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An Analytical Approach for Fractional Hyperbolic Telegraph Equation Using Shehu Transform in One, Two and Three Dimensions

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Abstract: In the present research paper, an iterative approach named the iterative Shehu transform method is implemented to solve time-fractional hyperbolic telegraph equations in one, two, and three dimensions, respectively. These equations are the prominent ones in the field of physics and in some other significant problems. The efficacy and authenticity of the proposed method are tested using a comparison of approximated and exact results in graphical form. Both 2D and 3D plots are provided to affirm the compatibility of approximated-exact results. The iterative Shehu transform method is a reliable and efficient tool to provide approximated and exact results to a vast class of ODEs, PDEs, and fractional PDEs in a simplified way, without any discretization or linearization, and is free of errors. A convergence analysis is also provided in this research.

Keywords: fractional calculus; Shehu transform; iterative method; 1D; 2D; 3D fractional hyperbolic telegraph equation; convergence analysis

MSC: 26A33; 42B10; 65B99; 35N30

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1. Introduction

Integral transforms are the need for time to solve mathematical problems efficiently. A suitable selection of integral transforms might be helpful to convert several PDEs as well as fractional PDEs into an algebraic equation, which is easy to tackle. Integral transforms are a simple way to deal with the variety of complex-natured PDEs. In the last few decades, a lot of research work has been done using integral transforms. Several integral transforms are developed, such as: the Sumudu transform, Elzaki transform, Natural transform, Pourreza transform, G-transform, Sawi transform, Shehu transform, and others [1–9]. These transforms provided in the literature are applied to solve several integral equations, ODEs, PDEs, and fractional PDEs [10–17]. Fusion of these transforms with semi-analytical techniques such as ADM, DTM, HPM, and VIM can also create novel and efficient regimes to solve such equations [18–27]. A coupled non-linear Schrodinger-KdV and Maccari system is solved using q-HATM [28,29]. q-HATM is implemented to tackle the fractional telegraph equation using a Laplace transform [30]. Several integral transforms are provided in Table 1. Chart regarding Shehu transform and Inverse Shehu transform are provided via Tables 2 and 3 respectively.

Table 1. Integral transforms.

Integral Transform	Expression	Given by	Related References
Elzaki transform	$E[v] = v\int_0^\infty f(t)e^{-\frac{t}{v}}dt$	T. M. Elzaki	[31–37]
Sumudu transform	$S[f(t)] = \int_0^\infty f(ut)e^{-t}dt$	Watugala	[38–46]
Natural transform	$N[f(t)] = \int_{-\infty}^\infty e^{-st}f(ut)dt$	-	[47–50]
Shehu transform	$S[f(t)] = \int_0^\infty e^{-\frac{st}{v}}f(t)dt$	Shehu Maitama and Weidong Zhao	[51–57]
Sawi transform	$S[f(t)] = \frac{1}{v^2}\int_0^\infty e^{-\frac{t}{v}}f(t)dt$	Abdelrahim	[58–61]
Pourreza transform	$T[f(t)] = v\int_0^\infty e^{-v^2t}f(t)dt$	S. A. Pourreza Ahmadi	[62]
Ara transform	$T[f(t)] = v\int_0^\infty t^{n-1}e^{-vt}f(t)dt$	Rania Saadeh, Ahmad Qazza, Aliaa Burqan	[63]
Laplace transform	$L[f(t)] = \int_0^\infty e^{-st}f(t)dt$	Laplace	[64–66]
Sadik transform	$S[f(t)] = \frac{1}{v^\beta}\int_0^\infty e^{-v^\beta t}f(t)dt$	Sadikali Latif Shaikh	[67–70]

Preliminaries.

Definition 1. *The Shehu transform of Caputo fractional derivative*:

$$S[D_t^\alpha u] = \left(\frac{s}{v}\right)^\alpha S[u] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\alpha-r-1}u^r(0)$$

Definition 2. *The Shehu transform is defined as follows [71]:*

$$S[Q(t)] = \int_0^\infty e^{(-\frac{st}{v})}Q(t)dt$$

where S is considered as the Shehu transform operator.

- *The Shehu transform will be transformed into the Laplace transform by considering* $v = 1$ [71]
- *The Shehu transform will be transformed into the Yang transform by considering* $s = 1$ [71]

Definition 3.

Let $S[Q(t)] = J(s,v)$ *and* $S^{-1}[J(s,v)] = Q(t)$
then $Q(t) = S^{-1}[J(s,v)] = \frac{1}{2\pi i}\int_{-\beta+\infty}^{\beta+\infty}\frac{e^{-st}}{v}J(s,v)ds$
where s, v *are considered as the Shehu transform variables.*

Table 2. Chart regarding the Shehu transform [57].

	$Q(t)$	$S[Q(t)]=J(s,v)$
1.	1	$\frac{v}{s}$
2.	t	$\frac{v^2}{s^2}$
3.	$t^m, m \in N$	$\angle m\left(\frac{v}{s}\right)^{m+1}$
4.	$t^m, m > -1$	$\Gamma(m+1)\left(\frac{v}{s}\right)^{m+1}$
5.	e^{at}	$\frac{v}{s-av}$
6.	$sin(mt)$	$\frac{mv^2}{s^2+m^2v^2}$
7.	$cos(mt)$	$\frac{sv^2}{s^2+m^2v^2}$
8.	$sinh(mt)$	$\frac{mv^2}{s^2-m^2v^2}$
9.	$cosh(mt)$	$\frac{sv^2}{s^2-m^2v^2}$

Table 3. Chart regarding the inverse Shehu transform [19].

	$J(s,\nu)$	$Q(t)=S^{-1}[J(s,\nu)]$
1.	$\frac{\nu}{s}$	1
2.	$\frac{\nu^2}{s^2}$	t
3.	$\left(\frac{\nu}{s}\right)^{m+1}$	$\frac{t^m}{\angle m}$
4.	$\Gamma(m+1)\left(\frac{\nu}{s}\right)^{m+1}$	$\frac{t^m}{\Gamma(m+1)}$
5.	$\frac{\nu}{s-a\nu}$	e^{at}
6.	$\frac{m\nu^2}{s^2+m^2\nu^2}$	$sin(mt)$
7.	$\frac{s\nu^2}{s^2+m^2\nu^2}$	$cos(mt)$
8.	$\frac{m\nu^2}{s^2-m^2\nu^2}$	$sinh(mt)$
9.	$\frac{s\nu^2}{s^2-m^2\nu^2}$	$cosh(mt)$

The notion of fractional calculus is a well-known concept, such as the fractional derivative and fractional integral. A letter was written by L′ Hospital to Leibnitz in 1695 regarding "How do we calculate $\frac{d^n y}{dx^n}$, when$n = \frac{1}{2}$?", the meaning of which is "What will happen if we consider n to be fractional?" The reply of Leibnitz to L′ Hospital was "$d^{\frac{1}{2}}x = x\sqrt{dx} : x$. However, the reply is an apparent paradox; from this apparent paradox, one day, the useful result might be drawn" [72–74]. Afterward, several researchers found numerous applications of fractional calculus in natural science and engineering, such as: signal processing, image processing, viscoelastic materials modeling, random walk, and anomalous diffusion [75–84]. It is cumbersome to fetch the solution of fractional differential equations usually. A great effort has been employed by researchers to develop novel techniques regarding the computation of approximated and exact solutions. In previous years, numerous techniques have been developed to tackle such PDEs, such as: HPM [85], HPSTM [86], HAM [87], ADM [88], RDTM [89], FRDTM [90,91], and VIM [92].

In recent years, fractional PDEs have emerged as the most important topic from the perspective of scientists and researchers due to their applicability in various fields of engineering and science. The degree of flexibility is very high for the fractional derivative in the associated models, which produces an excellent tool for describing the variable history and the hereditary characteristics of the various prototypes. Major scale research is completed to develop the analytical and numerical solutions of linear and non-linear FPDEs.

- **1D fractional hyperbolic telegraph equation** [93].

$$u_t^{\alpha} + \rho u(x,t) + \nu u_t = u_{xx} + g(x,t)$$

where $\rho, \nu \rightarrow$ arbitrary constants. $u(x,t)$ is the unknown function.

If $\rho > 0, \nu = 0$, then the damp wave equation model will be obtained.

If $\rho > 0, \nu > 0$, then the telegraph equation model will be obtained.

The model of the telegraph equation is mainly and mostly used in signal processing for the propagation of transmission of the electric impulses and wave theory process. A series of implementations is noticed of such models in the biomedical sciences and aerospace. The attention of researchers is drawn toward the solution of fractional derivative problems. The linear PDEs of the integer order are a specific model of the fractional-order PDEs. The fractional-order schemes converge to the results of the integer-order regime.

- **2D fractional hyperbolic telegraph equation** [93].

$$D_t^{2\alpha} u + 2\alpha D_t^{\alpha} + \beta^2 u = u_{xx} + u_{yy} + g(x,y,t)$$

I.C.: $u(x,y,0) = f_1(x,y)$ and $u_t(x,y,0) = f_2(x,y)$

- **3D fractional hyperbolic telegraph equation** [93].

$$D_t^{2\alpha}u + 2\alpha D_t^{\alpha} + \beta^2 u = u_{xx} + u_{yy} + u_{zz} + g(x,y,z,t)$$

I.C.: $u(x,y,z,0) = f_1(x,y,z)$ and $u_t(x,y,z,0) = f_2(x,y,z)$

2. Outline of Paper

This research is subdivided into different sections for a better understanding of the projected work.

- In Section 3.1, the general formula is developed regarding the 1D time-fractional HT equation, whereas Sections 3.2 and 3.3 are related to the generalization of the formula of 2D and 3D time-fractional HT equations mentioned in the Appendices A and B.
- In Section 4, seven examples are verified to validate the accuracy and efficacy of the proposed scheme. In this section, the mentioned examples are solved in detail. Examples 1–3 are related to the 1D time-fractional HT equation. Example 4 is related to the notion of the 1D space-fractional HT equation. Example 5 is concerned with the notion of the 1D time-fractional HT equation. Example 6 is provided regarding the 1D time-fractional HT equation. Example 6 is provided regarding the 2D time-fractional HT equation. Example 7 is concerned with the notion of the 3D time-fractional HT equation. For each and every mentioned example, a series solution is developed using the projected scheme.
- In Section 5, a graphical and tabular presentation is provided, along with an error and convergence analysis. Application is also provided.
- In Section 6, the crux of the research is provided as a conclusion.

3. Development of the Formulae

3.1. Implementation of Proposed Regime upon 1D Time-Fractional Hyperbolic Telegraph Equation

1D fractional Hyperbolic telegraph equation is considered as follows [57,93]:

$$D_t^{\alpha}u(x,t) + L[u(x,t)] + N[u(x,t)] = q(x,t)$$

where D_t^{α} is the Caputo derivative. L is the linear operator. N is the non-linear operator.

