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Article

Application to Lipschitzian and Integral Systems via a Quadruple Coincidence Point in Fuzzy Metric Spaces

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Abstract: In this paper, the results of a quadruple coincidence point (QCP) are established for commuting mapping in the setting of fuzzy metric spaces (FMSs) without using a partially ordered set. In addition, several related results are presented in order to generalize some of the prior findings in this area. Finally, to support and enhance our theoretical ideas, non-trivial examples and applications for finding a unique solution for Lipschitzian and integral quadruple systems are discussed.

Keywords: quadruple coincidence point; commuting mapping; Lipschitzian mappings; an integral equation; fuzzy metric spaces

MSC: 47H10; 54H25

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1. Introduction

Fixed-point (FP) theory has many applications, not only in nonlinear analysis and its trends—including solutions of differential and integral equations, functional equations arising from dynamic programming, topologies, and dynamic systems—but also in economics, game theory, biological sciences, computer sciences, chemistry, etc. [1–4].

There is no doubt that the study of fuzzy sets is extremely important for their many applications, such as in the control of ill-defined, complex, and non-linear systems. It is more common to find solutions for control problems that are difficult to solve with the classical control theory. Fuzzy set theory is becoming an increasingly important tool, especially in the rapidly evolving discipline of artificial intelligence, such as in expert systems and neural networks. It creates completely new opportunities for the application of fuzzy sets in chemical engineering [5–8].

The concept of fuzzy sets was initiated by Zadeh [9] in 1965. Many mathematicians used these sets to introduce interesting concepts into the field of mathematics, such as fuzzy logic, fuzzy differential equations, and fuzzy metric spaces. It is known that an FMS is an important generalization of an ordinary metric space where the topological definitions are extended, and there are possible applications in several areas. Many mathematicians have considered this problem in many ways. For example, the authors of [10] modified the concept of an FMS that was initiated by Kramosil and Michalek [11] and defined the Hausdorff topology of an FMS. For more details about this idea, we advise the reader to see [12–17].

In 2011, the coupled fixed-point (FP) [18] result was extended to a tripled FP in partially ordered metric spaces by Berinde and Borcut [19]. Using these spaces, they introduced exciting results of tripled FP theorems. For more details, see [20–24].

In the setting of FMSs, coupled FP results were presented and some important theorems were given by Zhu and Xiao [25] and Hu [26]. Elagan et al. [27] studied the existence of an FP in a locally convex topology generated by fuzzy $n-$normed spaces.

Motivated by the results of the notions of coupled and tripled FPs in partially ordered metric spaces, Karapinar [28] suggested the concept of a quadruple FP and proved some related consequences of FPs in the same spaces.

Based on the last two paragraphs, in this publication, a QCP is considered, and some new and relevant FP results in FMSs are reported. Our paper's strength is determined by two factors. First, we can adapt it to complete metric spaces (CMSs) so as to achieve Karapinar's results [28] (in non-fuzzy sets). So, our paper covers and unifies a large number of outcomes in the same direction. Secondly, we can apply the theoretical conclusions to Lipschitzian and integral quadruple systems in order to discover a unique solution. Finally, non-trivial examples are mentioned and discussed.

2. Preliminaries

Hereafter, we will refer to ζ as a non-empty set, $\Omega(\rho,\sigma,\tau,\upsilon)$ as $\Omega_{\rho\sigma\tau\upsilon}$, $\Psi(\rho,\sigma,\kappa)$ as $\Psi_{\rho\sigma}(\kappa)$, and $\omega(\rho,\sigma)$ as $\omega_{\rho\sigma}$.

The usual metric space is a non-empty set ζ equipped with a function $\omega:\zeta\times\zeta\to\mathbb{R}^+$ such that for all $\rho,\sigma,\tau\in\zeta$, the following conditions are true:

- $\omega_{\rho\sigma}\geq 0$,
- $\omega_{\rho\sigma}=0$ if $\rho=\sigma$,
- $\omega_{\rho\sigma}\leq\omega_{\rho\tau}+\omega_{\tau\sigma}$.

The pair (ζ,ω) is called an MS.

A mapping $\urcorner:\zeta\to\zeta$ on an MS (ζ,ω) is called Lipschitzian if there is $\varpi\geq 0$ such that

$$\omega_{\urcorner\rho\urcorner\sigma}\leq\varpi\omega_{\rho\sigma},\ \forall\rho,\sigma\in\zeta.$$

The smallest constant ϖ—denoted by $\varpi_{\urcorner}$—that satisfies the above inequality is called the Lipschitz constant for $\urcorner$. It is clear that a Lipschitzian mapping (LM) is a contraction with $\varpi_{\urcorner}<1$.

Theorem 1 ([29]). *Let (ζ,ω) be a complete MS and let $Q:\zeta\to\zeta$ be a contraction mapping, that is, the following inequality is true:*

$$\omega(Qx,Qy)\leq k\omega(Qx,Qy),\ \textit{for all}\ x,y\in\zeta,$$

where $k\in[0,1)$. Then, Q has a unique FP x^* in ζ. Moreover, for $x_0\in\zeta$, the sequence $(Q^n x_0)_{n\in\mathbb{N}}$ converges to x^*.

For examples on LMs, let $\zeta=\mathbb{R}$ and let $\urcorner_i:\zeta\to\zeta$ be defined by $\urcorner_1(\rho)=\Lambda$, $\urcorner_2(\rho)=\mu\rho$, $\urcorner_3(\rho)=\cos\rho$, $\urcorner_4(\rho)=\frac{1}{1+\rho}$, $\urcorner_5(\rho)=\frac{1}{(1+\rho)^2}$, and $\urcorner_6(\rho)=\arcsin\rho$, where $\Lambda,\mu\in\mathbb{R}$.

Definition 1 ([30]). *A mapping $\star:[0,1]^2\to[0,1]$ is called a κ-norm if it is nondecreasing in both arguments, associative, commutative, and has 1 as identity. For all $\ell\in[0,1]$, the sequence $\{\star^m\ell\}_{m=1}^{\infty}$ is inductively defined by $\star^1\ell=\ell$, $\star^m\ell=\left(\star^{m-1}\ell\right)\star\ell$. A triangular norm $\star$ is of Y-type if $\{\star^m\ell\}_{m=1}^{\infty}$ is equicontinuous at $\ell=1$, that is, for each $\epsilon\in(0,1)$, there is $\varkappa\in(0,1)$ such that if $\ell\in(1-\varkappa,1]$, then $\star^m\ell>1-\epsilon$ for each $m\in\mathbb{N}$.*

The most famous continuous κ-norm of the Y-type is $\star=\min$, which satisfies $\min(\ell_1,\ell_2)\geq\ell_1\ell_2$ for all $\ell_1,\ell_2\in[0,1]$.

