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Stability Results of Quadratic-Additive Functional Equation Based on Hyers Technique in Matrix Paranormed Spaces

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Abstract: In this work, we introduce a mixed type of quadratic-additive (QA) functional equation and obtain its general solution. The objective of this work is to investigate the Ulam–Hyers stability of this quadratic-additive (QA) functional equation in matrix paranormed spaces (briefly, MP spaces) using the Hyers method for the factor *sum of norms*.

Keywords: matrix paranormed spaces; Ulam–Hyers stability; quadratic-additive functional equation; Hyers method

MSC: 39B82; 39B52; 46L07; 47L25

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1. Introduction

The problem of Ulam [1] about the stability of group homomorphisms was originated by the stability problem of functional equations: suppose that A is a group, $B(d)$ is a metric group, and $\psi : A \to B$. For any $\varepsilon > 0$, does there exists a $\delta > 0$ such that

$$d(\psi(ab), \psi(a)\psi(b)) < \delta, \quad \forall\, a, b \in A,$$

holds and which gives a unique homomorphism $W : A \to B$ is such that

$$d(\psi(a), W(a)) < \varepsilon,$$

for all $a \in A$? If the answer is affirmative, we can say that the Cauchy equation $\psi(ab) = \psi(a)\psi(b)$ is stable.

In 1941, Hyers [2] provided the case of approximately additive mapping $F : A \to A^{'}$, where A and $A^{'}$ are Banach spaces and F satisfies the below *Hyers inequality*

$$\|F(a+b) - F(a) - F(b)\| \le \varepsilon, \quad \forall\, a, b \in A.$$

This proved that the limit

$$B(a) = \lim_{n\to\infty} \frac{F(2^n a)}{2^n}, \quad \forall\, a \in A,$$

exists with the unique additive mapping $B : A \to A'$, which satisfies

$$\|F(a) - B(a)\| \le \varepsilon, \quad \forall\, a \in A.$$

Moreover, if $\psi(\sigma a)$ is continuous in σ for each fixed $a \in A$, then the function B is linear.

The result declared was that the Cauchy functional equation is stable for any pair of Banach spaces. The method that was provided by Hyers formed the additive function $B(a)$ called the Direct Method. This is called the stability (Hyers–Ulam stability) of the Cauchy additive functional equation.

Every solution of the following Cauchy additive functional equation

$$\psi(u+v) = \psi(u) + \psi(v), \tag{1}$$

is known as *additive*. The functional equation

$$\psi(u+v) + \psi(u-v) = 2\psi(u) + 2\psi(v), \tag{2}$$

is connected to a symmetric bi-additive. A quadratic function is a name given to each solution of the functional Equation (2). It is widely known that real vector space ψ is quadratic if and only if a single symmetric bi-additive function B exists for all a and $\psi(u) = B(u,u)$. The function B is presumptively assumed by

$$B(u,v) = \frac{1}{4}(\psi(u+v) - \psi(u-v)).$$

Skof explored the Ulam–Hyers stability problem for the functional Equation (2) for the mapping ψ between a normed space and a Banach space. The stability results of a cubic-additive functional equation have been established by Jun [3]. Najati [4] has investigated stability of a quadratic-additive, in quasi-Banach spaces. After this, Najati [5] introduced an additive-cubic functional equation and examined its stability for a mapping between two quasi-Banach spaces.

In 2012, Choonkil Park [6] examined the stability of the functional Equations (1) and (2) in paranormed spaces. In 2013, Choonkil Park extended this work to examine (Ref. [7]) the stability results of the functional Equations (1) and (2) and the below Cauchy additive functional inequality

$$\psi(a) + \psi(b) + \psi(c) \le \psi(a+b+c),$$

in matrix paranormed spaces. Based on these two works, Murali et al. [8] investigated the stability for the quadratic and cubic functional equations in matrix paranormed spaces. Moreover, Murali et al. [9] investigated the Hyers–Ulam stability of the quartic mappings in the same space. Tamilvanan et al., who developed this work, explored numerous functional equations in various normed spaces [10–12].

2. Quadratic-Additive Functional Equation and Its General Solution

We introduce a new mixed type of quadratic-additive (in brief, QA) functional equation:

$$\begin{aligned} \psi\Big(\sum_{h=1}^{l} ha_h\Big) \quad &= \sum_{1\le h<g\le l} \psi\Big(ha_h + ga_g\Big) - (l-2)\sum_{h=1}^{l} h^2\left[\frac{\psi(a_h)+\psi(-a_h)}{2}\right] \\ &\quad -(l-2)\sum_{h=1}^{l} h\left[\frac{\psi(a_h)-\psi(-a_h)}{2}\right], \end{aligned} \tag{3}$$

where $l \ge 4$, and we obtain its general solution. The main objective of this work is to investigate the Ulam–Hyers stability results of this functional equation in matrix paranormed spaces by using the Hyers method for the factor *sum of norms*.

We utilize some notions from [7,13] as follows:

- $M_l(V)$ is the set of all $l \times l$-matrices in V;
- $e_i \in M_{1,l}(\mathbb{C})$ is the i^{th} element, which is 1, and the remaining elements are 0;
- $E_{hi} \in M_l(\mathbb{C})$ indicates that the (h,i)-element is 1 and the remaining elements are 0;
- $E_{hi} \otimes a \in M_l(V)$ indicates that the (h,i)-element is a and the remaining elements are 0.

For $a \in M_l(V)$, $b \in M_m(V)$,

$$a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

We remark that $(V, \|\cdot\|_l)$ is a matrix normed space if and only if $(M_l(V), \|\cdot\|_l)$ is a normed space for every integer $l > 0$ and

$$\|EaF\|_m \leq \|E\|\|F\|\|a\|_l,$$

holds for all $E \in M_{m,l}$, $a = [a_{hi}] \in M_l(V)$ and all $F \in M_{l,m}$, and that $(V, \|\cdot\|_l)$ is a matrix Banach space if and only if $(V, \|\cdot\|_l)$ is a matrix normed space, where V is a Banach space.