Apply the Shehu transform upon the 1D time-fractional hyperbolic telegraph equation:

$$S[D_t^{\alpha}u(x,t)]+S[L[u(x,t)] + N[u(x,t)]] = S[q(x,t)]$$

$$\left(\frac{s}{v}\right)^{\alpha} S[u(x,t)] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\alpha-r-1} u^r(x,0) = S[q(x,t)] - S[L[u(x,t)] - S[N[u(x,t)]]]$$

$$\left(\frac{s}{v}\right)^{\alpha} S[u(x,t)] = \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\alpha-r-1} u^r(x,0) + S[q(x,t)] - S[L[u(x,t)] - S[N[u(x,t)]]]$$

$$S[u(x,t)] = \left(\frac{v}{s}\right)^{\alpha}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\alpha-r-1} u^r(x,0) + \left(\frac{v}{s}\right)^{\alpha}[S[q(x,t)] - S[L[u(x,t)] - S[N[u(x,t)]]]]$$

$$\begin{gathered} u(x,t) = S^{-1}\left[\left(\tfrac{v}{s}\right)^{\alpha}\left\{\sum_{r=0}^{\theta-1}\left(\tfrac{s}{v}\right)^{\alpha-r-1}u^r(x,0) + S[q(x,t)]\right\}\right] \\ -S^{-1}\left[\left(\tfrac{v}{s}\right)^{\alpha}[S[L[u(x,t)] + S[N[u(x,t)]]]]\right] \end{gathered}$$

where,

$$L[u] = L\left[\sum_{r=0}^{\infty} u_r(x,t)\right] = L[u_0(x,t)] + \sum_{r=1}^{\infty}\left[L\left(\sum_{j=0}^{r} u_j(x,t)\right) - L\left(\sum_{j=0}^{r-1} u_j(x,t)\right)\right]$$

$$N[u] = N\left[\sum_{r=0}^{\infty} u_r(x,t)\right] = N[u_0(x,t)] + \sum_{r=1}^{\infty}\left[N\left(\sum_{j=0}^{r} u_j(x,t)\right) - N\left(\sum_{j=0}^{r-1} u_j(x,t)\right)\right]$$

Hence,

$$\sum_{k=0}^{\infty} u_k(x,t) = S^{-1}\left[\left(\tfrac{v}{s}\right)^{\alpha}\left\{\sum_{r=0}^{\theta-1}\left(\tfrac{s}{v}\right)^{\alpha-r-1} u^r(x,0) + S[q(x,t)]\right\}\right]$$
$$-S^{-1}\left[\left(\tfrac{v}{s}\right)^{\alpha} S\left\{L[u_0] + N(u_0) + \sum_{r=1}^{\infty} L[u_r(x,t)] + \sum_{r=1}^{\infty} N[u_r(x,t)]\right\}\right]$$

$$\sum_{k=0}^{\infty} u_k(x,t) = S^{-1}\left[\left(\tfrac{v}{s}\right)^{\alpha}\left\{\sum_{r=0}^{\theta-1}\left(\tfrac{s}{v}\right)^{\alpha-r-1} u^r(x,0) + S[q(x,t)]\right\}\right]$$
$$-S^{-1}\left[\left(\tfrac{v}{s}\right)^{\alpha} S\{L[u_0(x,t)] + N[u_0(x,t)]\}\right]$$
$$-S^{-1}\left[\left(\tfrac{v}{s}\right)^{\alpha} S\left\{\sum_{r=1}^{\infty} L[u_r(x,t)] + N\left(\sum_{j=0}^{r} u_r(x,t)\right) - N\left(\sum_{j=0}^{r-1} u_r(x,t)\right)\right\}\right]$$

$$u_0(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\alpha}\left\{\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\alpha-r-1} u^r(x,0) + S[q(x,t)]\right\}\right]$$

$$u_1(x,t) = -S^{-1}\left[\left(\frac{v}{s}\right)^{\alpha} S\{L[u_0(x,t)] + N[u_0(x,t)]\}\right]$$

$$u_{r+1}(x,t) = -S^{-1}\left[\left(\tfrac{v}{s}\right)^{\alpha} S\left\{\sum_{r=1}^{\infty} L[u_r(x,t)] + N\left(\sum_{j=0}^{r} u_r(x,t)\right) - N\left(\sum_{j=0}^{r-1} u_r(x,t)\right)\right\}\right],$$
$$r = 1,2,3,\ldots$$

3.2. *Implementation of Proposed Regime upon 2D Time-Fractional Hyperbolic Telegraph Equation*

2D fractional Hyperbolic telegraph equation is considered as follows:

$$D_t^{\alpha} u(x,y,t) + L[u(x,y,t)] + N[u(x,y,t)] = q(x,y,t)$$

See Appendix A.

3.3. *Implementation of Proposed Regime upon 3D Time-Fractional Hyperbolic Telegraph Equation*

3D fractional Hyperbolic telegraph equation is considered as follows:

$$D_t^{\alpha} u(x,y,z,t) + L[u(x,y,z,t)] + N[u(x,y,z,t)] = q(x,y,z,t)$$

See Appendix B.

4. Examples

Example 1. *Consider the one-dimensional hyperbolic telegraph equation as follows [93]:*

$$u_t^{\alpha} = u - 2u_t - u_{xx} \tag{1}$$

I.C.: $u(x,0) = e^x$ *and* $u_t(x,0) = -2e^x$, $0 < \alpha \le 2$

$$u_0(x,t) = u(x,0) + tu_t(x,0) = e^x - 2te^x = e^x(1-2t)$$

Apply the Shehu transform in Equation (1):

$$S[u_t^{\alpha}] = S[u - 2u_t - u_{xx}] \Rightarrow S[D_t^{\alpha} u(x,t)] = S[u - 2u_t - u_{xx}]$$

$$\Rightarrow u(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\alpha}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\alpha-r-1} u^r(0)\right] + S^{-1}\left[\left(\frac{v}{s}\right)^{\alpha} S[u - 2u_t - u_{xx}]\right]$$

$$u_0(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1}u^r(0)\right] \tag{2}$$

$$u_{r+1}(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}S\{R[u_r]\}\right] \tag{3}$$

where $r = 0,1,2,3,\ldots$

$$R[u(x,t)] = u - 2u_t - u_{xx}, R[u_0(x,t)] = u_0 - 2(u_0)_t - (u_0)_{xx} = 4e^x$$

$$R[u_1(x,t)] = u_1 - 2(u_1)_t - (u_1)_{xx} = -8e^x\alpha\frac{t^{\alpha-1}}{\Gamma(\alpha+1)}$$

$$R[u_2(x,t)] = u_2 - 2(u_2)_t - (u_2)_{xx} = 16e^x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{2\alpha-1}{\Gamma(2\alpha)}t^{2\alpha-2}$$

Consider $\theta = 1$*: From Equation (2):*

$$u_0(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\left(\frac{s}{\nu}\right)^{\alpha-1}u(0)\right] \Rightarrow u_0 = S^{-1}\left[\left(\frac{\nu}{s}\right)u(0)\right] \Rightarrow u_0(x,t) = u(0) = e^x(1-2t)$$

From Equation (3):

$$\begin{gathered}u_1(x,t) = S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}S[R[u_0(x,t)]]\right] \Rightarrow u_1 = S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}S[4e^x]\right] \\ \Rightarrow u_1(x,t) = 4e^xS^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}S[1]\right] \Rightarrow u_1 = 4e^xS^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha+1}\right] \Rightarrow u_1(x,t) \\ = 4e^x\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}\end{gathered}$$

From Equation (3):

$$\begin{gathered}u_2(x,t) = S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}S\left[-8e^x\alpha\tfrac{t^{\alpha-1}}{\Gamma(\alpha+1)}\right]\right] \Rightarrow u_2(x,t) = -8e^x\alpha S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}S\left[\tfrac{t^{\alpha-1}}{\Gamma(\alpha+1)}\right]\right] \\ \Rightarrow u_2(x,t) = -8e^x\tfrac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}\right] \\ \Rightarrow u_2(x,t) = -8e^x\tfrac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\tfrac{t^{2\alpha-1}}{\Gamma(2\alpha)}\end{gathered}$$

From Equation (3):

$$u_3(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}S\left\{16e^x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{(2\alpha-1)}{\Gamma(2\alpha)}t^{2\alpha-2}\right\}\right]$$

$$\Rightarrow u_3(x,t) = 16e^x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{(2\alpha-1)}{\Gamma(2\alpha)}S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}S\left\{t^{2\alpha-2}\right\}\right]$$

$$\Rightarrow u_3(x,t) = 16e^x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{(2\alpha-1)}{\Gamma(2\alpha)}S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\Gamma(2\alpha-1)\left(\frac{\nu}{s}\right)^{2\alpha-1}\right]$$

$$\Rightarrow u_3(x,t) = 16e^x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{(2\alpha-1)\Gamma(2\alpha-1)}{\Gamma(2\alpha)}S^{-1}\left[\left(\frac{\nu}{s}\right)^{3\alpha-1}\right]$$

$$\Rightarrow u_3(x,t) = 16e^x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{(2\alpha-1)\Gamma(2\alpha-1)}{\Gamma(2\alpha)}\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}$$

$$\Rightarrow u_3(x,t) = 16e^x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{(2\alpha-1)\Gamma(2\alpha-1)}{\Gamma(2\alpha)\Gamma(3\alpha-1)}t^{3\alpha-2}$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$\begin{gathered}\Rightarrow u(x,t) = e^x(1-2t) + 4e^x\tfrac{t^{\alpha}}{\Gamma(\alpha+1)} - 8e^x\tfrac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\tfrac{t^{2\alpha-1}}{\Gamma(2\alpha)} \\ +16e^x\tfrac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\tfrac{(2\alpha-1)\Gamma(2\alpha-1)}{\Gamma(2\alpha)\Gamma(3\alpha-1)}t^{3\alpha-2} - \cdots\end{gathered}$$

$$\text{Consider } \alpha = 2\text{: } \Rightarrow u(x,t) = e^x\left[1 - \frac{2t}{\angle 1} + \frac{(2t)^2}{\angle 2} - \frac{(2t)^3}{\angle 3} + \cdots\right] \Rightarrow u = e^{x-2t}$$

Example 2. *Consider the 1D time-fractional hyperbolic telegraph equation as follows [93]:*

$$D_t^\alpha u = u - u_t - u_{xx} \tag{4}$$

I.C.: $u(x,0) = e^x$ *and* $u_t(x,0) = -e^x$*, Where,* $0 < \alpha \le 2$

$$u_0(x,t) = u(x,0) + tu_t(x,0) = e^x - te^x = e^x(1-t)$$

$$R[u(x,t)] = u - u_t - u_{xx}, R[u_0(x,t)] = u_0 - (u_0)_t - (u_0)_{xx} = e^x$$

$$R[u_1(x,t)] = u_1 - (u_1)_t - (u_1)_{xx} = -e^x\alpha\frac{t^{\alpha-1}}{\Gamma(\alpha+1)}$$

$$R[u_2(x,t)] = u_2 - (u_2)_t - (u_2)_{xx} = e^x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{2\alpha-1}{\Gamma(2\alpha)}t^{2\alpha-2}$$