The results below include a wide range of κ-norms of the Y-type.

Lemma 1 ([30]). *Assume that $\star$ is a κ-norm and $\varrho \in (0,1]$ is a real number. Define $\star_\varrho$ by $\rho \star_\varrho \sigma = \rho \star \sigma$ if $\max\{\rho,\sigma\} \leq 1-\varrho$, and $\rho \star_\varrho \sigma = \min\{\rho,\sigma\}$ if $\max\{\rho,\sigma\} > 1-\varrho$. Then, $\star_\varrho$ is a κ-norm of the Υ-type.*

Definition 2 ([11]). *Let $\zeta \neq \emptyset$ be an arbitrary set, let $\star$ be a continuous κ-norm, and let $\Psi : \zeta \times \zeta \times [0,\infty) \to [0,1]$ be a fuzzy set. We say that $(\zeta,\Psi,\star)$ is an FMS if the function Ψ satisfies the hypotheses below for each $\rho,\sigma,\tau \in \zeta$, and $\kappa,\mu > 0$:*

(fms 1) $\Psi_{\rho\sigma}(0) = 0$;
(fms 2) $\Psi_{\rho\sigma}(\kappa) = 1 \Leftrightarrow \rho = \sigma$;
(fms 3) $\Psi_{\rho\sigma}(\kappa) = \Psi_{\sigma\rho}(\kappa)$;
(fms 4) $\Psi_{\rho\sigma}(.) : [0,\infty) \to [0,1]$ *is left continuous;*
(fms 5) $\Psi_{\rho\sigma}(\kappa) \star \Psi_{\sigma\tau}(\mu) \leq \Psi_{\rho\tau}(\kappa+\mu)$.

Here, we also consider (ζ,Ψ) an FMS under $\star$, and we will only consider the FMS that verifies:

(D) $\lim_{\kappa\to\infty} \Psi_{\rho\sigma}(\kappa) = 1, \forall \rho,\sigma \in \zeta.$

Lemma 2 ([12]). *On the infinite set $[0,\infty)$, $\Psi_{\rho\sigma}(.)$ is a non-decreasing function.*

Definition 3 ([10]). *Assume that (ζ,Ψ) is an FMS under some κ-norm; a sequence $\{\rho_m\} \subset \zeta$ is called:*

- *Convergent to $\rho \in \zeta$, and we write $\lim_{m\to\infty} \rho_m = \rho$ if, for every $\epsilon > 0$, $\kappa > 0$, there is $m_0 \in \mathbb{N}$ such that $\Psi_{\rho_m\rho}(\kappa) > 1-\epsilon$ for all $m \geq m_0$.*
- *A Cauchy sequence if, for every $\epsilon > 0$, $\kappa > 0$, there is $m_0 \in \mathbb{N}$ such that $\Psi_{\rho_m\rho_j}(\kappa) > 1-\epsilon$ for all $m, j \geq m_0$.*
- *An FMS is called complete if every Cauchy sequence is convergent.*

Definition 4 ([11]). *We say that a function $\urcorner : \zeta \to \zeta$ defined on an FMS is continuous at $\rho_0 \in \zeta$ if $\lim_{m\to\infty} \urcorner\rho_m = \urcorner\rho_0$ for any $\{\rho_m\} \in \zeta$ such that $\lim_{m\to\infty} \rho_m = \rho_0$. As is familiar, for $\rho_0 \in \zeta$, we will denote $\urcorner^{-1}(\rho_0) = \{\rho \in \zeta : \urcorner\rho = \rho_0\}$.*

Remark 1 ([11]). *If $\ell_1 \leq \ell_2$, then $\rho^{\ell_1} \geq \rho^{\ell_2}$ provided that $\rho \in [0,1]$ and $\ell_1, \ell_2 \in (0,\infty)$. This fact will be expressed here as follows: $0 < \ell_1 \leq \ell_2 \leq 1$ implies that $\Psi_{\rho\sigma}(\kappa)^{\ell_1} \geq \Psi_{\rho\sigma}(\kappa)^{\ell_2} \geq \Psi_{\rho\sigma}(\kappa)$.*

For any κ-norm $\star$, it is obvious that $\star \leq \min$. So, if (ζ,Ψ) is an FMS via min, then (ζ,Ψ) is an FMS under any κ-norm.

In the examples below, we only define $\Psi_{\rho\sigma}(\kappa)$ for $\kappa > 0$ and $\rho \neq \sigma$.

Example 1 ([10]). *For $\kappa > 0$ and $\rho \neq \sigma$, we define an FMS in different ways from an MS (ζ,ω) as follows:*

$$\bullet\ \Psi^{\omega}_{\rho\sigma}(\kappa) = \frac{\kappa}{\kappa+\omega_{\rho\sigma}} \quad \bullet\ \Psi^{e}_{\rho\sigma}(\kappa) = e^{-\frac{\omega_{\rho\sigma}}{\kappa}} \quad \bullet\ \Psi^{o}_{\rho\sigma}(\kappa) = \begin{cases} 0, & \text{if } \kappa \leq \omega_{\rho\sigma}, \\ 1, & \text{if } \kappa > \omega_{\rho\sigma}. \end{cases}$$

It is obvious that, under the product $\star = .$, (ζ,Ψ^{ω}) is an FMS, which is called the standard FMS on (ζ,ω). In addition, (ζ,Ψ^{ω}), (ζ,Ψ^{e}), and (ζ,Ψ^{o}) are FMSs under min. *This is a standard method for seeing the MS (ζ,ω) as an FMS, though it is not as well known.*

Moreover, (ζ,ω) is a CMS iff (ζ,Ψ^{ω}), (ζ,Ψ^{e}), or (ζ,Ψ^{o}) is a complete FMS.

3. Main Results

We begin this section with the following simple definition.