Lemma 1. *If an even mapping $\psi : V \to W$, which satisfies the functional Equation (3) for all $a_1, a_2, \cdots, a_l \in V$, then the mapping $\psi : V \to W$ is quadratic.*

Proof. In terms of the evenness of ψ, we obtain $\psi(-a) = \psi(a)$. Now, Equation (3) becomes

$$\psi\Big(\sum_{h=1}^{l} ha_h\Big) = \sum_{1\leq h<g\leq l} \psi\Big(ha_h + ga_g\Big) - (l-2)\sum_{h=1}^{l} h^2\psi(a_h), \tag{4}$$

for all $a_1, a_2, \cdots, a_l \in V$. Replacing $(a_1, a_2, \cdots, a_l)$ by $(0, 0, \cdots, 0)$ in (4), we have $\psi(0) = 0$. Now, replacing $(a_1, a_2, \cdots, a_l)$ by $(0, a, 0, \cdots, 0)$ in (4), we obtain

$$\psi(2a) = 2\psi(a), \tag{5}$$

for all $a \in V$. Replacing a by $2a$ in (5), we obtain

$$\psi(2^2 a) = 2^4\psi(a), \tag{6}$$

for all $a \in V$. Replacing a by $2a$ in (6), we obtain

$$\psi(2^3 a) = 2^6\psi(a), \tag{7}$$

for all $a \in V$. Finally, we conclude that, for any non-negative integer l, we obtain

$$\psi(2^l a) = 2^{2l}\psi(a), \tag{8}$$

for all $a \in V$. Now, replacing $(a_1, a_2, \cdots, a_l)$ by $\Big(0, \frac{a}{2}, 0, \cdots, 0\Big)$ in (4), we have

$$\psi\Big(\frac{a}{2}\Big) = \frac{1}{2^2}\psi(a), \tag{9}$$

for all $a \in V$. Replacing a by $\frac{a}{2}$ in (9), we obtain

$$\psi\Big(\frac{a}{2^2}\Big) = \frac{1}{2^4}\psi(a), \tag{10}$$

for all $a \in V$. Replacing a by $\frac{a}{2}$ in (10), we obtain

$$\psi\Big(\frac{a}{2^3}\Big) = \frac{1}{2^6}\psi(a),$$

for all $a \in V$. Finally, we conclude that, for any non-negative integer i, we obtain

$$\psi\left(\frac{a}{2^i}\right) = \frac{1}{2^{2i}}\psi(a),$$

for all $a \in V$. Replacing $\left(a_1, a_2, \cdots, a_l\right)$ by $\left(a, \frac{b}{2}, -\frac{a}{3}, -\frac{b}{4}, 0, \cdots, 0\right)$ in (4), we obtain (2) for all $a, b \in V$. Therefore, the mapping $\psi : V \to W$ is quadratic. □

Lemma 2. *If an odd mapping $\psi : V \to W$, which satisfies the functional Equation (3) for all $a_1, a_2, \cdots, a_l \in V$, then the mapping $\psi : V \to W$ is additive.*

Proof. In terms of the evenness of ψ, we obtain $\psi(-a) = -\psi(a)$. Now, Equation (3) becomes

$$\psi\left(\sum_{h=1}^{l} ha_h\right) = \sum_{1\le h<g\le l} \psi\left(ha_h + ga_g\right) - (l-2)\sum_{h=1}^{l} h\psi(a_h), \tag{11}$$

for all $a_1, a_2, \cdots, a_l \in V$. Now, replacing $(a_1, a_2, \cdots, a_l)$ by $(0, 0, \cdots, 0)$ in (11), we have $\psi(0) = 0$. Next, replacing $(a_1, a_2, \cdots, a_l)$ by $(0, a, 0, \cdots, 0)$ in (11) and using the oddness of ψ, we obtain

$$\psi(2a) = 2\psi(a), \tag{12}$$

for all $a \in V$. Again, replacing a by $2a$ in (12), we have

$$\psi(2^2 a) = 2^2\psi(a), \tag{13}$$

for all $a \in V$. Replacing a by $2a$ in (13), we obtain

$$\psi(2^3 a) = 2^3\psi(a),$$

for all $a \in V$. Finally, we conclude that for any non-negative integer l, we have

$$\psi(2^l a) = 2^l\psi(a),$$

for all $a \in V$. Now, replacing $(a_1, a_2, \cdots, a_l)$ by $\left(0, \frac{a}{2}, 0, \cdots, 0\right)$ in (11), we obtain

$$\psi\left(\frac{a}{2}\right) = \frac{1}{2}\psi(a), \tag{14}$$

for all $a \in V$. Replacing a by $\frac{a}{2}$ in (14), we obtain

$$\psi\left(\frac{a}{2^2}\right) = \frac{1}{2^2}\psi(a), \tag{15}$$

for all $a \in V$. Replacing a by $\frac{a}{2}$ in (15), we obtain

$$\psi\left(\frac{a}{2^3}\right) = \frac{1}{2^3}\psi(a),$$

for all $a \in V$. Finally, we conclude that for any non-negative integer i, we have

$$\psi\left(\frac{a}{2^i}\right) = \frac{1}{2^i}\psi(a),$$

for all $a \in V$. Replacing $\left(a_1, a_2, \cdots, a_l\right)$ by $\left(a, \frac{b}{2}, \frac{a}{3}, \frac{b}{4}, 0, \cdots, 0\right)$ in (11), we reach (1) for all $a, b \in V$. Therefore, the mapping $\psi : V \to W$ is additive. □

Lemma 3. *A mapping $\psi : V \to W$ such that $\psi(0) = 0$ and (3) for all $a_1, a_2, \cdots, a_l \in V$ if and only if there exist a mapping $Y : V \times V \to W$ which is symmetric bi-additive and a mapping $X : V \to W$ which is additive such that $\psi(a) = Y(a, a) + X(a)$ for all $a \in V$.*

Proof. Let ψ with $\psi(0) = 0$ satisfy (3). Now, split ψ into an even part and odd part by taking

$$\psi_e(a) = \frac{1}{2}(\psi(a) + \psi(-a)) \quad \text{and} \quad \psi_o(a) = \frac{1}{2}(\psi(a) - \psi(-a)),$$

for all $a \in V$. Clearly, $\psi(a) = \psi_e(a) + \psi_o(a)$ for all $a \in V$. Thus, ψ_e and ψ_o satisfy the functional Equation (3). From Lemmas 1 and 2, we obtain that ψ_e (quadratic) and ϕ_o (additive). Thus, there exists $Y : V \times V \to W$, which satisfies $\psi_e(a) = B(a, a)$ for all $a \in V$. Therefore,

$$\psi(a) = X(a) + Y(a, a), \quad a \in V,$$

where $X(a) = \psi_o(a)$.

