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An Application of Urysohn Integral Equation via Complex Partial Metric Space

Rajagopalan Ramaswamy [1,*], Gunaseelan Mani [2], Arul Joseph Gnanaprakasam [3], Ola A. Ashour Abdelnaby [1,4] and Stojan Radenović [5,*]

[1] Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam Bin Abdulaziz University, AlKharj 11942, Saudi Arabia
[2] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, India; mathsguna@yahoo.com
[3] Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur 603203, India; aruljoseph.alex@gmail.com
[4] Department of Mathematics, Cairo University, Cairo 12613, Egypt; ola.a.ashour@gmail.com
[5] Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Belgrad, Serbia
* Correspondence: r.gopalan@psau.edu.sa (R.R.); radens@beotel.net (S.R.)

Abstract: Metric fixed point theory has vast applications in various domain areas, as it helps in finding analytical solutions under various contractive conditions, including non-linear integral-type contractions. In our present work, we have established fixed point results in the setting of complex valued partial metric space. Our results extend the results proven in literature. Using our main result, we have provided an application to find the solution to the Urysohn-type integral equation.

Keywords: Urysohn integral equations; common fixed points; complex partial metric space

MSC: 47H10; 54H25; 54C30

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1. Introduction

Metric fixed point theory has its roots in the famous Banach Contraction Principle [1] of 1922. The principle has been applied in the setting of various metric spaces for the past several decades to establish fixed point results. In the past decade, many researchers have reported fixed point results for conformal mappings in the setting of various topological spaces, such as partial metric space, cone metric space, cone b-metric space and so on—see [2–16]. In the sequel, Azam et al. [17] introduced complex valued metric spaces, which is a special class of cone metric spaces, and established the following fixed point result for mappings satisfying rational inequality.

Theorem 1. *Let (X, d) be a complete complex-valued metric space and $S,T : X \to X$ be two mappings. If S and T satisfy*

$$d(S_x, T_y) \preceq \lambda d(x, y) + \frac{\mu d(x, S_x), d(y, T_y)}{1 + d(x, , y)}$$

for all $x, y \in X$, where λ, μ are non-negative reals with $\lambda + \mu < 1$, then S and T have a unique common fixed point in X.

The above theorem paved the way for the study of the existence of fixed point theorems in the setting of complex valued metric spaces. The results of Azam et al. [17] were generalized by Fayyaz et al. [18] and Sintunavarat et al. [19]. Subsequently, Rao et al. [20] proposed complex b-metric spaces and studied certain fixed point results in the setting of complex b-metric spaces.

Later, Dhivya and Marudhai [21] introduced the concept of complex partial metric spaces and studied the associated topologies and proved some fixed point results in the setting of complex partial metric spaces. Then, Gunaseelan et al. [22] introduced the concept of complex partial b-metric spaces and proved fixed point theorems thereon. Fixed point results using (CLR) and (E.A.) properties in complex partial b-metric spaces were studied by Leema et al. [23]. Gunaseelan et al. [24] proved some fixed point theorems in the setting of complex partial b-metric spaces, generalizing proven results.

A variety of real-world problems are described through integral equations. The Fredholm linear integral equation or its non-linear counterparts—the Hammerstein integral equation and its generalization—and the Urysohn integral equation are most commonly used to describe many scientific problems. Many authors have studied various types of integral equations and associated theories, cf. [25–28]. In [27], sufficient conditions for the existence of a principal solution of a non-linear Volterra integral equation of the second kind on the half-line and on a finite interval were obtained. In [28], Sidorov et al. established the uniform convergence for non-linear Hammerstein integral equations (a class of Urysohn-type integral equations) in the neighborhood about the bifurcation point using the implicit function theorem and the Schmidt lemma. The techniques used in [27,28] can be applied to study operator equations in Banach spaces.

Various researchers have reported the application of fixed point results to find analytical solutions of various types of integral contractions. Recently, Debanath et al. [29] reported the application of metric fixed point theory to solve real-world problems in various domain areas, such as science, engineering and behavioral science, etc. In 2013, Sintunavarat et al. [30] generalized the contractive conditions in [17] and presented an application to study the existence of a solution to Urysohn integral equations in the setting of complex metric spaces, cf. [19,30]. In the recent past, Rajagopalan et al. [31] established the existence of an analytical solution to non-linear integral equations of Voltera type, while Fahad et al. [32] applied the fixed point results to examine the analytical solutions of the integral equation of Caratheodory-type functions in modular metric spaces. In the recent past, Abood et al. [33] analyzed the existence of analytical and approximate solutions for a fractional quadratic integral equation, while Sarim et al. [34] introduced the concept of fuzzy cone metric spaces called fuzzy integrable functions and ξ fuzzy cone integrable functions and established fixed point results in these spaces. More recently, Aslam et al. [35] studied the application of fixed point results to find the solutions of Urysohn-type integral equations in the setting of complex valued b-metric spaces.

Inspired by the above, in this article, we establish fixed point results in the setting of complex partial metric spaces, extending the results of [21]. The achieved result has been supported with a suitable example. We have also presented an application to find a unique solution to a Urysohn integral equation. Throughout this paper, $\mathcal{CPMS}$ refers to complex partial metric space.

