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Finite-Time Contractively Bounded Control of Positive Linear Systems under H_∞ Performance and Its Application to Pest Management

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Abstract: This paper investigates the finite-time contractively bounded control issue for positive linear systems under H_∞ performance. The notion of H_∞ finite-time contractive boundedness is first extended to positive systems. Finite-time contractively bounded control is considered to ensure the H_∞ finite-time contractive boundedness of the considered positive systems. A state feedback finite-time contractively bounded controller design method is proposed. The corresponding sufficient condition for the existence of the desired controller is derived by using the Lyapunov function method and the matrix inequality technique. Moreover, a computable scheme for solving the controller gain is established by employing the cone complementary linearization approach. Finally, a numerical example and an application example about pest management are used to validate the effectiveness of proposed conditions.

Keywords: finite-time contractive boundedness; positive linear systems; H_∞ performance; cone complementary linearization

MSC: 93B36; 93B50; 93B52; 93C95

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1. Introduction

Positive systems are a particular class of systems whose states and outputs always have to be nonnegative for any nonnegative inputs and nonnegative initial conditions. The nonnegative characteristic of positive systems is common in nature and that is exactly the reason why such kind of systems is widely applied in numerous fields, including biomedicine [1], industrial engineering [2], and ecology [3]. Moreover, the analyses for positive systems may not be addressed by the well-established approach developed for general systems, since the states of positive systems are not defined on linear spaces but cones [4]. In this case, a large number of innovative investigations of positive systems have been reported in the past decades [5–11]. Here, to just name a few, robust stability analysis problems were discussed for uncertain positive linear systems under L_1-gain and L_∞-gain performance in [7]. In addition, the stability and L_p-gain characterization of positive linear time-varying systems on general time scale were analyzed in [9]. Also, a new delay-dependent stability criterion was developed in [10] for impulsive positive delayed systems by employing the impulse-time-dependent discretized copositive Lyapunov-Krasovskii function. Moreover, an analytical method that can solve the exact value of the ℓ_1-norm of discrete-time positive linear systems was introduced in [11].

The issue of stability plays an important role in system performance analysis. The stability in the Lyapunov sense describes the steady performance of systems over an infinite-time interval, while the transient behavior of systems in a fixed finite-time interval can't be characterized well by it. In this case, notions like finite-time stability (FTS) [12], finite-time annular domain stability (FTADS) [13,14] as well as finite-time contractive stability

(FTCS) [15] could be considered to characterize the transient performance of systems in a prescribed time interval. In fact, it has to be noted that the three kinds of stability notions are different from each other. More specifically, a system is finite-time stable if, given the fixed time interval as well as the bound on the initial state, the trajectories of the states will lie in a fixed bound [16]. While, if the bound on the initial condition is given, the system states of an FTAD-stable system cannot exceed an upper bound and they will not be less than a lower bound [17]. Besides, for a finite-time contractively stable system, if the bound on the initial condition is given, not only do the system states not escape from a prescribed bound, but they will also contract in a lower bound than the initial one before reaching the ending time of the fixed time interval [18]. Hence, compared with the FTS as well as the FTADS, it is more practical to consider the FTCS of systems when both the "boundedness" and the "contraction" of system states are required. For example, in pest management, the number of pests on a farm where the crops are growing is generally desired to be controlled at a lower enough bound in a fixed time interval. In this case, the FTCS control which can specifically constrain the system states in a smaller bound than the initial bound before arriving at the ending time of the time interval can be considered. In addition, it has to be pointed out that the notion of FTCS can be extended to finite time contractive boundedness (FTCB) for the concerned systems when exogenous disturbances are taken into account. In this aspect, the attenuation ability to exogenous disturbances is generally considered, and several input-output properties can be adopted to evaluate it, such as L_1-gain performance [7], H_∞ performance [19], etc.

In recent years, the issues about the FTS and FTADS of positive systems have gradually become a greater concern of researchers. For instance, the finite-time boundedness of positive switched systems with delays under the L_1-gain performance was discussed in [20]. The FTS criterion for positive impulsive systems was established in [21] based on the time-varying copositive Lyapunov function and the average impulsive interval approach. In addition, the finite-time annular domain stability and stabilization problems for T-S fuzzy positive interval systems were investigated in [22]. However, to the best of the author's knowledge, although the FTS and FTADS of positive systems have been discussed before, there are few investigations about the FTCS issue of positive systems in the existing literature.

Motivated by the aforementioned discussions, the FTCB issue for positive linear systems is investigated in this paper. Besides, the H_∞ performance of such systems is analyzed to evaluate the attenuation ability to exogenous disturbances. The main contributions of this paper are highlighted in the three aspects below. (1) The definition of H_∞ FTCB is first extended to positive systems. Compared with the H_∞ FTCB control problem for general systems in the existing literature, the positivity constraints of positive systems are fully taken into account in this paper. (2) Unlike the FTS and FTADS issues of positive systems, the investigated FTCB issues in this paper not only consider the "boundedness" but also care about the "contraction" of system states. (3) A sufficient condition for the existence of state feedback controllers is established by using the Lyapunov function method. Furthermore, a cone complementary linearization algorithm is designed to solve the obtained conditions.

The rest of the paper is organized as follows. In Section 2, some preliminaries are showed. Furthermore, a state feedback controller is designed in Section 3. A numerical example and an application example are studied in Section 4. Finally, we give the conclusions in Section 5.