Conversely, suppose that there exist mappings $Y : V \times V \to W$ and $X : V \to W$ satisfies

$$\psi(a) = B(a, a) + X(a), \quad a \in V.$$

Easily, we can prove that the mappings $a \mapsto Y(a, a)$ and X satisfy the functional Equation (3). Thus, the mapping ψ satisfies the functional Equation (3). □

3. Stability Results in Matrix Paranormed Spaces

Here, we take $(V, \|\cdot\|_l)$ as a matrix Banach space and $(W, P_l(\cdot))$ as a matrix Frechet space. For a mapping $\psi : V \to W$, define $D\psi : V^l \to W$ and $D\psi_l : M_l(V^l) \to M_l(W)$ by

$$\begin{aligned} D\psi(a_1, a_2, \cdots, a_l) \quad := \quad & \psi\Big(\sum_{h=1}^{l} ha_h\Big) - \sum_{1\le h<g\le l} \psi\Big(ha_h + ga_g\Big) \\ & +(l-2)\sum_{h=1}^{l} h^2\left[\frac{\psi(a_h) + \psi(-a_h)}{2}\right] \\ & +(l-2)\sum_{h=1}^{l} h\left[\frac{\psi(a_h) - \psi(-a_h)}{2}\right], \end{aligned}$$

and

$$\begin{aligned} D\psi_l\big((a_1)_{ij}], [(a_2)_{ij}], \cdots, [(a_l)_{ij}]\big) \quad := \quad & \psi_l\Big(\sum_{h=1}^{l} h[(a_h)_{ij}]\Big) \\ & - \sum_{1\le h<g\le l} \psi_l\Big(h[(a_h)_{ij}] + g[(a_g)_{ij}]\Big) \\ & +(l-2)\sum_{h=1}^{l} h^2\left[\frac{\psi_l([(a_h)_{ij}]) + \psi_l([(-a_h)_{ij}])}{2}\right] \\ & +(l-2)\sum_{h=1}^{l} h\left[\frac{\psi_l([(a_h)_{ij}]) - \psi_l([(-a_h)_{ij}])}{2}\right], \end{aligned}$$

for all $a_1, a_2, \cdots, a_l \in V$ and all $a_t = [(a_t)_{ij}] \in M_l(V), t = 1, 2, \cdots, l$.

Note that $P(2a) \le 2P(a)$ for all $a \in W$.

Theorem 1. *Let $\alpha, \lambda \in \mathbb{R}_+$ with $\alpha > 2$. If an even mapping $\psi : V \to W$ such that*

$$P_l\Big(D\psi_l\Big([(a_1)_{ij}], [(a_2)_{ij}], \cdots, [(a_l)_{ij}]\Big)\Big) \le \sum_{i,j=1}^{l} \lambda\left(\sum_{t=1}^{l} \|(a_t)_{ij}\|^{\alpha}\right), \tag{16}$$

for all $a_t = [(a_t)_{ij}] \in M_l(V), t = 1, 2, \cdots, l$, *then there exists a unique quadratic mapping* $Q : V \to W$ *satisfying*

$$P_l\left(\psi_l([a_{ij}]) - Q_l([a_{ij}])\right) \leq \sum_{i,j=1}^{l} \frac{\lambda}{(l-2)(2^\alpha - 2^2)} \|a_{ij}\|^\alpha,$$

for all $a = [a_{ij}] \in M_l(V)$.

Proof. Let $l = 1$ in (16). Then, the inequality (16) becomes

$$P(D\psi(a_1, a_2, \cdots, a_l)) \leq \lambda \left(\sum_{t=1}^{l} \|a_t\|^\alpha \right), \tag{17}$$

for all $a_1, a_2, \cdots, a_l \in V$. Replacing $(a_1, a_2, \cdots, a_l)$ by $(0, 0, 0, \cdots, 0)$ in (17), we obtain

$$P\left(\left(\frac{2l^4 - l^3 - 7l^2 - l + 6}{6} \right) \psi(0) \right) \leq 0.$$

Therefore, $\psi(0) = 0$. Replacing $(a_1, a_2, \cdots, a_l)$ by $(0, a, 0, \cdots, 0)$ in (17), we have

$$P((l-2)\psi(2a) - 4(l-2)\psi(a)) \leq \lambda \|a\|^\alpha, \tag{18}$$

for all $a \in V$. From inequality (18), we have

$$P(\psi(2a) - 4\psi(a)) \leq \frac{\lambda}{(l-2)} \|a\|^\alpha,$$

and so

$$P\left(\psi(a) - 2^2\psi\left(\frac{a}{2}\right)\right) \leq \frac{\lambda}{(l-2)2^\alpha} \|a\|^\alpha,$$

for all $a \in V$. Clearly,

$$P\left(2^{2r}\psi\left(\frac{a}{2^r}\right) - 2^{2s}\psi\left(\frac{a}{2^s}\right)\right) \leq \sum_{\delta=r}^{s-1} \frac{2^{2\delta}\lambda}{(l-2)2^{(\delta+1)\alpha}} \|a\|^\alpha, \tag{19}$$

for all $a \in V$ and all positive integers r, s with $r < s$. From (19), the sequence $\left\{2^{2\delta}\psi\left(\frac{a}{2^\delta}\right)\right\}$ is Cauchy for every $a \in V$.

Since W is complete, the sequence $\left\{2^{2\delta}\psi\left(\frac{a}{2^\delta}\right)\right\}$ converges. Next, we can define a mapping $Q : V \to W$ by

$$Q(a) = \lim_{\delta \to \infty} 2^{2\delta}\psi\left(\frac{a}{2^\delta}\right),$$

for all $a \in V$. Setting $r = 0$ and taking the limit s that tends to ∞ in (19), we have

$$P(\psi(a) - Q(a)) \leq \frac{\lambda}{(l-2)(2^\alpha - 2^2)} \|a\|^\alpha, \tag{20}$$

for all $a \in V$. From inequality (17),

$$\begin{aligned} P\left(2^{2\delta}\left(D\psi\left(\frac{a_1}{2^\delta}, \frac{a_2}{2^\delta}, \cdots, \frac{a_l}{2^\delta}\right)\right)\right) &\leq 2^{2\delta} P\left(D\psi\left(\frac{a_1}{2^\delta}, \frac{a_2}{2^\delta}, \cdots, \frac{a_l}{2^\delta}\right)\right) \\ &\leq \frac{2^{2\delta}\lambda}{2^{\delta\alpha}}\left(\sum_{t=1}^{l} \|a_t\|^\alpha\right) \\ &\to 0 \quad \text{as} \quad \delta \to \infty. \end{aligned}$$