The rest of the paper is organized as follows. In Section 2, we review certain basic concepts and monographs reported in the literature. In Section 3, we present a fixed point theorem and prove a corollary satisfying the contractive condition in the setting of complex partial metric space and supplement the obtained results with a suitable example. In Section 4, we present an application to find the analytical solution of a Urysohn-type integral equation in the setting of complex partial metric space, using our main result.

2. Preliminaries

The following are required in the sequel.

Let $\mathbb{C}$ be the set of complex numbers and $Z_1, Z_2, Z_3 \in \mathbb{C}$. Let the partial order $\preceq$ on $\mathbb{C}$ be defined as:

$Z_1 \preceq Z_2$ if and only if $Re(Z_1) \leq Re(Z_2)$, $Im(Z_1) \leq Im(Z_2)$.

It is thus clear that $Z_1 \preceq Z_2$ if one of the following holds:

(i) $Re(Z_1) = Re(Z_2), Im(Z_1) < Im(Z_2)$,
(ii) $Re(Z_1) < Re(Z_2), Im(Z_1) = Im(Z_2)$,

(iii) $Re(Z_1) < Re(Z_2)$, $Im(Z_1) < Im(Z_2)$,
(iv) $Re(Z_1) = Re(Z_2)$, $Im(Z_1) = Im(Z_2)$.

Precisely, we can say $Z_1 \precnsim Z_2$ if $Z_1 \neq Z_2$ and any one of (i), (ii) and (iii) holds and we say $Z_1 \prec Z_2$ if (iii) alone holds.

It may also be noted that

(a) $0 \preceq Z_1 \precnsim Z_2 \implies |Z_1| < |Z_2|$,
(b) $Z_1 \preceq Z_2$ and $Z_2 \prec Z_3 \implies Z_1 \prec Z_3$,
(c) $\eta, \gamma \in \mathbb{R}$ and $\eta \leq \gamma \implies \eta Z_1 \preceq \gamma Z_1$ for all $0 \preceq Z_1 \in \mathbb{C}$.

Here, $\mathbb{C}_+ (= \{(\zeta, \wp) | \zeta, \wp \in \mathbb{R}_+\})$ represents non-negative complex numbers, while $\mathbb{R}_+ (= \{\zeta \in \mathbb{R} | \zeta \geq 0\})$ represents non-negative reals.

Usually, in a metric space, the self distance $d(x, x) = 0$, whereas in the case of a partial metric space, it need not be equal to zero. Using this, Dhivya et al. [21] defined the complex partial metric space given as below.

Definition 1 ([21])**.** *Let $X \neq \emptyset$ and $d_{cp} : X \times X \to \mathbb{C}_+$ be a map, such that for all $\Gamma, \Upsilon, \mathfrak{Z} \in X$:*

(i) $0 \preceq d_{cp}(\Gamma, \Gamma) \preceq d_{cp}(\Gamma, \Upsilon)$;
(ii) $d_{cp}(\Gamma, \Upsilon) = d_{cp}(\Upsilon, \Gamma)$;
(iii) $d_{cp}(\Gamma, \Gamma) = d_{cp}(\Gamma, \Upsilon) = d_{cp}(\Upsilon, \Upsilon)$ *if and only if* $\Gamma = \Upsilon$;
(iv) $d_{cp}(\Gamma, \Upsilon) \preceq d_{cp}(\Gamma, \mathfrak{Z}) + d_{cp}(\mathfrak{Z}, \Upsilon) - d_{cp}(\mathfrak{Z}, \mathfrak{Z})$.

Then, d_{cp} is a complex partial metric on X and the pair (X, d_{cp}) is called a $\mathcal{CPMS}$.

Definition 2 ([21])**.** *Let (X, d_{cp}) be a $\mathcal{CPMS}$. Let $\{\zeta_n\}$ be any sequence in X.*

(i) *$\{\zeta_n\}$ converges to ζ, if $\lim_{n\to+\infty} d_{cp}(\zeta_n, \zeta) = d_{cp}(\zeta, \zeta)$.*
(ii) *$\{\zeta_n\}$ is $\mathcal{CP}$-Cauchy in (X, d_{cp}) if $\lim_{n,m\to+\infty} d_{cp}(\zeta_n, \zeta_m)$ exists and is finite.*
(iii) *(X, d_{cp}) is a complete $\mathcal{CPMS}$ if for every $\mathcal{CP}$-Cauchy sequence $\{\zeta_n\}$ in X if there exists $\zeta \in X$ such that $\lim_{n,m\to+\infty} d_{cp}(\zeta_n, \zeta_m) = \lim_{n\to+\infty} d_{cp}(\zeta_n, \zeta) = d_{cp}(\zeta, \zeta)$.*

Definition 3 ([21])**.** *Let $X \neq \emptyset$ and let Φ and Ψ be self maps on it. A point $\zeta \in X$ is called a common fixed point of Φ and Ψ if $\zeta = \Phi\zeta = \Psi\zeta$.*

The following theorem is the main result of Dhivya et al. [21].

