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Ćirić-Type Operators and Common Fixed Point Theorems

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Abstract: In the context of a complete metric space, we will consider the common fixed point problem for two self operators. The operators are assumed to satisfy a general contraction type condition inspired by the Ćirić fixed point theorems. Under some appropriate conditions we establish existence, uniqueness and approximation results for the common fixed point. In the same framework, the second problem is to study various stability properties. More precisely, we will obtain sufficient conditions assuring that the common fixed point problem is well-posed and has the Ulam–Hyers stability, as well as the Ostrowski property for the considered problem. Some examples and applications are finally given in order to illustrate the abstract theorems proposed in the first part of the paper. Our results extend and complement some theorems in the recent literature.

Keywords: metric space; fixed point; common fixed point; pair of Ćirić-type operators; well-posedness; Ulam–Hyers stability; Ostrowski property

MSC: 47H10; 54H25

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1. Introduction and Preliminaries

Let (X, d) be a metric space and $f : X \to X$ be an operator. Throughout this paper we denote by $Fix(f) := \{x \in X : x = f(x)\}$ the fixed point set of f and by $Graph(f) := \{(x, f(x)) : x \in X\}$ the graph of the operator f.

In this context, let $f : X \to X$ be an α-contraction, in the sense that there exists $\alpha \in (0, 1)$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y), \text{ for all } (x, y) \in X \times X.$$

The Banach–Caccioppoli Contraction Principle states that if (X, d) is a complete metric space, then any α-contraction $f : X \to X$ has a unique fixed point, and the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ of Picard iterates starting from any element $x \in X$ converges to the unique fixed point.

If the operator $f : X \to X$ satisfies the above condition for every $(x, y) \in Graph(f)$, then f is called a graph α-contraction.

It is also known that any graph α-contraction $f : X \to X$ on a complete metric space (X, d) that has a closed graph (i.e., the set $Graph(f)$ is closed) has at least one fixed point and, for each $x \in X$, the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ of Picard iterates converges to a fixed point of f.

The conclusions of the above two fixed point theorems generated the following two important notions.

Let (X, d) be a metric space and $f : X \to X$ be an operator. Then, by definition, f is called a weakly Picard operator if the sequence $(f^n(x))_{n\in\mathbb{N}}$ of Picard iterates, starting from any point $x \in X$, converges to a fixed point $x^*(x)$ of f. If, in particular, in the above definition f has a unique fixed point, then f is called a Picard operator. By the definition of a weakly Picard operator, the following set retraction is generated $f^\infty : X \to Fix(f)$, $f^\infty(x) = \lim_{n\to+\infty} f^n(x)$.

If $f : X \to X$ is a weakly Picard operator for which there exists $c > 0$ such that

$$d(x, f^\infty(x)) \leq cd(x, f(x)), \text{ for all } x \in X,$$

then f is called a weakly c-Picard operator. If a Picard operator satisfies the above condition (with $f^\infty(x) = \{x^*\}, x \in X$), then f is called a c-Picard operator.

It is easy to see that, in the context of a complete metric space, any self graph α-contraction with a closed graph is a weakly $\frac{1}{1-k}$-Picard operator, while any α-contraction is a $\frac{1}{1-k}$-Picard operator.

The following general result will be very useful in applications, see [1].

Theorem 1. *Let (X, d) be a metric space and $f : X \to X$ be a c-Picard operator with $x^* \in X$ its unique fixed point. Then:*

(a) the fixed point equation $x = f(x)$ is well-posed in the sense of Reich and Zaslawski (see [2]), i.e., $Fix(f) = \{x^*\}$ and for any sequence $(v_n)_{n\in\mathbb{N}} \subset X$ with $d(v_n, f(v_n)) \to 0$ as $n \to +\infty$, we have that $v_n \to x^*$ as $n \to +\infty$;

(b) the fixed point equation $x = f(x)$ is Ulam–Hyers stable, i.e., there exists $C > 0$ such that for every $\varepsilon > 0$ and for every $\tilde{x} \in X$ satisfying $d(\tilde{x}, f(\tilde{x})) \leq \varepsilon$, we have that $d(x^*, \tilde{x}) \leq C\varepsilon$.