Therefore, $P(Q(a_1, a_2, \cdots, a_l)) = 0$. Therefore, the function Q satisfies the functional Equation (3). Hence, the function Q is quadratic. Now, we prove that Q is unique. Consider Q' to be another quadratic function which satisfies the functional Equation (3). Hence,

$$\begin{aligned} P\left(Q(a) - Q'(a)\right) &= P\left(2^{2s} Q\left(\frac{a}{2^s}\right) - 2^{2s} Q'\left(\frac{a}{2^s}\right)\right) \\ &\leq P\left(2^{2s}\left(Q\left(\frac{a}{2^s}\right) - \psi\left(\frac{a}{2^s}\right)\right)\right) + P\left(2^{2s}\left(\psi\left(\frac{a}{2^s}\right) - Q'\left(\frac{a}{2^s}\right)\right)\right) \\ &\leq \frac{2^{2s+1}\lambda}{(l-2)(2^\alpha - 2^2)2^{s\alpha}} \|a\|^\alpha \\ &\to 0 \quad \text{as} \quad s \to \infty, \end{aligned}$$

for all $a \in V$. Thus, $Q(a) = Q'(a)$ for all $a \in V$. This shows that Q is a unique function. By Lemma 2.1 in [5] and (20), we can conclude that

$$P_l(\psi_l([a_{ij}]) - Q_l([a_{ij}])) \leq \sum_{i,j=1}^{l} P(\psi(a_{ij}) - Q(a_{ij})) \leq \sum_{i,j=1}^{l} \frac{\lambda}{(l-2)(2^\alpha - 2^2)} \|a_{ij}\|^\alpha,$$

for all $a = [a_{ij}] \in M_l(V)$. Hence, the proof of the theorem is now completed. □

Theorem 2. *Let $\alpha, \lambda \in \mathbb{R}_+$ with $\alpha < 2$. If an even mapping $\psi : W \to V$ such that*

$$\left\| D\psi_l\left([(a_1)_{ij}], [(a_2)_{ij}], \cdots, [(a_l)_{ij}]\right) \right\|_l \leq \sum_{i,j=1}^{l} \lambda\left(\sum_{t=1}^{l} P((a_t)_{ij})^\alpha\right), \tag{21}$$

for all $a_t = [(a_t)_{ij}] \in M_l(W), t = 1, 2, \cdots, l$, then there exists a unique quadratic mapping $Q : W \to V$ satisfying

$$\|\psi_l([a_{ij}]) - Q_l([a_{ij}])\|_l \leq \sum_{i,j=1}^{l} \frac{\lambda}{(l-2)(2^2 - 2^\alpha)} P(a_{ij})^\alpha,$$

for all $a = [a_{ij}] \in M_l(W)$.

Proof. Assume that $l = 1$ in (21). Then, the inequality (21) becomes

$$\|D\psi(a_1, a_2, \cdots, a_l)\| \leq \lambda\left(\sum_{t=1}^{l} P(a_t)^\alpha\right), \tag{22}$$

for all $a_1, a_2, \cdots, a_l \in W$. Replacing $(a_1, a_2, \cdots, a_l)$ by $(0, 0, 0, \cdots, 0)$ in (22), we have

$$\left\| \left(\frac{2l^4 - l^3 - 7l^2 - l + 6}{6}\right)\psi(0) \right\| \leq 0.$$

Therefore, $\psi(0) = 0$. Replacing $(a_1, a_2, \cdots, a_l)$ by $(0, a, 0, \cdots, 0)$ in (22), we have

$$\|(l-2)\psi(2a) - 4(l-2)\psi(a)\| \leq \lambda P(a)^\alpha, \tag{23}$$

for all $a \in W$. It follows from the inequality (23) that we obtain

$$\left\| \frac{\psi(2a)}{2^2} - \psi(a) \right\| \leq \frac{\lambda}{2^2(l-2)} P(a)^{\alpha},$$

for all $a \in W$. Replacing a by $2a$ and dividing by 2^2 in inequality (23), we obtain

$$\left\| \frac{\psi(2^2 a)}{2^4} - \frac{\psi(2a)}{2^2} \right\| \leq \frac{2^{\alpha}\lambda}{2^4(l-2)} P(a)^{\alpha},$$

for all $a \in W$. Clearly,

$$\left\| \frac{\psi(2^r a)}{2^{2r}} - \frac{\psi(2^s a)}{2^{2s}} \right\| \leq \sum_{\delta=s}^{r-1} \frac{2^{\alpha\delta}\lambda}{2^{2(\delta+1)}(l-2)} P(a)^{\alpha}, \tag{24}$$

for all $a \in W$ and all positive integers r, s with $s < r$. From (24), the sequence $\left\{ \frac{\psi(2^{\delta} a)}{2^{2\delta}} \right\}$ is a Cauchy sequence for all $a \in W$.

Since V is complete, the sequence $\left\{ \frac{\psi(2^{\delta} a)}{2^{2\delta}} \right\}$ converges. Next, we can define a mapping $Q : W \to V$ by

$$Q(a) = \lim_{\delta \to \infty} \frac{\psi(2^{\delta} a)}{2^{2\delta}},$$

for all $a \in W$. Now, setting $s = 0$ and taking the limit $r \to \infty$ in (24), we have

$$\|\psi(a) - Q(a)\| \leq \frac{\lambda}{(l-2)(2^2 - 2^{\alpha})} P(a)^{\alpha}, \tag{25}$$

for all $a \in W$. From inequality (22),

$$\begin{aligned} \left\| \frac{1}{2^{2\delta}} \left(D\psi\left(2^{\delta} a_1, 2^{\delta} a_2, \cdots, 2^{\delta} a_l\right) \right) \right\| &\leq \frac{1}{2^{2\delta}} \left\| D\psi\left(2^{\delta} a_1, 2^{\delta} a_2, \cdots, 2^{\delta} a_l\right) \right\| \\ &\leq \frac{2^{\delta\alpha}}{2^{2\delta}} \lambda \left(\sum_{t=1}^{l} \|a_t\|^{\alpha} \right) \\ &\to 0 \quad \text{as} \quad \delta \to \infty. \end{aligned}$$

Therefore, $\|Q(a_1, a_2, \cdots, a_l)\| = 0$. Thus, the function Q satisfies the functional Equation (3). Hence, the function Q is quadratic. Now, we prove that the quadratic function Q is unique. Consider Q' to be another quadratic function which satisfies the functional Equation (3). Hence,

$$\begin{aligned} \left\| Q(a) - Q'(a) \right\| &= \left\| \frac{Q(2^r a)}{2^{2r}} - \frac{Q'(2^r a)}{2^{2r}} \right\| \leq \left\| \frac{Q(2^r a)}{2^{2r}} - \frac{\psi(2^r a)}{2^{2r}} \right\| + \left\| \frac{\psi(2^r a)}{2^{2r}} - \frac{Q'(2^r a)}{2^{2r}} \right\| \\ &\leq \frac{2^{(r\alpha+1)}\lambda}{(l-2)(2^2 - 2^{\alpha})2^{2r}} P(a)^{\alpha} \quad \to 0 \quad \text{as} \quad r \to \infty, \end{aligned}$$

for all $a \in W$. Thus, $Q(a) = Q'(a)$ for all $a \in W$. This proves that the function Q is unique. By Lemma 2.2 in [5] and (25), we can conclude that

$$\|\psi_l([a_{ij}]) - Q_l([a_{ij}])\|_l \leq \sum_{i,j=1}^{l} \|\psi(a_{ij}) - Q(a_{ij})\| \leq \sum_{i,j=1}^{l} \frac{\lambda}{(l-2)(2^2 - 2^{\alpha})} P(a_{ij})^{\alpha},$$

