mathematics

Article

Some New Quantum Hermite–Hadamard Inequalities for Co-Ordinated Convex Functions

Fongchan Wannalookkhee [1], Kamsing Nonlaopon [1,*], Sotiris K. Ntouyas [2,3], Mehmet Zeki Sarikaya [4], Hüseyin Budak [4] and Muhammad Aamir Ali [5]

[1] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand; fongchan_wanna@kkumail.com
[2] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece; sntouyas@uoi.gr
[3] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[4] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey; sarikayamz@gmail.com (M.Z.S.); huseyinbudak@duzce.edu.tr (H.B.)
[5] Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China; mahr.muhammad.aamir@gmail.com
* Correspondence: nkamsi@kku.ac.th; Tel.: +668-6642-1582

Abstract: In this paper, we establish some new versions of Hermite–Hadamard type inequalities for co-ordinated convex functions via q_1, q_2-integrals. Since the inequalities are newly proved, we therefore consider some examples of co-ordinated convex functions and show their validity for particular choices of $q_1, q_2 \in (0, 1)$. We hope that the readers show their interest in these results.

Keywords: Hermite–Hadamard inequalities; convex functions; co-ordinated convex functions; quantum calculus

MSC: 05A30; 26D10; 26D15; 26A51; 26B25; 81P68

Citation: Wannalookkhee, F.; Nonlaopon, K.; Ntouyas, S.K.; Sarikaya, M.Z.; Budak, H.; Ali, M.A. Some New Quantum Hermite–Hadamard Inequalities for Co-Ordinated Convex Functions. *Mathematics* **2022**, *10*, 1962. https://doi.org/10.3390/math10121962

Academic Editors: Juan Eduardo Nápoles Valdes, Miguel Vivas-Cortez, Janusz Brzdęk and Shanhe Wu

Received: 24 March 2022
Accepted: 5 June 2022
Published: 7 June 2022

1. Introduction

Quantum calculus (sometimes is called q-calculus) is known as the study of calculus with no limits. Note that q-calculus can be reduced to ordinary calculus if we take $\lim_{q\to 1}$. It was firstly studied by the famous mathematician Euler (1707–1783). In 1910, F. H. Jackson [1] determined the definite q-integral known as the q-Jackson integral. Quantum calculus has many applications in several mathematical areas such as combinatorics, number theory, orthogonal polynomials, basic hypergeometric functions, mechanics, quantum theory, and theory of relativity, see, for instance, [2–7] and the references therein. The book by V. Kac and P. Cheung [8] covers the fundamental knowledge and also the basic theoretical concepts of quantum calculus.

In 2013, J. Tariboon and S. K. Ntouyas [9] defined the q-derivative and q-integral of a continuous function on finite intervals and proved some of its properties. Many well-known integral inequalities such as Hölder, Hermite–Hadamard, trapezoid, Ostrowski, Cauchy–Bunyakovsky–Schwarz, Grüss, and Grüss–Čebyšev inequalities have been studied in the concept of q-calculus, see [10] for more details. Based on these results, there are many outcomes concerning q-calculus. For example, in [11], some new Hermite–Hadamard type inequalities were established for co-ordinated convex functions and Simpson's type inequalities for co-ordinated convex functions were established in [12]. In [13], Kalsoom et al. used co-ordinated n-polynomial preinvexity and proved some Ostrowski type inequalities for quantum integrals. In [14,15], the authors used quantum integrals for the functions of two variables and proved some new Hermite–Hadamard type inequalities for co-ordinated convex functions.

In 2020, S. Bermudo et al. [16] newly defined the q-derivative and q-integral of a continuous function on finite intervals, called q^b-calculus, while the definition of J. Tariboon and S. K. Ntouyas is called q_a-calculus. Moreover, in their paper, they proved Hermite–Hadamard inequalities for convex functions and h-convex functions by using such the new definition.

The Hermite–Hadamard inequality is a classical inequality stated as: If $f:[a,b]\to\mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Inequality (1) was introduced by C. Hermite [17] in 1883 and was investigated by J. Hadamard [18] in 1893.

In [19], Alp et al. proved the following quantum Hermite–Hadamard inequality for convex functions using the following quantum integrals:

Theorem 1. *If $f:[a,b]\to\mathbb{R}$ is a convex function, then we have*

$$f\left(\frac{qa+b}{1+q}\right) \le \frac{1}{b-a}\int_a^b f(x)\,{}_a d_q x \le \frac{qf(a)+f(b)}{1+q}. \tag{2}$$

Bermudo et al. also proved the corresponding Hermite–Hadamard inequality for q^b-integrals, as follows:

Theorem 2. *If $f:[a,b]\to\mathbb{R}$ is a convex function, then we have*

$$f\left(\frac{a+qb}{1+q}\right) \le \frac{1}{b-a}\int_a^b f(x)\,{}^b d_q x \le \frac{f(a)+qf(b)}{1+q}. \tag{3}$$

Recently, Sitthiwirattham et al. proved some new quantum Hermite–Hadamard inequalities for convex functions by using their new techniques.

Theorem 3. *If $f:[a,b]\to\mathbb{R}$ is a convex function, then we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\,{}^{\frac{a+b}{2}} d_q x + \int_{\frac{a+b}{2}}^b f(x)\,{}_{\frac{a+b}{2}} d_q x\right] \le \frac{f(a)+f(b)}{2}. \tag{4}$$

Moreover, Ali et al. proved the following new version of quantum Hermite–Hadamard inequality involving a q_a-integral and q^b-integral. They also proved some inequalities for estimations of the left and right hand sides of this inequality.

Theorem 4. *If $f:[a,b]\to\mathbb{R}$ is a convex function, then we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\,{}_a d_q x + \int_{\frac{a+b}{2}}^b f(x)\,{}^b d_q x\right] \le \frac{f(a)+f(b)}{2}. \tag{5}$$

When $f:\triangle\to\mathbb{R}$ is a co-ordinated convex function, S. Dragomir [20] presented the Hermite–Hadamard type inequalities in 2001 as follows:

Theorem 5. *If $f : \triangle \to \mathbb{R}$ is a co-ordinated convex function, then we have*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)\, dy\, dx \\ &\leq \frac{1}{2}\left[\frac{1}{b-a}\left(\int_a^b f(x,c)\, dx + \int_a^b f(x,d)\, dx\right)\right. \\ &\quad \left. + \frac{1}{d-c}\left(\int_c^d f(a,y)\, dy + \int_c^d f(b,y)\, dy\right)\right] \\ &\leq \frac{1}{4}[f(a,c) + f(b,c) + f(a,d) + f(b,d)]. \end{aligned} \tag{6}$$

Inspired by the ongoing studies, we prove some new versions of quantum Hermite–Hadamard inequalities for co-ordinated convex functions. We also show the validity of newly established inequalities with some examples for particular choices of $q_1, q_2 \in (0,1)$.

The structure of this paper is as follows: The fundamentals of q-calculus for one and two variable functions, as well as other relevant topics in this field, are briefly discussed in Section 2. In Section 3, we establish new variants of the q-Hermite–Hadamard inequality for co-ordinated convex functions. We present some examples in Section 4 to illustrate the newly established inequalities. Section 5 concludes with some research suggestions for the future.

2. Preliminaries

Throughout this paper, we let $\triangle := [a,b] \times [c,d] \subseteq \mathbb{R} \times \mathbb{R}$, $0 < q < 1$ and $0 < q_i < 1$ for $i = 1,2$. The definitions of q-calculus, co-ordinated functions, and q-calculus for co-ordinates are given in [9,14–16,20].

Definition 1 ([9])**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then, the q_a-derivative of f at $x \in (a,b]$ is defined by*

$$_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}.$$

The q_a-integral is defined by

$$\int_a^x f(t)\, _ad_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a).$$

Example 1. *Let $f(x) = x^2$ for $x \in [0,1]$. Then, we have*

$$\begin{aligned} _0D_q f(x) &= \frac{f(x) - f(qx + (1-q)0))}{(1-q)(x-0)} \\ &= \frac{f(x) - f(qx)}{(1-q)x} \\ &= \frac{x^2 - q^2x^2}{(1-q)x} \\ &= (1+q)x, \qquad \textit{for } x \in (0,1]. \end{aligned}$$

Example 2. *Let $f(x) = x$ for $x \in [0,1]$. Then, we have*

$$\begin{aligned} \int_0^{1/2} f(x)\, _0d_q x &= (1-q)\left(\frac{1}{2} - 0\right)\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{1}{2}\right) + (1-q^n)0\right) \\ &= \frac{1-q}{2}\sum_{n=0}^{\infty} q^n\left(\frac{q^n}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1-q}{4}\sum_{n=0}^{\infty} q^{2n} \\
&= \frac{1-q}{4}\left(\frac{1}{1-q^2}\right) \\
&= \frac{1}{4(1+q)}.
\end{aligned}
$$

Definition 2 ([16]). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then, the q^b-derivative of f at $x \in [a,b)$ is defined by*

$$
{}^{b}D_q f(x) = \frac{f(x) - f(qx + (1-q)b)}{(1-q)(x-b)}.
$$

The q^b-integral is defined by

$$
\int_x^b f(t)\ {}^{b}d_q t = (1-q)(b-x)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b).
$$