Remark 1. *Let (X, d) be a metric space and $f : X \to X$ be a weakly c-Picard operator. Then, the fixed point equation $x = f(x)$ is Ulam–Hyers stable.*

For our next result, we recall the notion of quasi-contraction. Let (X, d) be a metric space and $f : X \to X$ be an operator such that $Fix(f) = \{x^*\}$. Then, f is said to be a β-quasi-contraction if $\beta \in]0, 1[$ and

$$d(f(x), x^*) \leq \beta d(x, x^*), \text{ for every } x \in X.$$

The concept was extended by I.A. Rus [3] to the case of weakly Picard operators as follows. Let (X, d) be a metric space and $f : X \to X$ be a weakly Picard operator. Then, f is said to be a γ-quasi-contraction if $\gamma \in]0, 1[$ and

$$d(f(x), f^\infty(x)) \leq \gamma d(x, f^\infty(x)), \text{ for every } x \in X.$$

Theorem 2. *Let (X, d) be a metric space and $f : X \to X$ be a β-quasi-contraction such that $Fix(f) = \{x^*\}$. Then, f has the Ostrowski stability property, i.e., $Fix(f) = \{x^*\}$ and any sequence $(w_n)_{n\in\mathbb{N}}$ in X with $d(w_{n+1}, f(w_n)) \to 0$ has the property that $w_n \to x^*$ as $n \to +\infty$.*

For other details on Picard and weakly Picard operator theory, see [4–6] and the references therein. For the above concepts and for related notions and results, see [7–12].

If (X, d) is a metric space and $f, g : X \to X$ are two operators, then a common fixed point for f and g is an element $x^* \in X$ with the property $x^* = f(x^*) = g(x^*)$. The common fixed point set for f and g is denoted by $ComFix(f, g)$. Notice that $ComFix(f, g) = Fix(f) \cap Fix(g)$. In the paper [1] (see also [13]), the following open problems are given:

Suppose there exists $\alpha \in]0, 1[$ such that, for every $x, y \in X$ we have

$$d(f(x), g(y)) \leq \alpha \max\{d(x, y), d(x, f(x)), d(y, g(y)), \frac{1}{2}[d(x, g(y)) + d(y, f(x))]\}.$$

I. Does the above metric condition of Ćirić (see [14]) on f and g imply all the following conclusions:

1. $Fix(f^n) = Fix(g^n) = \{x^*\}$, for $n \in \mathbb{N}^*$;

2. for each $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_{2n+1} = f(x_{2n}),\ x_{2n+2} = g(x_{2n+1}),$$

converges to x^*;
3. for each $y_0 \in X$, the sequence $(y_n)_{n\in\mathbb{N}}$ defined by

$$y_{2n+1} = g(y_{2n}),\ y_{2n+2} = f(y_{2n+1}),$$

converges to x^*;
4. for each $x_0 \in X$, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x^*;
5. for each $x_0 \in X$, the sequence $(g^n(x_0))_{n\in\mathbb{N}}$ converges to x^*.

II. Under which additional conditions, some other stability properties can be obtained?

In this work, we establish existence, uniqueness and approximation results for the common fixed point. In the same framework, we will obtain sufficient conditions assuring that the common fixed point problem is well-posed and has the Ulam–Hyers stability, as well as the Ostrowski property for the considered problem. Some examples and applications are finally given in order to illustrate the abstract theorems proposed in the first part of the paper. Our results extend and complement some theorems in the recent literature [1,14–20].

2. Main Results

Our first main result is the following common fixed point theorem for a pair of Ćirić-type operators.