for all $a = [a_{ij}] \in M_l(W)$, which ends the proof. $\square$

Theorem 3. *Let $\alpha, \lambda \in \mathbb{R}_+$ with $\alpha > 1$. If an odd mapping $\psi : V \to W$ such that*

$$P_l\Big(D\psi_l\Big([(a_1)_{ij}], [(a_2)_{ij}], \cdots, [(a_l)_{ij}]\Big)\Big) \leq \sum_{i,j=1}^{l} \lambda \left(\sum_{t=1}^{l} \|(a_t)_{ij}\|^\alpha\right), \tag{26}$$

for all $a_t = [(a_t)_{ij}] \in M_l(V), t = 1, 2, \cdots, l$, then there exists a unique additive mapping $A : V \to W$ satisfying

$$P_l(\psi_l([a_{ij}]) - A_l([a_{ij}])) \leq \sum_{i,j=1}^{l} \frac{\lambda}{(l-2)(2^\alpha - 2)} \|a_{ij}\|^\alpha \tag{27}$$

for all $a = [a_{ij}] \in M_l(V)$.

Proof. Assume that $l = 1$ in (26). Then, the inequality (26) becomes

$$P(D\psi(a_1, a_2, \cdots, a_l)) \leq \lambda \left(\sum_{t=1}^{l} \|a_t\|^\alpha\right), \tag{28}$$

for all $a_1, a_2, \cdots, a_l \in V$. Replacing $(a_1, a_2, \cdots, a_l)$ by $(0, a, 0, \cdots, 0)$ in (28), we have

$$P((l-2)\psi(2a) - 4(l-2)\psi(a)) \leq \lambda \|a\|^\alpha, \tag{29}$$

for all $a \in V$. From the inequality (29), we obtain

$$P(\psi(2a) - 2\psi(a)) \leq \frac{\lambda}{(l-2)} \|a\|^\alpha,$$

and so

$$P\left(\psi(a) - 2\psi\left(\frac{a}{2}\right)\right) \leq \frac{\lambda}{(l-2)2^\alpha} \|a\|^\alpha, \tag{30}$$

for all $a \in V$. Replacing a by $\frac{a}{2}$ and multiply by 2 in (30), we obtain

$$P\left(2\psi\left(\frac{a}{2}\right) - 2^2\psi\left(\frac{a}{2^2}\right)\right) \leq \frac{2\lambda}{(l-2)2^{2\alpha}} \|a\|^\alpha,$$

for all $a \in V$. It is easy to show that

$$P\left(2^r\psi\left(\frac{a}{2^r}\right) - 2^s\psi\left(\frac{a}{2^s}\right)\right) \leq \sum_{\delta=r}^{s-1} \frac{2^\delta \lambda}{(l-2)2^{(\delta+1)\alpha}} \|a\|^\alpha, \tag{31}$$

for all $a \in V$ and all positive integers r, s with $r < s$. From (31), the sequence $\left\{2^\delta\psi\left(\frac{a}{2^\delta}\right)\right\}$ is a Cauchy sequence for all $a \in V$.

Since W is complete, the sequence $\left\{2^\delta\psi\left(\frac{a}{2^\delta}\right)\right\}$ converges. Next, we can define a mapping $A : V \to W$ by

$$A(a) = \lim_{\delta \to \infty} 2^\delta \psi\left(\frac{a}{2^\delta}\right),$$

for all $a \in V$. Now, taking $r = 0$ and the limit s that tends to ∞ in (31), we arrive at

$$P(\psi(a) - A(a)) \leq \frac{\lambda}{(l-2)(2^\alpha - 2)} \|a\|^\alpha, \tag{32}$$

for all $a \in V$. From inequality (28),

$$\begin{aligned} P\Big(2^\delta\Big(D\psi\Big(\frac{a_1}{2^\delta}, \frac{a_2}{2^\delta}, \cdots, \frac{a_l}{2^\delta}\Big)\Big)\Big) &\le 2^\delta P\Big(D\psi\Big(\frac{a_1}{2^\delta}, \frac{a_2}{2^\delta}, \cdots, \frac{a_l}{2^\delta}\Big)\Big) \\ &\le \frac{2^\delta \lambda}{2^{\delta\alpha}}\left(\sum_{t=1}^{l} \|a_t\|^\alpha\right) \\ &\to 0 \quad \text{as} \quad \delta \to \infty. \end{aligned}$$

Therefore, $P(A(a_1, a_2, \cdots, a_l)) = 0$. That is, the function A satisfies the functional Equation (3). Thus, the function A is additive. Now, we want to prove that the function A is unique. Consider A' as another additive function which satisfies the functional Equation (3). Hence,

$$\begin{aligned} P\Big(A'(a) - A(a)\Big) &= P\Big(2^s A\Big(\frac{a}{2^s}\Big) - 2^s A'\Big(\frac{a}{2^s}\Big)\Big) \\ &\le P\Big(2^s\Big(A\Big(\frac{a}{2^s}\Big) - \psi\Big(\frac{a}{2^s}\Big)\Big)\Big) + P\Big(2^s\Big(\psi\Big(\frac{a}{2^s}\Big) - A'\Big(\frac{a}{2^s}\Big)\Big)\Big) \\ &\le \frac{2^{s+1}\lambda}{(l-2)(2^\alpha - 2)2^{s\alpha}}\|a\|^\alpha \quad \to 0 \quad \text{as} \quad s \to \infty, \end{aligned}$$

for all $a \in V$. Thus, $A(a) = A'(a)$ for all $a \in V$. This proves that the function A is a unique function. By Lemma 2.1 in [5] and (32), we can conclude that

$$P_l(\psi_l([a_{ij}]) - A_l([a_{ij}])) \le \sum_{i,j=1}^{l} P(\psi(a_{ij}) - A(a_{ij})) \le \sum_{i,j=1}^{l} \frac{\lambda}{(l-2)(2^\alpha - 2)}\|a_{ij}\|^\alpha,$$

for all $a = [a_{ij}] \in M_l(V)$. Hence, the proof of the theorem is now completed. □

Theorem 4. *Let $\alpha, \lambda \in \mathbb{R}_+$ with $\alpha < 1$. If an odd mapping $\psi : W \to V$ such that*

$$\Big\|D\psi_l\Big([(a_1)_{ij}], [(a_2)_{ij}], \cdots, [(a_l)_{ij}]\Big)\Big\|_l \le \sum_{i,j=1}^{l} \lambda\left(\sum_{t=1}^{l} P\big((a_t)_{ij}\big)^\alpha\right), \tag{33}$$

for all $a_t = [(a_t)_{ij}] \in M_l(W), t = 1, 2, \cdots, l$, then there exists a unique additive mapping $A : W \to V$ satisfying

$$\|\psi_l([a_{ij}]) - A_l([a_{ij}])\|_l \le \sum_{i,j=1}^{l} \frac{\lambda}{(l-2)(2-2^\alpha)} P(a_{ij})^\alpha,$$

for all $a = [a_{ij}] \in M_l(W)$.