Notations. Matrix or vector $G \succ 0$ ($\succeq 0$) denotes that the elements in G are all positive (nonnegative). Matrix $H > 0 (\geq 0)$ represents that matrix H is a positive definite matrix (positive semi-definite matrix). Matrix $G \in \mathcal{M}$ means matrix G is one of the matrices whose all entries are nonnegative. $\mathrm{tr}(G)$ represents the sum of diagonal elements of matrix G. Moreover, it is assumed that matrices have compatible dimensions, if their dimensions are not explicitly stated in advance. "w.r.t" represents the phrase "with regard to".

2. Problem Formulation

Consider the following linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + C\omega(t), \\ z(t) = Dx(t) + Eu(t) + F\omega(t), \end{cases} \tag{1}$$

where $x(t), u(t), \omega(t)$, and $z(t)$ denote the system state, system input, disturbance input as well as controlled output, respectively. A, B, C, D, E and F are known system matrices.

Assumption 1. *The disturbance input $\omega(t)$ satisfies*

$$\int_0^{T_e} \omega^{\mathrm{T}}(t)\omega(t)dt \le \varrho, \tag{2}$$

where scalar $\varrho > 0$ and $t \in [0, T_e]$, T_e is the terminal time of the concerned time-interval.

Lemma 1 ([23]). *For matrices L, M and $N > 0$, the following relationship holds.*

$$L^{\mathrm{T}}M + M^{\mathrm{T}}L \le L^{\mathrm{T}}NL + M^{\mathrm{T}}N^{-1}M. \tag{3}$$

Definition 1 ([24]). *System (1) is positive, if $x(t) \succeq 0$ and $z(t) \succeq 0$ always hold for any $u(t) \succeq 0$, $x(0) \succeq 0$, and $\omega(t) \succeq 0$, for $t \ge 0$.*

Lemma 2 ([25]). *Matrix $A \in \mathcal{M}$, if there is a constant $\imath$ satisfying*

$$A + \imath I \succeq 0. \tag{4}$$

Lemma 3 ([24]). *System (1) is positive, if and only if $A \in \mathcal{M}$, $B \succeq 0, C \succeq 0$, $D \succeq 0$, $E \succeq 0$ as well as $F \succeq 0$.*

Lemma 4 ([26]). *For Matrices $P > 0$ and $M > 0$, if and only if the following conditions hold*

$$tr(PM) = n, \tag{5}$$

$$\begin{bmatrix} P & I \\ I & M \end{bmatrix} \ge 0, \tag{6}$$

$PM = I$ holds, where n is the dimension of matrix P.

Definition 2. *Positive linear system (PLS) (1) is finite-time contractively bounded w.r.t ($\phi_1, \phi_2, \phi_3, R, \varrho, t_s, T_e$), if $x^{\mathrm{T}}(0)Rx(0) < \phi_1$ implies that $x^{\mathrm{T}}(t)Rx(t) < \phi_2$, $\forall t \in [0, T_e]$, furthermore, $x^{\mathrm{T}}(t)Rx(t) < \phi_3$, $\forall t \in [t_s, T_e]$, where $\phi_3 < \phi_1 < \phi_2$, $0 < t_s < T_e$, $\varrho > 0$, and $R > 0$.*

Definition 3 ([27,28]). *PLS (1) is finite-time bounded w.r.t ($\phi_1, \phi_2, R, \varrho, T_e$), if $x^{\mathrm{T}}(0)Rx(0) < \phi_1$ implies that $x^{\mathrm{T}}(t)Rx(t) < \phi_2$, $\forall t \in [0, T_e]$, where $\phi_1 < \phi_2$, $\varrho > 0$, and $R > 0$.*

Remark 1. *Compared with the traditional finite-time boundedness described in Definition 3, not only does the FTCB shown in Definition 2 consider the boundedness of states in the time interval $[0, T_e]$, but also it cares about the contractive dynamic of states in the time interval $[t_s, T_e]$. Moreover, if letting $t_s = 0$ and $c_3 \ge c_2$, Definition 2 can reduce to the style of the traditional FTS in Definition 3.*

Remark 2. *The illustration of Definition 2 is shown in Figure 1. Moreover, according to Definition 1, state x(t) of positive systems has to be nonnegative. Hence, the trajectory of $x^{\mathrm{T}}(t)Rx(t)$ of the finite-time contractive bounded positive systems always lie in the first orthant which is different from that of the finite-time contractive bounded general systems.*

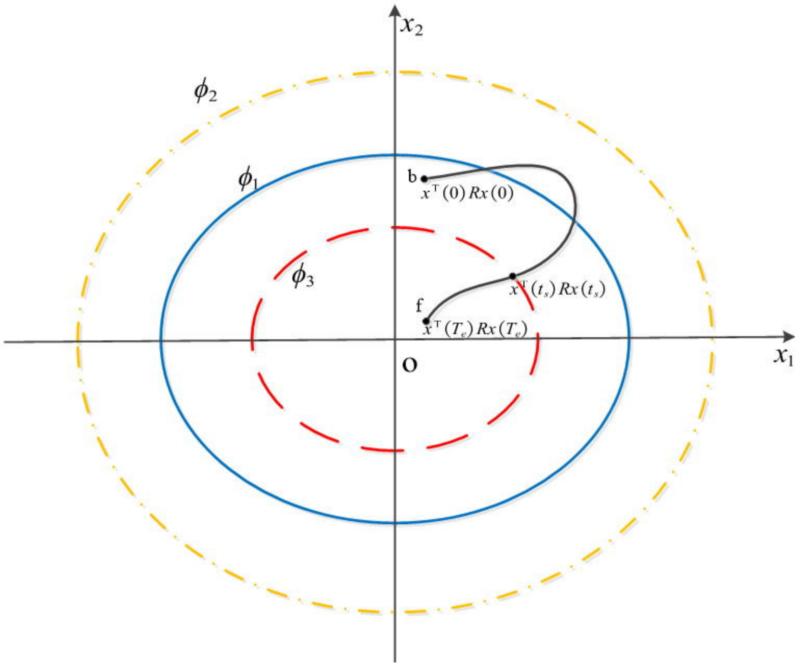

Figure 1. Illustration of Definition 2.

