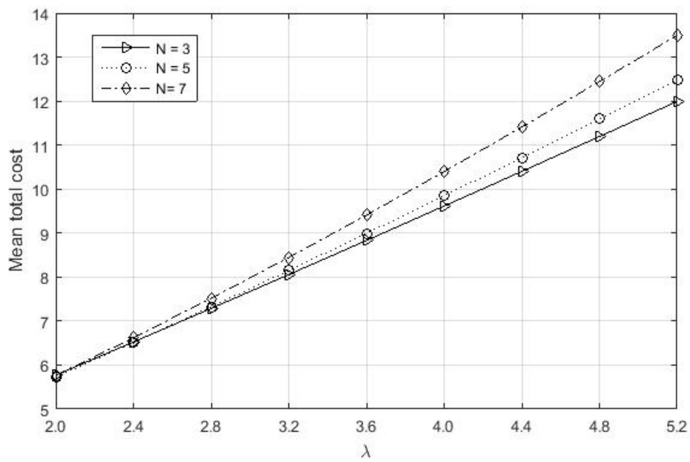

Figure 6. λ vs. N on MTC.

Example 3. *Table 2 shows the different characteristics of MTC and Δ_{11} with varying combinations Q, s, S_2 and q with set up cost dependent lead time for commodity 1.*

- *Basically the lead time of any receiving order depends upon the ordering quantity. Normally the lead time of an order decreases with increasing the order quantity. In addition, the set up cost per order may increase if the transportation cost increases with Q.*
- *But as β increases, MTC decreases due to high sensitive nature of lead time. β increases means that the average time of an ordered item to reach the system will be reduced. According to this fact the unnecessary delay of receiving the ordered items can be avoided. Relaxation of this delay time helps to reduce the MTC.*
- *As S_2 increases, MTC has both decreasing and increasing property under the given random environmental conditions. It possess the optimal MTC. As per the changes due to S_2, the C_{s2} will play the same role as S_2 on MTC.*
- *If we increase q, then the customer decides to leave the system without entering into the orbit. This leads to increase of mean customer lost. At the same time, it helps to reduce the mean total cost whenever it increases.*

Table 2. Mean total cost vs. Customer lost.

N	S_2	C_{s2}	$Q=18, s=2, q=0.5$ $\beta=1, C_{s1}=6$		$Q=18, s=2, q=0.75$ $\beta=1, C_{s1}=6$		$Q=16, s=4, q=0.5$ $\beta=0.75, C_{s1}=5.5$		$Q=16, s=4, q=0.75$ $\beta=0.75, C_{s1}=5.5$	
			MTC	Δ_{11}	MTC	Δ_{11}	MTC	Δ_{11}	MTC	Δ_{11}
2	2	3	8.7536	1.5519	7.5094	0.7532	9.0950	1.5882	8.0698	0.7743
	3	3.5	8.6646	1.5519	7.1303	0.7532	8.7645	1.5882	7.3157	0.7743
	4	4	8.7541	1.5519	7.0804	0.7532	8.7776	1.5882	7.1392	0.7743
4	2	3	10.3059	1.5886	9.9867	0.7724	12.5089	1.6150	13.0528	0.7870
	3	3.5	9.2156	1.5886	8.1644	0.7724	10.0960	1.6150	9.5449	0.7870
	4	4	9.1308	1.5886	7.8589	0.7724	9.7125	1.6150	8.7821	0.7870

Example 4. *From Table 3, Under the given cost structure and some parameters associated with service and any kind of arrivals, the comparisons of results of queue dependent with non-queue dependent service policies, the merits of the proposed model are studied and stated below:*

- *As μ increases, Δ_{11} and Δ_{14} increases. This is because the average service time per customer is reduced. So the overflowing of a customer in the waiting hall has been controlled by the server. Thus the attempt of an retrial customer to enter into the waiting hall becomes successful. In the mean time the customer lost is also controlled.*
- *At any given rate of primary arrival and retrial, the total cost curve is either decreased or convex under the given range of μ with non-queue dependent service time but it is lower at any given μ with queue dependent service time. Since the queue dependent service rate will reduce the average service time per customer, the optimal MTC of the proposed model is attained in the queue dependent service time cases only.*
- *As we expected $E[W_p], E[W_p], \Delta_{14}$ and MTC all increases as λ increases. The number of incoming arrival into the system generally increases the expected waiting time of a primary as well as retrial customer, MTC and customer lost when it increases.*
- *Further all these measures are much lower at queue dependent service time except Δ_{14}.*
- *As μ increases, $MTC, E[W_p]$ and $E[W_o]$ decreases and Δ_{11} and Δ_{14} decreases. This is due to the decreases of the mean service while increasing μ which reduces the number of customers in the waiting hall.*
- *We can notice that Δ_{14} decreases and others decrease on increasing the parameter vale of λ. Because the number of customers in the waiting hall increases if λ is increased.*
- *On comparing, the measures $E[W_p], E[W_o]$ and MTC under the parameter variation, Queue dependent service time is more effective than the non-queue dependent service time. For the others, non-queue dependent service time is effective.*

Table 3. Queue dependent service time Vs Non queue dependent service time.