Example 3. *Let $f(x) = x^2$ for $x \in [0,1]$. Then, we have*

$$
\begin{aligned}
{}^{1}D_q f(x) &= \frac{f(x) - f(qx + (1-q)1))}{(1-q)(x-1)} \\
&= \frac{x^2 - (qx - q + 1)^2}{(1-q)(x-1)} \\
&= (1+q)x + 1 - q, \qquad \text{for } x \in [0,1).
\end{aligned}
$$

Example 4. *Let $f(x) = x$ for $x \in [0,1]$. Then, we have*

$$
\begin{aligned}
\int_{1/2}^{1} f(x)\ {}^{1}d_q x &= (1-q)\left(1 - \frac{1}{2}\right)\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{1}{2}\right) + (1-q^n)1\right) \\
&= \frac{1-q}{2}\sum_{n=0}^{\infty} q^n\left(1 - \frac{q^n}{2}\right) \\
&= \frac{1-q}{2}\left[\sum_{n=0}^{\infty} q^n - \frac{1}{2}\sum_{n=0}^{\infty} q^{2n}\right] \\
&= \frac{1-q}{2}\left[\frac{1}{1-q} - \frac{1}{2}\left(\frac{1}{1-q^2}\right)\right] \\
&= \frac{1}{2} - \frac{1}{4(1+q)}.
\end{aligned}
$$

Definition 3 ([20]). *A function $f : \triangle \to \mathbb{R}$ is said to be co-ordinated convex (or convex on co-ordinates) if the partial mappings*

$$
f_x : [c,d] \ni v \mapsto f(x,v) \in \mathbb{R} \quad \text{and} \quad f_y : [a,b] \ni u \mapsto f(u,y) \in \mathbb{R}
$$

are convex for all $x \in (a,b)$ and $y \in (c,d)$.

A formal definition for co-ordinated convex functions may be stated as follows:

Definition 4 ([21]). *A function $f : \triangle \to \mathbb{R}$ is said to be convex on co-ordinates if*

$$
\begin{aligned}
f(tx + (1-t)z, \lambda y + (1-\lambda)w) \le{}& t\lambda f(x,y) + t(1-\lambda)f(x,w) + (1-t)\lambda f(z,y) \\
&+ (1-t)(1-\lambda)f(z,w)
\end{aligned} \tag{7}
$$

holds for all $t, \lambda \in [0,1]$ and $(x,y), (z,w) \in \triangle$.

Definition 5 ([14])**.** *Suppose that $f:\triangle\to\mathbb{R}$ is a continuous function of two variables. Then, the definite integral is given by*

$$\int_a^x\int_c^y f(t,s)\ {}_c d_{q_2}s\ {}_a d_{q_1}t=(1-q_1)(1-q_2)(x-a)(y-c)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f(q_1^n x+(1-q_1^n)a,q_2^m y+(1-q_2^m)c).$$

Definition 6 ([15])**.** *Suppose that $f:\triangle\to\mathbb{R}$ is a continuous function of two variables. Then, the definite integrals are given by*

$$\int_a^x\int_y^d f(t,s)\ {}^d d_{q_2}s\ {}_a d_{q_1}t=(1-q_1)(1-q_2)(x-a)(d-y)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f(q_1^n x+(1-q_1^n)a,q_2^m y+(1-q_2^m)d),$$

$$\int_x^b\int_c^y f(t,s)\ {}_c d_{q_2}s\ {}^b d_{q_1}t=(1-q_1)(1-q_2)(b-x)(y-c)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f(q_1^n x+(1-q_1^n)b,q_2^m y+(1-q_2^m)c)$$

and

$$\int_x^b\int_y^d f(t,s)\ {}^d d_{q_2}s\ {}^b d_{q_1}t=(1-q_1)(1-q_2)(b-x)(d-y)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f(q_1^n x+(1-q_1^n)b,q_2^m y+(1-q_2^m)d).$$

3. Main Results

In this section, we prove some new Hermite–Hadamard inequalities for co-ordinated convex functions.

Theorem 6. *Let $f:\triangle\to\mathbb{R}$ be a co-ordinated convex function. Then, we have*

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{(b-a)(d-c)}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}f(x,y)\ {}^{\frac{c+d}{2}}d_{q_2}y\ {}^{\frac{a+b}{2}}d_{q_1}x\right.\\&\quad+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}f(x,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\ {}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}f(x,y)\ {}^{\frac{c+d}{2}}d_{q_2}y\ {}_{\frac{a+b}{2}}d_{q_1}x\\&\quad\left.+\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}f(x,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\ {}_{\frac{a+b}{2}}d_{q_1}x\right]\\&\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned}\tag{8}$$

Proof. Since f is co-ordinated convex, we have

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&=f\left(\frac{1}{2}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+\frac{1}{2}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right),\frac{1}{2}\left(\frac{1+s}{2}c+\frac{1-s}{2}d\right)+\frac{1}{2}\left(\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right)\\&\le\frac{1}{4}\left[f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right.\\&\quad\left.+f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)+f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right],\end{aligned}\tag{9}$$

where $t,s\in[0,1]$.

$q_{a,c}$-Integrating both sides of (9) over $[0,1]\times[0,1]$, we obtain

$$
\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{4}\Bigg[&\int_0^1\int_0^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t\\
&+\int_0^1\int_0^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t\\
&+\int_0^1\int_0^1 f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t\\
&+\int_0^1\int_0^1 f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t\Bigg]\\
=:&\ \frac{1}{4}(I_1+I_2+I_3+I_4).
\end{aligned}
$$

By Definitions 5 and 6, we have

$$
\begin{aligned}
I_1 &= \int_0^1\int_0^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t\\
&= (1-q_1)(1-q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m f\left(\frac{1+q_1^n}{2}a+\frac{1-q_1^n}{2}b,\frac{1+q_2^m}{2}c+\frac{1-q_2^m}{2}d\right)\\
&= (1-q_1)(1-q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m f\left(q_1^n a+(1-q_1^n)\frac{a+b}{2},q_2^m c+(1-q_2^m)\frac{c+d}{2}\right)\\
&= \frac{4}{(b-a)(d-c)}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)\ {}^{\frac{c+d}{2}}d_{q_2}y\ {}^{\frac{a+b}{2}}d_{q_1}x.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
I_2 &= \int_0^1\int_0^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t\\
&= \frac{4}{(b-a)(d-c)}\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\ {}^{\frac{a+b}{2}}d_{q_1}x,\\
I_3 &= \int_0^1\int_0^1 f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t\\
&= \frac{4}{(b-a)(d-c)}\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}} f(x,y)\ {}^{\frac{c+d}{2}}d_{q_2}y\ {}_{\frac{a+b}{2}}d_{q_1}x
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \int_0^1\int_0^1 f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t\\
&= \frac{4}{(b-a)(d-c)}\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\ {}_{\frac{a+b}{2}}d_{q_1}x.
\end{aligned}
$$

Replacing I_1, I_2, I_3, and I_4 in (10), we obtain

$$
\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\Bigg[&\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)\ {}^{\frac{c+d}{2}}d_{q_2}y\ {}^{\frac{a+b}{2}}d_{q_1}x\\
&+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\ {}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}} f(x,y)\ {}^{\frac{c+d}{2}}d_{q_2}y\ {}_{\frac{a+b}{2}}d_{q_1}x\\
&+\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\ {}_{\frac{a+b}{2}}d_{q_1}x\Bigg].
\end{aligned}
$$

Thus, the first inequality of (8) holds.

Next, by co-ordinated convexity of f again, we have

$$f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\leq\frac{(1+t)(1+s)}{4}f(a,c)+\frac{(1+t)(1-s)}{4}f(a,d)$$
$$+\frac{(1-t)(1+s)}{4}f(b,c)+\frac{(1-t)(1-s)}{4}f(b,d), \tag{10}$$

$$f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\leq\frac{(1+t)(1-s)}{4}f(a,c)+\frac{(1+t)(1+s)}{4}f(a,d)$$
$$+\frac{(1-t)(1-s)}{4}f(b,c)+\frac{(1-t)(1+s)}{4}f(b,d), \tag{11}$$

$$f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\leq\frac{(1-t)(1+s)}{4}f(a,c)+\frac{(1-t)(1-s)}{4}f(a,d)$$
$$+\frac{(1+t)(1+s)}{4}f(b,c)+\frac{(1+t)(1-s)}{4}f(b,d) \tag{12}$$

and

$$f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\leq\frac{(1-t)(1-s)}{4}f(a,c)+\frac{(1-t)(1+s)}{4}f(a,d)$$
$$+\frac{(1+t)(1-s)}{4}f(b,c)+\frac{(1+t)(1+s)}{4}f(b,d). \tag{13}$$

Summing (10)–(13), we obtain

$$\begin{aligned}&f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)+f\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\\&+f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)+f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\\&\leq f(a,c)+f(a,d)+f(b,c)+f(b,d).\end{aligned} \tag{14}$$

$q_{a,c}$-Integrating both sides of (14) over $[0,1]$, the second inequality of (8) holds. The proof is completed. □