Proof. Assume that $l = 1$ in (33). Then, the equality (33) becomes

$$\|D\psi(a_1, a_2, \cdots, a_l)\| \le \lambda\left(\sum_{t=1}^{l} P(a_t)^\alpha\right), \tag{34}$$

for all $a_1, a_2, \cdots, a_l \in W$. Replacing $(a_1, a_2, \cdots, a_l)$ by $(0, a, 0, \cdots, 0)$ in (34), we obtain

$$\|(l-2)\psi(2a) - 2(l-2)\psi(a)\| \le \lambda P(a)^\alpha, \tag{35}$$

for all $a \in W$. It follows from the inequality (35) that we obtain

$$\left\|\frac{\psi(2a)}{2} - \psi(a)\right\| \le \frac{\lambda}{2(l-2)} P(a)^\alpha, \tag{36}$$

for all $a \in W$. Replacing a by $2a$ and dividing by 2 in (36), we obtain

$$\left\| \frac{\psi(2^2 a)}{2^2} - \frac{\psi(2a)}{2} \right\| \leq \frac{2^\alpha \lambda}{2^2(l-2)} P(a)^\alpha,$$

for all $a \in W$. It is easy to show that

$$\left\| \frac{\psi(2^r a)}{2^r} - \frac{\psi(2^s a)}{2^s} \right\| \leq \sum_{\delta=s}^{r-1} \frac{2^{\alpha\delta} \lambda}{2^{(\delta+1)}(l-2)} P(a)^\alpha, \tag{37}$$

for all $a \in W$ and all positive integers r, s with $s < r$. From (37), the sequence $\left\{ \frac{\psi(2^\delta a)}{2^\delta} \right\}$ is a Cauchy sequence for all $a \in W$.

Since V is complete, the sequence $\left\{ \frac{\psi(2^\delta a)}{2^\delta} \right\}$ converges. Next, we can define a mapping $A : W \to V$ by

$$A(a) = \lim_{\delta \to \infty} \frac{\psi(2^\delta a)}{2^\delta},$$

for all $a \in W$. Now, setting $s = 0$ and taking the limit $r \to \infty$ in (37), we obtain

$$\|\psi(a) - A(a)\| \leq \frac{\lambda}{(l-2)(2-2^\alpha)} P(a)^\alpha, \tag{38}$$

for all $a \in W$. From inequality (34),

$$\begin{aligned} \left\| \frac{1}{2^\delta} \left(D\psi\left(2^\delta a_1, 2^\delta a_2, \cdots, 2^\delta a_l\right) \right) \right\| &\leq \frac{1}{2^\delta} \left\| D\psi\left(2^\delta a_1, 2^\delta a_2, \cdots, 2^\delta a_l\right) \right\| \\ &\leq \frac{2^{\delta\alpha}}{2^\delta} \lambda \left(\sum_{t=1}^{l} \|a_t\|^\alpha \right) \\ &\to 0 \quad \text{as} \quad \delta \to \infty. \end{aligned}$$

Therefore, $\|A(a_1, a_2, \cdots, a_l)\| = 0$. That is, the function A satisfies (3). Hence, the function A is additive. Now, we want to prove that the additive function A is unique. Consider A' as another additive function which satisfies the functional Equation (3). Hence,

$$\begin{aligned} \left\| A(a) - A'(a) \right\| &= \left\| \frac{A(2^r a)}{2^r} - \frac{A'(2^r a)}{2^r} \right\| \\ &\leq \left\| \frac{A(2^r a)}{2^r} - \frac{\psi(2^r a)}{2^r} \right\| + \left\| \frac{\psi(2^r a)}{2^r} - \frac{A'(2^r a)}{2^r} \right\| \\ &\leq \frac{2^{(r\alpha+1)} \lambda}{(l-2)(2-2^\alpha)2^r} P(a)^\alpha \\ &\quad \to 0 \quad \text{as} \quad r \to \infty, \end{aligned}$$

for all $a \in W$. Thus, $A(a) = A'(a)$ for all $a \in W$. This shows that A is a unique function. By Lemma 2.2 in [5] and (38), we can conclude that

$$\left\| \psi_l([a_{ij}]) - A_l([a_{ij}]) \right\|_l \leq \sum_{i,j=1}^{l} \left\| \psi(a_{ij}) - A(a_{ij}) \right\| \leq \sum_{i,j=1}^{l} \frac{\lambda}{(l-2)(2-2^\alpha)} P(a_{ij})^\alpha,$$

for all $a = [a_{ij}] \in M_l(W)$, which ends the proof. $\square$

Proposition 1. *Let $\alpha, \lambda \in \mathbb{R}_+$ with $\alpha > 2$ or $\alpha > 1$. Let $\psi : V \to W$ be a mapping with $\psi(0) = 0$ such that (16) holds for all $a_t = [(a_t)_{ij}] \in M_l(V), t = 1, 2, \cdots, l$. Then, there exists a unique quadratic mapping $Q : V \to W$ and a unique additive mapping $A : V \to W$ that satisfies*

$$P_l\big(\psi_l([a_{ij}]) - Q_l([a_{ij}]) - A_l([a_{ij}])\big) \le \sum_{i,j=1}^{l} \frac{\lambda}{(l-2)} \|a_{ij}\|^{\alpha} \Big[\frac{1}{(2^{\alpha}-2^{2})} + \frac{1}{(2^{\alpha}-2)}\Big],$$

for all $a = [a_{ij}] \in M_l(V)$.