Definition 4. *PLS (1) is H_∞ finite-time contractive bounded, if the following two conditions hold*

(1) PLS (1) is finite-time contractively bounded w.r.t $(\phi_1, \phi_2, \phi_3, R, \varrho, t_s, T_e)$;
(2) Under the zero initial condition, the following relationship is satisfied

$$\int_0^{T_e} z(t)^T z(t)dt < \gamma^2 \int_0^{T_e} \omega(t)^T \omega(t)dt. \tag{7}$$

3. State Feedback Finite-Time Contractively Bounded Controller Synthesis

Consider the following state feedback control law for PLS (1)

$$u = Kx(t), \tag{8}$$

where K is the controller gain to be designed. The corresponding closed-loop system can be obtained as below

$$\begin{cases} \dot{x}(t) = (A + BK)x(t) + C\omega(t), \\ z(t) = (D + EK)x(t) + F\omega(t). \end{cases} \tag{9}$$

A state feedback finite-time contractively bounded controller design method for system (1) is given in the following theorem.

Theorem 1. *For given scalars $\alpha > 0$, $\varrho > 0$, $\phi_3 < \phi_1 < \phi_2$, $0 < t_s < T_e$, and a matrix $R > 0$, system (9) is positive and H_∞ finite-time contractively bounded w.r.t $(\phi_1, \phi_2, \phi_3, R, \varrho, t_s, T_e)$, if there exist diagonal matrices $M > 0$, $P > 0$, and a matrix W as well as scalars $\eta > 0$, ς, and γ such that*

$$\begin{bmatrix} \Lambda + \Lambda^{\mathrm{T}} + \alpha M & C & (DM + EW)^{\mathrm{T}} \\ * & -\gamma^2 I & F^{\mathrm{T}} \\ * & * & -I \end{bmatrix} < 0, \tag{10}$$

$$R < P < \eta R, \tag{11}$$

$$\eta\phi_1 + \gamma^2\varrho < \phi_2, \tag{12}$$

$$e^{-\alpha t_s}\eta\phi_1 + \gamma^2\varrho < \phi_3, \tag{13}$$

$$AM + BW + \varsigma I \succeq 0, \tag{14}$$

$$DM + EW \succeq 0, \tag{15}$$

with the equation restriction

$$PM = I, \tag{16}$$

where $\Lambda = AM + BW$, $M = P^{-1}$. *Furthermore, the controller gain is formulated as* $K = WM^{-1}$ *in this case.*

Remark 3. *In general, the states of positive systems are always nonnegative, while that of general systems are not required to be nonnegative. Hence, different from the* H_∞ *finite-time contractively bounded control issue of general systems [19], the positivity characteristic needs to be especially considered when the* H_∞ *finite-time contractively bounded control issue of the positive systems is discussed. By Lemma 3, if system (9) is positive, the conditions "*$A + BK \in \mathcal{M}$*" and "*$D + EK \succeq 0$*" which contain the controller gain K have to be satisfied. Thus, (15) is derived. Moreover, (14) is developed by Lemma 2.*

Proof. The positivity and H_∞ FTCB of system (9) are proved by the two steps below.

Step 1. Let us prove that system (9) is positive. Post-multiplying both sides of (14) by M^{-1}, and then one has

$$A + BWM^{-1} + \varsigma M^{-1} I \succeq 0. \tag{17}$$

Let $\varsigma^{'} = \varsigma m'$. In view of $\varsigma m^{'} I \succeq \varsigma M^{-1}$, we have

$$A + BWM^{-1} + \varsigma^{'} I \succeq A + BWM^{-1} + \varsigma M^{-1} I \succeq 0. \tag{18}$$

Then, substituting WM^{-1} for K yields

$$A + BK + \varsigma^{'} I \succeq A + BK + \varsigma M^{-1} I \succeq 0, \tag{19}$$

where m' is the maximal element of M^{-1}.

According to Lemma 2, (19) indicates that the matrix $A + BK \in \mathcal{M}$. Similarly, post-multiplying both sides of (15) by M^{-1}, it can be derived that $D + EWM^{-1} \succeq 0$. Letting $K = WM^{-1}$, then $D + EK \succeq 0$ holds. Hence, we can conclude that system (9) is positive by Lemma 3.

Step 2. According to the properties of negative definite matrix, the following inequality can be obtained from (10)

$$\begin{bmatrix} \Lambda + \Lambda^{\mathrm{T}} + \alpha M & C \\ * & -\gamma^2 I \end{bmatrix} < 0. \tag{20}$$

Pre- and post-multiplying (20) by diag $\{P,\ I\}$, then it can be obtained that

$$\begin{bmatrix} \Lambda'^{\mathrm{T}} + \Lambda' + \alpha P & PC \\ * & -\gamma^2 I \end{bmatrix} < 0, \tag{21}$$

where $\Lambda' = PA + PBK$.