Theorem 7. *Let $f:\triangle\to\mathbb{R}$ be a co-ordinated convex function. Then, we have*

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\leq\frac{1}{(b-a)(d-c)}\Bigg[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}f(x,y)\ {}_cd_{q_2}y\ {}_ad_{q_1}x\\&\quad+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}f(x,y)\ {}^dd_{q_2}y\ {}_ad_{q_1}x+\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}f(x,y)\ {}_cd_{q_2}y\ {}^bd_{q_1}x\\&\quad+\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}f(x,y)\ {}^dd_{q_2}y\ {}^bd_{q_1}x\Bigg]\\&\leq\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned} \tag{15}$$

Proof. Since f is co-ordinated convex, we have

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&=f\left(\frac{1}{2}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+\frac{1}{2}\left(\frac{2-t}{2}a+\frac{t}{2}b\right),\frac{1}{2}\left(\frac{s}{2}c+\frac{2-s}{2}d\right)+\frac{1}{2}\left(\frac{2-s}{2}c+\frac{s}{2}d\right)\right)\\&\leq\frac{1}{4}\Bigg[f\left(\frac{t}{2}a+\frac{2-t}{2}b,\frac{s}{2}c+\frac{2-s}{2}d\right)+f\left(\frac{t}{2}a+\frac{2-t}{2}b,\frac{2-s}{2}c+\frac{s}{2}d\right)\\&\quad+f\left(\frac{2-t}{2}a+\frac{t}{2}b,\frac{s}{2}c+\frac{2-s}{2}d\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b,\frac{2-s}{2}c+\frac{s}{2}d\right)\Bigg],\end{aligned} \tag{16}$$

where $t,s\in[0,1]$.

$q_{a,c}$-Integrating both sides of (16) over $[0,1]\times[0,1]$, we obtain

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{4}\Bigg[&\int_0^1\int_0^1 f\left(\frac{t}{2}a+\frac{2-t}{2}b,\frac{s}{2}c+\frac{2-s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t \\ &+\int_0^1\int_0^1 f\left(\frac{t}{2}a+\frac{2-t}{2}b,\frac{2-s}{2}c+\frac{s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t \\ &+\int_0^1\int_0^1 f\left(\frac{2-t}{2}a+\frac{t}{2}b,\frac{s}{2}c+\frac{2-s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t \\ &+\int_0^1\int_0^1 f\left(\frac{2-t}{2}a+\frac{t}{2}b,\frac{2-s}{2}c+\frac{s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t\Bigg] \\ =:\ &\frac{1}{4}(I_5+I_6+I_7+I_8). \end{aligned} \tag{17}$$

By Definitions 5 and 6, we have

$$\begin{aligned} I_5 &= \int_0^1\int_0^1 f\left(\frac{t}{2}a+\frac{2-t}{2}b,\frac{s}{2}c+\frac{2-s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t \\ &= (1-q_1)(1-q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m f\left(\frac{q_1^n}{2}a+\frac{2-q_1^n}{2}b,\frac{q_2^m}{2}c+\frac{2-q_2^m}{2}d\right) \\ &= (1-q_1)(1-q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m f\left(q_1^n\left(\frac{a+b}{2}\right)+(1-q_1^n)b,q_2^m\left(\frac{c+d}{2}\right)+(1-q_2^m)d\right) \\ &= \frac{4}{(b-a)(d-c)}\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}^d d_{q_2}y\,{}^b d_{q_1}x. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} I_6 &= \int_0^1\int_0^1 f\left(\frac{t}{2}a+\frac{2-t}{2}b,\frac{2-s}{2}c+\frac{s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t \\ &= \frac{4}{(b-a)(d-c)}\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}} f(x,y)\,{}_c d_{q_2}y\,{}^b d_{q_1}x, \\ I_7 &= \int_0^1\int_0^1 f\left(\frac{2-t}{2}a+\frac{t}{2}b,\frac{s}{2}c+\frac{2-s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t \\ &= \frac{4}{(b-a)(d-c)}\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}^d d_{q_2}y\,{}_a d_{q_1}x \end{aligned}$$

and

$$\begin{aligned} I_8 &= \int_0^1\int_0^1 f\left(\frac{2-t}{2}a+\frac{t}{2}b,\frac{2-s}{2}c+\frac{s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t \\ &= \frac{4}{(b-a)(d-c)}\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}} f(x,y)\,{}_c d_{q_2}y\,{}_a d_{q_1}x. \end{aligned}$$

Substituting I_5, I_6, I_7, and I_8 in (17), we obtain

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)}\Bigg[&\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}} f(x,y)\,{}_c d_{q_2}y\,{}_a d_{q_1}x \\ &+\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}^d d_{q_2}y\,{}_a d_{q_1}x+\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}} f(x,y)\,{}_c d_{q_2}y\,{}^b d_{q_1}x \\ &+\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}^d d_{q_2}y\,{}^b d_{q_1}x\Bigg]. \end{aligned}$$

Thus, the first inequality of (15) holds.

Next, by co-ordinated convexity of f again, we have

$$\begin{aligned} f\left(\frac{t}{2}a+\frac{2-t}{2}b,\frac{s}{2}c+\frac{2-s}{2}d\right) &\leq \frac{ts}{4}f(a,c)+\frac{t(2-s)}{4}f(a,d) \\ &\quad +\frac{(2-t)s}{4}f(b,c)+\frac{(2-t)(2-s)}{4}f(b,d), \end{aligned} \tag{18}$$

$$\begin{aligned} f\left(\frac{t}{2}a+\frac{2-t}{2}b,\frac{2-s}{2}c+\frac{s}{2}d\right) &\leq \frac{t(2-s)}{4}f(a,c)+\frac{ts}{4}f(a,d) \\ &\quad +\frac{(2-t)(2-s)}{4}f(b,c)+\frac{(2-t)s}{4}f(b,d), \end{aligned} \tag{19}$$

$$\begin{aligned} f\left(\frac{2-t}{2}a+\frac{t}{2}b,\frac{s}{2}c+\frac{2-s}{2}d\right) &\leq \frac{(2-t)s}{4}f(a,c)+\frac{(2-t)(2-s)}{4}f(a,d) \\ &\quad +\frac{ts}{4}f(b,c)+\frac{t(2-s)}{4}f(b,d) \end{aligned} \tag{20}$$

and

$$\begin{aligned} f\left(\frac{2-t}{2}a+\frac{t}{2}b,\frac{2-s}{2}c+\frac{s}{2}d\right) &\leq \frac{(2-t)(2-s)}{4}f(a,c)+\frac{(2-t)s}{4}f(a,d) \\ &\quad +\frac{t(2-s)}{4}f(b,c)+\frac{ts}{4}f(b,d). \end{aligned} \tag{21}$$

Summing (18)–(21), we obtain

$$\begin{aligned} & f\left(\frac{t}{2}a+\frac{2-t}{2}b,\frac{s}{2}c+\frac{2-s}{2}d\right)+f\left(\frac{t}{2}a+\frac{2-t}{2}b,\frac{2-s}{2}c+\frac{s}{2}d\right) \\ &\quad +f\left(\frac{2-t}{2}a+\frac{t}{2}b,\frac{s}{2}c+\frac{2-s}{2}d\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b,\frac{2-s}{2}c+\frac{s}{2}d\right) \\ &\leq f(a,c)+f(a,d)+f(b,c)+f(b,d). \end{aligned} \tag{22}$$

$q_{a,c}$-Integrating both sides of (22) over $[0,1]$ and then multiplying by $\frac{1}{4}$, the second inequality of (15) holds. The proof is completed. □

Theorem 8. *Let $f : \triangle \to \mathbb{R}$ be a co-ordinated convex function. Then, we have*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}} f\left(x,\frac{c+d}{2}\right)\ {}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^b f\left(x,\frac{c+d}{2}\right)\ {}_{\frac{a+b}{2}}d_{q_1}x\right] \\ &\quad +\frac{1}{2(d-c)}\left[\int_c^{\frac{c+d}{2}} f\left(\frac{a+b}{2},y\right)\ {}^{\frac{c+d}{2}}d_{q_2}y+\int_{\frac{c+d}{2}}^d f\left(\frac{a+b}{2},y\right)\ {}_{\frac{c+d}{2}}d_{q_2}y\right] \\ &\leq \frac{1}{(b-a)(d-c)}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)\ {}^{\frac{c+d}{2}}d_{q_2}y\ {}^{\frac{a+b}{2}}d_{q_1}x\right. \\ &\quad +\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d f(x,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\ {}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}} f(x,y)\ {}^{\frac{c+d}{2}}d_{q_2}y\ {}_{\frac{a+b}{2}}d_{q_1}x \\ &\quad \left.+\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d f(x,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\ {}_{\frac{a+b}{2}}d_{q_1}x\right] \\ &\leq \frac{1}{4(b-a)}\left[\int_a^{\frac{a+b}{2}} f(x,c)+f(x,d)\ {}^{\frac{a+b}{2}}d_{q_1}x\right]+\frac{1}{4(b-a)}\left[\int_{\frac{a+b}{2}}^b f(x,c)+f(x,d)\ {}_{\frac{a+b}{2}}d_{q_1}x\right] \\ &\quad +\frac{1}{4(d-c)}\left[\int_c^{\frac{c+d}{2}} f(a,y)+f(b,y)\ {}^{\frac{c+d}{2}}d_{q_2}y\right]+\frac{1}{4(d-c)}\left[\int_{\frac{c+d}{2}}^d f(a,y)+f(b,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\right] \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{23}$$