Theorem 2 ([21])**.** *Let $(X, \preceq)$ be a partially ordered set. Let d_{cp} be a complex partial metric on X such that (X, d_{cp}) is a complete $\mathcal{CPMS}$. Let $\mho, \amalg : X \to X$ be a pair of weakly increasing mappings and suppose that, for every comparable $\zeta, \wp \in X$, we have either*

$$d_{cp}(\mho\zeta, \amalg\wp) \preceq a\frac{d_{cp}(\zeta, \mho\zeta) d_{cp}(\wp, \amalg\wp)}{d_{cp}(\zeta, \wp)} + b d_{cp}(\zeta, \wp),$$

whenever $d_{cp}(\zeta, \wp) \neq 0$, $a \geq 0, b \geq 0$ and $a + b < 1$, or

$$d_c(\mho\zeta, \amalg\wp) = 0 \text{ if } d_{cp}(\zeta, \wp) = 0.$$

$\vartheta \in X$ is a common fixed point of $\mho$ and $\amalg$ with $d_{cp}(\vartheta, \vartheta) = 0$, if either $\mho$ or $\amalg$ is continuous.

Now, we present our main result.

3. Main Results

Throughout this paper, $\sqcap$ represents the class of functions $\pounds\colon \mathbb{C}_+ \to [0,1)$ so that

$$\pounds(\zeta_\alpha) \to 1 \quad \Rightarrow \quad |\zeta_\alpha| \to 0,$$

for any sequences $\{\zeta_\alpha\}$ in $\mathbb{C}_+$.

Theorem 3. *Let (X, d_{cp}) be a complete $\mathcal{CPMS}$ and let $\mho, \amalg\colon X \to X$ be two maps. Consider the two maps $\mathfrak{e}, \mathfrak{g}\colon \mathbb{C}_+ \to [0,1)$, such that, for all $\zeta, \wp \in X$,*

(i) $\mathfrak{e}(\zeta) + \mathfrak{g}(\zeta) < 1$;

(ii) *the mapping $\pounds\colon \mathbb{C}_+ \to [0,1)$ defined by $\pounds(\zeta) = \dfrac{\mathfrak{e}(\zeta)}{1-\mathfrak{g}(\zeta)}$ belongs to $\sqcap$;*

(iii) $d_{cp}(\mho\zeta, \amalg\wp) \preceq \mathfrak{e}(d_{cp}(\zeta,\wp))d_{cp}(\zeta,\wp) + \dfrac{\mathfrak{g}(d_{cp}(\zeta,\wp))d_{cp}(\zeta,\mho\zeta)d_{cp}(\wp,\amalg\wp)}{1+d_{cp}(\zeta,\wp)}.$

Then, there exists a unique common fixed point for $\mho$ and $\amalg$ in X.

Proof. Let $\zeta_0 \in X$ be arbitrary. Consider a sequence $\{\zeta_\alpha\}$ in X such that

$$\zeta_{2\alpha+1} = \mho\zeta_{2\alpha}, \quad \zeta_{2\alpha+2} = \amalg\zeta_{2\alpha+1}, \ \ \forall \alpha \in \mathbb{N}\cup\{0\}. \tag{1}$$

By using Equation (1), we obtain

$$\begin{aligned}
&d_{cp}(\zeta_{2\alpha+1}, \zeta_{2\alpha+2})\\
&= d_{cp}(\mho\zeta_{2\alpha}, \amalg\zeta_{2\alpha+1})\\
&\preceq \mathfrak{e}(d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1})\\
&\quad + \frac{\mathfrak{g}(d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha},\mho\zeta_{2\alpha})d_{cp}(\zeta_{2\alpha+1},\amalg\zeta_{2\alpha+1})}{1+d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1})}\\
&= \mathfrak{e}(d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1})\\
&\quad + \frac{\mathfrak{g}(d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1})d_{cp}(\zeta_{2\alpha+1},\zeta_{2\alpha+2})}{1+d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1})}\\
&= \mathfrak{e}(d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1})\\
&\quad + \mathfrak{g}(d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha+1},\zeta_{2\alpha+2})\left(\frac{d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1})}{1+d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1})}\right)\\
&\preceq \mathfrak{e}(d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1}) + \mathfrak{g}(d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha+1},\zeta_{2\alpha+2}),
\end{aligned}$$

which implies that

$$\begin{aligned}
d_{cp}(\zeta_{2\alpha+1},\zeta_{2\alpha+2}) &\preceq \left(\frac{\mathfrak{e}(d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1}))}{1-\mathfrak{g}(d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1}))}\right)d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1})\\
&= \pounds\big(d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1})\big)d_{cp}(\zeta_{2\alpha},\zeta_{2\alpha+1}).
\end{aligned} \tag{2}$$

Similarly,

$$
\begin{aligned}
& d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+3}) \\
& = d_{cp}(\zeta_{2\alpha+3}, \zeta_{2\alpha+2}) \\
& = d_{cp}(\mho\zeta_{2\alpha+2}, \amalg\zeta_{2\alpha+1}) \\
& \\
& \preceq \mathfrak{e}(d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}) \\
& \quad + \frac{\mathfrak{g}(d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha+2}, \mho\zeta_{2\alpha+2})d_{cp}(\zeta_{2\alpha+1}, \amalg\zeta_{2\alpha+1})}{1 + d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1})} \\
& = \mathfrak{e}(d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}) \\
& \quad + \frac{\mathfrak{g}(d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+3})d_{cp}(\zeta_{2\alpha+1}, \zeta_{2\alpha+2})}{1 + d_{cp}(\zeta_{2\alpha+1}, \zeta_{2\alpha+2})} \\
& = \mathfrak{e}(d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}) \\
& \quad + \mathfrak{g}(d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+3})\left(\frac{d_{cp}(\zeta_{2\alpha+1}, \zeta_{2\alpha+2})}{1 + d_{cp}(\zeta_{2\alpha+1}, \zeta_{2\alpha+2})}\right) \\
& \preceq \mathfrak{e}(d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}) \\
& \quad + \mathfrak{g}(d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}))d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+3}),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+3}) & \preceq \left(\frac{\mathfrak{e}(d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}))}{1 - \mathfrak{g}(d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}))}\right)d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}) \\
& = \pounds\big(d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1})\big)d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+1}). \qquad (3)
\end{aligned}
$$