Apply the Shehu transform in Equation (4):

$$S[D_t^\alpha u(x,t)] = S[u - u_t - u_{xx}]$$

$$\Rightarrow \left(\frac{s}{\nu}\right)^\alpha S[u(x,t)] - \sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1}u^r(0) = S[u - u_t - u_{xx}]$$

$$\Rightarrow u(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1}u^r(0)\right] + S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha S[u - u_t - u_{xx}]\right]$$

$$u_0(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1}u^r(0)\right] \tag{5}$$

$$u_{r+1}(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha S\{R[u_r]\}\right] \tag{6}$$

where $r = 0,1,2,3,\ldots$ *Consider* $\theta = 1$*:*

From Equation (5):

$$u_0(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha\left(\frac{s}{\nu}\right)^{\alpha-1}u(0)\right] \Rightarrow u_0 = S^{-1}\left[\left(\frac{\nu}{s}\right)u(0)\right]$$

$$\Rightarrow u_0(x,t) = u(0) = e^x(1-t)$$

From Equation (6):

$$u_1(x,t) = S^{-1}\left[\left(\tfrac{\nu}{s}\right)^\alpha S\{R[u_0(x,t)]\}\right] \Rightarrow u_1(x,t) = S^{-1}\left[\left(\tfrac{\nu}{s}\right)^\alpha S\{e^x\}\right] \Rightarrow u_1 = e^x S^{-1}\left[\left(\tfrac{\nu}{s}\right)^\alpha S\{1\}\right]$$

$$\Rightarrow u_1(x,t) = e^x S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha\left(\frac{\nu}{s}\right)\right] \Rightarrow u_1 = e^x S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha+1}\right] \Rightarrow u_1(x,t) = e^x\frac{t^\alpha}{\Gamma(\alpha+1)}$$

From Equation (6):

$$u_2(x,t) = S^{-1}\left[\left(\tfrac{\nu}{s}\right)^\alpha S\left\{-e^x\alpha\tfrac{t^{\alpha-1}}{\Gamma(\alpha+1)}\right\}\right] \Rightarrow u_2 = -e^x\tfrac{\alpha}{\Gamma(\alpha+1)}S^{-1}\left[\left(\tfrac{\nu}{s}\right)^\alpha S\left\{t^{\alpha-1}\right\}\right]$$

$$\Rightarrow u_2(x,t) = -e^x\frac{\alpha}{\Gamma(\alpha+1)}S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha\Gamma\alpha\left(\frac{\nu}{s}\right)^\alpha\right] \Rightarrow u_2 = -e^x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\right]$$

$$\Rightarrow u_2(x,t) = -e^x \frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)} \frac{t^{2\alpha-1}}{\Gamma(2\alpha)}$$

From Equation (6):

$$u_3(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha S\left\{e^x \frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)} \frac{(2\alpha-1)}{\Gamma(2\alpha)} t^{2\alpha-2}\right\}\right]$$

$$\Rightarrow u_3(x,t) = e^x \frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)} \frac{(2\alpha-1)}{\Gamma(2\alpha)} S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha S\left\{t^{2\alpha-2}\right\}\right]$$

$$\Rightarrow u_3(x,t) = e^x \frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)} \frac{(2\alpha-1)}{\Gamma(2\alpha)} S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha \Gamma(2\alpha-1)\left(\frac{\nu}{s}\right)^{2\alpha-1}\right]$$

$$\Rightarrow u_3(x,t) = e^x \frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)} \frac{(2\alpha-1)\Gamma(2\alpha-1)}{\Gamma(2\alpha)} S^{-1}\left[\left(\frac{\nu}{s}\right)^{3\alpha-1}\right]$$

$$\Rightarrow u_3(x,t) = e^x \frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)} \frac{(2\alpha-1)\Gamma(2\alpha-1)}{\Gamma(2\alpha)} \frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$\begin{gathered}\Rightarrow u(x,t) = e^x(1-t) + e^x \tfrac{t^\alpha}{\Gamma(\alpha+1)} - e^x \tfrac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)} \tfrac{t^{2\alpha-1}}{\Gamma(2\alpha)} \\ +e^x \tfrac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)} \tfrac{(2\alpha-1)\Gamma(2\alpha-1)}{\Gamma(2\alpha)} \tfrac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} - \cdots\end{gathered}$$

Consider $\alpha = 1$*:* $\Rightarrow u(x,t) = e^x\left[1 - \frac{t}{\angle 1} + \frac{t^2}{\angle 2} - \frac{t^3}{\angle 3} + \cdots\right] \Rightarrow u(x,t) = e^{x-t}$.

Example 3. *Consider a 1D non-linear time-fractional hyperbolic telegraph equation as follows [94]:*

$$D_t^\alpha u = u_{xx} + u_t - u^2 + xuu_x \tag{7}$$

where $0 < \alpha \leq 2$.

I.C.:

$$\begin{gathered}u(x,0) = xandu_t(x,0) = x, u_0(x,t) = u(x,0) + tu_t(x,0) = x + xt = x(1+t) \\ R[u(x,t)] = u_{xx} + u_t, N[u(x,t)] = xuu_x - u^2 \\ R[u_0(x,t)] = (u_0)_{xx} + (u_0)_t = x, N[u_0(x,t)] = xu_0(u_0)_x - (u_0)^2 = 0 \\ R[u_1(x,t)] = (u_1)_{xx} + (u_1)_t = x\tfrac{\alpha}{\Gamma(\alpha+1)}t^{\alpha-1}, [u_1(x,t)] = xu_1(u_1)_x - (u_1)^2 = 0 \\ R[u_2(x,t)] = (u_2)_{xx} + (u_2)_t = x\tfrac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\tfrac{(2\alpha-1)}{\Gamma(2\alpha)}t^{2\alpha-2} \\ N[u_2(x,t)] = xu_2(u_2)_x - (u_2)^2 = 0\end{gathered}$$

Apply the Shehu transform upon Equation (7):

$$S[D_t^\alpha u(x,t)] = S\left[u_{xx} + u_t - u^2 + xuu_x\right]$$

$$\Rightarrow \left(\frac{s}{\nu}\right)^\alpha S[u(x,t)] - \sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1} u^r(0) = S\left[u_{xx} + u_t - u^2 + xuu_x\right]$$

$$\Rightarrow \left(\frac{s}{\nu}\right)^\alpha S[u(x,t)] = \sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1} u^r(0) + S\left[u_{xx} + u_t - u^2 + xuu_x\right]$$

$$\Rightarrow u(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha \sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1} u^r(0)\right] + S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha S\left[u_{xx} + u_t - u^2 + xuu_x\right]\right]$$

$$\Rightarrow u(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha \sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1} u^r(0)\right] + S^{-1}\left[\left(\frac{\nu}{s}\right)^\alpha S[R[u] + N[u]]\right]$$

$$\Rightarrow u(x,t) = S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}\sum_{r=0}^{\theta-1}\left(\tfrac{s}{\nu}\right)^{\alpha-r-1}u^{r}(0)\right]$$
$$+S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}S[R[u_0]+N[u_0]]+\sum_{r=1}^{\infty}\left\{R[u_r]+\sum_{j=0}^{n}N\left(u_j\right)-\sum_{j=0}^{n-1}N\left(u_j\right)\right\}\right]$$

$$u_0(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1}u^{r}(0)\right] \tag{8}$$

$$u_1(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}S[R[u_0]+N[u_0]]\right] \tag{9}$$

$$u_{r+1}(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}S\left\{R[u_r]+\sum_{j=0}^{n}N\left(u_j\right)-\sum_{j=0}^{n-1}N\left(u_j\right)\right\}\right] \tag{10}$$

where $r = 0,1,2,3,\ldots$ Consider $\theta = 1$:

From Equation (8):

$$u_0(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\left(\frac{s}{\nu}\right)^{\alpha-1}u(0)\right] \Rightarrow u_0 = S^{-1}\left[\left(\frac{\nu}{s}\right)u(0)\right] \Rightarrow u_0(x,t) = u(0) = x(1+t)$$

From Equation (9):

$$\Rightarrow u_1(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}S[x]\right] \Rightarrow u_1 = xS^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}S[1]\right]$$

$$\Rightarrow u_1(x,t) = xS^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\left(\frac{\nu}{s}\right)\right] \Rightarrow u_1 = xS^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha+1}\right] \Rightarrow u_1(x,t) = x\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

From Equation (10):

$$u_2(x,t) = S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}S[R[u_1(x,t)]+N[u_1(x,t)]]\right] \Rightarrow u_2$$
$$= S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}S\left[x\tfrac{\alpha}{\Gamma(\alpha+1)}t^{\alpha-1}\right]\right]$$

$$\Rightarrow u_2(x,t) = x\frac{\alpha}{\Gamma(\alpha+1)}S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}S\left[t^{\alpha-1}\right]\right]$$

$$\Rightarrow u_2(x,t) = x\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha+1)}S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\left(\frac{\nu}{s}\right)^{\alpha}\right] \Rightarrow u_2(x,t) = x\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha+1)}S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\right]$$

$$\Rightarrow u_2(x,t) = x\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha+1)}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}$$

From Equation (10):

$$u_3(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}S[R[u_2(x,t)]+N[u_2(x,t)]]\right]$$

$$\Rightarrow u_3(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}S\left[x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{(2\alpha-1)}{\Gamma(2\alpha)}t^{2\alpha-2}\right]\right]$$

$$\Rightarrow u_3(x,t) = x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{(2\alpha-1)}{\Gamma(2\alpha)}S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\Gamma(2\alpha-1)\left(\frac{\nu}{s}\right)^{2\alpha-1}\right]$$

$$\Rightarrow u_3(x,t) = x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{(2\alpha-1)\Gamma(2\alpha-1)}{\Gamma(2\alpha)}S^{-1}\left[\left(\frac{\nu}{s}\right)^{3\alpha-1}\right]$$

$$\Rightarrow u_3(x,t) = x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{(2\alpha-1)\Gamma(2\alpha-1)}{\Gamma(2\alpha)}\frac{t^{3\alpha-1}}{\Gamma(3\alpha-1)}$$

$$u(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots$$

$$\Rightarrow u(x,t) = x(1+t) + x\frac{t^{\alpha}}{\Gamma(\alpha+1)} + x\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha+1)}\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}$$
$$+x\frac{\alpha\Gamma\alpha}{\Gamma(\alpha+1)}\frac{(2\alpha-1)\Gamma(2\alpha-1)}{\Gamma(2\alpha)}\frac{t^{3\alpha-1}}{\Gamma(3\alpha-1)} + \cdots$$

Consider $\alpha = 2$: $\Rightarrow u(x,t) = x\left[1 + \frac{t}{\angle 1} + \frac{t^2}{\angle 2} + \frac{t^3}{\angle 3} + \cdots\right] \Rightarrow u(x,t) = xe^t$.