Proof. Let $g_y : [a,b] \to \mathbb{R}$ be a function defined by $g_y(x) = f(x,y)$. Then, g_y is convex on $[a,b]$ because f is co-ordinated convex on $\triangle$. By Theorem 3, we can write

$$g_y\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} g_y(x) \ {}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} g_y(x) \ {}_{\frac{a+b}{2}}d_{q_1}x\right] \leq \frac{g_y(a)+g_y(b)}{2}.$$

That is,

$$f\left(\frac{a+b}{2},y\right) \leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x,y) \ {}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f(x,y) \ {}_{\frac{a+b}{2}}d_{q_1}x\right] \leq \frac{f(a,y)+f(b,y)}{2}. \tag{24}$$

q^d-Integrating both sides of (24) over $[c, \frac{c+d}{2}]$ and then dividing by $(d-c)$, we have

$$\begin{aligned}
&\frac{1}{d-c}\int_c^{\frac{c+d}{2}} f\left(\frac{a+b}{2},y\right) \ {}^{\frac{c+d}{2}}d_{q_2}y \\
&\leq \frac{1}{(b-a)(d-c)}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y) \ {}^{\frac{c+d}{2}}d_{q_2}y \ {}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}} f(x,y) \ {}^{\frac{c+d}{2}}d_{q_2}y \ {}_{\frac{a+b}{2}}d_{q_1}x\right] \\
&\leq \frac{1}{2(d-c)}\int_c^{\frac{c+d}{2}} f(a,y)+f(b,y) \ {}^{\frac{c+d}{2}}d_{q_2}y.
\end{aligned} \tag{25}$$

Similarly, q_c-integrating both sides of (24) over $[\frac{c+d}{2}, d]$ and then dividing by $(d-c)$, we have

$$\begin{aligned}
&\frac{1}{d-c}\int_{\frac{c+d}{2}}^{d} f\left(\frac{a+b}{2},y\right) \ {}_{\frac{c+d}{2}}d_{q_2}y \\
&\leq \frac{1}{(b-a)(d-c)}\left[\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y) \ {}_{\frac{c+d}{2}}d_{q_2}y \ {}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y) \ {}_{\frac{c+d}{2}}d_{q_2}y \ {}_{\frac{a+b}{2}}d_{q_1}x\right] \\
&\leq \frac{1}{2(d-c)}\int_{\frac{c+d}{2}}^{d} f(a,y)+f(b,y) \ {}_{\frac{c+d}{2}}d_{q_2}y.
\end{aligned} \tag{26}$$

Let $h_x : [c,d] \to \mathbb{R}$ be defined by $h_x(y) = f(x,y)$. Then, h_x is convex on $[c,d]$ because f is co-ordinated convex on $\triangle$. By Theorem 3, we can write

$$h_x\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\left[\int_c^{\frac{c+d}{2}} h_x(y) \ {}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} h_x(y) \ {}_{\frac{c+d}{2}}d_{q_2}y\right] \leq \frac{h_x(c)+h_x(d)}{2}.$$

That is,

$$f\left(x,\frac{c+d}{2}\right) \leq \frac{1}{d-c}\left[\int_c^{\frac{c+d}{2}} f(x,y) \ {}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f(x,y) \ {}_{\frac{c+d}{2}}d_{q_2}y\right] \leq \frac{f(x,c)+f(x,d)}{2}. \tag{27}$$

q^b-Integrating both sides of (27) over $[a, \frac{a+b}{2}]$ and then dividing by $(b-a)$, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\left(x,\frac{c+d}{2}\right) \ {}^{\frac{a+b}{2}}d_{q_1}x \\
&\leq \frac{1}{(b-a)(d-c)}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y) \ {}^{\frac{c+d}{2}}d_{q_2}y \ {}^{\frac{a+b}{2}}d_{q_1}x + \int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y) \ {}_{\frac{c+d}{2}}d_{q_2}y \ {}^{\frac{a+b}{2}}d_{q_1}x\right] \\
&\leq \frac{1}{2(b-a)}\int_a^{\frac{a+b}{2}} f(x,c)+f(x,d) \ {}^{\frac{a+b}{2}}d_{q_1}x.
\end{aligned} \tag{28}$$

Similarly, q_a-integrating both sides of (27) over $[\frac{a+b}{2}, b]$ and then dividing by $(b-a)$, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f\left(x,\frac{c+d}{2}\right)\ {}_{\frac{a+b}{2}}d_{q_1}x = \\
&\le \frac{1}{(b-a)(d-c)}\left[\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}} f(x,y)\ {}^{\frac{c+d}{2}}d_{q_2}y\ {}_{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\ {}_{\frac{a+b}{2}}d_{q_1}x\right] = \\
&\le \frac{1}{2(b-a)}\int_{\frac{a+b}{2}}^{b} f(x,c)+f(x,d)\ {}_{\frac{a+b}{2}}d_{q_1}x.
\end{aligned} \tag{29}$$

Summing (25), (26), (28), and (29), we derive

$$\begin{aligned}
&\frac{1}{2(b-a)}\left[\int_{a}^{\frac{a+b}{2}} f\left(x,\frac{c+d}{2}\right)\ {}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f\left(x,\frac{c+d}{2}\right)\ {}_{\frac{a+b}{2}}d_{q_1}x\right] \\
&+ \frac{1}{2(d-c)}\left[\int_{c}^{\frac{c+d}{2}} f\left(\frac{a+b}{2},y\right)\ {}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f\left(\frac{a+b}{2},y\right)\ {}_{\frac{c+d}{2}}d_{q_2}y\right] \\
&\le \frac{1}{(b-a)(d-c)}\left[\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}} f(x,y)\ {}^{\frac{c+d}{2}}d_{q_2}y\ {}^{\frac{a+b}{2}}d_{q_1}x + \int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\ {}^{\frac{a+b}{2}}d_{q_1}x\right. \\
&\left.+ \int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}} f(x,y)\ {}^{\frac{c+d}{2}}d_{q_2}y\ {}_{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\ {}_{\frac{a+b}{2}}d_{q_1}x\right] \\
&\le \frac{1}{4(b-a)}\left[\int_{a}^{\frac{a+b}{2}} f(x,c)+f(x,d)\ {}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f(x,c)+f(x,d)\ {}_{\frac{a+b}{2}}d_{q_1}x\right] \\
&+ \frac{1}{4(d-c)}\left[\int_{c}^{\frac{c+d}{2}} f(a,y)+f(b,y)\ {}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f(a,y)+f(b,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\right].
\end{aligned} \tag{30}$$

Now, the second and the third inequalities of (23) hold.

For the first inequality, by the left hand side of the inequality (24) with $y=\frac{c+d}{2}$, we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{b-a}\left[\int_{a}^{\frac{a+b}{2}} f\left(x,\frac{c+d}{2}\right)\ {}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f\left(x,\frac{c+d}{2}\right)\ {}_{\frac{a+b}{2}}d_{q_1}x\right] \tag{31}$$

and by the left hand side of the inequality (27) with $x=\frac{a+b}{2}$, we obtain

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{d-c}\left[\int_{c}^{\frac{c+d}{2}} f\left(\frac{a+b}{2},y\right)\ {}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f\left(\frac{a+b}{2},y\right)\ {}_{\frac{c+d}{2}}d_{q_2}y\right]. \tag{32}$$

Combining (31) and (32), we obtain the first inequality of (23).

Using the right hand side of (24) with $y=c$ and $y=d$, we obtain

$$\frac{1}{4(b-a)}\left[\int_{a}^{\frac{a+b}{2}} f(x,c)\ {}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f(x,c)\ {}_{\frac{a+b}{2}}d_{q_1}x\right] \le \frac{f(a,c)+f(b,c)}{8} \tag{33}$$

and

$$\frac{1}{4(b-a)}\left[\int_{a}^{\frac{a+b}{2}} f(x,d)\ {}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f(x,d)\ {}_{\frac{a+b}{2}}d_{q_1}x\right] \le \frac{f(a,d)+f(b,d)}{8}, \tag{34}$$

respectively. Using the right hand side of (27) with $x=a$ and $x=b$, we have

$$\frac{1}{4(d-c)}\left[\int_{c}^{\frac{c+d}{2}} f(a,y)\ {}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f(a,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\right] \le \frac{f(a,c)+f(a,d)}{8} \tag{35}$$

and

$$\frac{1}{4(d-c)}\left[\int_{c}^{\frac{c+d}{2}} f(b,y)\ {}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f(b,y)\ {}_{\frac{c+d}{2}}d_{q_2}y\right] \leq \frac{f(b,c)+f(b,d)}{8}, \tag{36}$$

respectively. Replacing (33)–(36) in (30), we obtain the last inequality of (23). The proof is completed. □

Theorem 9. *Let $f : \triangle \to \mathbb{R}$ be a co-ordinated convex function. Then, we have*