Definition 5. *Assume that $\Omega : \zeta^4 \to \zeta$ and $\urcorner : \zeta \to \zeta$ are two mappings.*

- *We say that Ω and $\urcorner$ are commuting if $\urcorner\Omega_{\rho\sigma\tau\upsilon} = \Omega_{\urcorner\rho\urcorner\sigma\urcorner\tau\urcorner\upsilon}$, $\forall \ell,\sigma,\rho,\upsilon \in \zeta$.*

- *We say that* $(\rho, \sigma, \tau, \upsilon) \in \zeta^4$ *is a QCP of* Ω *and* $\urcorner$ *if*

$$\Omega_{\rho\sigma\tau\upsilon} = \urcorner_\rho,\ \Omega_{\sigma\tau\upsilon\rho} = \urcorner_\sigma,\ \Omega_{\tau\upsilon\rho\sigma} = \urcorner_\tau \text{ and } \Omega_{\upsilon\rho\sigma\tau} = \urcorner_\upsilon.$$

Theorem 2. *Assume that* $\star$ *is a* κ*-norm of the* Υ*-type such that* $\mu \star \kappa \geq \mu\kappa$ *for all* $\mu, \kappa \in [0,1]$. *Suppose that* $(\zeta, \Psi, \star)$ *is a complete FMS and* $\Omega : \zeta^4 \to \zeta$, $\urcorner : \zeta \to \zeta$ *are two mappings such that*

(a) $\Omega(\zeta^4) \subseteq \urcorner(\zeta)$,
(b) $\urcorner$ *is continuous,*
(c) $\urcorner$ *is commuting with* Ω,
(d) *for all* $\rho, \sigma, \tau, \upsilon, \widehat{\rho}, \widehat{\sigma}, \widehat{\tau}, \widehat{\upsilon} \in \zeta$,

$$\Psi_{\Omega_{\rho\sigma\tau\upsilon}\Omega_{\widehat{\rho}\widehat{\sigma}\widehat{\tau}\widehat{\upsilon}}}(\kappa\varpi) \geq \Psi_{\urcorner\rho\urcorner\widehat{\rho}}(\kappa)^{\ell_1} \star \Psi_{\urcorner\sigma\urcorner\widehat{\sigma}}(\kappa)^{\ell_2} \star \Psi_{\urcorner\tau\urcorner\widehat{\tau}}(\kappa)^{\ell_3} \star \Psi_{\urcorner\upsilon\urcorner\widehat{\upsilon}}(\kappa)^{\ell_4}, \tag{1}$$

where $\varpi \in (0,1)$ *and* $\ell_1, \ell_2, \ell_3, \ell_4$ *are real numbers in* $[0,1]$ *such that* $\ell_1 + \ell_2 + \ell_3 + \ell_4 \leq 1$. *Then, the following conclusions hold.*

(1) *There is a unique* $\rho \in \zeta$ *such that* $\rho = \urcorner_\rho = \Omega_{\rho\rho\rho\rho}$. *In particular,*
(2) *There is at least a QCP for the mappings* $\urcorner$ *and* Ω*; moreover, in the case of* $\Omega = \rho_0$, *there is a constant on* ζ^4. *This holds only if the inverse of the mapping* $\urcorner$ *exists and it satisfies* $.\urcorner^{-1}(\rho_0) = \{\rho_0\}$*; then, we have*
(3) (ρ, ρ, ρ, ρ) *is a unique QCP of* $\urcorner$ *and* Ω.

Note that, to avoid the unidentified quantity 0^0, we consider here $\Psi_{\urcorner\rho\urcorner\widehat{\rho}}(\kappa)^0 = 1$ for all $\kappa > 0$ and all $\rho, \widehat{\rho} \in \zeta$.

Proof. We divide the proof into two cases:

Case 1. When $\Omega \subseteq \zeta$ is constant, that is, there is $\rho_0 \in \zeta$ such that, for all $\rho, \sigma, \tau, \upsilon \in \zeta$, $\Omega_{\rho\sigma\tau\upsilon} = \rho_0$. Since Ω and $\urcorner$ are commuting, one can write $\urcorner\rho_0 = \urcorner\Omega_{\rho\sigma\tau\upsilon} = \Omega_{\urcorner\rho\urcorner\sigma\urcorner\tau\urcorner\upsilon} = \rho_0$. Therefore, $\rho_0 = \urcorner\rho_0 = \Omega_{\rho_0\rho_0\rho_0\rho_0}$ and $(\rho_0, \rho_0, \rho_0, \rho_0)$ is a QCP of Ω and $\urcorner$. On the other hand, assume that $\urcorner^{-1}(\rho_0) = \{\rho_0\}$ and $(\rho, \sigma, \tau, \upsilon) \in \zeta^4$ is another QCP of Ω and $\urcorner$. Then, $\urcorner\rho = \Omega_{\rho\sigma\tau\upsilon} = \rho_0$, so $\rho \in \urcorner^{-1}(\rho_0) = \{\rho_0\}$. In the same manner, we can write $\rho = \sigma = \tau = \upsilon = \rho_0$; hence, $(\rho_0, \rho_0, \rho_0, \rho_0)$ is a unique QCP of Ω and $\urcorner$.

Case 2. Assume that $\Omega \in \zeta$ is not constant; for this, let $(\ell_1, \ell_2, \ell_3, \ell_4) \neq (0,0,0,0)$. In this case, we consider j and m to be non-negative integers and $\kappa \in [0, \infty)$. This case is divided into five steps.