Theorem 3. *Let (X,d) be a complete metric space and $f,g : X \to X$ be two operators for which there exists $\alpha \in (0,1)$ such that, for each $x,y \in X$, the following condition holds:*

$$d(f(x),g(y)) \le \alpha \max\{d(x,y), d(x,f(x)), d(y,g(y)), \frac{1}{2}[d(x,g(y)) + d(y,f(x))]\}. \quad (1)$$

Then we have the following conclusions:

(a) $ComFix(f,g) = Fix(f) = Fix(g) = \{x^*\}$;

(b) for every $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ given by

$$x_{2n+1} = f(x_{2n}), \quad x_{2n+2} = g(x_{2n+1}), \quad \text{for all } n \in \mathbb{N},$$

converges to x^* as $n \to +\infty$;

(c) for every $y_0 \in X$, the sequence $(y_n)_{n\in\mathbb{N}}$ given by

$$y_{2n+1} = g(y_{2n}), \quad y_{2n+2} = f(y_{2n+1}), \quad \text{for all } n \in \mathbb{N},$$

converges to x^* as $n \to +\infty$;

(d) if $\alpha < \sqrt{5} - 2$, then f and g are graph contractions;

(e) if $\alpha < \frac{1}{3}$, then f and g are quasi-contractions;

(f) if $\alpha < \sqrt{5} - 2$, then f and g are c-Picard operators, with $c := \frac{(1-\alpha)^2}{1-4\alpha-\alpha^2}$;

(g) if $\alpha < \sqrt{5} - 2$, then the fixed point equation $x = f(x)$ and the fixed point equation $x = g(x)$ are well-posed in the sense of Reich and Zaslavski;

(h) if $\alpha < \sqrt{5} - 2$, then the fixed point equation $x = f(x)$ and the fixed point equation $x = g(x)$ are Ulam–Hyers stable;

(i) if $\alpha < \frac{1}{3}$, then f and g have the Ostrowski stability property.

Proof. (a) Let us prove that $Fix(f) = Fix(g)$. Let us consider first $x^* \in Fix(f)$. Then, by (1), we have

$$d(x^*, g(x^*)) = d(f(x^*), g(x^*)) \le \alpha d(x^*, g(x^*)).$$

Thus $x^* \in Fix(g)$.

We will now prove that f and g have at most one fixed point. Indeed, if we suppose that $x^*, y^* \in ComFix(f,g) = F_f \cap F_g$, then: Then, by (1), we have

$$d(x^*, y^*) = d(f(x^*), g(y^*)) \leq \alpha \max\{d(x^*, y^*)\}.$$

Hence $d(x^*, y^*) = 0$.

(b) For arbitrary $x_0 \in X$ we consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined in (b). Then

$$\begin{aligned} d(x_1, x_2) = d(f(x_0), g(x_1)) &\leq \alpha \max\{d(x_0, x_1), d(x_0, f(x_0)), d(x_1, g(x_1)), \frac{1}{2}d(x_0, g(x_1))\} \\ &= \alpha \max\{d(x_0, x_1), d(x_1, x_2), \frac{1}{2}d(x_0, x_2)\} \\ &\leq \alpha \max\{d(x_0, x_1), d(x_1, x_2), \frac{1}{2}(d(x_0, x_1) + d(x_1, x_2))\} \\ &= \alpha \max\{d(x_0, x_1), d(x_1, x_2)\}. \end{aligned}$$

Since $\alpha < 1$, we obtain that

$$d(x_1, x_2) \leq \alpha d(x_0, x_1).$$

By induction, we get

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1), \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Using the above expression, we get that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $x^* \in X$ be its limit. We have

$$\begin{aligned} d(x^*, f(x^*)) &\leq d(x^*, x_{2n+2}) + d(x_{2n+2}, f(x^*)) \\ &\leq d(x^*, x_{2n+2}) + d(g(x_{2n+1}), f(x^*)) \\ &\leq d(x^*, x_{2n}) + \alpha \max\{d(x^*, x_{2n+1}), d(x^*, f(x^*)), d(x_{2n+1}, g(x_{2n+1})), \\ &\quad \frac{1}{2}(d(x^*, g(x_{2n+1})) + d(x_{2n+1}, f(x^*)))\} \\ &= d(x^*, x_{2n}) + \alpha \max\{d(x^*, x_{2n+1}), d(x^*, f(x^*)), d(x_{2n+1}, x_{2n+2}), \\ &\quad \frac{1}{2}(d(x^*, x_{2n+2}) + d(x_{2n+1}, f(x^*)))\}. \end{aligned}$$