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{2(b-a)}\left[\int_{a}^{\frac{a+b}{2}} f\left(x,\frac{c+d}{2}\right)\ {}_{a}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f\left(x,\frac{c+d}{2}\right)\ {}^{b}d_{q_1}x\right]\\
&\quad + \frac{1}{2(d-c)}\left[\int_{c}^{\frac{c+d}{2}} f\left(\frac{a+b}{2},y\right)\ {}_{c}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f\left(\frac{a+b}{2},y\right)\ {}^{d}d_{q_2}y\right]\\
&\leq \frac{1}{(b-a)(d-c)}\left[\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}} f(x,y)\ {}_{c}d_{q_2}y\ {}_{a}d_{q_1}x\right.\\
&\quad + \int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\ {}^{d}d_{q_2}y\ {}_{a}d_{q_1}x + \int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}} f(x,y)\ {}_{c}d_{q_2}y\ {}^{b}d_{q_1}x\\
&\quad \left. + \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\ {}^{d}d_{q_2}y\ {}^{b}d_{q_1}x\right]\\
&\leq \frac{1}{4(b-a)}\left[\int_{a}^{\frac{a+b}{2}} [f(x,c)+f(x,d)]\ {}_{a}d_{q_1}x\right] + \frac{1}{4(b-a)}\left[\int_{\frac{a+b}{2}}^{b} [f(x,c)+f(x,d)]\ {}^{b}d_{q_1}x\right]\\
&\quad + \frac{1}{4(d-c)}\left[\int_{c}^{\frac{c+d}{2}} [f(a,y)+f(b,y)]\ {}_{c}d_{q_2}y\right] + \frac{1}{4(d-c)}\left[\int_{\frac{c+d}{2}}^{d} [f(a,y)+f(b,y)]\ {}^{d}d_{q_2}y\right]\\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned} \tag{37}$$

Proof. Let $g_y : [a,b] \to \mathbb{R}$ be a function defined by $g_y(x) = f(x,y)$. Then, g_y is convex on $[a,b]$ because f is co-ordinated convex on $\triangle$. By Theorem 4, we can write

$$g_y\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\int_{a}^{\frac{a+b}{2}} g_y(x)\ {}_{a}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} g_y(x)\ {}^{b}d_{q_1}x\right] \leq \frac{g_y(a)+g_y(b)}{2}.$$

That is,

$$f\left(\frac{a+b}{2},y\right) \leq \frac{1}{b-a}\left[\int_{a}^{\frac{a+b}{2}} f(x,y)\ {}_{a}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f(x,y)\ {}^{b}d_{q_1}x\right] \leq \frac{f(a,y)+f(b,y)}{2}. \tag{38}$$

q_c-Integrating both sides of (38) over $[c,\frac{c+d}{2}]$ and then dividing by $(d-c)$, we have

$$\begin{aligned}
&\frac{1}{d-c}\int_{c}^{\frac{c+d}{2}} f\left(\frac{a+b}{2},y\right)\ {}_{c}d_{q_2}y\\
&\quad \leq \frac{1}{(b-a)(d-c)}\left[\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}} f(x,y)\ {}_{c}d_{q_2}y\ {}_{a}d_{q_1}x + \int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}} f(x,y)\ {}_{c}d_{q_2}y\ {}^{b}d_{q_1}x\right]\\
&\quad \leq \frac{1}{2(d-c)}\int_{c}^{\frac{c+d}{2}} [f(a,y)+f(b,y)]\ {}_{c}d_{q_2}y.
\end{aligned} \tag{39}$$

Similarly, q^d-integrating both sides of (38) over $[\frac{c+d}{2}, d]$ and then dividing by $(d-c)$, we have

$$\begin{aligned}
&\frac{1}{d-c}\int_{\frac{c+d}{2}}^{d} f\left(\frac{a+b}{2}, y\right)\, {}^{d}d_{q_2}y \\
&\le \frac{1}{(b-a)(d-c)}\left[\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\, {}^{d}d_{q_2}y\, {}_{a}d_{q_1}x + \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\, {}^{d}d_{q_2}y\, {}^{b}d_{q_1}x\right] \\
&\le \frac{1}{2(d-c)}\int_{\frac{c+d}{2}}^{d} [f(a,y)+f(b,y)]\, {}^{d}d_{q_2}y. \qquad (40)
\end{aligned}$$

On the other hand, let $h_x : [c,d] \to \mathbb{R}$ be defined by $h_x(y) = f(x,y)$. Then, h_x is convex on $[c,d]$ since f is co-ordinated convex on $\triangle$. By Theorem 4, we can write

$$h_x\left(\frac{c+d}{2}\right) \le \frac{1}{d-c}\left[\int_{c}^{\frac{c+d}{2}} h_x(y)\, {}_{c}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} h_x(y)\, {}^{d}d_{q_2}y\right] \le \frac{h_x(c)+h_x(d)}{2}.$$

That is,

$$f\left(x, \frac{c+d}{2}\right) \le \frac{1}{d-c}\left[\int_{c}^{\frac{c+d}{2}} f(x,y)\, {}_{c}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f(x,y)\, {}^{d}d_{q_2}y\right] \le \frac{f(x,c)+f(x,d)}{2}. \qquad (41)$$

q_a-Integrating both sides of (41) over $[a, \frac{a+b}{2}]$ and then dividing by $(b-a)$, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_{a}^{\frac{a+b}{2}} f\left(x, \frac{c+d}{2}\right)\, {}_{a}d_{q_1}x \\
&\le \frac{1}{(b-a)(d-c)}\left[\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}} f(x,y)\, {}_{c}d_{q_2}y\, {}_{a}d_{q_1}x + \int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\, {}^{d}d_{q_2}y\, {}_{a}d_{q_1}x\right] \qquad (42) \\
&\le \frac{1}{2(b-a)}\int_{a}^{\frac{a+b}{2}} [f(x,c)+f(x,d)]\, {}_{a}d_{q_1}x.
\end{aligned}$$

Similarly, q^b-integrating both sides of (41) over $[\frac{a+b}{2}, b]$ and then dividing by $(b-a)$, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f\left(x, \frac{c+d}{2}\right)\, {}^{b}d_{q_1}x \\
&\le \frac{1}{(b-a)(d-c)}\left[\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}} f(x,y)\, {}_{c}d_{q_2}y\, {}^{b}d_{q_1}x + \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\, {}^{d}d_{q_2}y\, {}^{b}d_{q_1}x\right] \qquad (43) \\
&\le \frac{1}{2(b-a)}\int_{\frac{a+b}{2}}^{b} [f(x,c)+f(x,d)]\, {}^{b}d_{q_1}x.
\end{aligned}$$

Summing (39), (40), (42), and (43), we derive

$$\begin{aligned}
&\frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}} f\left(x,\frac{c+d}{2}\right)\, {}_a d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f\left(x,\frac{c+d}{2}\right)\, {}^b d_{q_1}x\right] \\
&+ \frac{1}{2(d-c)}\left[\int_c^{\frac{c+d}{2}} f\left(\frac{a+b}{2},y\right)\, {}_c d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f\left(\frac{a+b}{2},y\right)\, {}^d d_{q_2}y\right] \\
&\leq \frac{1}{(b-a)(d-c)}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)\, {}_c d_{q_2}y\, {}_a d_{q_1}x + \int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\, {}^d d_{q_2}y\, {}_a d_{q_1}x\right. \\
&\left. + \int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}} f(x,y)\, {}_c d_{q_2}y\, {}^b d_{q_1}x + \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\, {}^d d_{q_2}y\, {}^b d_{q_1}x\right] \\
&\leq \frac{1}{4(b-a)}\left[\int_a^{\frac{a+b}{2}} [f(x,c)+f(x,d)]\, {}_a d_{q_1}x + \int_{\frac{a+b}{2}}^{b} [f(x,c)+f(x,d)]\, {}^b d_{q_1}x\right] \\
&+ \frac{1}{4(d-c)}\left[\int_c^{\frac{c+d}{2}} [f(a,y)+f(b,y)]\, {}_c d_{q_2}y + \int_{\frac{c+d}{2}}^{d} [f(a,y)+f(b,y)]\, {}^d d_{q_2}y\right].
\end{aligned} \tag{44}$$

Now, the second and the third inequalities of (37) hold.

For the first inequality, by the left hand side of the inequality (38) with $y = \frac{c+d}{2}$, we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f\left(x,\frac{c+d}{2}\right)\, {}_a d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f\left(x,\frac{c+d}{2}\right)\, {}^b d_{q_1}x\right] \tag{45}$$

and by the left hand side of the inequality (41) with $x = \frac{a+b}{2}$, we obtain

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{d-c}\left[\int_c^{\frac{c+d}{2}} f\left(\frac{a+b}{2},y\right)\, {}_c d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f\left(\frac{a+b}{2},y\right)\, {}^d d_{q_2}y\right]. \tag{46}$$

By combining (45)–(46), we obtain the first inequality of (37).