From Equations (2) and (3), we have

$$d_{cp}(\zeta_\alpha, \zeta_{\alpha+1}) \preceq \pounds\big(d_{cp}(\zeta_{\alpha-1}, \zeta_\alpha)\big)d_{cp}(\zeta_{\alpha-1}, \zeta_\alpha), \forall \alpha \in \mathbb{N}.$$

Finally, we obtain

$$|d_{cp}(\zeta_\alpha, \zeta_{\alpha+1})| \leq \pounds\big(d_{cp}(\zeta_{\alpha-1}, \zeta_\alpha)\big)|d_{cp}(\zeta_{\alpha-1}, \zeta_\alpha)| \leq |d_{cp}(\zeta_{\alpha-1}, \zeta_\alpha)|, \forall \alpha \in \mathbb{N}. \qquad (4)$$

This implies that the sequence $\{|d_{cp}(\zeta_{\alpha-1}, \zeta_\alpha)|\}_{\alpha\in\mathbb{N}}$ is monotonically non-increasing and bounded below. Hence, $|d_{cp}(\zeta_{\alpha-1}, \zeta_\alpha)| \to g$ for some $g \geq 0$. We claim that $g = 0$.

Suppose not. Let us assume $g > 0$. Letting $\alpha \to +\infty$ in (4), we obtain $\pounds\big(d_{cp}(\zeta_{\alpha-1}, \zeta_\alpha)\big) \to 1$. Since $\pounds \in \sqcap$, we obtain $|d_{cp}(\zeta_{\alpha-1}, \zeta_\alpha)| \to 0$. This is a contradiction. Thus, $g = 0$, that is

$$|d_{cp}(\zeta_{\alpha-1}, \zeta_\alpha)| \to 0. \qquad (5)$$

To show that $\{\zeta_\alpha\}$ is a $\mathcal{CP}$-Cauchy, we shall prove that the subsequence $\{\zeta_{2\alpha}\}$ is a $\mathcal{CP}$-Cauchy sequence based on Equation (5). Let us suppose that $\{\zeta_{2\alpha}\}$ is not a $\mathcal{CP}$-Cauchy. Then, there exists $\mu \in \mathbb{C}$ with $0 \prec \mu$, and for all $i \in \mathbb{N} \cup \{0\}$, there exists $\beta_k > \alpha_k \geq k$ such that

$$d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k}) \succeq \mu. \qquad (6)$$

Further, corresponding to α_k, we can choose β_k in such a way that it is the smallest integer with $\beta_k > \alpha_k \geq k$ satisfying Equation (6), and,

$$d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k-2}) \prec \mu.$$

By the definition of a $\mathcal{CPMS}$, we derive that

$$\mu \preceq d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k}) \tag{7}$$

$$\prec c + d_{cp}(\zeta_{2\beta_k-2}, \zeta_{2\beta_k-1}) + d_{cp}(\zeta_{2\beta_k-1}, \zeta_{2\beta_k}). \tag{8}$$

This implies

$$|\mu| \leq |d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k})| \leq |\mu| + |d_{cp}(\zeta_{2\beta_k-2}, \zeta_{2\beta_k-1})| + |d_{cp}(\zeta_{2\beta_k-1}, \zeta_{2\beta_k})|.$$

Therefore, we have

$$|\mu| \leq \lim_{k\to+\infty} |d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k})| \leq |\mu|. \tag{9}$$

Further, we have

$$\begin{aligned} d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}) &\preceq d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k}) + d_{cp}(\zeta_{2\beta_k+1}, \zeta_{2\beta_k}) - d_{cp}(\zeta_{2\beta_k}, \zeta_{2\beta_k}) \\ &\preceq d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k}) + d_{cp}(\zeta_{2\beta_k+1}, \zeta_{2\beta_k}) \end{aligned}$$

and

$$|d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})| \leq |d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k})| + |d_{cp}(\zeta_{2\beta_k+1}, \zeta_{2\beta_k})|.$$

By using Equations (5) and (9) and as $k \to +\infty$, we obtain

$$|d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})| \to |\mu|. \tag{10}$$