Proof. Let us define $\psi([a_{ij}]) = \psi_e([a_{ij}]) + \psi_o([a_{ij}])$, where

$$\psi_e([a_{ij}]) = \frac{\psi([a_{ij}]) + \psi_e(-[a_{ij}])}{2} \quad \text{and} \quad \psi_0([a_{ij}]) = \frac{\psi([a_{ij}]) - \psi_e(-[a_{ij}])}{2}$$

are even and odd functions, respectively. Hence,

$$\begin{aligned} &P_l\big(\psi_l([a_{ij}]) - Q_l([a_{ij}]) - A_l([a_{ij}])\big) \\ &\quad = P_l\big((\psi_l)_e([a_{ij}]) + (\psi_l)_o([a_{ij}]) - Q_l([a_{ij}]) - A_l([a_{ij}])\big) \\ &\quad \le P_l\big((\psi_l)_e([a_{ij}]) - Q_l([a_{ij}])\big) + P_l\big((\psi_l)_o([a_{ij}]) - A_l([a_{ij}])\big) \end{aligned}$$

for all $[a_{ij}] \in M_l(V)$. The remaining proof is followed by the results of Theorem 1 and Theorem 3. □

Proposition 2. *Let $\alpha, \lambda \in \mathbb{R}_+$ with $\alpha < 2$ or $\alpha < 1$. Let $\psi : W \to V$ be a mapping with $\psi(0) = 0$ such that (16) holds for all $a_t = [(a_t)_{ij}] \in M_l(W), t = 1, 2, \cdots, l$. Then, there exists a unique quadratic mapping $Q : W \to V$ and a unique additive mapping $A : W \to V$ such that*

$$\big\|\psi_l([a_{ij}]) - Q_l([a_{ij}]) - A_l([a_{ij}])\big\|_l \le \sum_{i,j=1}^{l} \frac{\lambda}{(l-2)} P(a_{ij})^{\alpha} \Big[\frac{1}{(2^{2}-2^{\alpha})} + \frac{1}{(2-2^{\alpha})}\Big],$$

for all $a = [a_{ij}] \in M_l(W)$.

Proof. Let us define $\psi([a_{ij}]) = \psi_e([a_{ij}]) + \psi_o([a_{ij}])$, where

$$\psi_e([a_{ij}]) = \frac{\psi([a_{ij}]) + \psi_e(-[a_{ij}])}{2} \quad \text{and} \quad \psi_0([a_{ij}]) = \frac{\psi([a_{ij}]) - \psi_e(-[a_{ij}])}{2}$$

are even and odd functions, respectively. Hence,

$$\begin{aligned} &\big\|\psi_l([a_{ij}]) - Q_l([a_{ij}]) - A_l([a_{ij}])\big\|_l \\ &\quad = \big\|(\psi_l)_e([a_{ij}]) + (\psi_l)_o([a_{ij}]) - Q_l([a_{ij}]) - A_l([a_{ij}])\big\|_l \\ &\quad \le \big\|(\psi_l)_e([a_{ij}]) - Q_l([a_{ij}])\big\|_l + \big\|(\psi_l)_o([a_{ij}]) - A_l([a_{ij}])\big\|_l, \end{aligned}$$

for all $[a_{ij}] \in M_l(W)$. The remaining proof is followed by the results of Theorem 2 and Theorem 4. □

4. Illustrative Example

We use a suitable example to show that the functional Equation (3) fails to be stable in the singular situation. In response to Gajda's excellent example in [14], we give the following counter-example, which demonstrates the instability in Theorem 1 of Equation (3) under specific conditions $\alpha = 2$.

Remark 1. *If an even mapping $\psi : \mathbb{R} \to V$ satisfies the functional Equation (3), then the below assertions hold:*

(1) $\psi(m^{c/2}a) = m^c\phi(a)$, *for all* $a \in \mathbb{R}, m \in \mathbb{Q}$ *and* $c \in \mathbb{Z}$.
(2) $\psi(a) = a^2\psi(1)$, *for all* $a \in \mathbb{R}$ *if the function* ψ *is continuous.*

Example 1. *Let an even mapping* $\psi : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\psi(a) = \sum_{p=0}^{\infty} \frac{\phi(2^p a)}{2^{2p}}, \tag{39}$$

where

$$\phi(a) = \begin{cases} \lambda a^2, & -1 < a < 1 \\ \lambda, & \text{else.} \end{cases}$$

Suppose that the function ψ *defined in* (39) *satisfies*

$$|D\psi(a_1, a_2, \cdots, a_l)| \le \left(\frac{2l^4 - l^3 - 2l^2 - 5l + 6}{6}\right)\frac{8}{3}\lambda\left(\sum_{j=1}^{l}|a_j|^2\right), \tag{40}$$

for all $a_1, a_2, \cdots, a_l \in \mathbb{R}$. *We show that there does not exist a quadratic mapping* $Q : \mathbb{R} \to \mathbb{R}$ *such that*

$$|\psi(a) - Q(a)| \le \delta|a|^2, \tag{41}$$

for all $a \in \mathbb{R}$, *where* λ *and* δ *are constants.*

We can easily find that ψ *is bounded by* $\frac{2^2}{3}\lambda$ *on* $\mathbb{R}$. *If* $\sum_{j=1}^{l}|a_j|^2 \ge \frac{1}{2^2}$ *or 0, then*

$$|D\psi(a_1, a_2, \cdots, a_l)| < \left(\frac{2l^4 - l^3 - 2l^2 - 5l + 6}{6}\right)\frac{2^2}{3}\lambda.$$

Thus, (40) *is valid. Next, suppose that*

$$0 < \sum_{j=1}^{s}|a_j|^2 < \frac{1}{2^2},$$

and then there is an integer $s > 0$ *that satisfies*

$$\frac{1}{2^{2(s+2)}} \le \sum_{j=1}^{l}|a_j|^2 < \frac{1}{2^{2(s+1)}}. \tag{42}$$

Thus, $2^{2s}|a_1| < \frac{1}{2^2}, 2^{2s}|a_2| < \frac{1}{2^2}, 2^{2s}|a_3| < \frac{1}{2^2}, \cdots, 2^{2s}|a_l| < \frac{1}{2^2}$ *and*

$$\left.\begin{array}{r} 2^t a_1, 2^t a_2, \cdots, 2^t a_l \\ \left(\sum_{h=1}^{l} 2^t h a_h\right) \\ \sum_{1 \le h < g \le l}\left(2^t h a_h + 2^t g a_g\right) \\ \sum_{h=1}^{l} h^2(2^t a_h) \end{array}\right\} \in]-1, 1[, \quad t = 0, 1, \cdots, s-1.$$

Moreover, for $t = 0, 1, \cdots, s-1$

$$\begin{aligned}\Psi(a_1, a_2, \cdots, a_l) &= \phi\Big(\sum_{h=1}^{l} ha_h\Big) - \sum_{1\leq h<g\leq l} \phi\Big(ha_h + ga_g\Big) \\ &\quad +(l-2)\sum_{h=1}^{l} h^2\phi(a_h) \\ &= 0.\end{aligned}$$

Next, by inequality (42)*, we obtain that*

$$\begin{aligned}|D\psi(a_1, a_2, \cdots, a_l)| &\leq \sum_{t=0}^{\infty} \frac{1}{2^{2t}} |\Psi(2^t a_1, 2^t a_2, \cdots, 2^t a_l)| \\ &\leq \sum_{t=s}^{\infty} \frac{1}{2^{2t}} \left(\frac{2l^4 - l^3 - 2l^2 - 5l + 6}{6}\right)\lambda.\end{aligned}$$