Choose a Lyapunov function $V(x(t)) = x^{\mathrm{T}}(t)Px(t)$. Taking the time derivative of $V(x(t))$ along the trajectory of system (9), it follows that

$$\dot{V}(x(t)) = x^{\mathrm{T}}(t)\left(\Lambda'^{\mathrm{T}} + \Lambda'\right)x(t) + \omega(t)^{\mathrm{T}}\Gamma + \Gamma^{\mathrm{T}}\omega(t), \tag{22}$$

where $\Gamma = C^{\mathrm{T}}Px(t)$. By using the schur complement lemma, (21) equals

$$\Lambda' + \Lambda'^{\mathrm{T}} < -\alpha P - PC\left(\gamma^2 I\right)^{-1}C^{\mathrm{T}}P, \tag{23}$$

from which we have

$$\dot{V}(x(t)) < -\alpha V(x(t)) - x^{\mathrm{T}}(t)\Psi x + \omega(t)^{\mathrm{T}}\Gamma + \Gamma^{\mathrm{T}}\omega(t), \tag{24}$$

where $\Psi = PC(\gamma^2 I)^{-1}C^{\mathrm{T}}P$. Moreover, according to Lemma 1, for matrices $\omega(t)$, Γ and $\gamma^2 I > 0$, the following inequality holds

$$\omega(t)^{\mathrm{T}}\Gamma + \Gamma^{\mathrm{T}}\omega(t) < \omega(t)^{\mathrm{T}}(\gamma^2 I)\omega(t) + x^{\mathrm{T}}(t)\Psi x(t).$$

Then, we can further obtain

$$\dot{V}(x(t)) + \alpha V(x(t)) < \gamma^2\omega(t)^{\mathrm{T}}\omega(t). \tag{25}$$

Multiplying both sides of (25) by $\mathrm{e}^{\alpha t}$, and then integrating both sides of it from 0 to t for $t \in [0,\ T_e]$ yields

$$\mathrm{e}^{\alpha t}V(x(t)) - V(x(0)) < \gamma^2\int_0^t \mathrm{e}^{\alpha\tau}\omega^{\mathrm{T}}(\tau)\omega(\tau)\mathrm{d}\tau, \tag{26}$$

from which one has

$$V(x(t)) < \mathrm{e}^{-\alpha t}V(x(0)) + \gamma^2\int_0^t \mathrm{e}^{-\alpha(t-\tau)}\omega^{\mathrm{T}}(\tau)\omega(\tau)\mathrm{d}\tau. \tag{27}$$

Moreover, (11) yeilds that $V(x(0)) = x^{\mathrm{T}}(0)Px(0) < \eta x^{\mathrm{T}}(0)Rx(0) < \eta\phi_1$. Hence, in view of $\mathrm{e}^{-(t-\tau)} < 1$, $\mathrm{e}^{-\alpha t} < 1$ and (2), the following inequality is obtained for $t \in [0,\ T_e]$

$$\begin{aligned} V(x(t)) &< V(x(0)) + \gamma^2\int_0^t \omega^{\mathrm{T}}(\tau)\omega(\tau)\mathrm{d}\tau \\ &< \eta\phi_1 + \gamma^2\varrho. \end{aligned} \tag{28}$$

Considering (12) and $V(x(t)) = x^{\mathrm{T}}(t)Px(t) > x^{\mathrm{T}}(t)Rx(t)$, (28) implies that $x^{\mathrm{T}}(t)Rx(t) < \phi_2, \forall t \in [0,\ T_e]$.

Similarly, for $t \in [t_s,\ T_e]$, we have

$$V(x(t)) < \mathrm{e}^{-\alpha t_s}\eta\phi_1 + \gamma^2\varrho. \tag{29}$$

Then, by using (13), (29) implies that $x^{\mathrm{T}}(t)Rx(t) < \phi_3$, for $t \in [t_s,\ T_e]$.

Next, we verify that the system (9) satisfies the H_∞ performance defined in (7).

Pre and post-multiplying (10) by $\mathrm{diag}\{P,\ I,\ I\}$ yields that

$$\begin{bmatrix} \Lambda' + \Lambda'^{\mathrm{T}} + \alpha P & PC & (D+EK)^{\mathrm{T}} \\ * & -\gamma^2 I & F^{\mathrm{T}} \\ * & * & -I \end{bmatrix} < 0, \tag{30}$$

from which the following relationship can be developed by using the schur complement lemma

$$\dot{V}(x(t)) + \alpha V(x(t)) + z^{\mathrm{T}}(t)z(t) - \gamma^2\omega^{\mathrm{T}}(t)\omega(t) < 0. \tag{31}$$

Integrating both sides of (31) from 0 to t, then one has

$$\int_0^{T_e}\dot{V}(x(t))\mathrm{d}t + \alpha\int_0^{T_e}V(x(t))\mathrm{d}t + \int_0^{T_e}z^{\mathrm{T}}(t)z(t)\mathrm{d}t - \gamma^2\int_0^{T_e}\omega^{\mathrm{T}}(t)\omega(t)\mathrm{d}t < 0. \tag{32}$$

Since $V(x(t)) > 0$, for $t \in (0,\ T_e]$, we can verify that (7) is satisfied from (32) under the zero initial condition. Hence, the proof is completed. □