By the definition of a $\mathcal{CPMS}$, we derive that

$$\begin{aligned} & d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}) \\ & \preceq d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1}) + d_{cp}(\zeta_{2\alpha_k+1}, \zeta_{2\beta_k+2}) + d_{cp}(\zeta_{2\beta_k+2}, \zeta_{2\beta_k+1}) \\ & \quad - d_{cp}(\zeta_{2\beta_k+2}, \zeta_{2\beta_k+2}) - d_{cp}(\zeta_{2\alpha_k+1}, \zeta_{2\alpha_k+1}) \\ & \preceq d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1}) + d_{cp}(\zeta_{2\alpha_k+1}, \zeta_{2\beta_k+2}) + d_{cp}(\zeta_{2\beta_k+2}, \zeta_{2\beta_k+1}) \\ & = d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1}) + d_{cp}(\mho\zeta_{2\alpha_k}, \amalg\zeta_{2\beta_k+1}) + d_{cp}(\zeta_{2\beta_k}, \zeta_{2\beta_k+1}) \\ & \preceq d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1}) + \mathfrak{e}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}) \\ & \quad + \frac{\mathfrak{g}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))d_{cp}(\zeta_{2\alpha_k}, \mho\zeta_{2\alpha_k})d_{cp}(\zeta_{2\beta_k+1}, \amalg\zeta_{2\beta_k+1})}{1 + d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})} \\ & \quad + d_{cp}(\zeta_{2\beta_k}, \zeta_{2\beta_k+1})\} \\ & = d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1}) + \mathfrak{e}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}) \\ & \quad + \frac{\mathfrak{g}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1})d_{cp}(\zeta_{2\beta_k+1}, \zeta_{2\beta_k+2})}{1 + d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})} \\ & \quad + d_{cp}(\zeta_{2\beta_k}, \zeta_{2\beta_k+1}), \end{aligned}$$

which implies that

$$\begin{aligned}
&|d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})| \\
&\leq |d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1})| + \mathfrak{e}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))|d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})| \\
&\quad + \mathfrak{g}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))\left|\frac{d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1})d_{cp}(\zeta_{2\beta_k+1}, \zeta_{2\beta_k+2})}{1 + d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})}\right| \\
&\quad + |d_{cp}(\zeta_{2\beta_k}, \zeta_{2\beta_k+1})| \\
&\leq |d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1})| + \mathfrak{e}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))|d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})| \\
&\quad + \mathfrak{g}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))\left|\frac{d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1})d_{cp}(\zeta_{2\beta_k+1}, \zeta_{2\beta_k+2})}{1 + d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})}\right| \\
&\quad + |d_{cp}(\zeta_{2\beta_k}, \zeta_{2\beta_k+1})| \\
&\leq |d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1})| + \frac{\mathfrak{e}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))}{1 - \mathfrak{g}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))}|d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})| \\
&\quad + \mathfrak{g}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))\left|\frac{d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1})d_{cp}(\zeta_{2\beta_k+1}, \zeta_{2\beta_k+2})}{1 + d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})}\right| \\
&\quad + |d_{cp}(\zeta_{2\beta_k}, \zeta_{2\beta_k+1})| \\
&\leq |d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1})| + \pounds(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))|d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})| \\
&\quad + \mathfrak{g}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))\left|\frac{d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1})d_{cp}(\zeta_{2\beta_k+1}, \zeta_{2\beta_k+2})}{1 + d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})}\right| \\
&\quad + |d_{cp}(\zeta_{2\beta_k}, \zeta_{2\beta_k+1})| \\
&\leq |d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1})| + \pounds(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))|d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})| \\
&\quad + \mathfrak{g}(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))\left|\frac{d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\alpha_k+1})d_{cp}(\zeta_{2\beta_k+1}, \zeta_{2\beta_k+2})}{1 + d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})}\right| \\
&\quad + |d_{cp}(\zeta_{2\beta_k}, \zeta_{2\beta_k+1})|.
\end{aligned}$$

As $k \to +\infty$, we have

$$|\mu| \leq \left(\lim_{k\to+\infty} \pounds(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1}))\right)|\mu| \leq |\mu|.$$

That is

$$\lim_{k\to+\infty} \pounds(d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})) = 1.$$

Since, $\pounds \in \sqcap$, we obtain $|d_{cp}(\zeta_{2\alpha_k}, \zeta_{2\beta_k+1})| \to 0$, which contradicts the fact that $0 \prec \mu$. Hence, $\{\zeta_{2\alpha}\}$ is a $\mathcal{CP}$-Cauchy, which proves that $\{\zeta_\alpha\}$ is a $\mathcal{CP}$-Cauchy sequence. By the completeness of X, there exists a point $\vartheta \in X$ such that $\zeta_\alpha \to \vartheta$ as $\alpha \to +\infty$ and

$$d_{cp}(\vartheta, \vartheta) = \lim_{n\to+\infty} d_{cp}(\zeta_n, \vartheta) = \lim_{n\to+\infty} d_{cp}(\zeta_n, \zeta_m). \tag{11}$$

Next, we claim that $\mho\vartheta = \vartheta$. On the contrary, if $\mho\vartheta \neq \vartheta$, then $d_{cp}(\vartheta, \mho\vartheta) > 0$. Then, we have