Letting $n \to +\infty$, we get $d(x^*, f(x^*)) \leq \alpha d(x^*, f(x^*))$, which yields that $x^* = f(x^*)$. Moreover, by (2), we obtain

$$d(x_n, x_{n+p}) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1), \quad \text{for all } n \in \mathbb{N},\ p \in \mathbb{N}^*.$$

Letting $p \to +\infty$ and taking $n = 0$, we obtain:

$$d(x_0, x^*) \leq \frac{1}{1-\alpha} d(x_0, x_1) = \frac{1}{1-\alpha} d(x_0, f(x_0)), \tag{3}$$

which is a retraction-displacement-type condition, see [4].

(c) Consider $y_0 \in X$ arbitrary chosen and the sequence $(y_n)_{n\in\mathbb{N}}$ defined as in (c). Thus

$$\begin{aligned} d(y_1, y_2) = d(g(y_0), f(y_1)) &\leq \alpha \max\{d(y_1, y_0), d(y_1, f(y_1)), d(y_0, g(y_0)), \frac{1}{2}d(y_0, f(y_1))\} \\ &= \alpha \max\{d(y_0, y_1), d(y_1, y_2), \frac{1}{2}d(y_0, y_2)\} \\ &\leq \alpha \max\{d(y_0, y_1), d(y_1, y_2), \frac{1}{2}(d(y_0, y_1) + d(y_1, y_2))\} \end{aligned}$$

$$= \alpha \max\{d(y_0, y_1), d(y_1, y_2)\}.$$

Since $\alpha < 1$, we deduce that

$$d(y_1, y_2) \leq \alpha d(y_0, y_1).$$

By induction, we obtain

$$d(y_n, y_{n+1}) \leq \alpha^n d(y_0, y_1), \quad \text{for all } n \in \mathbb{N}. \tag{4}$$

By (4), it results that the sequence $(y_n)_{n\in\mathbb{N}}$ is Cauchy, hence convergent in (X, d). Let $y^* \in X$ be its limit. We have

$$\begin{aligned} d(y^*, g(y^*)) &\leq d(y^*, y_{2n+2}) + d(y_{2n+2}, g(y^*)) \\ &\leq d(y^*, y_{2n+2}) + d(f(y_{2n+1}), g(y^*)) \\ &\leq d(y^*, y_{2n}) + \alpha \max\{d(y^*, y_{2n+1}), d(y^*, g(y^*)), d(y_{2n+1}, f(y_{2n+1})), \\ &\quad \frac{1}{2}(d(y^*, f(y_{2n+1})) + d(y_{2n+1}, g(y^*)))\} \\ &= d(y^*, y_{2n}) + \alpha \max\{d(y^*, y_{2n+1}), d(y^*, g(y^*)), d(y_{2n+1}, y_{2n+2}), \\ &\quad \frac{1}{2}(d(y^*, y_{2n+2}) + d(y_{2n+1}, g(y^*)))\} \end{aligned}$$

Letting $n \to +\infty$, we get $d(y^*, g(y^*)) \leq \alpha d(y^*, g(y^*))$ and then $y^* = g(y^*)$. Since $ComFix(f, g) = Fix(f) = Fix(g)$ we get that $y^* = x^*$.