St$_1$. Deriving four sequences $\{\rho_m\}$, $\{\sigma_m\}$, $\{\tau_m\}$, and $\{\upsilon_m\}$: Suppose that $\rho_0, \sigma_0, \tau_0, \upsilon_0$ are arbitrary points in ζ. As $\Omega(\zeta^4) \subseteq \urcorner(\zeta)$, we can select $\rho_1, \sigma_1, \tau_1, \upsilon_1 \in \zeta$ so that $\urcorner\rho_1 = \Omega_{\rho_0\sigma_0\tau_0\upsilon_0}$, $\urcorner\sigma_1 = \Omega_{\sigma_0\tau_0\upsilon_0\rho_0}$, $\urcorner\tau_1 = \Omega_{\tau_0\upsilon_0\rho_0\sigma_0}$ and $\urcorner\upsilon_1 = \Omega_{\upsilon_0\rho_0\sigma_0\tau_0}$. Again, with $\Omega(\zeta^4) \subseteq \urcorner(\zeta)$, we can select $\rho_2, \sigma_2, \tau_2, \upsilon_2 \in \zeta$ so that $\urcorner\rho_2 = \Omega_{\rho_1\sigma_1\tau_1\upsilon_1}$, $\urcorner\sigma_2 = \Omega_{\sigma_1\tau_1\upsilon_1\rho_1}$, $\urcorner\tau_2 = \Omega_{\tau_1\upsilon_1\rho_1\sigma_1}$, and $\urcorner\upsilon_2 = \Omega_{\upsilon_1\rho_1\sigma_1\tau_1}$. Continuing with the same scenario, we can construct $\{\rho_m\}$, $\{\sigma_m\}$, $\{\tau_m\}$, and $\{\upsilon_m\}$ so that for $m \geq 0$, $\urcorner\rho_{m+1} = \Omega_{\rho_m\sigma_m\tau_m\upsilon_m}$, $\urcorner\sigma_{m+1} = \Omega_{\sigma_m\tau_m\upsilon_m\rho_m}$, $\urcorner\tau_{m+1} = \Omega_{\tau_m\upsilon_m\rho_m\sigma_m}$, and $\urcorner\upsilon_{m+1} = \Omega_{\upsilon_m\rho_m\sigma_m\tau_m}$.

St$_2$. $\{\rho_m\}$, $\{\sigma_m\}$, $\{\tau_m\}$, and $\{\upsilon_m\}$ are Cauchy sequences. For $m \geq 0$ and all $\kappa > 0$, we define

$$\Xi_m(\kappa) = \Psi_{\urcorner\rho_m\urcorner\rho_{m+1}}(\kappa) \star \Psi_{\urcorner\sigma_m\urcorner\sigma_{m+1}} \star \Psi_{\urcorner\tau_m\urcorner\tau_{m+1}} \star \Psi_{\urcorner\upsilon_m\urcorner\upsilon_{m+1}}.$$

Ξ_m is a non-decreasing function and $\kappa - \kappa\varpi \leq \kappa \leq \frac{\kappa}{\varpi}$, so we get

$$\Xi_m(\kappa - \kappa\varpi) \leq \Xi_m(\kappa) \leq \Xi_m\left(\frac{\kappa}{\varpi}\right), \text{ for all } \kappa > 0 \text{ and } m \geq 0. \tag{2}$$

It follows from (1) that, for all $m \in \mathbb{N}$ and all $\kappa \geq 0$,

$$\begin{aligned} \Psi_{\urcorner\rho_m\urcorner\rho_{m+1}}(\kappa) &= \Psi_{\Omega_{\rho_{m-1}\sigma_{m-1}\tau_{m-1}\upsilon_{m-1}}\Omega_{\rho_m\sigma_m\tau_m\upsilon_m}}(\kappa) \\ &\geq \Psi_{\urcorner\rho_{m-1}\urcorner\rho_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_1} \star \Psi_{\urcorner\sigma_{m-1}\urcorner\sigma_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_2} \\ &\quad \star \Psi_{\urcorner\tau_{m-1}\urcorner\tau_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_3} \star \Psi_{\urcorner\upsilon_{m-1}\urcorner\upsilon_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_4}; \end{aligned} \tag{3}$$

$$\begin{aligned} \Psi_{\neg\sigma_m\neg\sigma_{m+1}}(\kappa) &= \Psi_{\Omega_{\sigma_{m-1}\tau_{m-1}\upsilon_{m-1}\rho_{m-1}}\Omega_{\sigma_m\tau_m\upsilon_m\rho_m}}(\kappa) \\ &\geq \Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_1} \star \Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_2} \\ &\quad \star \Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_3} \star \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_4}; \end{aligned} \tag{4}$$

$$\begin{aligned} \Psi_{\neg\tau_m\neg\tau_{m+1}}(\kappa) &= \Psi_{\Omega_{\tau_{m-1}\upsilon_{m-1}\rho_{m-1}\sigma_{m-1}}\Omega_{\tau_m\upsilon_m\rho_m\sigma_m}}(\kappa) \\ &\geq \Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_1} \star \Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_2} \\ &\quad \star \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_3} \star \Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_4}; \end{aligned} \tag{5}$$

$$\begin{aligned} \Psi_{\upsilon_m\neg\upsilon_{m+1}}(\kappa) &= \Psi_{\Omega_{\upsilon_{m-1}\rho_{m-1}\sigma_{m-1}\tau_{m-1}}\Omega_{\upsilon_m\rho_m\sigma_m\tau_m}}(\kappa) \\ &\geq \Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_1} \star \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_2} \\ &\quad \star \Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_3} \star \Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\tfrac{\kappa}{\varpi}\right)^{\ell_4}; \end{aligned} \tag{6}$$

It follows from (3)–(6) and Remark 1 that

$$\begin{aligned} &\Psi_{\neg\rho_m\neg\rho_{m+1}}(\kappa) \\ \geq\ & \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1} \star \Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2} \star \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3} \star \Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4} \\ \geq\ & \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right) \star \Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right) \star \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right) \star \Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right) \\ =\ & \Xi_{m-1}(\frac{\kappa}{\varpi}); \end{aligned}$$

$$\begin{aligned} &\Psi_{\neg\sigma_m\neg\sigma_{m+1}}(\kappa) \\ \geq\ & \Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1} \star \Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2} \star \Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3} \star \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4} \\ \geq\ & \Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right) \star \Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right) \star \Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right) \star \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right) \\ =\ & \Xi_{m-1}(\frac{\kappa}{\varpi}); \end{aligned}$$

$$\begin{aligned} &\Psi_{\neg\tau_m\neg\tau_{m+1}}(\kappa) \\ \geq\ & \Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1} \star \Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2} \star \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3} \star \Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4} \\ \geq\ & \Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right) \star \Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right) \star \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right) \star \Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right) \\ =\ & \Xi_{m-1}(\frac{\kappa}{\varpi}); \end{aligned}$$

and

$$\begin{aligned} &\Psi_{\upsilon\tau_m\neg\upsilon_{m+1}}(\kappa) \\ \geq\ & \Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1} \star \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2} \star \Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3} \star \Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4} \\ \geq\ & \Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right) \star \Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right) \star \Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right) \star \Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right) \\ =\ & \Xi_{m-1}(\frac{\kappa}{\varpi}). \end{aligned}$$