Example 4. *Space-fractional telegraph equation is as follows [95]:*

$$D_x^{\alpha} u = u_{tt} + u_t + u \tag{11}$$

where $0 < \alpha \le 2$,

$$u(0,t) = e^{-t}, u_x(0,t) = e^{-t}, u(x,0) = e^x \Rightarrow u_0(x,t) = e^{-t} + xe^{-t}$$

Taking the Shehu transform upon Equation (11):

$$[D_x^{\alpha} u(x,t)] = S[u_{tt} + u_t + u] \Rightarrow \left(\frac{s}{\nu}\right)^{\alpha} S[u(x,t)]$$
$$-\sum_{n=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1} u^r(0) = S[u_{tt} + u_t + u]$$

$$S \Rightarrow u(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\sum_{n=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1} u^r(0)\right] + S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha} S[u_{tt} + u_t + u]\right]$$

where

$$u_0(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\sum_{n=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1} u^r(0)\right] \tag{12}$$

$$u_{r+1}(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha} S[R(u_r)]\right], r = 0,1,2,3,\ldots \tag{13}$$

where $R[u_r(x,t)] = u_{tt} + u_t + u$, $R[u_0(x,t)] = (u_0)_{tt} + (u_0)_t + u_0 = e^{-t} + xe^{-t}$

$$R[u_1(x,t)] = (u_1)_{tt} + (u_1)_t + u_1 = e^{-t}\left[\frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right]$$

$$R[u_2(x,t)] = (u_2)_{tt} + (u_2)_t + u_2 = e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]$$

From Equation (12): Considering $\theta = 1$:

$$u_0(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\left(\frac{s}{\nu}\right)^{\alpha-1} u(0)\right] \Rightarrow u_0(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right) u(0)\right]$$

$$\Rightarrow u_0(x,t) = u(0) = e^{-t} + xe^{-t}$$

From Equation (13):

$$\Rightarrow u_1(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha} S[R(u_0)]\right] \Rightarrow u_1(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha} S\left[e^{-t} + xe^{-t}\right]\right]$$

$$\Rightarrow u_1(x,t) = e^{-t}S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha} S[1+x]\right] \Rightarrow u_1(x,t) = e^{-t}S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}[S[1] + S[x]]\right]$$

$$\Rightarrow u_1(x,t) = e^{-t}S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\left[\left(\frac{\nu}{s}\right) + \left(\frac{\nu}{s}\right)^2\right]\right] \Rightarrow u_1(x,t) = e^{-t}S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha+1} + \left(\frac{\nu}{s}\right)^{\alpha+2}\right]$$

$$\Rightarrow u_1(x,t) = e^{-t}\left[S^{-1}\left(\frac{\nu}{s}\right)^{\alpha+1} + S^{-1}\left(\frac{\nu}{s}\right)^{\alpha+2}\right] \Rightarrow u_1(x,t) = e^{-t}\left[\frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right]$$

From Equation (13):

$$u_2(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha} S[R(u_1)]\right] \Rightarrow u_2(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha} S\left[e^{-t}\left[\frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right]\right]\right]$$

$$\Rightarrow u_2(x,t) = e^{-t} S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\left[S\left(\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right) + S\left(\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right)\right]\right]$$

$$\Rightarrow u_2(x,t) = e^{-t} S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\left[\left(\frac{\nu}{s}\right)^{\alpha+1} + \left(\frac{\nu}{s}\right)^{\alpha+2}\right]\right]$$

$$\Rightarrow u_2(x,t) = e^{-t}\left[S^{-1}\left(\frac{\nu}{s}\right)^{2\alpha+1} + S^{-1}\left(\frac{\nu}{s}\right)^{2\alpha+2}\right] \Rightarrow u_2(x,t) = e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]$$

From Equation (13):

$$u_3(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha} S[R(u_2)]\right] \Rightarrow u_3(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha} S\left[e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\right]\right]$$

$$\Rightarrow u_3(x,t) = e^{-t} S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha} S\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\right]$$

$$\Rightarrow u_3(x,t) = e^{-t} S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\left[S\left(\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\right) + S\left(\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)\right]\right]$$

$$\Rightarrow u_3(x,t) = e^{-t} S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\left[\left(\frac{\nu}{s}\right)^{2\alpha+1} + \left(\frac{\nu}{s}\right)^{2\alpha+2}\right]\right] \Rightarrow u_3(x,t) = e^{-t} S^{-1}\left[\left(\frac{\nu}{s}\right)^{3\alpha+1} + \left(\frac{\nu}{s}\right)^{3\alpha+2}\right]$$

$$\Rightarrow u_3(x,t) = e^{-t}\left[\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}\right]$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

Considering $\alpha = 1$: $u(x,t) = e^{-t}\left[1 + \frac{x}{\angle 1} + \frac{x^2}{\angle 2} + \frac{x^3}{\angle 3} + \cdots\right] \Rightarrow u(x,t) = e^{x-t}$.

Example 5. *Consider the fractional telegraph equation as follows [23]:*

$$D_t^{2\alpha} u + 2D_t^{\alpha} u + u = u_{xx} \tag{14}$$

where

$$u(x,0) = e^x, u_t(x,0) = -2e^x$$

$$u_0(x,t) = u(x,0) + t u_t(x,0) = e^x + t(-2e^x) = e^x(1-2t)$$

Applying the Shehu transform upon Equation (14):

$$S\left[D_t^{2\alpha} u + 2D_t^{\alpha} u + u\right] = S[u_{xx}] \Rightarrow S\left[D_t^{2\alpha} u(x,t)\right] = -S[2D_t^{\alpha} u + u - u_{xx}]$$

$$\Rightarrow \left(\frac{s}{\nu}\right)^{2\alpha} S[u(x,t)] - \sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{2\alpha-r-1} u^r(0) = -S[2D_t^{\alpha} u + u - u_{xx}]$$

$$\Rightarrow u(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{2\alpha-r-1} u^r(0)\right] - S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S[2D_t^{\alpha} u + u - u_{xx}]\right]$$

where

$$u_0(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{2\alpha-r-1} u^r(0)\right] \tag{15}$$

Considering $\theta = 1 : u_0(x,t) = S^{-1}\left[\left(\frac{\nu}{s}\right) u(0)\right] \Rightarrow u_0(x,t) = u(0) = (1-2t)e^x$

$$u_1(x,t) = -S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S[2D_t^{\alpha}u_0 + u_0 - (u_0)_{xx}]\right] \Rightarrow u_1(x,t) = -S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S[2D_t^{\alpha}u_0]\right]$$
$$\Rightarrow u_1(x,t) = -2S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S[D_t^{\alpha}u_0]\right] \tag{16}$$

where

$$S[D_t^{\alpha}u_0(x,t)] = \left(\frac{s}{\nu}\right)^{\alpha} S[u_0] - \left(\frac{s}{\nu}\right)^{\alpha-1} u_0(x,0)$$
$$\Rightarrow S[D_t^{\alpha}u_0(x,t)] = \left(\frac{s}{\nu}\right)^{\alpha} S[u_0] - \left(\frac{s}{\nu}\right)^{\alpha-1} e^x$$
$$\Rightarrow S[D_t^{\alpha}u_0(x,t)] = \left(\frac{s}{\nu}\right)^{\alpha} e^x\left[\frac{\nu}{s} - 2\left(\frac{\nu}{s}\right)^2\right] - \left(\frac{s}{\nu}\right)^{\alpha-1} e^x \tag{17}$$
$$\Rightarrow S[D_t^{\alpha}u_0(x,t)] = e^x\left[\left(\frac{s}{\nu}\right)^{\alpha-1} - 2\left(\frac{s}{\nu}\right)^{\alpha-2}\right] - \left(\frac{s}{\nu}\right)^{\alpha-1} e^x$$
$$\Rightarrow S[D_t^{\alpha}u_0(x,t)] = -2e^x\left(\frac{s}{\nu}\right)^{\alpha-2}$$

Using Equation (17) in Equation (16): $u_1(x,t) = -2S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\left(-2e^x\left(\frac{s}{\nu}\right)^{\alpha-2}\right)\right]$

$$\Rightarrow u_1(x,t) = 4e^x S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\left(\left(\frac{s}{\nu}\right)^{\alpha-2}\right)\right] \Rightarrow u_1(x,t) = 4e^x S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha+2}\right]$$
$$\Rightarrow u_1(x,t) = 4e^x \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \tag{18}$$
$$\Rightarrow u_2(x,t) = -2S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S[D_t^{\alpha}u_1]\right]$$

where

$$S[D_t^{\alpha}u_1(x,t)] = \left(\frac{s}{\nu}\right)^{\alpha} S[u_1] - \left(\frac{s}{\nu}\right)^{\alpha-1} u_1(x,0)$$
$$\Rightarrow S[D_t^{\alpha}u_1(x,t)] = \left(\frac{s}{\nu}\right)^{\alpha} S[u_1] \Rightarrow S[D_t^{\alpha}u_1(x,t)] = \left(\frac{s}{\nu}\right)^{\alpha} S\left[4e^x\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right]$$
$$\Rightarrow S[D_t^{\alpha}u_1(x,t)] = 4e^x\left(\frac{s}{\nu}\right)^{\alpha} S\left[\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right] \Rightarrow S[D_t^{\alpha}u_1(x,t)] = 4e^x\left(\frac{s}{\nu}\right)^{\alpha}\left(\frac{\nu}{s}\right)^{\alpha+2} \tag{19}$$
$$\Rightarrow S[D_t^{\alpha}u_1(x,t)] = 4e^x\left(\frac{\nu}{s}\right)^2$$

Using Equation (19) in Equation (18):

$$u_2(x,t) = -2S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} 4e^x\left(\frac{\nu}{s}\right)^2\right] \Rightarrow u_2(x,t) = -2(4e^x)S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\left(\frac{\nu}{s}\right)^2\right]$$
$$\Rightarrow u_2(x,t) = -(8e^x)S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha+2}\right]$$
$$\Rightarrow u_2(x,t) = -(8e^x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$
$$u_3(x,t) = -2S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S[D_t^{\alpha}u_2]\right] \tag{20}$$

where

$$S[D_t^{\alpha}u_2(x,t)] = \left(\frac{s}{\nu}\right)^{\alpha} S[u_2] - \left(\frac{s}{\nu}\right)^{\alpha-1} u_2(x,0)$$
$$\Rightarrow S[D_t^{\alpha}u_2(x,t)] = -(8e^x)\left(\frac{s}{\nu}\right)^{\alpha} S\left[\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] \Rightarrow S[D_t^{\alpha}u_2(x,t)] = -(8e^x)\left(\frac{s}{\nu}\right)^{\alpha}\left(\frac{\nu}{s}\right)^{2\alpha+2}$$
$$\Rightarrow S[D_t^{\alpha}u_2(x,t)] = -(8e^x)\left(\frac{\nu}{s}\right)^{\alpha+2} \tag{21}$$