Remark 4. *Actually, on one hand, the nonlinear term PBK in the condition (30) makes it difficult to solve the constraint conditions in Theorem 1. Hence, separating P from K in this term is certainly a feasible and computable scheme. On this basis, the condition (10) in Theorem 1 is derived from (30) by employing the variable substitution method. The matrix M represents P^{-1}, matrix W donates KM in (10).*

On the other hand, we can not obtain a qualified control gain K yet by solving the constraints (10)–(15) directly, since the existence of the equality restriction (16). More specifically, although a set of feasible solution $(M, P, W, \eta, \varsigma, \gamma)$ can be found by solving the matrix inequality conditions (10)–(15), the obtained P and M may not satisfy the potential condition "$PM = I$". Then, the calculated $K = WM^{-1}$ is not appropriate. In order to solve such a non-convex problem, the cone complementary linearization approach can be adopted to turn it into an equivalent nonlinear minimization problem with linear matrix inequality constraints shown in Problem 1.

Problem 1.

$$\begin{gathered} \min tr(PM) \\ \text{s.t. (10)–(15) and (6)} \end{gathered}$$

Remark 5. (6) *holds yields that $tr(PM) \geq n$, then, if and only if $tr(PM) = n$, we can obtain that $tr(PM)$ is minimum, moreover, it can be concluded that "$PM = I$" holds by Lemma 4. Hence, the conditions (10)–(15) with the equation restriction (16) is feasible when the solution of Problem 1 is n. Furthermore, a qualified controller gain $K = WM^{-1}$ can be obtained according to the obtained feasible set.*

Inspired by the work in [29], the corresponding linearization Algorithm 1 is designed to solve Problem 1 as follows.

Algorithm 1 Cone complementary linearization algorithm

Step 1. Given α, ϱ, R, c_1, c_2, c_3, t_s, and T_e. Moreover, Given a small enough scalar ε and let $i = 1$ and $N = 100$, N is the maximum number of iterations.
Step 2. Solve the conditions (10)–(15) and (6). If they are feasible, go to **Step 3**; else, exit.
Step 3. Let $(M_i, P_i, W_i, \eta_i, \varsigma_i, \gamma_i) = (M, P, W, \eta, \varsigma, \gamma)$, where $(M, P, W, \eta, \varsigma, \gamma)$ is the feasible set obtained in **Step 2**. Solve the optimization problem as follows

$$\begin{gathered} \min tr(P_i M + P M_i) \\ \text{s.t. (6) and (10)–(15)} \end{gathered}$$

Step 4. Compare $tr(P_i M + P M_i)$ with $2n$, where n is the rank of matrix P. If $|tr(P_i M + P M_i) - 2\text{n}| < \varepsilon$, output the value of $K = WM^{-1}$ and then exit; else, $i = i + 1$, compare i with N, if $i \leq 100$, go to **Step3**; else, exit.

4. Examples

4.1. Example 1

Consider system (1) with

$$\begin{gathered} A = \begin{bmatrix} -0.4 & 0.6 \\ 0.35 & -0.5 \end{bmatrix}, \ B = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, \ C = \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix}, \\ D = \begin{bmatrix} 0.7 & 0.6 \\ 0.4 & 0.8 \end{bmatrix}, \ E = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}, \ F = \begin{bmatrix} 0.9 \\ 0.8 \end{bmatrix}. \end{gathered} \tag{33}$$

The system (33) can be deduced to be a positive system by Lemma 3. Moreover, for simulation purpose, the following disturbance input $\omega(t)$ is considered here.

$$\omega(t) = \begin{cases} 2|\sin(t)|e^{-2t}, & t \in [0,\ 3], \\ 0, & else. \end{cases}$$

The evolution of $\omega(t)$ is shown in Figure 2. In addition, according to Assumption 1, a qualified value of $\varrho = 0.1001 > \int_0^6 \omega^{\mathrm{T}}(t)\omega(t)\mathrm{d}t$ is chosen.

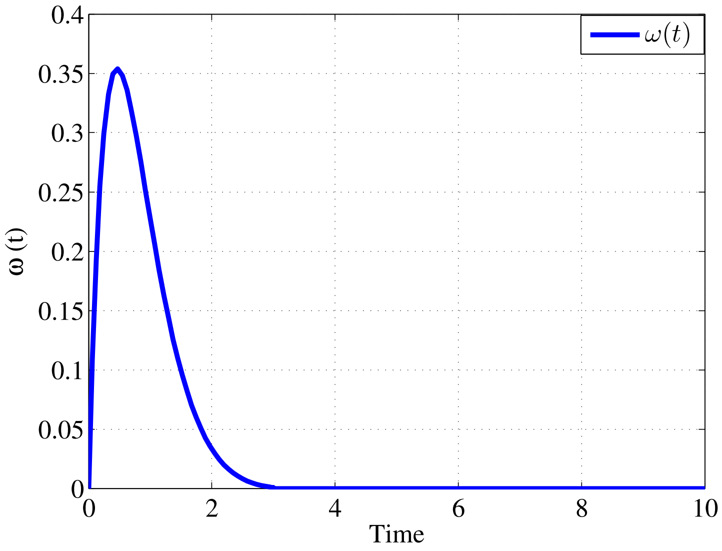

Figure 2. Disturbance $\omega(t)$.