This proves that, for all $\kappa > 0$ and all $m \geq 0$,

$$\Psi_{\neg\rho_m\neg\rho_{m+1}}(\kappa), \Psi_{\neg\sigma_m\neg\sigma_{m+1}}(\kappa), \Psi_{\neg\tau_m\neg\tau_{m+1}}(\kappa), \Psi_{\upsilon\tau_m\neg\upsilon_{m+1}}(\kappa) \geq \Xi_{m-1}(\frac{\kappa}{\varpi}) \geq \Xi_{m-1}(\kappa). \tag{7}$$

Putting $\kappa - \varpi\kappa$ instead of κ, we obtain, for all $\kappa > 0$ and all $m \geq 0$, that

$$\begin{aligned}&\Psi_{\neg\rho_m\neg\rho_{m+1}}(\kappa-\varpi\kappa),\Psi_{\neg\sigma_m\neg\sigma_{m+1}}(\kappa-\varpi\kappa),\Psi_{\neg\tau_m\neg\tau_{m+1}}(\kappa-\varpi\kappa),\Psi_{\upsilon\tau_m\neg\upsilon_{m+1}}(\kappa-\varpi\kappa)\\ \geq\ &\Xi_{m-1}(\kappa-\varpi\kappa).\end{aligned}\tag{8}$$

Since $\star$ is commutative and $\star\geq .$, using (3)–(6), we deduce that

$$\begin{aligned}\Xi_m(\kappa)&=\Psi_{\neg\rho_m\neg\rho_{m+1}}(\kappa)\star\Psi_{\neg\sigma_m\neg\sigma_{m+1}}\star\Psi_{\neg\tau_m\neg\tau_{m+1}}\star\Psi_{\neg\upsilon_m\neg\upsilon_{m+1}}\\
&\geq\left(\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1}\star\Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}\star\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3}\star\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}\right)\\
&\quad\star\left(\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}\star\Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3}\star\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}\star\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1}\right)\\
&\quad\star\left(\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3}\star\Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}\star\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1}\star\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}\right)\\
&\quad\star\left(\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}\star\Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1}\star\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}\star\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3}\right)\\
&=\left(\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1}\star\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}\star\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3}\star\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}\right)\\
&\quad\star\left(\Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}\star\Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3}\star\Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}\star\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}\right)\\
&\quad\star\left(\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3}\star\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}\star\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1}\star\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}\right)\\
&\quad\star\left(\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}\star\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1}\star\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}\star\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3}\right).\end{aligned}$$

It follows that

$$\begin{aligned}\Xi_m(\kappa)&\geq\left(\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1}.\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}.\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3}.\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}\right)\\
&\quad\star\left(\Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}.\Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3}.\Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}.\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}\right)\\
&\quad\star\left(\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3}.\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}.\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1}.\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}\right)\\
&\quad\star\left(\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_4}.\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1}.\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_2}.\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_3}\right)\\
&=\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1+\ell_2+\ell_3+\ell_4}\star\Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1+\ell_2+\ell_3+\ell_4}\\
&\quad\star\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1+\ell_2+\ell_3+\ell_4}\star\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)^{\ell_1+\ell_2+\ell_3+\ell_4}\\
&\geq\Psi_{\neg\rho_{m-1}\neg\rho_m}\left(\frac{\kappa}{\varpi}\right)\star\Psi_{\neg\sigma_{m-1}\neg\sigma_m}\left(\frac{\kappa}{\varpi}\right)\star\Psi_{\neg\tau_{m-1}\neg\tau_m}\left(\frac{\kappa}{\varpi}\right)\star\Psi_{\neg\upsilon_{m-1}\neg\upsilon_m}\left(\frac{\kappa}{\varpi}\right)\\
&=\Xi_{m-1}(\frac{\kappa}{\varpi})\end{aligned}$$

By using (2), one can write

$$\Xi_m(\kappa)\geq\Xi_{m-1}(\frac{\kappa}{\varpi})\geq\Xi_{m-1}(\kappa)\geq\Xi_{m-1}(\kappa-\kappa\varpi),\ \forall\kappa>0,\ \text{and}\ m\geq 1.\tag{9}$$

By continuing in the same manner, we have

$$\Xi_m(\kappa)\geq\Xi_{m-1}(\frac{\kappa}{\varpi})\geq\Xi_{m-2}(\frac{\kappa}{\varpi^2})\geq\ldots\geq\Xi_0(\frac{\kappa}{\varpi^m}),\ \forall\kappa>0,\ \text{and}\ m\geq 1,$$

which leads to find that for all $\kappa>0$

$$\lim_{m\to\infty}\Xi_m(\kappa)\geq\lim_{m\to\infty}\Xi_0(\frac{\kappa}{\varpi^m})=1\Rightarrow\lim_{m\to\infty}\Xi_m(\kappa)=1.\tag{10}$$

From (7) and (9), we have

$$\Psi_{\daleth\rho_m\daleth\rho_{m+1}}(\kappa), \Psi_{\daleth\sigma_m\daleth\sigma_{m+1}}(\kappa), \Psi_{\daleth\tau_m\daleth\tau_{m+1}}(\kappa), \Psi_{\upsilon\tau_m\daleth\upsilon_{m+1}}(\kappa) \geq \Xi_m(\kappa) \geq \Xi_{m-1}(\kappa - \kappa\varpi). \quad (11)$$

After that, we will prove that, for all $\kappa > 0$ and all $m, r \geq 1$,

$$\Psi_{\daleth\rho_m\daleth\rho_{m+r}}(\kappa), \Psi_{\daleth\sigma_m\daleth\sigma_{m+r}}(\kappa), \Psi_{\daleth\tau_m\daleth\tau_{m+r}}(\kappa), \Psi_{\upsilon\tau_m\daleth\upsilon_{m+r}}(\kappa) \geq \star^r\Xi_{m-1}(\kappa - \kappa\varpi). \quad (12)$$