Using Equation (21) in Equation (20):

$$u_3(x,t) = -2S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\left(-(8e^x)\left(\frac{\nu}{s}\right)^{\alpha+2}\right)\right]$$
$$\Rightarrow u_3(x,t) = -2(-(8e^x))S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\left(\left(\frac{\nu}{s}\right)^{\alpha+2}\right)\right]$$
$$\Rightarrow u_3(x,t) = 16e^x S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\left(\left(\frac{\nu}{s}\right)^{\alpha+2}\right)\right] \Rightarrow u_3(x,t) = 16e^x S^{-1}\left[\left(\left(\frac{\nu}{s}\right)^{3\alpha+2}\right)\right]$$

$$\Rightarrow u_3(x,t) = 16e^x \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

Considering $\alpha = 1$: $u(x,t) = e^x[1-2t] + 4e^x \frac{t^2}{\Gamma 3} - 8e^x \frac{t^3}{\Gamma 4} + 16e^x \frac{t^4}{\Gamma 5} - \cdots$

$$\Rightarrow u(x,t) = e^x\left[1 - \frac{2t}{\angle 1} + \frac{(2t)^2}{\angle 2} - \frac{(2t)^3}{\angle 3} + \frac{(2t)^4}{\angle 4} - \cdots\right] \Rightarrow u = e^{x-2t}$$

Example 6. *Consider the 2D time-fractional telegraph equation as follows [23]:*

$$D_t^{2\alpha}u + 3D_t^{\alpha}u + 2u = u_{xx} + u_{yy} \tag{22}$$

where $0 < \alpha \le 1$

I.C.: $u(x,y,0) = e^{x+y}$ *and* $u_t(x,y,0) = -3e^{x+y}$

$$\Rightarrow u_0 = u(x,y,0) + tu_t(x,y,0) = e^{x+y} - 3te^{x+y} = e^{x+y}(1-3t)$$

Applying the Shehu transform in Equation (22):

$$S[D_t^{2\alpha}u + 3D_t^{\alpha}u + 2u] = S[u_{xx} + u_{yy}]$$

$$\Rightarrow S\left[D_t^{2\alpha}u\right] = S\left[u_{xx} + u_{yy} - 2u - 3D_t^{\alpha}u\right] \Rightarrow S\left[D_t^{2\alpha}u\right] = -S\left[3D_t^{\alpha}u + 2u - u_{xx} - u_{yy}\right]$$

$$\Rightarrow u(x,y,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{2\alpha-r-1}u^r(0)\right] - S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}S\left\{3D_t^{\alpha}u + 2u - u_{xx} - u_{yy}\right\}\right]$$

where

$$u_0(x,y,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{2\alpha-r-1}u^r(0)\right]$$

$$u_1(x,y,t) = -S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}S\left\{3D_t^{\alpha}u_0 + 2u_0 - (u_0)_{xx} - (u_0)_{yy}\right\}\right]$$

$$u_2(x,y,t) = -S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}S\left\{3D_t^{\alpha}u_1 + 2u_1 - (u_1)_{xx} - (u_1)_{yy}\right\}\right]$$

$$u_3(x,y,t) = -S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}S\left\{3D_t^{\alpha}u_2 + 2u_2 - (u_2)_{xx} - (u_2)_{yy}\right\}\right]$$

Considering $\theta = 1$: $u_0(x,y,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\left(\frac{s}{\nu}\right)^{2\alpha-1}u(0)\right] \Rightarrow u_0(x,y,t) = u(0) = e^{x+y}(1-3t)$

$$\begin{aligned} u_1(x,y,t) = -S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}S\left\{3D_t^{\alpha}u_0 + 2u_0 - (u_0)_{xx} - (u_0)_{yy}\right\}\right] \\ \Rightarrow u_1(x,y,t) = -S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}S\{3D_t^{\alpha}u_0\}\right] \\ \Rightarrow u_1(x,y,t) = -3S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}S\{D_t^{\alpha}u_0\}\right] \end{aligned} \tag{23}$$

where

$$\begin{aligned} S[D_t^{\alpha}u_0] = \left(\tfrac{s}{\nu}\right)^{\alpha}S[u_0] - \left(\tfrac{s}{\nu}\right)^{\alpha-1}u_0(x,0) \\ \Rightarrow S[D_t^{\alpha}u_0] = \left(\tfrac{s}{\nu}\right)^{\alpha}S[e^{x+y}(1-3t)] - \left(\tfrac{s}{\nu}\right)^{\alpha-1}e^{x+y} \\ \Rightarrow S[D_t^{\alpha}u_0] = \left(\tfrac{s}{\nu}\right)^{\alpha}e^{x+y}\left[\tfrac{\nu}{s} - 3\left(\tfrac{\nu}{s}\right)^2\right] - \left(\tfrac{s}{\nu}\right)^{\alpha-1}e^{x+y} \\ \Rightarrow S[D_t^{\alpha}u_0] = -3\left(\tfrac{\nu}{s}\right)^{2-\alpha}e^{x+y} \end{aligned} \tag{24}$$

Using Equation (24) in Equation (23):

$$\begin{gathered}
u_1(x,y,t) = -3S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}\left\{-3\left(\tfrac{\nu}{s}\right)^{2-\alpha}e^{x+y}\right\}\right] \\
\Rightarrow u_1(x,y,t) = 9e^{x+y}S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}\left\{\left(\tfrac{\nu}{s}\right)^{2-\alpha}\right\}\right] \Rightarrow u_1(x,y,t) = 9e^{x+y}S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha+2}\right] \\
\Rightarrow u_1(x,y,t) = 9e^{x+y}\tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\
u_2(x,y,t) = -S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}S\left\{3D_t^\alpha u_1 + 2u_1 - (u_1)_{xx} - (u_1)_{yy}\right\}\right] \\
\Rightarrow u_2(x,y,t) = -3S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}S\{D_t^\alpha u_1\}\right]
\end{gathered} \tag{25}$$

where

$$\begin{gathered}
S[D_t^\alpha u_1] = \left(\tfrac{s}{\nu}\right)^{\alpha}S[u_1] - \left(\tfrac{s}{\nu}\right)^{\alpha-1}u_1(x,0) \\
\Rightarrow S[D_t^\alpha u_1] = \left(\tfrac{s}{\nu}\right)^{\alpha}S[u_1] \Rightarrow S[D_t^\alpha u_1] = \left(\tfrac{s}{\nu}\right)^{\alpha}S\left[9e^{x+y}\tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right] \\
\Rightarrow S[D_t^\alpha u_1] = 9e^{x+y}\left(\tfrac{s}{\nu}\right)^{\alpha}S\left[\tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right] \Rightarrow S[D_t^\alpha u_1] = 9e^{x+y}\left(\tfrac{s}{\nu}\right)^{\alpha}\left(\tfrac{\nu}{s}\right)^{\alpha+2} \\
\Rightarrow S[D_t^\alpha u_1] = 9e^{x+y}\left(\tfrac{\nu}{s}\right)^{2}
\end{gathered} \tag{26}$$

Using Equation (26) in Equation (25):

$$\begin{gathered}
u_2(x,y,t) = -3S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}\left\{9e^{x+y}\left(\tfrac{\nu}{s}\right)^{2}\right\}\right] \Rightarrow u_2(x,y,t) \\
= -3(9e^{x+y})S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}\left\{\left(\tfrac{\nu}{s}\right)^{2}\right\}\right] \\
\Rightarrow u_2(x,y,t) = -(27e^{x+y})S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}\left\{\left(\tfrac{\nu}{s}\right)^{2}\right\}\right] \Rightarrow u_2(x,y,t) \\
= -(27e^{x+y})S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha+2}\right] \\
\Rightarrow u_2(x,y,t) = -(27e^{x+y})\tfrac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\
u_3(x,y,t) = -S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}S\left\{3D_t^\alpha u_2 + 2u_2 - (u_2)_{xx} - (u_2)_{yy}\right\}\right] \\
\Rightarrow u_3(x,y,t) = -3S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}S\{D_t^\alpha u_2\}\right]
\end{gathered} \tag{27}$$

where

$$\begin{gathered}
S[D_t^\alpha u_2] = \left(\tfrac{s}{\nu}\right)^{\alpha}S[u_2] - \left(\tfrac{s}{\nu}\right)^{\alpha-1}u_2(x,0) \\
\Rightarrow S[D_t^\alpha u_2] = \left(\tfrac{s}{\nu}\right)^{\alpha}S[u_2] \Rightarrow S[D_t^\alpha u_2] = \left(\tfrac{s}{\nu}\right)^{\alpha}S\left[-(27e^{x+y})\tfrac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] \\
\Rightarrow S[D_t^\alpha u_2] = -(27e^{x+y})\left(\tfrac{\nu}{s}\right)^{\alpha+2}
\end{gathered} \tag{28}$$

Using Equation (28) in Equation (27):

$$u_3(x,y,t) = -3S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\left\{-(27e^{x+y})\left(\frac{\nu}{s}\right)^{\alpha+2}\right\}\right] \Rightarrow u_3(x,y,t) = 81e^{x+y}\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$\begin{gathered}
u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t) + \cdots \\
\Rightarrow u(x,y,t) = e^{x+y}(1-3t) + 9e^{x+y}\tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)} - (27e^{x+y})\tfrac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\
+81e^{x+y}\tfrac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - \cdots
\end{gathered}$$

Considering $\alpha = 1$*:*

$$u(x,y,t) = e^{x+y}\left[1 - \frac{3t}{\angle 1} + \frac{(3t)^2}{\angle 2} - \frac{(3t)^3}{\angle 3} + \cdots\right] \Rightarrow u(x,y,t) = e^{x+y}e^{-3t} = e^{x+y-3t}$$

Example 7. *Consider the 3D time-fractional telegraph equation as follows [23]:*

$$D_t^{2\alpha}u + 2D_t^\alpha u + 3u = u_{xx} + u_{yy} + u_{zz} \tag{29}$$

where $0 < \alpha \leq 1$.