Let $x(0) = \begin{bmatrix} 2 & 2.1 \end{bmatrix}$, $R = I$, $\phi_1 = 10$, $\phi_2 = 20$, $\phi_3 = 4$, $t_s = 4$, $T_e = 10$. Then, the plot of $x^{\mathrm{T}}(t)Rx(t)$ of system (33) shown in Figure 3 is obtained. It shows that for initial $x^{\mathrm{T}}(0)Rx(0) = 8.41 < 10$, the value of $x^{\mathrm{T}}(t)Rx(t) < 20$ holds, $\forall t \in [0, 10]$. However, for $t \in [4, 10]$, $x^{\mathrm{T}}(t)Rx(t) < 4$ is not satisfied in this case. It suggests that system (33) is finite-time bounded w.r.t $(10, 20, I, 0.1001, 10)$ not finite-time contractively bounded for the given parameters above. Next, the finite-time contractively bounded control is taken into account to ensure the H_∞ FTCB of system (33).

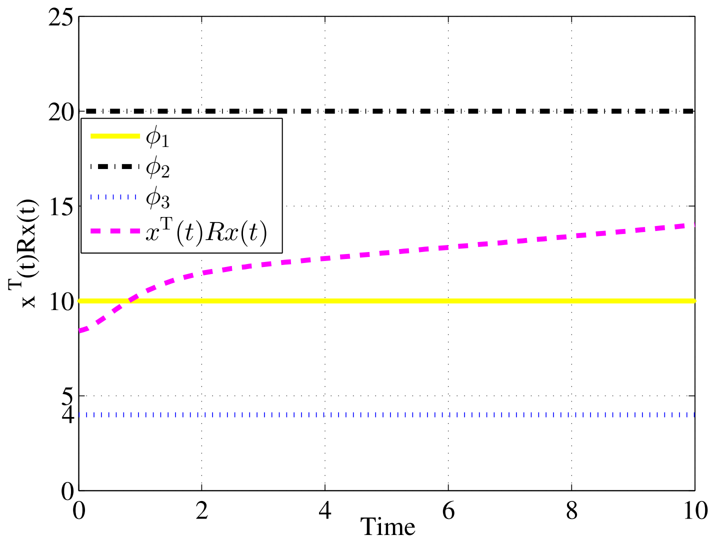

Figure 3. Illustration of $x^{\mathrm{T}}(t)Rx(t)$ of system (33).

Run Algorithm 1 to solve Problem 1 by using the Yalmip toolbox in MATLAB [30]. The running results suggest that Problem 1 is feasible only if the value of α is taken from the interval [0.2321 1.5232]. Then, a feasible solution can be obtained as follows when $\alpha = 1$

$$M = \begin{bmatrix} 0.7116 & 0 \\ 0 & 0.6721 \end{bmatrix}, \quad P = \begin{bmatrix} 1.4053 & 0 \\ 0 & 1.4879 \end{bmatrix},$$
$$W = \begin{bmatrix} -1.0957 & -1.1812 \end{bmatrix}, \quad \eta = 1.8379, \; \gamma = 2.4437.$$

Furthermore, the control gain K can be attained as

$$K = WM^{-1} = [-1.5398 \ \ -1.7575],$$

from which one has

$$A + BK = \begin{bmatrix} -0.8619 & 0.0727 \\ 0.0420 & -0.8515 \end{bmatrix} \in \mathcal{M}, \quad D + EK = \begin{bmatrix} 0.2381 & 0.0727 \\ 0.2460 & 0.6242 \end{bmatrix} \succ 0.$$

Thus, the positivity of the corresponding closed-loop system can be verified by Lemma 3. Furthermore, the evolution of $x^{\mathrm{T}}(t)Rx(t)$ of the closed-loop system is illustrated in Figure 4. It demonstrates that if $x^{\mathrm{T}}(0)Rx(0) < 10$, the value of $x^{\mathrm{T}}(t)Rx(t) < 20$, $\forall t \in [0, 10]$, moreover, $x^{\mathrm{T}}(t)Rx(t) < 4$, $\forall t \in [4, 10]$ for the closed-loop system. In addition, the calculated $\gamma = 2.4437$ satisfies the relationship shown in (7). By Definition 4, it can be concluded that the closed-loop system is H_∞ finite-time contractively bounded w.r.t.$(10, 20, 4, I, 0.1001, 4, 10)$.

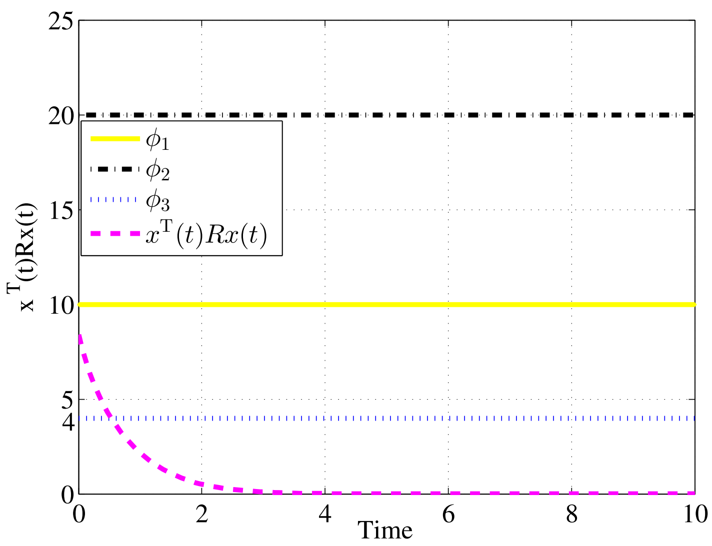

Figure 4. Evolution of $x^{\mathrm{T}}(t)Rx(t)$ of the corresponding closed-loop system.