On the other hand, by (4), we obtain

$$d(y_n, y_{n+p}) \leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1), \quad \text{for all } n \in \mathbb{N},\ p \in \mathbb{N}^*.$$

Letting $p \to +\infty$ and considering $n = 0$, we obtain again a retraction-displacement-type condition:

$$d(y_0, x^*) \leq \frac{1}{1-\alpha} d(y_0, y_1) = \frac{1}{1-\alpha} d(y_0, f(y_0)). \tag{5}$$

(d) We will show now that f is a graph contraction. Let $x \in X$ be arbitrary chosen. We have

$$\begin{aligned} d(f^2(x), f(x)) &\leq d(f^2(x), g(x^*)) + d(f(x), g(x^*)) \\ &\leq \alpha \max\{d(f(x), x^*), d(f(x), f^2(x)), \frac{1}{2}(d(f(x), g(x^*)) + d(x^*, f^2(x)))\} \\ &\quad + \alpha \max\{d(x, x^*), d(x, f(x)), \frac{1}{2}(d(x, g(x^*)) + d(x^*, f(x)))\} \\ &= \alpha \max\{d(f(x), x^*), d(f(x), f^2(x)), \frac{1}{2}(d(f(x), x^*) + d(x^*, f^2(x)))\} \\ &\quad + \alpha \max\{d(x, x^*), d(x, f(x)), \frac{1}{2}(d(x, x^*) + d(x^*, f(x)))\} \\ &\leq \alpha \max\{d(f(x), x^*), d(f(x), f^2(x)), \frac{1}{2}(2d(f(x), x^*) + d(f(x), f^2(x)))\} \\ &\quad + \alpha \max\{d(x, x^*), d(x, f(x)), \frac{1}{2}(2d(x, x^*) + d(x, f(x)))\} \\ &\leq \alpha \left[d(f(x), x^*) + d(f(x), f^2(x))\right] + \alpha[d(x, x^*) + d(x, f(x))]. \end{aligned}$$

Thus,

$$d(f^2(x), f(x)) \leq \frac{\alpha}{1-\alpha}[d(x, x^*) + d(f(x), x^*) + d(x, f(x))]. \tag{6}$$

On the other hand, by the above relations, we have

$$d(f(x), x^*) = d(f(x), g(x^*)) \leq \alpha[d(x, x^*) + d(x, f(x))] \tag{7}$$

and, then we get

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), x^*) \leq d(x, f(x)) + \alpha[d(x, x^*) + d(x, f(x))].$$

From the last relation, we deduce

$$d(x, x^*) \leq \frac{1+\alpha}{1-\alpha} d(x, f(x)). \tag{8}$$

Using (8) in (7) we obtain that

$$d(f(x), x^*) \leq \alpha[d(x, x^*) + d(x, f(x))]$$
$$\leq \frac{\alpha(1+\alpha)}{1-\alpha} d(x, f(x)) + \alpha d(x, f(x)) = \frac{2\alpha}{1-\alpha} d(x, f(x)).$$

Hence

$$d(f(x), x^*) \leq \frac{2\alpha}{1-\alpha} d(x, f(x)). \tag{9}$$

Now, using (8) and (9) in (6) we conclude that

$$d(f^2(x), f(x)) \leq \frac{2\alpha(1+\alpha)}{(1-\alpha)^2} d(x, f(x)).$$

Since

$$\Gamma := \frac{2\alpha(1+\alpha)}{(1-\alpha)^2} < 1$$

we get the desired conclusion.

(e) By (a) we know that f and g have a unique fixed point. We prove that f is quasi-contraction. Indeed, we have

$$d(f(x), x^*) = d(f(x), g(x^*))$$
$$\leq \alpha \max\{d(x, x^*), d(x, f(x)), d(x^*, g(x^*)), \frac{1}{2}(d(x, g(x^*)) + d(x^*, f(x)))\}$$
$$= \alpha \max\{d(x, x^*), d(x, f(x)), \frac{1}{2}(d(x, x^*) + d(x^*, f(x)))\}$$
$$\leq \alpha(d(x, x^*) + d(x, f(x)))$$
$$\leq \alpha d(x, x^*) + \alpha(d(x, x^*) + d(x^*, f(x))),$$

which implies

$$d(f(x), x^*) \leq \frac{2\alpha}{1-\alpha} d(x, x^*), \text{ for all } x \in X.$$

Thus, f is quasi-contraction. From the symmetry of condition (1), we also get that g is quasi-contraction.