$$\textbf{\textit{I.C.:}}\ u(x,y,z,0) = sinhx\ sinhy\ sinhz,\ u_t(x,y,z,0) = -sinhx\ sinhy\ sinhz$$

$$u_0 = u(x,y,z,0) + tu_t(x,y,z,0) = (1-t)sinhx\ sinhy\ sinhz$$

Applying the Shehu transform in Equation (29):

$$S\left[D_t^{2\alpha}u + 2D_t^{\alpha}u + 3u\right] = S[u_{xx} + u_{yy} + u_{zz}] \Rightarrow S\left[D_t^{2\alpha}u\right] = -S\left[2D_t^{\alpha}u + 3u - u_{xx} - u_{yy} - u_{zz}\right]$$

$$\Rightarrow \left(\frac{s}{\nu}\right)^{2\alpha} S[u(x,y,z,t)] - \sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{2\alpha-r-1} u^r(0) = -S\left[2D_t^{\alpha}u + 3u - u_{xx} - u_{yy} - u_{zz}\right]$$

$$\Rightarrow S[u(x,y,z,t)] = \left(\frac{\nu}{s}\right)^{2\alpha}\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{2\alpha-r-1} u^r(0) - \left(\frac{\nu}{s}\right)^{2\alpha} S\left[2D_t^{\alpha}u + 3u - u_{xx} - u_{yy} - u_{zz}\right]$$

$$\Rightarrow u = S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{2\alpha-r-1} u^r(0)\right] - S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S\left[2D_t^{\alpha}u + 3u - u_{xx} - u_{yy} - u_{zz}\right]\right]$$

$$u_0(x,y,z,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{2\alpha-r-1} u^r(0)\right]$$

$$u_1(x,y,z,t) = -S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S\left[2D_t^{\alpha}u_0 + 3u_0 - (u_0)_{xx} - (u_0)_{yy} - (u_0)_{zz}\right]\right]$$

$$u_2(x,y,z,t) = -S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S\left[2D_t^{\alpha}u_1 + 3u_1 - (u_1)_{xx} - (u_1)_{yy} - (u_1)_{zz}\right]\right]$$

$$u_3(x,y,z,t) = -S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S\left[2D_t^{\alpha}u_2 + 3u_2 - (u_2)_{xx} - (u_2)_{yy} - (u_2)_{zz}\right]\right]$$

Considering $\theta = 1$: $u_0(x,y,z,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\left(\frac{s}{\nu}\right)^{2\alpha-1}u(0)\right] \Rightarrow u_0(x,y,z,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)u(0)\right]$

$$\begin{gathered}\Rightarrow u_0(x,y,z,t) = u(0) = sinhx\ sinhy\ sinhz(1-t)\\ u_1(x,y,z,t) = -S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S\left[2D_t^{\alpha}u_0 + 3u_0 - (u_0)_{xx} - (u_0)_{yy} - (u_0)_{zz}\right]\right]\\ \Rightarrow u_1(x,y,z,t) = -2S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S[D_t^{\alpha}u_0]\right]\end{gathered} \tag{30}$$

where $S[D_t^{\alpha}u_0] = \left(\frac{s}{\nu}\right)^{\alpha} S[u_0] - \left(\frac{s}{\nu}\right)^{\alpha-1} u_0(x,0)$

$$\begin{gathered}\Rightarrow S[D_t^{\alpha}u_0] = \left(\frac{s}{\nu}\right)^{\alpha} S[sinhx\ sinhy\ sinhz(1-t)] - \left(\frac{s}{\nu}\right)^{\alpha-1} sinhx\ sinhy\ sinhz\\ \Rightarrow S[D_t^{\alpha}u_0] = sinhx\ sinhy\ sinhz\left[\left(\frac{s}{\nu}\right)^{\alpha} S[(1-t)] - \left(\frac{s}{\nu}\right)^{\alpha-1}\right]\\ \Rightarrow S[D_t^{\alpha}u_0] = sinhx\ sinhy\ sinhz\left[\left(\frac{s}{\nu}\right)^{\alpha-1} - \left(\frac{s}{\nu}\right)^{\alpha-2} - \left(\frac{s}{\nu}\right)^{\alpha-1}\right]\\ \Rightarrow S[D_t^{\alpha}u_0] = -\left(\frac{s}{\nu}\right)^{\alpha-2} sinhx\ sinhy\ sinhz\end{gathered} \tag{31}$$

Using Equation (31) in Equation (30):

$$u_1(x,y,z,t) = -2S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\left\{-\left(\frac{s}{\nu}\right)^{\alpha-2} sinhx\ sinhy\ sinhz\right\}\right]$$

$$\Rightarrow u_1(x,y,z,t) = 2sinhx\ sinhy\ sinhz\ S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha+2}\right]$$

$$\Rightarrow u_1(x,y,z,t) = 2sinhx\ sinhy\ sinhz\ \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$u_2(x,y,z,t) = -S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S\left[2D_t^{\alpha}u_1 + 3u_1 - (u_1)_{xx} - (u_1)_{yy} - (u_1)_{zz}\right]\right]$$

$$u_2(x,y,z,t) = -2S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha} S[D_t^\alpha u_1]\right] \tag{32}$$

where

$$\begin{aligned} S[D_t^\alpha u_1] = \left(\tfrac{s}{\nu}\right)^\alpha S[u_1] - \left(\tfrac{s}{\nu}\right)^{\alpha-1} u_1(x,0) \Rightarrow S[D_t^\alpha u_1] = \left(\tfrac{s}{\nu}\right)^\alpha S[u_1] \\ \Rightarrow S[D_t^\alpha u_1] = \left(\tfrac{s}{\nu}\right)^\alpha S\left[2\, sinhx\ sinhy\ sinhz\ \tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right] \\ \Rightarrow S[D_t^\alpha u_1] = 2sinhx\ sinhy\ sinhz\ \left(\tfrac{s}{\nu}\right)^\alpha \left(\tfrac{\nu}{s}\right)^{\alpha+2} \\ \Rightarrow S[D_t^\alpha u_1] = 2sinhx\ sinhy\ sinhz\ \left(\tfrac{\nu}{s}\right)^2 \end{aligned} \tag{33}$$

Using Equation (33) in Equation (32):

$$\begin{aligned} u_2(x,y,z,t) = -2S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha}\left\{2sinhx\ sinhy\ sinhz\left(\tfrac{\nu}{s}\right)^2\right\}\right] \\ \Rightarrow u_2(x,y,z,t) = -(4sinhx\ sinhy\ sinhz)S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha+2}\right] \\ \Rightarrow u_2(x,y,z,t) = -(4sinhx\ sinhy\ sinhz)\tfrac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ u_3(x,y,z,t) = -S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha} S\left[2D_t^\alpha u_2 + 3u_2 - (u_2)_{xx} - (u_2)_{yy} - (u_2)_{zz}\right]\right] \\ \Rightarrow u_3(x,y,z,t) = -2S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{2\alpha} S[D_t^\alpha u_2]\right] \end{aligned} \tag{34}$$

where

$$\begin{aligned} S[D_t^\alpha u_2] = \left(\tfrac{s}{\nu}\right)^\alpha S[u_2] - \left(\tfrac{s}{\nu}\right)^{\alpha-1} u_2(x,0) \Rightarrow S[D_t^\alpha u_2] = \left(\tfrac{s}{\nu}\right)^\alpha S[u_2] \\ \Rightarrow S[D_t^\alpha u_2] = -(4sinhx\ sinhy\ sinhz)\left(\tfrac{s}{\nu}\right)^\alpha \left(\tfrac{\nu}{s}\right)^{2\alpha+2} \\ \Rightarrow S[D_t^\alpha u_2] = -(4sinhx\ sinhy\ sinhz\ )\left(\tfrac{\nu}{s}\right)^{\alpha+2} \end{aligned} \tag{35}$$

Using Equation (35) in Equation (34):

$$\Rightarrow u_3(x,y,z,t) = -2S^{-1}\left[\left(\frac{\nu}{s}\right)^{2\alpha}\left\{-(4sinhx\ sinhy\ sinhz\ )\left(\frac{\nu}{s}\right)^{\alpha+2}\right\}\right]$$

$$\Rightarrow u_3(x,y,z,t) = (8sinhx\ sinhy\ sinhz)S^{-1}\left[\left(\frac{\nu}{s}\right)^{3\alpha+2}\right]$$

$$\Rightarrow u_3(x,y,z,t) = (8sinhx\ sinhy\ sinhz)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$u(x,y,z,t) = u_0(x,y,z,t) + u_1(x,y,z,t) + u_2(x,y,z,t) + u_3(x,y,z,t) + \cdots$$

$$\begin{aligned} \Rightarrow u(x,y,z,t) = sinhx\ sinhy\ sinhz(1-t) + 2sinhx\ sinhy\ sinhz\tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ -(4sinhx\ sinhy\ sinhz)\tfrac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ +(8sinhx\ sinhy\ sinhz)\tfrac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - \cdots \end{aligned}$$

Considering $\alpha = 1$*:* $\Rightarrow u(x,y,z,t) = sinhx\ sinhy\ sinhz\left[1 - t + \frac{2t^2}{\angle 2} - \frac{4t^3}{\angle 3} + \frac{8t^4}{\angle 4} - \cdots\right]$

5. Graphical and Tabular Discussion

In Figure 1, approx. and exact results are matched at t = 1, 2, and 3 regarding Example 1. In Figure 2, a comparison of approx. and exact results is given at t = 1, 2, and 3 regarding Example 2. In Figure 3, a comparison of approx. and exact profiles is provided at t = 1, 2, and 3 regarding Example 3. In Figure 4, approx. and exact profiles are compared at t = 1, 2, and 3 regarding Example 4. In Figure 5, approx. and exact profiles are matched at t = 1, 2, and 3 regarding Example 5. In Figure 6, contour and surface representations are provided at t = 1 regarding Example 6. In Figure 7, contour and surface representations are provided at t = 2 regarding Example 6. In Figure 8, contour and surface representations are provided at t = 3 regarding Example 6 With the aid of an analysis of the figures, it can be affirmed that the proposed regime is providing the good compatibility of the approx. and exact profiles for a wide range of time levels.

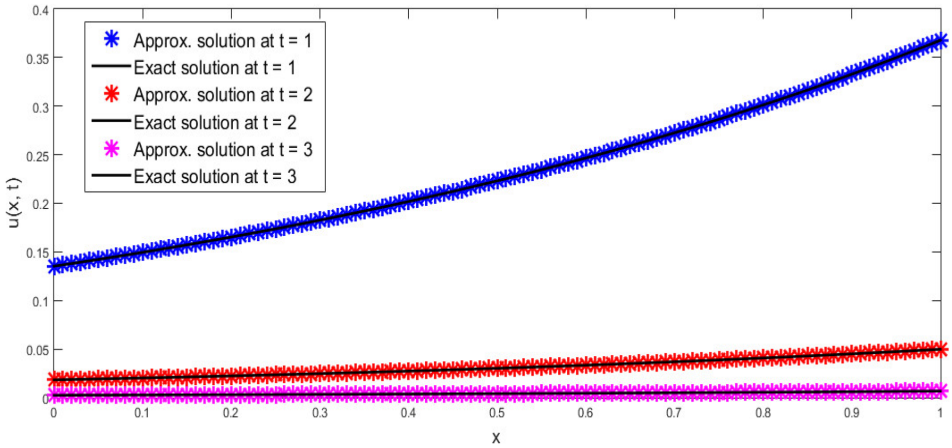

Figure 1. Comparison of approx. and exact profiles at t = 1, 2, 3 regarding Example 1.