Using the right hand side of (38) with $y = c$ and $y = d$, we obtain

$$\frac{1}{4(b-a)}\left[\int_a^{\frac{a+b}{2}} f(x,c)\, {}_a d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f(x,c)\, {}^b d_{q_1}x\right] \leq \frac{f(a,c)+f(b,c)}{8} \tag{47}$$

and

$$\frac{1}{4(b-a)}\left[\int_a^{\frac{a+b}{2}} f(x,d)\, {}_a d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f(x,d)\, {}^b d_{q_1}x\right] \leq \frac{f(a,d)+f(b,d)}{8}, \tag{48}$$

respectively. Using the right hand side of (41) with $x = a$ and $x = b$, we have

$$\frac{1}{4(d-c)}\left[\int_c^{\frac{c+d}{2}} f(a,y)\, {}_c d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f(a,y)\, {}^d d_{q_2}y\right] \leq \frac{f(a,c)+f(a,d)}{8} \tag{49}$$

and

$$\frac{1}{4(d-c)}\left[\int_c^{\frac{c+d}{2}} f(b,y)\, {}_c d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f(b,y)\, {}^d d_{q_2}y\right] \leq \frac{f(b,c)+f(b,d)}{8}, \tag{50}$$

respectively. Substituting (47)–(50) in (44), we obtain the last inequality of (37). The proof is completed. □

Remark 1. *If we take the limit $q_1 \to 1^-$ and $q_2 \to 1^-$ in Theorem 8 and 9, then Theorem 8 and 9 reduce to Theorem 5.*

4. Examples

Now, we give some examples of our main results to demonstrate our theorems.

Example 5. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a function defined by $f(x,y) = x^2y^2$. Then, f is co-ordinated convex on $[0,1] \times [0,1]$. By applying Theorem 6 with $q_1 = \frac{1}{4}$ and $q_2 = \frac{3}{4}$, the first inequality of (8) becomes*

$$\begin{aligned}
\frac{1}{16} &= \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = f\left(\frac{1}{2},\frac{1}{2}\right) = f\left(\frac{0+1}{2},\frac{0+1}{2}\right) \\
&\leq \frac{1}{(1-0)(1-0)}\Bigg[\int_0^{\frac{0+1}{2}}\int_0^{\frac{0+1}{2}} x^2y^2 \,{}^{\frac{0+1}{2}}d_{\frac{3}{4}}y \,{}^{\frac{0+1}{2}}d_{\frac{1}{4}}x \\
&\quad + \int_0^{\frac{0+1}{2}}\int_{\frac{0+1}{2}}^1 x^2y^2 \,{}_{\frac{0+1}{2}}d_{\frac{3}{4}}y \,{}^{\frac{0+1}{2}}d_{\frac{1}{4}}x + \int_{\frac{0+1}{2}}^1\int_0^{\frac{0+1}{2}} x^2y^2 \,{}^{\frac{0+1}{2}}d_{\frac{3}{4}}y \,{}_{\frac{0+1}{2}}d_{\frac{1}{4}}x \\
&\quad + \int_{\frac{0+1}{2}}^1\int_{\frac{0+1}{2}}^1 x^2y^2 \,{}_{\frac{0+1}{2}}d_{\frac{3}{4}}y \,{}_{\frac{0+1}{2}}d_{\frac{1}{4}}x\Bigg] \\
&= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} x^2y^2 \,{}^{\frac{1}{2}}d_{\frac{3}{4}}y \,{}^{\frac{1}{2}}d_{\frac{1}{4}}x + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 x^2y^2 \,{}_{\frac{1}{2}}d_{\frac{3}{4}}y \,{}^{\frac{1}{2}}d_{\frac{1}{4}}x \\
&\quad + \int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} x^2y^2 \,{}^{\frac{1}{2}}d_{\frac{3}{4}}y \,{}_{\frac{1}{2}}d_{\frac{1}{4}}x + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 x^2y^2 \,{}_{\frac{1}{2}}d_{\frac{3}{4}}y \,{}_{\frac{1}{2}}d_{\frac{1}{4}}x \\
&= \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^n\right)^2\left(1-\left(\frac{3}{4}\right)^m\right)^2 \\
&\quad + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^n\right)^2\left(1+\left(\frac{3}{4}\right)^m\right)^2 \\
&\quad + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(1+\left(\frac{1}{4}\right)^n\right)^2\left(1-\left(\frac{3}{4}\right)^m\right)^2 \\
&\quad + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(1+\left(\frac{1}{4}\right)^n\right)^2\left(1+\left(\frac{3}{4}\right)^m\right)^2 \\
&= \frac{85}{116{,}032} + \frac{11{,}339}{1{,}740{,}480} + \frac{1765}{116{,}032} + \frac{235{,}451}{1{,}740{,}480} = \frac{53}{336}.
\end{aligned}$$

We also have

$$\frac{f(0,0)+f(0,1)+f(1,0)+f(1,1)}{4} = \frac{1}{4}.$$

It is clear that

$$\frac{1}{16} \leq \frac{53}{336} \leq \frac{1}{4},$$

which demonstrates the result described in Theorem 6.

Example 6. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a function defined by $f(x,y) = x^2y^2$. Then, f is co-ordinated convex on $[0,1] \times [0,1]$. By applying Theorem 7 with $q_1 = \frac{1}{4}$ and $q_2 = \frac{3}{4}$, the first inequality of (15) becomes*

$$\begin{aligned}
\frac{1}{16} &= \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = f\left(\frac{1}{2},\frac{1}{2}\right) = f\left(\frac{0+1}{2},\frac{0+1}{2}\right) \\
&\leq \frac{1}{(1-0)(1-0)}\left[\int_0^{\frac{0+1}{2}}\int_0^{\frac{0+1}{2}} x^2y^2 \, {}_0d_{\frac{3}{4}}y \, {}_0d_{\frac{1}{4}}x\right. \\
&\quad + \int_0^{\frac{0+1}{2}}\int_{\frac{0+1}{2}}^1 x^2y^2 \, {}^1d_{\frac{3}{4}}y \, {}_0d_{\frac{1}{4}}x + \int_{\frac{0+1}{2}}^1\int_0^{\frac{0+1}{2}} x^2y^2 \, {}_0d_{\frac{3}{4}}y \, {}^1d_{\frac{1}{4}}x \\
&\quad \left. + \int_{\frac{0+1}{2}}^1\int_{\frac{0+1}{2}}^1 x^2y^2 \, {}^1d_{\frac{3}{4}}y \, {}^1d_{\frac{1}{4}}x\right] \\
&= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} x^2y^2 \, {}_0d_{\frac{3}{4}}y \, {}_0d_{\frac{1}{4}}x + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 x^2y^2 \, {}^1d_{\frac{3}{4}}y \, {}_0d_{\frac{1}{4}}x \\
&\quad + \int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} x^2y^2 \, {}_0d_{\frac{3}{4}}y \, {}^1d_{\frac{1}{4}}x + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 x^2y^2 \, {}^1d_{\frac{3}{4}}y \, {}^1d_{\frac{1}{4}}x \\
&= \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(\frac{1}{4}\right)^{2n}\left(\frac{3}{4}\right)^{2m} \\
&\quad + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\left(\frac{1}{4}\right)^{2n}\left(1-\frac{1}{2}\left(\frac{3}{4}\right)^m\right)^2 \\
&\quad + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\left(1-\frac{1}{2}\left(\frac{1}{4}\right)^n\right)^2\left(\frac{3}{4}\right)^{2m} \\
&\quad + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\left(1-\frac{1}{2}\left(\frac{1}{4}\right)^n\right)^2\left(1-\frac{1}{2}\left(\frac{3}{4}\right)^m\right)^2 \\
&= \frac{4}{777} + \frac{139}{5439} + \frac{41}{3885} + \frac{5699}{108{,}780} = \frac{10{,}187}{108{,}780}.
\end{aligned}$$

We also have

$$\frac{f(0,0) + f(0,1) + f(1,0) + f(1,1)}{4} = \frac{1}{4}.$$

It is clear that

$$\frac{1}{16} \leq \frac{10{,}187}{108{,}780} \leq \frac{1}{4},$$

which demonstrates the result described in Theorem 7.