(f) We prove that f and g are Picard operators. Indeed, being graph contractions with a unique fixed point x^*, by the Graph Contraction Principle (see Theorem 3 in [21]), we obtain that f and g are c-Picard operators with $c := \frac{1}{1-\Gamma} = \frac{(1-\alpha)^2}{1-4\alpha-\alpha^2}$.

(g) and (h) These two conclusions follow (f) via Theorem 1.

(i) The conclusion follows from (e) via Theorem 2. □

In the next example, we show the case of two operators, f and g, for which the main theorem in [1] is not applicable but which satisfies the above condition (1) and hence Theorem 3 applies.

Example 1. *Let $f, g : [0,2] \to [0,2]$ be given by*

$$f(x) = \begin{cases} \frac{x}{9}, \; x \in [0,1] \\ \frac{x}{10}, x \in (1,2]. \end{cases} \tag{10}$$

and

$$g(y) = \frac{y}{10}, \; y \in [0,2].$$

Choose $x = \frac{999}{1000}$ and $y = \frac{1001}{1000}$. Then $d(f(x), g(y)) = \frac{109}{10{,}000}$, while $d(x,y) = \frac{20}{10{,}000}$. If we suppose that there exists $\alpha < 1$ such that $d(f(x), g(y)) \le \alpha d(x,y)$ for all $x, y \in [0,2]$, then we get the contradiction $\alpha > \frac{109}{20}$. On the other hand, the pair f, g satisfies the condition (1) with $\alpha := \frac{1}{4}$. Moreover, f and g have a unique common fixed point $x^* = 0$.

Remark 2. *It could be of real interest to give a common fixed point theory for a pair of Ćirić-type operators in the context of generalized metric spaces (b-metric space, partial metric space, vector-valued metric space, ...). See also [7], Chapter 3.*

3. An Application

Let us consider the following operatorial problem: find $(x,y) \in X \times Y$ satisfying the following relations

$$\begin{cases} x = f(y) \\ y = g(x) \\ (x,y) = h(x,y), \end{cases} \tag{11}$$

where $f : Y \to X, g : X \to Y$ and $h = (h_1, h_2) : X \times Y \to X \times Y$ are given operators and X, Y are two nonempty and closed subsets of a metric space (M, d). Notice that the problem composed by the first two equations is also called the altering point problem, see [22].

We suppose the following hypotheses:

(i) there exists $\beta \in (0,1)$ such that

$$d(h_1(x_1,y_1), f(y_2)) \le \beta \max\{d(y_1,y_2), d(x_1, h_1(x_1,y_1)), \frac{1}{2} d(x_1, f(y_2))\},$$

for every $(x_1, y_1) \in X \times Y$ and $y_2 \in Y$;

(ii) there exists $\gamma \in (0,1)$ such that

$$d(h_2(x_1,y_1), g(x_2)) \le \gamma \max\{d(x_1,x_2), d(y_1, h_2(x_1,y_1)), \frac{1}{2} d(y_1, g(x_2))\},$$

for every $(x_1, y_1) \in X \times Y$ and $x_2 \in X$;

(iii) the space (M, d) is complete.

We introduce on $X \times Y$ the metric $\tilde{d}$ defined, for $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in X \times Y$, by

$$\tilde{d}(z_1, z_2) := \max\{d(x_1,x_2), d(y_1,y_2)\}.$$

We also denote $t : X \times Y \to X \times Y$, $t(x,y) = (f(y), g(x))$ and $\alpha := \max\{\beta, \gamma\}$.