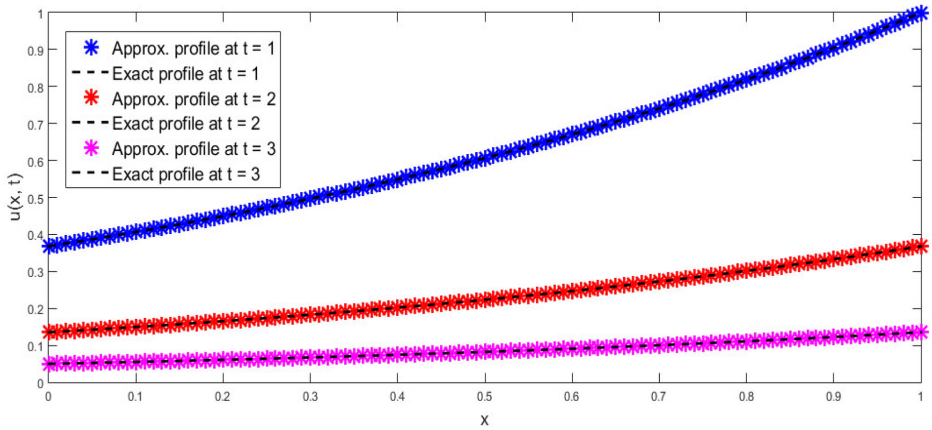

Figure 2. Comparison of approx. and exact profiles at t = 1, 2, 3 regarding Example 2.

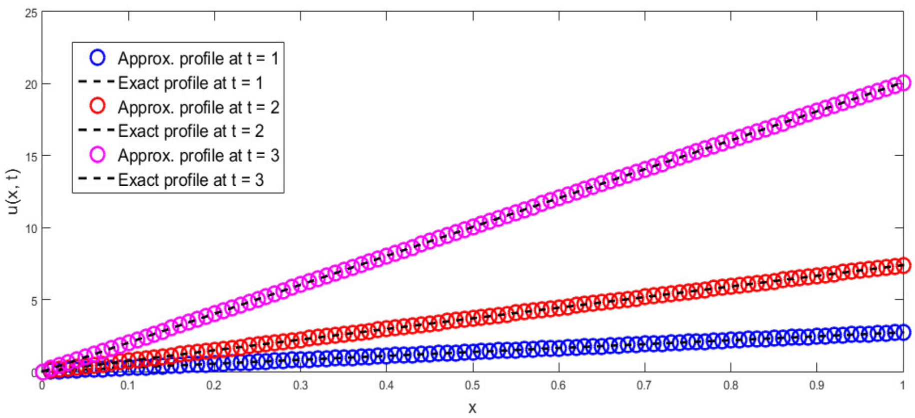

Figure 3. Comparison of approx. and exact profiles at t = 1, 2, 3 regarding Example 3.

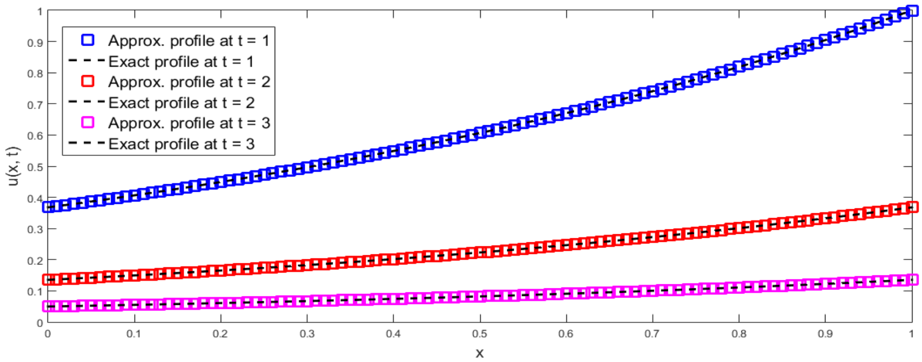

Figure 4. Comparison of approx. and exact profiles at $t = 1, 2, 3$ regarding Example 4.

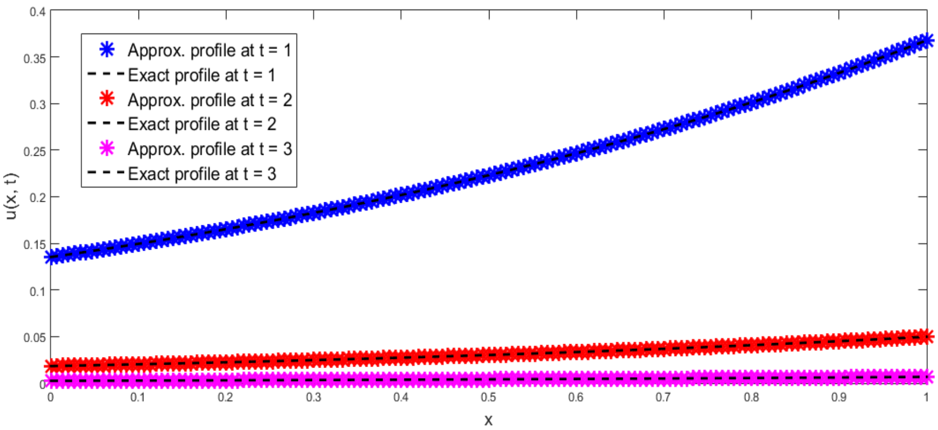

Figure 5. Comparison of approx. and exact profiles at $t = 1, 2, 3$ regarding Example 5.

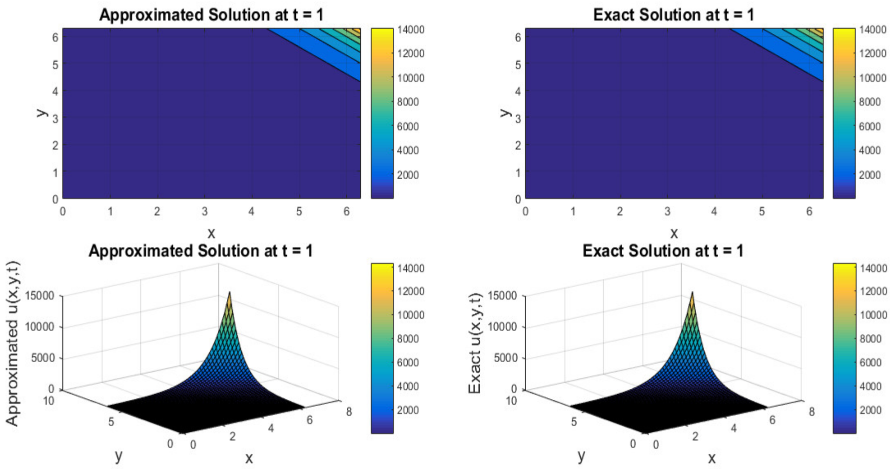

Figure 6. Comparison of approx. and exact profiles at $t = 1$ regarding Example 6.

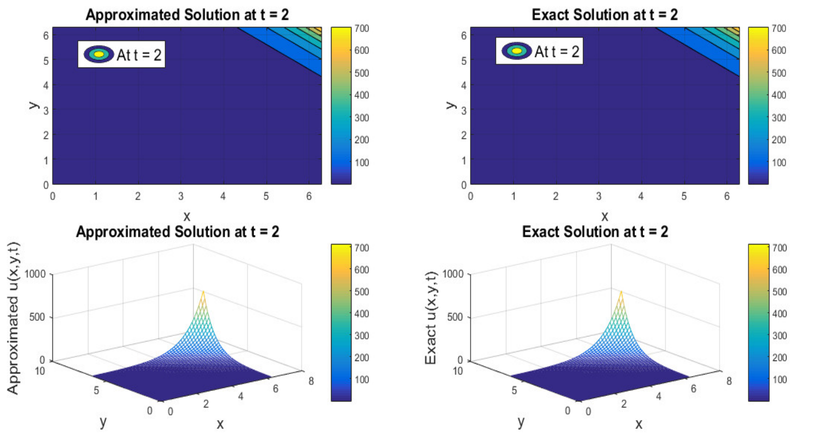

Figure 7. Comparison of approx. and exact profiles at t = 2 regarding Example 6.

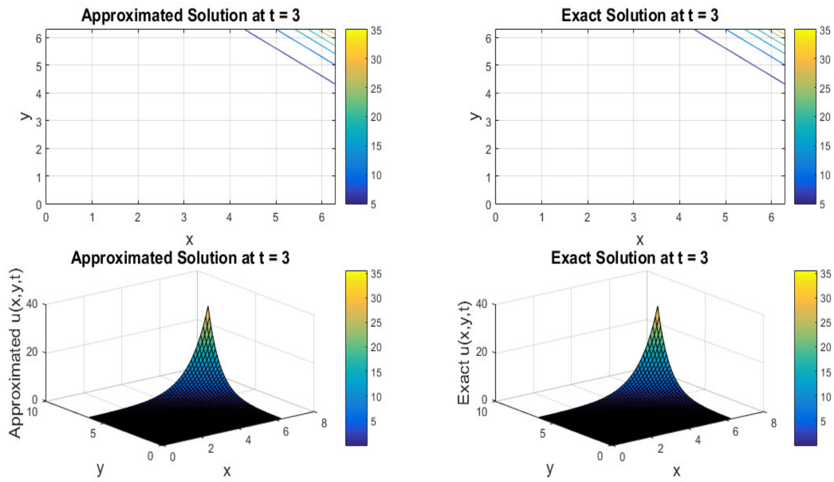

Figure 8. Comparison of approx. and exact profiles at t = 3 regarding Example 6.

In Table 4, the L_∞ error was provided at diverse time levels. It is noticeable that at each time level, the obtained error got reduced by increasing the value of N, which is a parameter of the convergence of the proposed scheme. In Table 5, L_∞ errors were fetched at t = 1.1, 1.2, and 1.3, where, upon increasing the value of N, obtained errors got reduced up to 10^{-16}. L_∞ errors were obtained at t = 1.1, 1.2, and 1.3, and on changing the value of N from 10 to 20, errors got reduced significantly up to 10^{-16}. In Table 6, the L_∞ error was evaluated at t = 1.1, 1.2, and 1.3, upon changed values of N, errors got reduced up to 10^{-15}. In Table 7, the L_∞ error was calculated at t = 0.1, 0.2, and 0.3; with the change in the value of N, the error got reduced up to 10^{-11}. Therefore, it is affirmed that the proposed regime is generating the convergent solution with a higher and acceptable order of convergence for a wide range of time levels.

Table 4. Error analysis regarding Example 1.

N	L_∞ **Error**		
	t **= 1**	t **= 1.3**	t **= 1.5**
10	1.3701×10^{-3}	1.8034×10^{-2}	7.3219×10^{-2}
20	3.1701×10^{-12}	5.8706×10^{-10}	1.0099×10^{-8}
30	2.2204×10^{-16}	-3.4417×10^{-15}	7.2164×10^{-16}
	↓ Convergenceupto10^{-16}	↓ Convergenceupto10^{-15}	↓ Convergenceupto10^{-16}

Table 5. Error analysis regarding Example 2 and 4.

N	L_∞ **Error**		
	t **= 1.1**	t**= 1.2**	t**= 1.3**
10	3.7337×10^{-6}	8.8389×10^{-6}	1.9517×10^{-5}
20	4.4409×10^{-16}	4.4409×10^{-16}	4.4409×10^{-16}

Table 6. Error analysis regarding Example 5.