We can show this by induction in $r \geq 1$ as follows: Inequality (12) holds if $r = 1$ for all $m \geq 1$ and all $\kappa > 0$ by (11). Assume that (12) is true for all $m \geq 1$ and all $\kappa > 0$ for some r. Now, we prove the relation for $r + 1$. It follows from (1), the induction assumption, and $\star \geq .$ that

$$\begin{aligned}
&\Psi_{\daleth\rho_{m+1}\daleth\rho_{m+r+1}}(\varpi\kappa) \\
=\ &\Psi_{\Omega_{\rho_m\sigma_m\tau_m\upsilon_m}\Omega_{\rho_{m+r}\sigma_{m+r}\tau_{m+r}\upsilon_{m+r}}}(\varpi\kappa) \\
\geq\ &\Psi_{\daleth\rho_m\daleth\rho_{m+r}}(\kappa)^{\ell_1} \star \Psi_{\daleth\sigma_m\daleth\sigma_{m+r}}(\kappa)^{\ell_2} \star \Psi_{\daleth\tau_m\daleth\tau_{m+r}}(\kappa)^{\ell_3} \star \Psi_{\daleth\upsilon_m\daleth\upsilon_{m+r}}(\kappa)^{\ell_4} \\
\geq\ &(\star^r\Xi_{m-1}(\kappa-\kappa\varpi))^{\ell_1} \star (\star^r\Xi_{m-1}(\kappa-\kappa\varpi))^{\ell_2} \star (\star^r\Xi_{m-1}(\kappa-\kappa\varpi))^{\ell_3} \star (\star^r\Xi_{m-1}(\kappa-\kappa\varpi))^{\ell_4} \\
\geq\ &(\star^r\Xi_{m-1}(\kappa-\kappa\varpi))^{\ell_1}.(\star^r\Xi_{m-1}(\kappa-\kappa\varpi))^{\ell_2}.(\star^r\Xi_{m-1}(\kappa-\kappa\varpi))^{\ell_3}.(\star^r\Xi_{m-1}(\kappa-\kappa\varpi))^{\ell_4} \\
=\ &(\star^r\Xi_{m-1}(\kappa-\kappa\varpi))^{\ell_1+\ell_2+\ell_3+\ell_4} \geq \star^r\Xi_{m-1}(\kappa-\kappa\varpi).
\end{aligned}$$

Similarly, we arrive at

$$(\Psi_{\daleth\rho_{m+1}\daleth\rho_{m+r+1}}(\varpi\kappa), \Psi_{\daleth\sigma_{m+1}\daleth\sigma_{m+r+1}}(\varpi\kappa), \Psi_{\daleth\tau_{m+1}\daleth\tau_{m+r+1}}(\varpi\kappa), \Psi_{\daleth\upsilon_{m+1}\daleth\upsilon_{m+r+1}}(\varpi\kappa)) \geq \star^r\Xi_{m-1}(\kappa-\kappa\varpi).$$

From Definition 2 *(fms 5)*, (8), and the induction assumption, we get

$$\begin{aligned}
\Psi_{\daleth\rho_{m+1}\daleth\rho_{m+r+1}}(\kappa) &= \Psi_{\daleth\rho_{m+1}\daleth\rho_{m+r+1}}(\kappa-\kappa\varpi+\kappa\varpi) \\
&\geq \Psi_{\daleth\rho_m\daleth\rho_{m+1}}(\kappa-\kappa\varpi) \star \Psi_{\daleth\rho_{m+1}\daleth\rho_{m+r+1}}(\kappa\varpi) \\
&\geq \Xi_{m-1}(\kappa-\kappa\varpi) \star (\star^r\Xi_{m-1}(\kappa-\kappa\varpi)) \\
&= \star^{r+1}\Xi_{m-1}(\kappa-\kappa\varpi).
\end{aligned}$$

In addition, the same result holds if we consider $\Psi_{\daleth\sigma_{m+1}\daleth\sigma_{m+r+1}}(\kappa)$, $\Psi_{\daleth\tau_{m+1}\daleth\tau_{m+r+1}}(\kappa)$, and $\Psi_{\daleth\upsilon_{m+1}\daleth\upsilon_{m+r+1}}(\kappa)$. This leads to (12) being true. This allows us to prove that $\{\daleth\rho_m\}$ is Cauchy. Assume that $\kappa > 0$ and $\varepsilon \in (0,1)$ are given. From this assumption, as $\star$ is a κ-norm of the Y-type, there is $\varphi \in (0,1)$ such that $\star^r\ell_1 > 1-\varepsilon$ for all $\ell_1 \in (1-\varphi, 1]$ and for all $r \geq 1$. From (10), $\lim_{m\to\infty}\Xi_m(\kappa) = 1$, so there is $m_0 \in \mathbb{N}$ such that

$$\Xi_m(\kappa-\kappa\varpi) > 1-\varphi,\ \forall m \geq m_0.$$

Hence, by (12), we have

$$\Psi_{\daleth\rho_m\daleth\rho_{m+r}}(\kappa), \Psi_{\daleth\sigma_m\daleth\sigma_{m+r}}(\kappa), \Psi_{\daleth\tau_m\daleth\tau_{m+r}}(\kappa), \Psi_{\upsilon\tau_m\daleth\upsilon_{m+r}}(\kappa) > 1-\varepsilon,\ \forall m \geq m_0 \text{ and } r \geq 1.$$

Thus, $\{\daleth\rho_m\}$ is a Cauchy sequence. Similarly, $\{\daleth\sigma_m\}$, $\{\daleth\tau_m\}$, and $\{\daleth\upsilon_m\}$ are also Cauchy sequences.

St$_3$. Proving that Ω and $\daleth$ have a QCP: As ζ is complete, there are $\rho, \sigma, \tau, \upsilon \in \zeta$ such that

$$\lim_{m\to\infty}\daleth\rho_m = \rho,\ \lim_{m\to\infty}\daleth\sigma_m = \sigma,\ \lim_{m\to\infty}\daleth\tau_m = \tau \text{ and } \lim_{m\to\infty}\daleth\upsilon_m = \upsilon.$$

The continuity of $\daleth$ implies that

$$\lim_{m\to\infty}\daleth\daleth\rho_m = \daleth\rho,\ \lim_{m\to\infty}\daleth\daleth\sigma_m = \daleth\sigma,\ \lim_{m\to\infty}\daleth\daleth\tau_m = \daleth\tau, \text{ and } \lim_{m\to\infty}\daleth\daleth\upsilon_m = \daleth\upsilon.$$