Example 7. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a function defined by $f(x,y) = x^2y^2$. Then, f is co-ordinated convex on $[0,1] \times [0,1]$. By applying Theorem 8 with $q_1 = \frac{1}{4}$ and $q_2 = \frac{3}{4}$, the first inequality of (23) becomes*

$$\begin{aligned}
\frac{1}{16} &= \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = f\left(\frac{1}{2},\frac{1}{2}\right) = f\left(\frac{0+1}{2},\frac{0+1}{2}\right) \\
&\leq \frac{1}{2(1-0)}\left[\int_0^{\frac{0+1}{2}} f\left(x,\frac{0+1}{2}\right) {}^{\frac{0+1}{2}}d_{\frac{1}{4}}x + \int_{\frac{0+1}{2}}^1 f\left(x,\frac{0+1}{2}\right) {}_{\frac{0+1}{2}}d_{\frac{1}{4}}x\right] \\
&\quad + \frac{1}{2(1-0)}\left[\int_0^{\frac{0+1}{2}} f\left(\frac{0+1}{2},y\right) {}^{\frac{0+1}{2}}d_{\frac{3}{4}}y + \int_{\frac{0+1}{2}}^1 f\left(\frac{0+1}{2},y\right) {}_{\frac{0+1}{2}}d_{\frac{3}{4}}y\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\int_0^{\frac{1}{2}} \frac{x^2}{4}\ {}^{\frac{1}{2}}d_{\frac{1}{4}}x + \int_{\frac{1}{2}}^1 \frac{x^2}{4}\ {}_{\frac{1}{2}}d_{\frac{1}{4}}x + \int_0^{\frac{1}{2}} \frac{y^2}{4}\ {}^{\frac{1}{2}}d_{\frac{3}{4}}y + \int_{\frac{1}{2}}^1 \frac{y^2}{4}\ {}_{\frac{1}{2}}d_{\frac{3}{4}}y\right] \\
&= \frac{1}{2}\left[\frac{3}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\frac{1}{4}\cdot\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^n\right)^2 + \frac{3}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\frac{1}{4}\cdot\frac{1}{4}\left(1+\left(\frac{1}{4}\right)^n\right)^2\right. \\
&\quad \left. + \frac{1}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\frac{1}{4}\cdot\frac{1}{4}\left(1-\left(\frac{3}{4}\right)^n\right)^2 + \frac{1}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\frac{1}{4}\cdot\frac{1}{4}\left(1+\left(\frac{3}{4}\right)^n\right)^2\right] \\
&= \frac{1}{2}\left[\frac{17}{3360}+\frac{353}{3360}+\frac{75}{8288}+\frac{667}{8288}\right] = \frac{1241}{12{,}432}.
\end{aligned}
$$

The third inequality of (23) *becomes*

$$
\begin{aligned}
\frac{53}{336} &= \frac{85}{116{,}032}+\frac{11{,}339}{1{,}740{,}480}+\frac{1765}{116{,}032}+\frac{235{,}451}{1{,}740{,}480} \\
&= \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^n\right)^2\left(1-\left(\frac{3}{4}\right)^m\right)^2 \\
&\quad + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^n\right)^2\left(1+\left(\frac{3}{4}\right)^m\right)^2 \\
&\quad + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(1+\left(\frac{1}{4}\right)^n\right)^2\left(1-\left(\frac{3}{4}\right)^m\right)^2 \\
&\quad + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(1+\left(\frac{1}{4}\right)^n\right)^2\left(1+\left(\frac{3}{4}\right)^m\right)^2 \\
&= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} x^2y^2\ {}^{\frac{1}{2}}d_{\frac{3}{4}}y\ {}^{\frac{1}{2}}d_{\frac{1}{4}}x + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 x^2y^2\ {}_{\frac{1}{2}}d_{\frac{3}{4}}y\ {}^{\frac{1}{2}}d_{\frac{1}{4}}x \\
&\quad + \int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} x^2y^2\ {}^{\frac{1}{2}}d_{\frac{3}{4}}y\ {}_{\frac{1}{2}}d_{\frac{1}{4}}x + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 x^2y^2\ {}_{\frac{1}{2}}d_{\frac{3}{4}}y\ {}_{\frac{1}{2}}d_{\frac{1}{4}}x \\
&= \frac{1}{(1-0)(1-0)}\left[\int_0^{\frac{0+1}{2}}\int_0^{\frac{0+1}{2}} x^2y^2\ {}^{\frac{0+1}{2}}d_{\frac{3}{4}}y\ {}^{\frac{0+1}{2}}d_{\frac{1}{4}}x\right. \\
&\quad + \int_0^{\frac{0+1}{2}}\int_{\frac{0+1}{2}}^1 x^2y^2\ {}_{\frac{0+1}{2}}d_{\frac{3}{4}}y\ {}^{\frac{0+1}{2}}d_{\frac{1}{4}}x + \int_{\frac{0+1}{2}}^1\int_0^{\frac{0+1}{2}} x^2y^2\ {}^{\frac{0+1}{2}}d_{\frac{3}{4}}y\ {}_{\frac{0+1}{2}}d_{\frac{1}{4}}x \\
&\quad \left. + \int_{\frac{0+1}{2}}^1\int_{\frac{0+1}{2}}^1 x^2y^2\ {}_{\frac{0+1}{2}}d_{\frac{3}{4}}y\ {}_{\frac{0+1}{2}}d_{\frac{1}{4}}x\right] \\
&\le \frac{1}{4(1-0)}\left[\int_0^{\frac{0+1}{2}} f(x,0)+f(x,1)\ {}^{\frac{0+1}{2}}d_{\frac{1}{4}}x\right] + \frac{1}{4(1-0)}\left[\int_{\frac{0+1}{2}}^1 f(x,0)+f(x,1)\ {}_{\frac{0+1}{2}}d_{\frac{1}{4}}x\right] \\
&\quad + \frac{1}{4(1-0)}\left[\int_0^{\frac{0+1}{2}} f(0,y)+f(1,y)\ {}^{\frac{0+1}{2}}d_{\frac{3}{4}}y\right] + \frac{1}{4(1-0)}\left[\int_{\frac{0+1}{2}}^1 f(0,y)+f(1,y)\ {}_{\frac{0+1}{2}}d_{\frac{3}{4}}y\right] \\
&= \frac{1}{4}\left[\int_0^{\frac{1}{2}} x^2\ {}^{\frac{1}{2}}d_{\frac{1}{4}}x + \int_{\frac{1}{2}}^1 x^2\ {}_{\frac{1}{2}}d_{\frac{1}{4}}x + \int_0^{\frac{1}{2}} y^2\ {}^{\frac{1}{2}}d_{\frac{3}{4}}y + \int_{\frac{1}{2}}^1 y^2\ {}_{\frac{1}{2}}d_{\frac{3}{4}}y\right] \\
&= \frac{1}{4}\left[\frac{3}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^n\right)^2 + \frac{3}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\frac{1}{4}\left(1+\left(\frac{1}{4}\right)^n\right)^2\right. \\
&\quad \left. + \frac{1}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\frac{1}{4}\left(1-\left(\frac{3}{4}\right)^n\right)^2 + \frac{1}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\frac{1}{4}\left(1+\left(\frac{3}{4}\right)^n\right)^2\right] \\
&= \frac{1}{4}\left[\frac{68}{3360}+\frac{1412}{3360}+\frac{300}{8288}+\frac{2668}{8288}\right] = \frac{1241}{6216}.
\end{aligned}
$$

We also have

$$
\frac{f(0,0)+f(0,1)+f(1,0)+f(1,1)}{4} = \frac{1}{4}.
$$

It is clear that

$$\frac{1}{16} \leq \frac{1241}{12{,}432} \leq \frac{53}{336} \leq \frac{1241}{6216} \leq \frac{1}{4},$$

which demonstrates the result described in Theorem 8.

Example 8. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a function defined by $f(x,y) = x^2y^2$. Then, f is co-ordinated convex on $[0,1] \times [0,1]$. By applying Theorem 9 with $q_1 = \frac{1}{4}$ and $q_2 = \frac{3}{4}$, the first inequality of (37) becomes*

$$\begin{aligned}
\frac{1}{16} &= \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = f\left(\frac{1}{2}, \frac{1}{2}\right) = f\left(\frac{0+1}{2}, \frac{0+1}{2}\right) \\
&\leq \frac{1}{2(1-0)}\left[\int_0^{\frac{0+1}{2}} f\left(x, \frac{0+1}{2}\right) {}_0d_{\frac{1}{4}}x + \int_{\frac{0+1}{2}}^1 f\left(x, \frac{0+1}{2}\right) {}^1d_{\frac{1}{4}}x\right] \\
&\quad + \frac{1}{2(1-0)}\left[\int_0^{\frac{0+1}{2}} f\left(\frac{0+1}{2}, y\right) {}_0d_{\frac{3}{4}}y + \int_{\frac{0+1}{2}}^1 f\left(\frac{0+1}{2}, y\right) {}^1d_{\frac{3}{4}}y\right] \\
&= \frac{1}{2}\left[\int_0^{\frac{1}{2}} \frac{x^2}{4} {}_0d_{\frac{1}{4}}x + \int_{\frac{1}{2}}^1 \frac{x^2}{4} {}^1d_{\frac{1}{4}}x + \int_0^{\frac{1}{2}} \frac{y^2}{4} {}_0d_{\frac{3}{4}}y + \int_{\frac{1}{2}}^1 \frac{y^2}{4} {}^1d_{\frac{3}{4}}y\right] \\
&= \frac{1}{2}\left[\frac{3}{4} \cdot \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \frac{1}{4} \cdot \frac{1}{4}\left(\frac{1}{4}\right)^{2n} + \frac{3}{4} \cdot \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \frac{1}{4}\left(1 - \frac{1}{2}\left(\frac{1}{4}\right)^n\right)^2\right. \\
&\quad \left. + \frac{1}{4} \cdot \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \frac{1}{4} \cdot \frac{1}{4}\left(\frac{3}{4}\right)^{2n} + \frac{1}{4} \cdot \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \frac{1}{4}\left(1 - \frac{1}{2}\left(\frac{3}{4}\right)^n\right)^2\right] \\
&= \frac{1}{2}\left[\frac{3}{126} + \frac{41}{840} + \frac{1}{74} + \frac{139}{2072}\right] = \frac{2381}{31{,}080}.
\end{aligned}$$