Under the above notations, our problem (11) becomes a common fixed point problem of the following form

$$(x,y) = t(x,y) = h(x,y), (x,y) \in X \times Y. \tag{12}$$

Then, for $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in X \times Y$, we have

$$\tilde{d}(h(z_1), t(z_2)) = \max\{d(h_1(x_1,y_1), f(y_2)), d(h_2(x_1,y_1), g(x_2))\}$$

$$\le \max\{\beta \max\{d(y_1,y_2), d(x_1,h_1(x_1,y_1)), \frac{1}{2}d(x_1,f(y_2))\},$$
$$\gamma \max\{d(x_1,x_2), d(x_2,h_2(x_1,y_1)), \frac{1}{2}d(y_1,g(x_2))\}\}$$
$$\le \alpha \max\{\max\{d(y_1,y_2), d(x_1,x_2)\}, \max\{d(x_1,h_1(x_1,y_1)), d(x_2,h_2(x_1,y_1))\},$$
$$\frac{1}{2}\max\{d(x_1,f(y_2)), d(y_1,g(x_2))\}\}$$
$$\le \alpha \max\{\tilde{d}(z_1,z_2), \tilde{d}(z_1,h(z_1)), \frac{1}{2}\tilde{d}(z_1,t(x_2))\}.$$

Thus, h and t satisfy the main assumptions of Theorem 3, and we can get the following conclusions for our problem: existence and uniqueness of the solution, convergence results for the corresponding sequences and stability theorems (under additional assumptions on β and γ).

For example, the above abstract model can be applied in the case of a hierarchical system of nonlinear variational inequality problems, which is defined as follows:

Find $(x^*,y^*) \in Fix(S)$ such that

$$\begin{cases} \langle aT_1(y^*) + x^* - y^*, x - x^* \rangle \ge 0, \\ \langle bT_2(x^*) + y^* - x^*, y - y^* \rangle \ge 0, \end{cases} \quad \text{for all } (x,y) \in Fix(S) \tag{13}$$

where $S : X \times Y \to X \times Y$ is given by $S(x,y) := (S_1(x), S_2(y))$, where $S_1 : X \to X, S_2 : Y \to Y$, $T_1 : X \to Y$ and $T_2 : Y \to X$ are given operators, $a, b > 0$ and X, Y are two nonempty closed convex subsets of a real Hilbert space H.

It is known that problem (13) is equivalent to the following problem:

Find $(x^*,y^*) \in X \times Y$ such that

$$\begin{cases} y^* = P_{Fix(S_2)}(I - aT_1)(x^*) \\ x^* = P_{Fix(S_1)}(I - bT_2)(y^*) \\ \quad (x^*,y^*) \in Fix(S), \end{cases} \tag{14}$$

where, for a nonempty, closed and convex set, $C \subset H$, the symbol P_C denotes the metric projection onto C, i.e., $P_C(u) := \{v \in C : \|u - v\| = \inf_{c \in C} \|u - c\|\}$, $u \in H$.

Notice that (14) is exactly the type of problem modeled by system (11). Thus, imposing adequate assumptions on S, T_1, T_2, on the parameters $a, b > 0$ and on the given sets X, Y we can obtain existence, uniqueness and stability results for the hierarchical system of nonlinear variational inequality problems (13). For other results of this type, see [22,23].

4. Conclusions

In this work, we have considered the common fixed point problem for a pair of operators satisfying a very general metric condition of Ćirić type. Under some mild assumptions, we proved several properties of the common fixed point problem: existence and uniqueness of the common fixed point, approximation of the common fixed point, well-posedness, Ulam–Hyers stability and Ostrowski stability of the common fixed point problem. Our results extend and generalize some recent results in the literature, and an example is given to illustrate the generality of our theorems. Moreover, an application to a system composed by an altering point problem and a fixed point problem is presented. A model for these kinds of applications is the hierarchical system of nonlinear variational inequality problems. As an open problem, we can propose the following one: construct a similar common fixed point theory for a pair of self operators in a complete metric space (X,d) satisfying, for every element x, y from the space X, the following condition:

$$d(f(x), g(y)) \le \alpha \max\{d(x,y), d(x,f(x)), d(y,g(y)), d(x,g(y)), d(y,f(x))\}. \tag{15}$$

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