4.2. Example 2

In this subsection, we consider the population control problem of the number of pests between two groups in an area to verify the effectiveness of the proposed method. It is described by the following Lotka–Volterra population model [31].

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -0.5905 & 0.505 \\ 0.505 & -0.39 \end{bmatrix} x(t) + \begin{bmatrix} 0.105 & 0.025 \\ 0.3 & 0.505 \end{bmatrix} u(t) + \begin{bmatrix} 0.205 \\ 0.11 \end{bmatrix} \omega(t), \\ z(t) = \begin{bmatrix} 0.1105 \\ 0.11 \end{bmatrix} x(t) + \begin{bmatrix} 0.11 \\ 0.11 \end{bmatrix} u(t), \end{cases} \tag{34}$$

where $x(t) = [x_1, \ x_2]^{\mathrm{T}}$, $x_1, \ x_2$ represent the population density of corresponding groups, $\omega(t)$ is regarded as the measure of the number of the pests from other areas and a set of appropriate system matrices is given.

For simulation purpose, the following disturbance input $\omega(t)$ is chosen, and the illustration of it is shown in Figure 5.

$$\omega(t) = e^{-t}|\cos(2t)|. \tag{35}$$

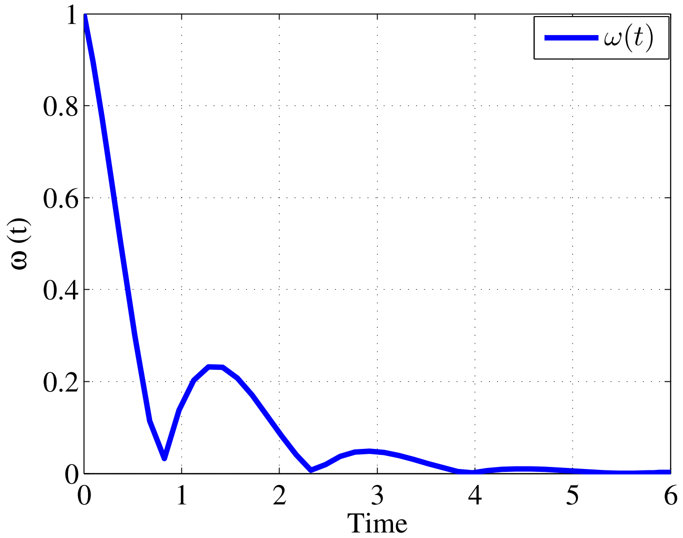

Figure 5. Disturbance $\omega(t)$.

Let $x(0) = \begin{bmatrix} 1.8 & 1.9 \end{bmatrix}$, $R = I$, $\phi_1 = 8$, $\phi_2 = 10$, $\phi_3 = 2$, $t_s = 3$, $T_e = 6$. Furthermore, according to Assumption 1, for $\omega(t) = e^{-t}|\cos(2t)|$, an appropriate value of $\varrho = 0.3001 > \int_0^6 \omega^{\mathrm{T}}(t)\omega(t)\mathrm{d}t$ is chosen. Figure 6 presents the evolution of $x^{\mathrm{T}}(t)Rx(t)$ of system (34).

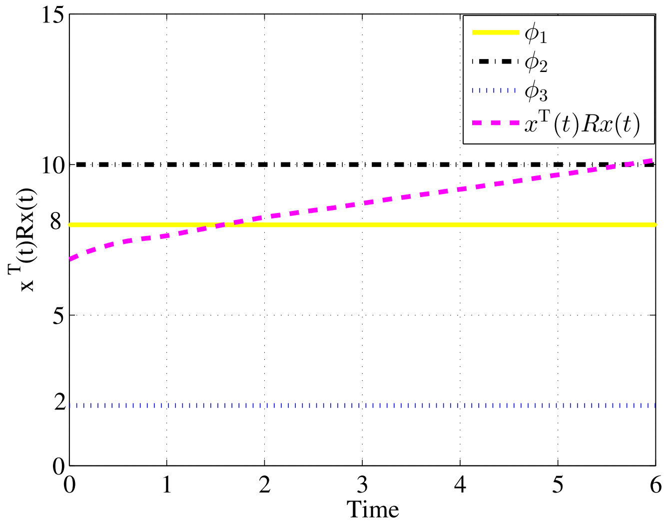

Figure 6. Illustration of $x^{\mathrm{T}}(t)Rx(t)$ of system (34).

It can be found that there does not exist $x^{\mathrm{T}}(t)Rx(t) < 2$, for $t \in [3\ 6]$. Hence, according to Definition 2, it obviously illustrates that system (34) is not finite-time contractively bounded w.r.t $(8, 10, 2, I, 0.3001, 3, 6)$. Furthermore, it indicates that the population of pests in the area increases sustainedly during the fixed time interval without any governance measurement. Our goal is to decrease the population of pests in this area into a lower enough range during a fixed time interval with the existence of the disturbance input $\omega(t)$. Here, this goal is viewed as a H_∞ finite-time contractively bounded control issue for system (34). Finite-time contractively bounded control is considered to guarantee the positivity and H_∞ FTCB of the corresponding closed-loop system.