The commutativity of Ω and $\daleth$ leads to

$$\daleth\daleth\rho_{m+1} = \daleth\Omega(\rho_m, \sigma_m, \tau_m, \upsilon_m) = \Omega(\daleth\rho_m, \daleth\sigma_m, \daleth\tau_m, \daleth\upsilon_m).$$

By (1), we get

$$\begin{aligned} \Psi_{\daleth\daleth\rho_{m+1}\Omega_{\rho\sigma\tau\upsilon}}(\kappa\varpi) &= \Psi_{\Omega_{\daleth\rho_m\sigma_m\tau_m\upsilon_m}\Omega_{\rho\sigma\tau\upsilon}}(\kappa\varpi) \\ &\geq \Psi_{\daleth\daleth\rho_m\daleth\rho}(\kappa)^{\ell_1} \star \Psi_{\daleth\daleth\sigma_m\daleth\sigma}(\kappa)^{\ell_2} \star \Psi_{\daleth\daleth\tau_m\daleth\tau}(\kappa)^{\ell_3} \star \Psi_{\daleth\daleth\upsilon_m\daleth\upsilon}(\kappa)^{\ell_4} \\ &\geq \Psi_{\daleth\daleth\rho_m\daleth\rho}(\kappa) \star \Psi_{\daleth\daleth\sigma_m\daleth\sigma}(\kappa) \star \Psi_{\daleth\daleth\tau_m\daleth\tau}(\kappa) \star \Psi_{\daleth\daleth\upsilon_m\daleth\upsilon}(\kappa). \end{aligned} \tag{13}$$

As $m \to \infty$, in (13), we find that

$$\lim_{m\to\infty} \daleth\daleth\rho_{m+1} = \Omega_{\rho\sigma\tau\upsilon} = \daleth\rho.$$

Similarly, we deduce that $\Omega_{\sigma\tau\upsilon\rho} = \daleth\sigma$, $\Omega_{\tau\upsilon\rho\sigma} = \daleth\tau$, $\Omega_{\upsilon\rho\sigma\tau} = \daleth\upsilon$. This shows that $(\rho, \sigma, \tau, \upsilon)$ is a QCP of Ω and $\daleth$.

$$\daleth\rho = \Omega_{\rho\sigma\tau\upsilon}, \ \daleth\sigma = \Omega_{\sigma\tau\upsilon\rho}, \ \daleth\tau = \Omega_{\tau\upsilon\rho\sigma}, \ \daleth\upsilon = \Omega_{\upsilon\rho\sigma\tau}. \tag{14}$$

St$_4$. Showing that $\rho = \Omega_{\rho\sigma\tau\upsilon}$, $\sigma = \Omega_{\sigma\tau\upsilon\rho}$, $\tau = \Omega_{\tau\upsilon\rho\sigma}$, and $\upsilon = \Omega_{\upsilon\rho\sigma\tau}$: From Stipulation (1), we get

$$\begin{aligned} &\Psi_{\daleth\rho\daleth\sigma_{m+1}}(\kappa\varpi) \\ = \ &\Psi_{\Omega_{\rho\sigma\tau\upsilon}\Omega_{\sigma_m\tau_m\upsilon_m\rho_m}}(\kappa\varpi) \\ \geq \ &\Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_1} \star \Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_2} \star \Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_3} \star \Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_4}; \end{aligned} \tag{15}$$

$$\begin{aligned} &\Psi_{\daleth\sigma\daleth\tau_{m+1}}(\kappa\varpi) \\ = \ &\Psi_{\Omega_{\sigma\tau\upsilon\rho}\Omega_{\tau_m\upsilon_m\rho_m\sigma_m}}(\kappa\varpi) \\ \geq \ &\Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_1} \star \Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_2} \star \Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_3} \star \Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_4}; \end{aligned} \tag{16}$$

$$\begin{aligned} &\Psi_{\daleth\tau\daleth\upsilon_{m+1}}(\kappa\varpi) \\ = \ &\Psi_{\Omega_{\tau\upsilon\rho\sigma}\Omega_{\upsilon_m\rho_m\sigma_m\tau_m}}(\kappa\varpi) \\ \geq \ &\Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_1} \star \Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_2} \star \Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_3} \star \Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_4}; \end{aligned} \tag{17}$$

$$\begin{aligned} &\Psi_{\daleth\upsilon\daleth\rho_{m+1}}(\kappa\varpi) \\ = \ &\Psi_{\Omega_{\upsilon\rho\sigma\tau}\Omega_{\rho_m\sigma_m\tau_m\upsilon_m}}(\kappa\varpi) \\ \geq \ &\Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_1} \star \Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_2} \star \Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_3} \star \Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_4}. \end{aligned} \tag{18}$$

We set $\nabla_m(\kappa\varpi) = \Psi_{\daleth\rho\daleth\sigma_m}(\kappa\varpi) \star \Psi_{\daleth\sigma\daleth\tau_m}(\kappa\varpi) \star \Psi_{\daleth\tau\daleth\upsilon_m}(\kappa\varpi) \star \Psi_{\daleth\upsilon\daleth\rho_m}(\kappa\varpi)$ for all $\kappa > 0$ and $m \geq 0$. It follows from (15)–(18) that

$$\begin{aligned} \nabla_{m+1}(\kappa\varpi) &= \Psi_{\daleth\rho\daleth\sigma_{m+1}}(\kappa\varpi) \star \Psi_{\daleth\sigma\daleth\tau_{m+1}}(\kappa\varpi) \star \Psi_{\daleth\tau\daleth\upsilon_{m+1}}(\kappa\varpi) \star \Psi_{\daleth\tau\daleth\upsilon_{m+1}}(\kappa\varpi) \\ &\geq \left(\Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_1} \star \Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_2} \star \Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_3} \star \Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_4}\right) \\ &\quad \star\left(\Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_1} \star \Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_2} \star \Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_3} \star \Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_4}\right) \\ &\quad \star\left(\Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_1} \star \Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_2} \star \Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_3} \star \Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_4}\right) \\ &\quad \star\left(\Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_1} \star \Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_2} \star \Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_3} \star \Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_4}\right) \\ &= \left(\Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_1} \star \Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_4} \star \Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_3} \star \Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_2}\right) \\ &\quad \star\left(\Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_2} \star \Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_1} \star \Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_4} \star \Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_3}\right) \\ &\quad \star\left(\Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_3} \star \Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_2} \star \Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_1} \star \Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_4}\right) \\ &\quad \star\left(\Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_4} \star \Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_3} \star \Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_2} \star \Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_1}\right), \end{aligned}$$

which implies that

$$\begin{aligned}
\nabla_{m+1}(\kappa\varpi) &\geq \left(\Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_1}.\Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_4}.\Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_3}.\Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_2}\right) \\
&\star\left(\Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_2}.\Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_1}.\Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_4}.\Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_3}\right) \\
&\star\left(\left(\Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_3}.\Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_2}.\Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_1}.\Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_4}\right)\right) \\
&\star\left(\Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_4}.\Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_3}.\Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_2}.\Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_1}\right) \\
&= \Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4}\star\Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \\
&\star\Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4}\star\Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \\
&\geq \Psi_{\daleth\rho\daleth\sigma_m}(\kappa)\star\Psi_{\daleth\sigma\daleth\tau_m}(\kappa)\star\Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)\star\Psi_{\daleth\upsilon\daleth\rho_m}(\kappa) = \nabla_m(\kappa).
\end{aligned}$$