It follows from (42) *that*

$$|D\psi(a_1, a_2, \cdots, a_l)| \leq \left(\frac{2l^4 - l^3 - 2l^2 - 5l + 6}{6}\right) \frac{8\lambda}{3} \left(\sum_{j=1}^{l} |a_j|^2\right).$$

Thus, the function ψ *satisfies the inequality* (40)*. Suppose, on the contrary, that there exists a quadratic mapping* $Q : \mathbb{R} \to \mathbb{R}$ *which satisfies* (41)*.*
From Remark 1, Q *must be* $Q(a) = ca^2$*,* $a \in \mathbb{R}$*. Thus, we obtain*

$$|\chi(a)| \leq (\delta + |c|)|a|^2, \; a \in \mathbb{R}.$$

However, we have a choice $s > 0$ *with* $s\lambda > \delta + |c|$*. If* $a \in \left(0, \frac{1}{2^{s-1}}\right)$*, then* $2^t a \in (0, 1)$ *for every* $t = 0, 1, \cdots, s-1$*, we have*

$$\psi(a) = \sum_{t=0}^{\infty} \frac{\phi(2^t a)}{2^{2t}} \geq \sum_{t=0}^{s-1} \frac{\lambda(2^t a)^2}{2^{2t}} = s\lambda a^2 > (\delta + |c|) \; |a^2|,$$

which contradicts. Thus, Equation (3) *is not stable.*

The upcoming counter-example shows the non-stability in a particular condition $\alpha = 1$ in Theorem 3 of the functional Equation (3).

Remark 2. *If an odd mapping* $\psi : \mathbb{R} \to V$ *satisfies the functional Equation* (3)*, then the below assertions hold:*

(1) $\psi(m^c a) = m^c \phi(a), \quad a \in \mathbb{R}, m \in \mathbb{Q}$ *and* $c \in \mathbb{Z}$*.*
(2) $\psi(a) = a\psi(1), \quad a \in \mathbb{R}$ *if the function* ψ *is continuous.*

Example 2. *Let an odd mapping* $\psi : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\psi(a) = \sum_{p=0}^{\infty} \frac{\phi(2^p a)}{2^p}, \tag{43}$$

where

$$\phi(a) = \begin{cases} \lambda a, & -1 < a < 1 \\ \lambda, & \text{else}. \end{cases}$$

Suppose that the function ψ is defined in (43) such that

$$|D\psi(a_1,a_2,\cdots,a_l)| \le \left(\frac{2l^4-l^3-2l^2-5l+6}{6}\right)4\lambda\left(\sum_{j=1}^{l}|a_j|^2\right) \tag{44}$$

for all $a_1,a_2,\cdots,a_l \in \mathbb{R}$. We show that there does not exist an additive mapping $A:\mathbb{R}\to\mathbb{R}$ satisfying

$$|\psi(a)-A(a)| \le \delta|a|, \quad a\in\mathbb{R}, \tag{45}$$

where λ and δ are constants.

We can easily find that ψ is bounded by 2λ on $\mathbb{R}$. If $\sum_{j=1}^{l}|a_j| \ge \frac{1}{2}$ or 0, then

$$|D\psi(a_1,a_2,\cdots,a_l)| < \left(\frac{2l^4-l^3-2l^2-5l+6}{6}\right)2\lambda.$$

Thus, (44) is valid. Next, suppose that

$$0<\sum_{j=1}^{s}|a_j|<\frac{1}{2},$$

and then there exists an integer $s>0$ that satisfies

$$\frac{1}{2^{(s+2)}} \le \sum_{j=1}^{l}|a_j| < \frac{1}{2^{(s+1)}}. \tag{46}$$

Thus, $2^s|a_1|<\frac{1}{2}, 2^s|a_2|<\frac{1}{2}, 2^s|a_3|<\frac{1}{2},\cdots,2^s|a_l|<\frac{1}{2}$ and

$$\left.\begin{array}{r} 2^t a_1, 2^t a_2, \cdots, 2^t a_l \\ \left(\sum_{h=1}^{l} 2^t h a_h\right) \\ \sum_{1\le h<g\le l}\left(2^t h a_h + 2^t g a_g\right) \\ \sum_{h=1}^{l} h(2^t a_h) \end{array}\right\} \in]-1,1[, \quad t=0,1,\cdots,s-1.$$

Moreover, for $t=0,1,\cdots,s-1$

$$\begin{aligned}\Psi(a_1,a_2,\cdots,a_l) &= \phi\left(\sum_{h=1}^{l}ha_h\right)-\sum_{1\le h<g\le l}\phi\left(ha_h+ga_g\right) \\ &\quad +(l-2)\sum_{h=1}^{l}h\phi(a_h) \\ &= 0.\end{aligned}$$

Next, by inequality (46), we obtain that

$$\begin{aligned}|D\psi(a_1,a_2,\cdots,a_l)| &\le \sum_{t=0}^{\infty}\frac{1}{2^t}|\Psi(2^ta_1,2^ta_2,\cdots,2^ta_l)| \\ &\le \sum_{t=s}^{\infty}\frac{1}{2^t}\left(\frac{2l^4-l^3-2l^2-5l+6}{6}\right)\lambda.\end{aligned}$$

It follows from (46) that

$$|D\psi(a_1, a_2, \cdots, a_l)| \le \left(\frac{2l^4 - l^3 - 2l^2 - 5l + 6}{6}\right) 4\lambda \left(\sum_{j=1}^{l} |a_j|\right). \tag{47}$$

Thus, the function ψ satisfies the inequality (44). Assume that, on the contrary, there is an additive mapping $A : \mathbb{R} \to \mathbb{R}$ which satisfies (45).

From Remark 2, A must be $A(a) = ca,\ a \in \mathbb{R}$. Thus, we have

$$|\chi(a)| \le (\delta + |c|)|a|,\ a \in \mathbb{R}.$$

However, we have the choice of $s > 0$ with $s\lambda > \delta + |c|$. If $a \in \left(0, \frac{1}{2^{s-1}}\right)$, then $2^t a \in (0,1)$ for all $t = 0, 1, \cdots, s-1$, and we have

$$\psi(a) = \sum_{t=0}^{\infty} \frac{\phi(2^t a)}{2^t} \ge \sum_{t=0}^{s-1} \frac{\lambda(2^t a)}{2^t} = s\lambda a > (\delta + |c|)\ |a|,$$

which contradicts. Thus, the functional Equation (3) is not stable.

5. Conclusions

In this work, we have introduced a new dimension to the finite variable QA functional Equation (3) and its general solution for the function ψ was derived. Mainly, Ulam–Hyers stability in the matrix paranormed spaces has been explored by employing the Hyers method for the *sum of norms* factor of the generalized finite variable QA functional Equation (3).

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