Running Algorithm 1 to solve Problem 1, it can be found that the Problem 1 is feasible only if the value of α is taken from the range $[0.4621\ 0.9846]$. Then, when $\alpha = 0.5$, a corresponding feasible solution is shown as below.

$$M = \begin{bmatrix} 0.9744 & 0 \\ 0 & 0.9742 \end{bmatrix}, \quad P = \begin{bmatrix} 1.0262 & 0 \\ 0 & 1.0265 \end{bmatrix},$$
$$W = \begin{bmatrix} -2.0080 & 1.1820 \\ 1.1820 & -2.0147 \end{bmatrix}, \quad \eta = 1.0581, \; \gamma = 0.5548.$$

Furthermore, the controller gain K can be resolved as

$$K = WM^{-1} = \begin{bmatrix} -2.0606 & 1.2134 \\ 1.2130 & -2.0682 \end{bmatrix}.$$

Moreover, it can be attained that $A + BK = \begin{bmatrix} -0.7765 & 0.5807 \\ 0.4994 & -1.0704 \end{bmatrix} \in \mathcal{M}$. Also, we can verify that $D + EK = [0.0173 \;\; 0.0160] \succeq 0$. Hence, it can be concluded that the corresponding closed-loop system is positive by Lemma 3. Moreover, trajectories of $x_1(t)$ and $x_2(t)$ in Figure 7 illustrate the evolution of the number of the pests in this area in the time interval $[0 \; 6]$. It indicates that the number of pests sustainably decreases through control measures, which corresponds to the desired goal above. The plot of $x^{\mathrm{T}}(t)Rx(t)$ of the closed-loop system is given in Figure 8.

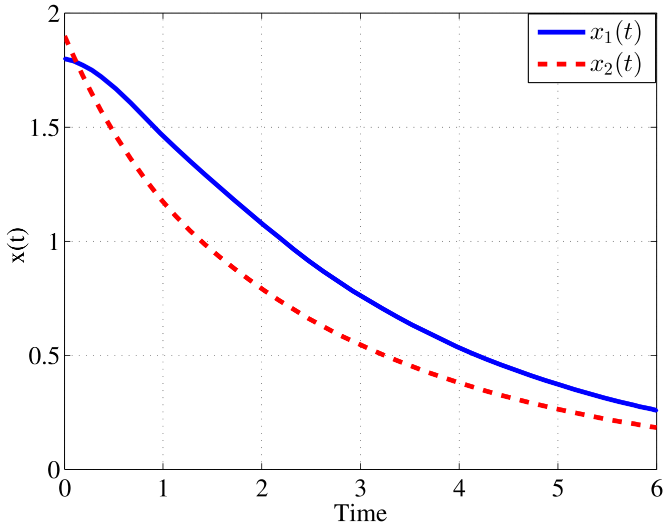

Figure 7. Illustration of system state $x(t)$.

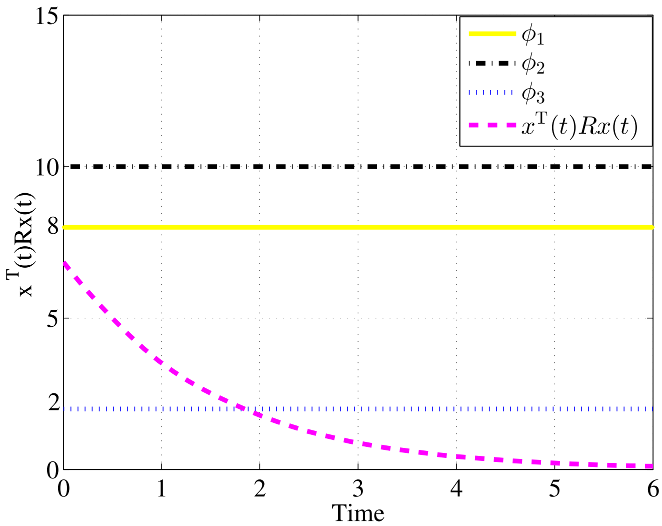

Figure 8. Evolution of $x^{\mathrm{T}}(t)Rx(t)$ of the closed-loop system.

As shown in Figure 8, for $x^{\mathrm{T}}(0)Rx(0) = 6.85 < 8$, the value of $x^{\mathrm{T}}(t)Rx(t) < 10$, for $\forall t \in [0\ 6]$, moreover, $x^{\mathrm{T}}(t)Rx(t) < 2$, for $\forall t \in [3\ 6]$. In addition, we can also verify that (7) is satisfied for the obtained $\gamma = 0.5548$ in this case. Hence, according to Definition 4, it can be concluded that the closed-loop system is H_∞ finite-time contractively bounded w.r.t $(8, 10, 2, I,\ 0.3001, 3, 6)$ under the H_∞ performance index $\gamma = 0.5548$.

5. Conclusions

The finite-time contractively bounded control issue for PLSs under H_∞ performance have been investigated in this paper. The definition of H_∞ FTCB has been extended to positive systems. Finite-time contractively bounded control has been considered to finite-time contractively stabilize the PLSs under H_∞ performance. The variable substitution method has been adopted to design the state feedback finite-time contractively bounded controller. The corresponding sufficient condition for the existence of such controllers has been derived in the form of matrix inequalities through the use of the Lyapunov function approach. Furthermore, the cone complementary linearization method has been used to solve the control gain and a corresponding algorithm has been designed. Finally, simulation results of the studied numerical example and the application example about pest management suggested the effectiveness of the finite-time contractively bounded control.

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