$$\begin{aligned}
&d_{cp}(\vartheta, \mho\vartheta) \\
&\preceq d_{cp}(\vartheta, \zeta_{2\alpha+2}) + d_{cp}(\zeta_{2\alpha+2}, \mho\vartheta) - d_{cp}(\zeta_{2\alpha+2}, \zeta_{2\alpha+2}) \\
&\preceq d_{cp}(\vartheta, \zeta_{2\alpha+2}) + d_{cp}(\zeta_{2\alpha+2}, \mho\vartheta) \\
&= d_{cp}(\vartheta, \zeta_{2\alpha+2}) + d_{cp}(\amalg\zeta_{2\alpha+1}, \mho\vartheta) \\
&= d_{cp}(\vartheta, \zeta_{2\alpha+2}) + d_{cp}(\mho\vartheta, \amalg\zeta_{2\alpha+1}) \\
&\preceq d_{cp}(\zeta_{2\alpha+2}, \vartheta) + \mathfrak{e}(d_{cp}(\vartheta, \zeta_{2\alpha+1}))d_{cp}(\vartheta, \zeta_{2\alpha+1}) \\
&\quad + \frac{\mathfrak{g}(d_{cp}(\vartheta, \zeta_{2\alpha+1}))d_{cp}(\vartheta, \mho\vartheta)d_{cp}(\zeta_{2\alpha+1}, \amalg\zeta_{2\alpha+1})}{1 + d_{cp}(\vartheta, \zeta_{2\alpha+1})} \\
&= d_{cp}(\zeta_{2\alpha+2}, \vartheta) + \mathfrak{e}(d_{cp}(\vartheta, \zeta_{2\alpha+1}))d_{cp}(\vartheta, \zeta_{2\alpha+1}) \\
&\quad + \frac{\mathfrak{g}(d_{cp}(\vartheta, \zeta_{2\alpha+1}))d_{cp}(\vartheta, \mho\vartheta)d_{cp}(\zeta_{2\alpha+1}, \zeta_{2\alpha+2})}{1 + d_{cp}(\vartheta, \zeta_{2\alpha+1})} \\
&\preceq d_{cp}(\zeta_{2\alpha+2}, \vartheta) + \mathfrak{e}(d_{cp}(\vartheta, \zeta_{2\alpha+1}))d_{cp}(\vartheta, \zeta_{2\alpha+1}) \\
&\quad + \mathfrak{g}(d_{cp}(\vartheta, \zeta_{2\alpha+1}))\frac{d_{cp}(\vartheta, \mho\vartheta)d_{cp}(\zeta_{2\alpha+1}, \zeta_{2\alpha+2})}{1 + d_{cp}(\vartheta, \zeta_{2\alpha+1})},
\end{aligned}$$

which implies that

$$\begin{aligned}
|d_{cp}(\vartheta, \mho\vartheta)| \leq{}& |d_{cp}(\zeta_{2\alpha+2}, \vartheta)| + \mathfrak{e}(d_{cp}(\vartheta, \zeta_{2\alpha+1}))|d_{cp}(\vartheta, \zeta_{2\alpha+1})| \\
&+ \mathfrak{g}(d_{cp}(\vartheta, \zeta_{2\alpha+1}))\left|\frac{d_{cp}(\vartheta, \mho\vartheta)d_{cp}(\zeta_{2\alpha+1}, \zeta_{2\alpha+2})}{1 + d_{cp}(\vartheta, \zeta_{2\alpha+1})}\right|.
\end{aligned}$$

As $\alpha \to +\infty$, we have $|d_{cp}(\vartheta, \mho\vartheta)| = 0$, which is a contradiction. Hence, $\mho\vartheta = \vartheta$. It follows that, similarly, $\amalg\vartheta = \vartheta$. Therefore, $\vartheta = \mho\vartheta = \amalg\vartheta$. Hence, ϑ is a common fixed point of $\mho$ and $\amalg$.

Let us suppose $\widehat{\vartheta}$ to be another fixed point, such that $\widehat{\vartheta} = \mho\widehat{\vartheta} = \amalg\widehat{\vartheta}$. We have

$$\begin{aligned}
d_{cp}(\vartheta, \widehat{\vartheta}) &= d_{cp}(\mho\vartheta, \amalg\widehat{\vartheta}) \\
&\preceq \mathfrak{e}(d_{cp}(\vartheta, \widehat{\vartheta}))d_{cp}(\vartheta, \widehat{\vartheta}) + \frac{\mathfrak{g}(d_{cp}(\vartheta, \widehat{\vartheta}))d_{cp}(\vartheta, \mho\vartheta)d_{cp}(\widehat{\vartheta}, \amalg\widehat{\vartheta})}{1 + d_{cp}(\vartheta, \widehat{\vartheta})} \\
&= \mathfrak{e}(d_{cp}(\vartheta, \widehat{\vartheta}))d_{cp}(\vartheta, \widehat{\vartheta}).
\end{aligned}$$

which means that $|d_{cp}(\vartheta, \widehat{\vartheta})| \leq \mathfrak{e}(d_{cp}(\vartheta, \widehat{\vartheta}))|d_{cp}(\vartheta, \widehat{\vartheta})|$. Since $0 \leq \mathfrak{e}(d_{cp}(\vartheta, \widehat{\vartheta})) < 1$, we obtain $|d_{cp}(\vartheta, \widehat{\vartheta})| = 0$. Therefore, $\vartheta = \widehat{\vartheta}$. □

Corollary 1. *Let (X, d_{cp}) be a $\mathcal{CPMS}$ and $\amalg\colon X \to X$ be a mapping. If there exist two maps $\mathfrak{e}, \mathfrak{g}\colon \mathtt{C}_+ \to [0,1)$ such that for all $\zeta, \wp \in X$,*

(i) $\mathfrak{e}(\zeta) + \mathfrak{g}(\zeta) < 1$;

(ii) *The mapping $\pounds\colon \mathtt{C}_+ \to [0,1)$ defined by $\pounds(\zeta) = \dfrac{\mathfrak{e}(\zeta)}{1 - \mathfrak{g}(\zeta)}$ belongs to $\sqcap$;*

(iii) $d_{cp}(\amalg\zeta, \amalg\wp) \preceq \mathfrak{e}(d_{cp}(\zeta, \wp))d_{cp}(\zeta, \wp) + \dfrac{\mathfrak{g}(d_{cp}(\zeta, \wp))d_{cp}(\zeta, \amalg\zeta)d_{cp}(\wp, \amalg\wp)}{1 + d_{cp}(\zeta, \wp)}$.