λ	λ_r	μ	Queue Dependent Service Time					Non Queue Dependent Service Time				
			$E[W_p]$	$E[W_o]$	Δ_{14}	MTC	Δ_{11}	$E[W_p]$	$E[W_o]$	Δ_{14}	MTC	Δ_{11}
3.2	2.2	5	0.0325	0.0545	0.9283	7.7969	1.2069	0.0785	0.2167	0.8799	11.2858	0.7562
		10	0.0269	0.0431	0.9569	8.1573	1.2751	0.0349	0.0656	0.8950	8.9257	1.0759
		15	0.0256	0.0404	0.9689	8.6277	1.2930	0.0288	0.0497	0.9171	8.9840	1.1803
		20	0.0250	0.0393	0.9756	9.1269	1.3008	0.0268	0.0447	0.9326	9.3530	1.2259
	3.2	5	0.0224	0.0356	0.9273	7.6068	1.2671	0.0549	0.1433	0.8766	10.7980	0.7821
		10	0.0185	0.0280	0.9564	7.7515	1.3431	0.0241	0.0433	0.8925	8.4550	1.1201
		15	0.0176	0.0263	0.9685	7.9831	1.3634	0.0198	0.0326	0.9151	8.314	1.2347
		20	0.0172	0.0255	0.9753	8.2393	1.3722	0.0185	0.0292	0.9308	8.4485	1.2862
	4.2	5	0.0171	0.0263	0.9272	7.5383	1.3009	0.0422	0.1066	0.8749	10.5441	0.7975
		10	0.0141	0.0207	0.9564	7.6030	1.3823	0.0184	0.0323	0.8916	8.2765	1.1440
		15	0.0134	0.0194	0.9685	7.7398	1.4044	0.0151	0.0242	0.9144	8.0709	1.2644
		20	0.0131	0.0188	0.9752	7.8979	1.4140	0.0141	0.0217	0.9301	8.1125	1.3198

Table 3. *Cont.*

λ	λ_r	μ	Queue Dependent Service Time					Non Queue Dependent Service Time				
			$E[W_p]$	$E[W_o]$	Δ_{14}	MTC	Δ_{11}	$E[W_p]$	$E[W_o]$	Δ_{14}	MTC	Δ_{11}
3.6	2.2	5	0.0398	0.0716	0.9291	8.5298	1.3357	0.1020	0.3253	0.8859	12.8286	0.8124
		10	0.0324	0.0557	0.9572	9.0293	1.4180	0.0435	0.0889	0.8972	10.0259	1.1803
		15	0.0308	0.0521	0.9690	9.6480	1.4392	0.0351	0.0653	0.9181	10.0798	1.3060
		20	0.0301	0.0505	0.9756	10.2973	1.4484	0.0325	0.0581	0.9331	10.5559	1.3608
	3.2	5	0.0275	0.0467	0.9279	8.3045	1.4044	0.0718	0.2146	0.8828	12.2445	0.8386
		10	0.0224	0.0362	0.9566	8.5384	1.4962	0.0302	0.0586	0.8946	9.4455	1.2293
		15	0.0212	0.0338	0.9686	8.8598	1.5204	0.0243	0.0429	0.9159	9.2578	1.3674
		20	0.0207	0.0328	0.9753	9.2066	1.5308	0.0224	0.0380	0.9312	9.4438	1.4295
	4.2	5	0.0210	0.0345	0.9276	8.2237	1.4435	0.0554	0.1594	0.8810	11.9324	0.8546
		10	0.0170	0.0267	0.9565	8.3560	1.5418	0.0231	0.0437	0.8935	9.2145	1.2560
		15	0.0162	0.0249	0.9686	8.5547	1.5683	0.0185	0.0318	0.9150	8.9471	1.4011
		20	0.0158	0.0241	0.9752	8.7743	1.5796	0.0171	0.0281	0.9304	9.0161	1.4680

7. Conclusions

This paper analyses a single server two commodity inventory system with queue-dependent services for finite queue and an optional retrial facility. We applied a Neuts matrix geometric approach to resolve the infinitesimal generator matrix and further we derived a stability condition and the steady state probability vector of the system. Upon computing the necessary system characteristics of the system, we obtained the mean total cost of the considered model. This model explored the queue dependent and non-queue dependent service polices in a two commodity retrial inventory system. The minimal optimum mean total cost is obtained for the queue-dependent service policy. Indeed, this policy also helped to reduce the rate of customer lost and his/her expected waiting time. Apart from that the retrial customers successful retrial rate also increased as we expected. As for the consequences, the queue dependent service is more profitable when the major product attached with its complimentary. As a result of the numerical survey, maximum stock levels of both products are economically controlled. Furthermore, the fixed stocking capacities of inventory and the preference level of customer entering into the orbit, the system may control the size of queue with the ordering quantity which corresponds to lead time and set up cost. However, the comparison of queue dependent and non-queue dependent service polices will give a great impact on the inventory management. The readers can easily understand these different service polices which one is adoptable to bring out the profitable business. This model will be extended into a multi server service system in the future. In such an environment, one can analyse the impact of customer lost and their waiting time with the single server model. This analysis also helps to the business people to develop their business.

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Conflicts of Interest: The authors declare no conflict of interest.

Notations

$\mathbf{0}$	Null matrix of an appropriate order
$\mathbf{e}$	A column vector of the each entries are one with an appropriate dimension
I	An identity matrix of an appropriate order
δ_{ij}	$\begin{cases} 1, & \text{if } j = i, \\ 0, & \text{otherwise} \end{cases}$
$H(z)$	$\begin{cases} 1, & \text{if} z \geq 0, \\ 0, & \text{otherwise} \end{cases}$
$\bar{\delta}_{ij}$	$1 - \delta_{ij}$

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