The third inequality of (37) *becomes*

$$
\begin{aligned}
\frac{10{,}187}{108{,}780} &= \frac{4}{777} + \frac{139}{5439} + \frac{41}{3885} + \frac{5699}{108{,}780} \\
&= \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(\frac{1}{4}\right)^{2n}\left(\frac{3}{4}\right)^{2m} \\
&\quad + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\left(\frac{1}{4}\right)^{2n}\left(1-\frac{1}{2}\left(\frac{3}{4}\right)^m\right)^2 \\
&\quad + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\left(1-\frac{1}{2}\left(\frac{1}{4}\right)^n\right)^2\left(\frac{3}{4}\right)^{2m} \\
&\quad + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\left(1-\frac{1}{2}\left(\frac{1}{4}\right)^n\right)^2\left(1-\frac{1}{2}\left(\frac{3}{4}\right)^m\right)^2 \\
&= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} x^2y^2 \ {}_0d_{\frac{3}{4}}y \ {}_0d_{\frac{1}{4}}x + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 x^2y^2 \ {}^1d_{\frac{3}{4}}y \ {}_0d_{\frac{1}{4}}x \\
&\quad + \int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} x^2y^2 \ {}_0d_{\frac{3}{4}}y \ {}^1d_{\frac{1}{4}}x + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 x^2y^2 \ {}^1d_{\frac{3}{4}}y \ {}^1d_{\frac{1}{4}}x \\
&= \frac{1}{(1-0)(1-0)}\left[\int_0^{\frac{0+1}{2}}\int_0^{\frac{0+1}{2}} x^2y^2 \ {}_0d_{\frac{3}{4}}y \ {}_0d_{\frac{1}{4}}x\right. \\
&\quad + \int_0^{\frac{0+1}{2}}\int_{\frac{0+1}{2}}^1 x^2y^2 \ {}^1d_{\frac{3}{4}}y \ {}_0d_{\frac{1}{4}}x + \int_{\frac{0+1}{2}}^1\int_0^{\frac{0+1}{2}} x^2y^2 \ {}_0d_{\frac{3}{4}}y \ {}^1d_{\frac{1}{4}}x \\
&\quad \left.+ \int_{\frac{0+1}{2}}^1\int_{\frac{0+1}{2}}^1 x^2y^2 \ {}^1d_{\frac{3}{4}}y \ {}^1d_{\frac{1}{4}}x\right] \\
&\le \frac{1}{4(1-0)}\left[\int_0^{\frac{0+1}{2}}[f(x,0)+f(x,1)] \ {}_0d_{\frac{1}{4}}x\right] + \frac{1}{4(1-0)}\left[\int_{\frac{0+1}{2}}^1[f(x,0)+f(x,1)] \ {}^1d_{\frac{1}{4}}x\right] \\
&\quad + \frac{1}{4(1-0)}\left[\int_0^{\frac{0+1}{2}}[f(0,y)+f(1,y)] \ {}_0d_{\frac{3}{4}}y\right] + \frac{1}{4(1-0)}\left[\int_{\frac{0+1}{2}}^1[f(0,y)+f(1,y)] \ {}^1d_{\frac{3}{4}}y\right] \\
&= \frac{1}{4}\left[\int_0^{\frac{1}{2}} x^2 \ {}_0d_{\frac{1}{4}}x + \int_{\frac{1}{2}}^1 x^2 \ {}^1d_{\frac{1}{4}}x + \int_0^{\frac{1}{2}} y^2 \ {}_0d_{\frac{3}{4}}y + \int_{\frac{1}{2}}^1 y^2 \ {}^1d_{\frac{3}{4}}y\right] \\
&= \frac{1}{4}\left[\frac{3}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\frac{1}{4}\left(\frac{1}{4}\right)^{2n} + \frac{3}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\left(1-\frac{1}{2}\left(\frac{1}{4}\right)^n\right)^2\right. \\
&\quad \left.+ \frac{1}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\frac{1}{4}\left(\frac{3}{4}\right)^{2n} + \frac{1}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\left(1-\frac{1}{2}\left(\frac{3}{4}\right)^n\right)^2\right] \\
&= \frac{1}{4}\left[\frac{12}{126} + \frac{164}{840} + \frac{4}{74} + \frac{556}{2072}\right] = \frac{2381}{15{,}540}.
\end{aligned}
$$

We also have

$$
\frac{f(0,0)+f(0,1)+f(1,0)+f(1,1)}{4} = \frac{1}{4}.
$$

It is clear that

$$
\frac{1}{16} \le \frac{2381}{31{,}080} \le \frac{10{,}187}{108{,}780} \le \frac{2381}{15{,}540} \le \frac{1}{4},
$$

which demonstrates the result described in Theorem 9.

5. Conclusions

In this paper, we proved some new Hermite–Hadamard inequalities for co-ordinated convex functions in q-calculus. We also gave some examples in order to demonstrate our main results. We can extend these results further to another convexities, post-quantum calculus, and fractional calculus. We can also use other techniques to improve the outcomes.

Author Contributions: Conceptualization, F.W. and K.N.; investigation, F.W., K.N., S.K.N., M.Z.S., H.B. and M.A.A.; methodology, F.W., K.N., S.K.N., M.Z.S., H.B. and M.A.A.; validation, F.W., K.N., S.K.N., M.Z.S., H.B. and M.A.A.; visualization, F.W., K.N., S.K.N., M.Z.S., H.B. and M.A.A.; writing—original draft, F.W. and K.N.; writing—review and editing, K.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This research was supported by the Fundamental Fund of Khon Kaen University, Thailand. The first author is supported by the Development and Promotion of Science and Technology talents project (DPST), Thailand. We would like to thank anonymous referees for their comments which are helpful for improvement in this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Jackson, F.H. On a q-definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
2. Jackson, F.H. q-difference equations. *Am. J. Math.* **1910**, *32*, 305–314. [CrossRef]
3. Bangerezaka, G. Variational q-calculus. *J. Math. Anal. Appl.* **2004**, *289*, 650–665. [CrossRef]
4. Annyby, H.M.; Mansour, S.K. *q-Fractional Calculus and Equations*; Springer: Berlin/Helidelberg, Germany, 2012.
5. Ernst, T. *A Comprehensive Treatment of q-Calculus*; Springer: Basel, Switzerland, 2012.
6. Ernst, T. *A History of q-Calculus and a New Method*; UUDM Report; Uppsala University: Uppsala, Sweden, 2000.
7. Gauchman, H. Integral inequalities in q-calculus. *J. Comput. Appl. Math.* **2002**, *47*, 281–300.
8. Kac,V.; Cheung, P. *Quantum Calculus*; Springer: New York, NY, USA, 2002.
9. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.* **2013**, *2013*, 282. [CrossRef]
10. Tariboon, J.; Ntouyas, S.K. Quantum integral inequalities on finite intervals. *J. Inequal. Appl.* **2014**, *2014*, 121. [CrossRef]
11. Kunt, M.; Latif, M.A.; Iscan, I.; Dragomir, S.S. Quantum Hermite-Hadamard type inequality and some estimates of quantum midpoint type inequalities for double integrals. *Sigma J. Eng. Nat. Sci.* **2019**, *37*, 207–223.
12. Kalsoom, H.; Wu, J.D.; Hussain, S.; Latif, M.A. Simpson's type inequalities for co-ordinated convex functions on quantum calculus. *Symmetry* **2019**, *11*, 768. [CrossRef]
13. Kalsoom, H.; Idrees, M.; Baleanu, D.; Chu, Y.-M. New estimates of q_1q_2-Ostrowski-type inequalities within a class of n-polynomial prevexity of functions. *J. Funct. Spaces* **2020**, *2020*, 3720798. [CrossRef]
14. Latif, M.A.; Dragomir, S.S.; Momoniat, E. Some q-analogues of Hermite-Hadamard inequality of functions of two variables on finite rectangles in the plane. *J. King Saud Univ. Sci.* **2017**, *29*, 263–273. [CrossRef]
15. Budak, H.; Ali, M.A.; Tarhanaci, M. Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions. *J. Optim. Theory Appl*. **2020**, *186*, 899–910. [CrossRef]
16. Bermudo, S.; Korus, P.; Valdes, J.E.N. On q-Hermite-Hadamard inequalities for general convex functions. *Acta Math. Hung.* **2020**, *162*, 364–374. [CrossRef]
17. Hermite, C. Sur deux limites d'une integrale de finie. *Mathesis* **1883**, *3*, 82.
18. Hadamard, J. Etude sur les fonctions entiees et en particulier d'une fonction consideree par Riemann. *J. Math. Pures Appl*. **1893**, 58, 171–215.
19. Alp, N.; Sarıkaya, M. Z.; Kunt, M.; İşcan, İ. q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *J. King Saud. Univ. Sci*. **2018**, *30*, 193–203. [CrossRef]
20. Dragomir, S.S. On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwan. J. Math.* **2001**, *5*, 775–788.
21. Latif, M.A.; Alomari, M. Hadamard-type inequalities for product of two convex functions on the co-ordinates. *Int. Math.* **2009**, *4*, 2327–2338.