Then, $\amalg$ has a unique fixed point in X.

Proof. The result follows by putting $\mho = \amalg$ in Theorem 3 . $\square$

Example 1. *Let $X = \{1, 2, 3, 4\}$ together with the order $\zeta \preceq \wp$ if $\zeta \leq \wp$. Then, $\preceq$ is a partial order in X. Define $d_{cp} : X \times X \to \mathbb{C}_+$ as follows:*

$(\zeta, \wp)$	$d_{cp}(\zeta, \wp)$
(1,1), (2,2)	0
(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(3,3)	e^{ik}
(1,4),(4,1),(2,4),(4,2),(3,4),(4,3),(4,4)	$3e^{ik}$

Obviously, (X, d_{cp}) is a complete $\mathcal{CPMS}$, for $k \in [0, \frac{\pi}{2}]$. Define $\mho, \amalg : X \to X$ by $\mho\zeta = 1$,

$$\amalg(\zeta) = \begin{cases} 1 & \text{if } \zeta \in \{1, 2, 3\} \\ 2 & \text{if } \zeta = 4. \end{cases}$$

Define $\mathfrak{e}, \mathfrak{g} \colon \mathbb{C}_+ \to [0, 1)$ by $\mathfrak{e}(\zeta) = \frac{1}{2}$, $\mathfrak{g}(\zeta) = \frac{1}{3}$. We have the following cases:

1. *$\zeta = 1$ with $\wp \in X - \{4\}$, $\implies$ $\mho(\zeta) = \mho(\wp) = 1$ and $d_{cp}(\mho(\zeta), \amalg(\wp)) = 0$ satisfying the conditions of Theorem 3.*
2. *If $\zeta = 1$, $\wp = 4$, then $\mho\zeta = 1$, $\amalg\wp = 2$,*

$$\begin{aligned} d_c(\mho\zeta, \amalg\wp) = e^{2ik} &\preceq \frac{3}{2}e^{ik} \\ &= \frac{3}{2}e^{ik} + \mathfrak{g}(d_{cp}(\zeta, \wp))\frac{(0)3e^{ik}}{3e^{ik}} \\ &= \mathfrak{e}(d_{cp}(\zeta, \wp))d_{cp}(\zeta, \wp) \\ &+ \mathfrak{g}(d_{cp}(\zeta, \wp))\frac{d_{cp}(\zeta, \mho\zeta)d_{cp}(\wp, \amalg\wp)}{1 + d_{cp}(\zeta, \wp)}. \end{aligned}$$

3. *If $\zeta = 2$, $\wp = 4$, then $\mho\zeta = 1$, $\amalg\wp = 2$,*

$$\begin{aligned} d_{cp}(\mho\zeta, \amalg\wp) = e^{ik} &\preceq \left(\frac{3}{2} + \frac{1}{3}\right)e^{ik} \\ &= \frac{3}{2}e^{ik} + \mathfrak{g}(d_{cp}(\zeta, \wp))\frac{(e^{ik})3e^{ik}}{3e^{ik}} \\ &= \mathfrak{e}(d_{cp}(\zeta, \wp))d_{cp}(\zeta, \wp) \\ &+ \mathfrak{g}(d_{cp}(\zeta, \wp))\frac{d_{cp}(\zeta, \mho\zeta)d_{cp}(\wp, \amalg\wp)}{1 + d_{cp}(\zeta, \wp)}. \end{aligned}$$

4. *If $\zeta = 3$, $\wp = 4$, then $\mho\zeta = 1$, $\amalg\wp = 2$,*

$$\begin{aligned} d_{cp}(\mho\zeta, \amalg\wp) = e^{ik} &\preceq \left(\frac{3}{2} + \frac{1}{3}\right)e^{ik} \\ &= \frac{3}{2}e^{ik} + \mathfrak{g}(d_{cp}(\zeta, \wp))\frac{(e^{ik})3e^{ik}}{3e^{ik}} \\ &= \mathfrak{e}(d_{cp}(\zeta, \wp))d_{cp}(\zeta, \wp) \\ &+ \mathfrak{g}(d_{cp}(\zeta, \wp))\frac{d_{cp}(\zeta, \mho\zeta)d_{cp}(\wp, \amalg\wp)}{1 + d_{cp}(\zeta, \wp)}. \end{aligned}$$

5. *If $\zeta = 4$, $\wp = 4$, then $\mho\zeta = 1$, $\amalg\wp = 2$,*

$$\begin{aligned} d_{cp}(\mho\zeta, \amalg\wp) = e^{ik} &\preceq 3\left(\frac{1}{2} + \frac{1}{3}\right)e^{ik} \\ &= \frac{3}{2}e^{ik} + \mathfrak{g}(d_{cp}(\zeta, \wp))\frac{(3e^{ik})3e^{ik}}{3e^{ik}} \\ &= \mathfrak{e}(d_{cp}(\zeta, \wp))d_{cp}(\zeta, \wp) \\ &+ \mathfrak{g}(d_{cp}(\zeta, \wp))\frac{d_{cp}(\zeta, \mho\zeta)d_{cp}(\wp, \amalg\wp)}{1 + d_{cp}(\zeta, \wp)}. \end{aligned}$$

Theorem 3 is satisfied. Hence, $\mho$ and $\amalg$ have the unique common fixed point 1.