N	L_∞ **Error**		
	t**= 1.1**	t**= 1.2**	t**= 1.3**
10	3.4987×10^{-3}	8.2241×10^{-3}	1.8034×10^{-2}
20	2.1144×10^{-11}	1.1944×10^{-10}	5.8706×10^{-10}
30	1.3323×10^{-15}	-1.4433×10^{-15}	-3.4417×10^{-15}

Table 7. Error analysis regarding Example 6.

N	L_∞ **Error**		
	t**= 0.1**	t**= 0.2**	t**= 0.3**
10	4.5419×10^{-7}	4.5299×10^{-4}	2.5457×10^{-2}
20	8.7311×10^{-11}	7.2760×10^{-11}	4.3656×10^{-11}

Application of the proposed regime [93].

Considered an infinitesimal piece of the telegraph cable wire as an electrical circuit [Figure 9], and consider that the cable has the perfect insulation, so that the capacitor and leakage to the floor are present. C is the capacitance to the ground; x is the distance from the end of cable; $u(x,t)$ is the voltage; G is the inductance; $i(x,t)$ is the current; L is the inductance of the cable.

Fractional derivative model equations are [93]:

$$c^2 D_x^\delta i = D_t^\beta i + (\theta + \phi) D_t^\alpha i + \theta\phi i$$

and

$$c^2 D_x^\delta u = D_t^\beta u + (\theta + \phi) D_t^\alpha u + \theta\phi u$$

and where $0 < \alpha \leq 1, 1 < \delta, \beta \leq 2$.

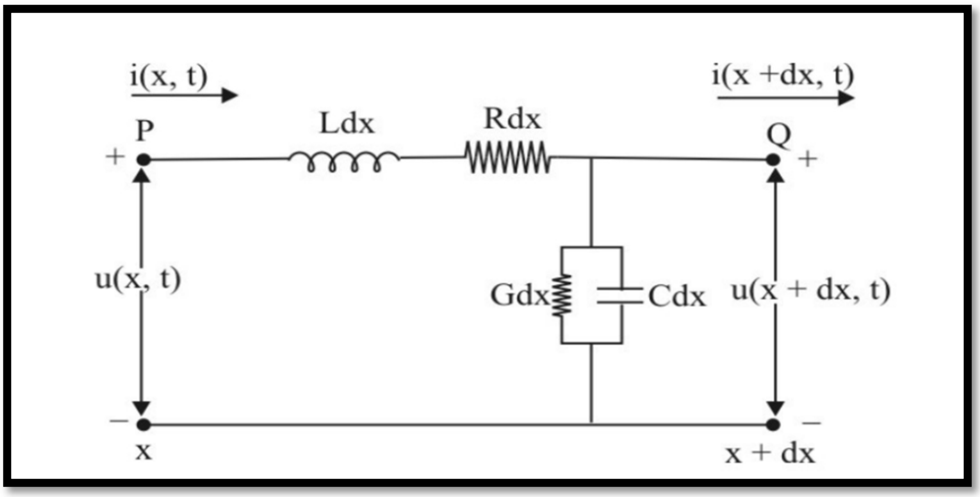

Figure 9. Telegraph transmission line with leakage.

6. Conclusions

In the present paper, the general formulae are provided to obtain the approximated and exact solutions of the HT equation in 1D, 2D, and 3D using the iterative Shehu transform method. With the aid of graphical representation, it is claimed that a good compatibility of results is obtained for a wide range of time levels. In the tabular discussion, it is noticed that on changing the value of grid points, the obtained L_∞ error was reduced up to a high and acceptable order of convergence. With the aid of the tabular form, the convergence of the proposed regime was claimed. The present regime might be one of the most useful regimes to tackle fractional differential equations and partial-integro differential equations.

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Appendix A

Implementation of proposed regime upon 2D time-fractional hyperbolic telegraph equation.

Applying the Shehu transform upon a 2D time-fractional hyperbolic telegraph equation:

$$S[D_t^\alpha u(x,y,t)]+S[L[u(x,y,t)]+N[u(x,y,t)]]=S[q(x,y,t)]$$

$$\left(\frac{s}{\nu}\right)^\alpha S[u(x,y,t)]-\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1}u^r(x,y,0)=S[q(x,y,t)]-S[L[u(x,y,t)]-S[N[u(x,y,t)]]$$

$$u(x,y,t) = S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}\left\{\sum_{r=0}^{\theta-1}\left(\tfrac{s}{\nu}\right)^{\alpha-r-1}u^{r}(x,y,0)+S[q(x,y,t)]\right\}\right] \\ -S^{-1}[\left(\tfrac{\nu}{s}\right)^{\alpha}[S[L[u(x,y,t)]+S[N[u(x,y,t)]]]]]$$

where, $L[u] = L[\sum_{r=0}^{\infty}u_r(x,y,t)] = L[u_0(x,y,t)] + \sum_{r=1}^{\infty}[L\left(\sum_{j=0}^{r}u_j(x,y,t)\right) - L\left(\sum_{j=0}^{r-1}u_j(x,y,t)\right)]$

$$N[u] = N\left[\sum_{r=0}^{\infty}u_r(x,y,t)\right] = N[u_0(x,y,t)] + \sum_{r=1}^{\infty}\left[N\left(\sum_{j=0}^{r}u_j(x,y,t)\right) - N\left(\sum_{j=0}^{r-1}u_j(x,y,t)\right)\right]$$

Hence,

$$\sum_{k=0}^{\infty}u_k(x,y,t) = S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}\left\{\sum_{r=0}^{\theta-1}\left(\tfrac{s}{\nu}\right)^{\alpha-r-1}u^{r}(x,y,0)+S[q(x,y,t)]\right\}\right] \\ -S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}S\{L[u_0(x,y,t)]+N[u_0(x,y,t)]\}\right] \\ -S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}S\left\{\sum_{r=1}^{\infty}L[u_r(x,y,t)]+N\left(\sum_{j=0}^{r}u_r(x,y,t)\right)-N\left(\sum_{j=0}^{r-1}u_r(x,y,t)\right)\right\}\right]$$

$$u_0(x,y,t) = S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}\left\{\sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1}u^{r}(x,y,0)+S[q(x,y,t)]\right\}\right]$$

$$u_1(x,y,t) = -S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}S\{L[u_0(x,y,t)]+N[u_0(x,y,t)]\}\right]$$

$$u_{r+1}(x,y,t) = -S^{-1}\left[\left(\frac{\nu}{s}\right)^{\alpha}S\left\{\sum_{r=1}^{\infty}L[u_r(x,y,t)]+N\left(\sum_{j=0}^{r}u_r(x,y,t)\right)-N\left(\sum_{j=0}^{r-1}u_r(x,y,t)\right)\right\}\right], r=1,2,3,$$

Appendix B

Implementation of proposed regime upon 3D time-fractional hyperbolic telegraph equation.

Applying the Shehu transform upon the 3D time-fractional hyperbolic telegraph equation:

$$S[D_t^{\alpha}u(x,y,z,t)]+S[L[u(x,y,z,t)]+N[u(x,y,z,t)]] = S[q(x,y,z,t)]$$

$$\left(\frac{s}{\nu}\right)^{\alpha}S[u(x,y,z,t)] = \sum_{r=0}^{\theta-1}\left(\frac{s}{\nu}\right)^{\alpha-r-1}u^{r}(x,y,z,0)+S[q(x,y,z,t)]-S[L[u(x,y,z,t)]-S[N[u(x,y,z,t)]]$$

$$u(x,y,z,t) = S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}\left\{\sum_{r=0}^{\theta-1}\left(\tfrac{s}{\nu}\right)^{\alpha-r-1}u^{r}(x,y,z,0)+S[q(x,y,z,t)]\right\}\right] \\ -S^{-1}[\left(\tfrac{\nu}{s}\right)^{\alpha}[S[L[u(x,y,z,t)]+S[N[u(x,y,z,t)]]]]]$$

where, $L[u] = L[\sum_{r=0}^{\infty}u_r(x,y,z,t)] = L[u_0(x,y,z,t)] + \sum_{r=1}^{\infty}[L\left(\sum_{j=0}^{r}u_j(x,y,z,t)\right) - L\left(\sum_{j=0}^{r-1}u_j(x,y,z,t)\right)]$

$$N[u] = N\left[\sum_{r=0}^{\infty}u_r(x,y,z,t)\right] = N[u_0(x,y,z,t)] + \sum_{r=1}^{\infty}\left[N\left(\sum_{j=0}^{r}u_j(x,y,z,t)\right) - N\left(\sum_{j=0}^{r-1}u_j(x,y,z,t)\right)\right]$$

Hence,

$$\sum_{k=0}^{\infty}u_k(x,y,z,t) = S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}\left\{\sum_{r=0}^{\theta-1}\left(\tfrac{s}{\nu}\right)^{\alpha-r-1}u^{r}(x,y,z,0)+S[q(x,y,z,t)]\right\}\right] \\ -S^{-1}\left[\left(\tfrac{\nu}{s}\right)^{\alpha}S\left\{L[u_0]+N(u_0)+\sum_{r=1}^{\infty}L[u_r(x,y,z,t)]+\sum_{r=1}^{\infty}N[u_r(x,y,z,t)]\right\}\right]$$

$$\sum_{k=0}^{\infty} u_k(x,y,z,t) = S^{-1}\left[\left(\tfrac{v}{s}\right)^{\alpha}\left\{\sum_{r=0}^{\theta-1}\left(\tfrac{s}{v}\right)^{\alpha-r-1}u^r(x,y,z,0) + S[q(x,y,z,t)]\right\}\right]$$
$$-S^{-1}\left[\left(\tfrac{v}{s}\right)^{\alpha}S\{L[u_0(x,y,z,t)] + N[u_0(x,y,z,t)]\}\right]$$
$$-S^{-1}\left[\left(\tfrac{v}{s}\right)^{\alpha}S\left\{\sum_{r=1}^{\infty} L[u_r(x,y,z,t)] + N\left(\sum_{j=0}^{r} u_r(x,y,z,t)\right) - N\left(\sum_{j=0}^{r-1} u_r(x,y,z,t)\right)\right\}\right]$$

$$u_0(x,y,z,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\alpha}\left\{\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\alpha-r-1}u^r(x,y,z,0) + S[q(x,y,z,t)]\right\}\right]$$

$$u_1(x,y,z,t) = -S^{-1}\left[\left(\frac{v}{s}\right)^{\alpha}S\{L[u_0(x,y,z,t)] + N[u_0(x,y,z,t)]\}\right]$$

$$u_{r+1}(x,y,z,t) = -S^{-1}\left[\left(\tfrac{v}{s}\right)^{\alpha}S\left\{\sum_{r=1}^{\infty} L[u_r(x,y,z,t)] + N\left(\sum_{j=0}^{r} u_r(x,y,z,t)\right) - N\left(\sum_{j=0}^{r-1} u_r(x,y,z,t)\right)\right\}\right], \quad r = 1,2,3,\ldots$$

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