4. Application

We now present our application to Urysohn-type integral equations. Consider the system

$$\begin{cases} \zeta(h) = a(h) + \int_x^y U_1(h, s, \zeta(s))ds \\ \wp(h) = a(h) + \int_x^y U_2(h, s, \wp(s))ds, \end{cases} \tag{12}$$

where

1. $\zeta(h)$ and $\wp(h)$ are unknown variables for each $h \in [x, y]$, $x > 0$,
2. $a(h)$ is the deterministic free term defined for $h \in [x, y]$,
3. $U_1(h, s)$ and $U_2(h, s)$ are deterministic kernels defined for $h, s \in [x, y]$.

Let $X = (C[x, y], \mathbb{R}^n)$, $q > 0$ and $d_{cp} : X \times X \to \mathbb{R}^n$ be defined by

$$d_{cp}(\zeta, \wp) = |\zeta - \wp| + 2 + i(|\zeta - \wp| + 2),$$

for all $\zeta, \wp \in X$.

Obviously, $(C[x, y], \mathbb{R}^n, d_{cp})$ is a complete $\mathcal{CPMS}$. We consider the Urysohn-type integral system as in Equation (12) with the following:

1. $a(h) \in X$;
2. There exist two mappings $\mathfrak{e}, \mathfrak{g}$: $\mathbb{C}_+ \to [0, 1)$ by $\mathfrak{e}(\zeta) = \frac{1}{2}$ and $\mathfrak{g}(\zeta) = 0$ such that $\mathfrak{e}(\zeta) + \mathfrak{g}(\zeta) < 1$;
3. $U_1, U_2 : [x, y] \times [x, y] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions such that

$$|U_1(h, s, \zeta(s)) - U_2(h, s, \wp(s))| \preceq \frac{|\zeta - \wp|}{2(y - x)} - \frac{2}{y - x}.$$

Theorem 4. *Let $(C[x, y], \mathbb{R}^n, \wp_{cp})$ be a complete $\mathcal{CPMS}$, and then the system in Equation (12), satisfying 1–3 above, has a unique common solution.*

Proof. For $\zeta, \wp \in X$ and $q \in [x, y]$, let us define continuous maps, $\mho, \amalg : X \to X$ by

$$\mho\zeta(h) = a(h) + \int_x^y U_1(h, s, \zeta(s))ds,$$

and

$$\amalg\wp(h) = a(h) + \int_x^y U_2(h, s, \wp(s))ds.$$

Next, we have

$$\begin{aligned} d_{cp}(\mho\zeta(h), \amalg\wp(h)) &= |\mho\zeta(h) - \amalg\wp(h)| + 2 + i(|\mho\zeta(h) - \amalg\wp(h)| + 2) \\ &= \int_x^y |U_1(h, s, \zeta(s)) - U_2(h, s, \wp(s))| ds + 2 \\ &+ i\left(\int_x^y |U_1(h, s, \zeta(s)) - U_2(h, s, \wp(s))| ds + 2 \right) \\ &\preceq \int_x^y \left(\frac{|\zeta - \wp|}{2(y-x)} - \frac{2}{y-x} \right) ds + 2 \\ &+ i\left(\int_x^y \left(\frac{|\zeta - \wp|}{2(y-x)} - \frac{2}{y-x} \right) ds + 2 \right) \\ &= \frac{|\zeta - \wp|}{2} + i\left(\frac{|\zeta - \wp|}{2} \right) \\ &\preceq \frac{|\zeta - \wp|}{2} + 1 + i\left(\frac{|\zeta - \wp|}{2} + 1 \right) \\ &= \mathfrak{e}(\zeta)(|\zeta - \wp| + 2 + i(|\zeta - \wp| + 2)) \\ &= \mathfrak{e}(\zeta) d_{cp}(\zeta, \wp) + \mathfrak{g}(d_{cp}(\zeta, \wp)) \frac{d_{cp}(\zeta, \mho\zeta) d_{cp}(\wp, \amalg\wp)}{1 + d_{cp}(\zeta, \wp)}. \end{aligned}$$

Thus, all the conditions of Theorem 3 are fulfilled and hence the system of Equation (12) has a unique common solution. □

5. Conclusions

It is a proven fact that the Banach contraction principle and its generalization in the setting of various topological spaces can be applied to find fixed point results and analytical solutions to various types of contractions, including integral-type contractions. In the first part of our paper, we established common fixed point theorems in the setting of a $\mathcal{CPMS}$. In the application section, we applied the derived result to find the solution of Urysohn-type integral equations, in the setting of the $\mathcal{CPMS}$. It is an open problem to further investigate the fixed point results for multi-valued contractions in the setting of complex valued partial metric spaces.

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