This implies that $\nabla_{m+1}(\kappa\varpi) \geq \nabla_m(\kappa)$ for all $m \geq 0$ and all $\kappa > 0$. Repeating this process,

$$\nabla_m(\kappa) \geq \nabla_{m-1}(\frac{\kappa}{\varpi}) \geq \nabla_{m-2}(\frac{\kappa}{\varpi^2}) \geq ... \geq \nabla_0(\frac{\kappa}{\varpi^m}),\ \forall\kappa > 0 \text{ and } m \geq 1. \tag{19}$$

From (15)–(19), we conclude that

$$\begin{aligned}
\Psi_{\daleth\rho\daleth\sigma_{m+1}}(\kappa\varpi) &\geq \Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_1}\star\Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_2}\star\Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_3}\star\Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_4} \\
&\geq \nabla_m(\kappa) \geq \nabla_0(\tfrac{\kappa}{\varpi^m});
\end{aligned} \tag{20}$$

$$\begin{aligned}
\Psi_{\daleth\sigma\daleth\tau_{m+1}}(\kappa\varpi) &\geq \Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_1}\star\Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_2}\star\Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_3}\star\Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_4} \\
&\geq \nabla_m(\kappa) \geq \nabla_0(\tfrac{\kappa}{\varpi^m});
\end{aligned} \tag{21}$$

$$\begin{aligned}
\Psi_{\daleth\tau\daleth\upsilon_{m+1}}(\kappa\varpi) &\geq \Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_1}\star\Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_2}\star\Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_3}\star\Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_4} \\
&\geq \nabla_m(\kappa) \geq \nabla_0(\tfrac{\kappa}{\varpi^m});
\end{aligned} \tag{22}$$

$$\begin{aligned}
\Psi_{\daleth\upsilon\daleth\rho_{m+1}}(\kappa\varpi) &\geq \Psi_{\daleth\upsilon\daleth\rho_m}(\kappa)^{\ell_1}\star\Psi_{\daleth\rho\daleth\sigma_m}(\kappa)^{\ell_2}\star\Psi_{\daleth\sigma\daleth\tau_m}(\kappa)^{\ell_3}\star\Psi_{\daleth\tau\daleth\upsilon_m}(\kappa)^{\ell_4} \\
&\geq \nabla_m(\kappa) \geq \nabla_0(\tfrac{\kappa}{\varpi^m}).
\end{aligned} \tag{23}$$

Thus,

$$\Psi_{\daleth\rho\daleth\sigma_{m+1}}(\kappa\varpi), \Psi_{\daleth\sigma\daleth\tau_{m+1}}(\kappa\varpi), \Psi_{\daleth\tau\daleth\upsilon_{m+1}}(\kappa\varpi), \Psi_{\daleth\upsilon\daleth\rho_{m+1}}(\kappa\varpi) \geq \nabla_0(\frac{\kappa}{\varpi^m}),\ \forall\kappa > 0 \text{ and } m \geq 1.$$

Taking the limit as $m \to \infty$ in (20)–(23) and using $\lim_{m\to\infty}\nabla_0(\frac{\kappa}{\varpi^m}) = 1$, for all $\kappa > 0$, we get $\lim_{m\to\infty}\daleth\rho_m = \daleth\upsilon$, $\lim_{m\to\infty}\daleth\sigma_m = \daleth\rho$, $\lim_{m\to\infty}\daleth\tau_m = \daleth\sigma$, and $\lim_{m\to\infty}\daleth\upsilon_m = \daleth\tau$. This shows, together with (14), that

$$\begin{aligned}
\Omega_{\rho\sigma\tau\upsilon} &= \daleth\rho = \lim_{m\to\infty}\daleth\sigma_m = \sigma,\ \Omega_{\sigma\tau\upsilon\rho} = \daleth\sigma = \lim_{m\to\infty}\daleth\tau_m = \tau, \\
\Omega_{\tau\upsilon\rho\sigma} &= \daleth\tau = \lim_{m\to\infty}\daleth\upsilon_m = \upsilon,\ \Omega_{\upsilon\rho\sigma\tau} = \daleth\upsilon = \lim_{m\to\infty}\daleth\rho_m = \rho.
\end{aligned}$$

St$_5$. We shall prove that $\rho = \sigma = \tau = \upsilon$. We set $\Pi(\kappa) = \Psi_{\rho\sigma}(\kappa)\star\Psi_{\sigma\tau}(\kappa)\star\Psi_{\tau\upsilon}(\kappa)\star\Psi_{\upsilon\rho}(\kappa)$ for all $\kappa > 0$. Then, according to (1), we can write

$$\begin{aligned}
\Psi_{\rho\sigma}(\kappa\varpi) &= \Psi_{\Omega_{\rho\sigma\tau\upsilon}\Omega_{\sigma\tau\upsilon\rho}}(\kappa\varpi) \geq \Psi_{\daleth\rho\daleth\sigma}(\kappa)^{\ell_1}\star\Psi_{\daleth\sigma\daleth\tau}(\kappa)^{\ell_2}\star\Psi_{\daleth\tau\daleth\upsilon}(\kappa)^{\ell_3}\star\Psi_{\daleth\upsilon\daleth\rho}(\kappa)^{\ell_4} \\
&= \Psi_{\sigma\tau}(\kappa)^{\ell_1}\star\Psi_{\tau\upsilon}(\kappa)^{\ell_2}\star\Psi_{\upsilon\rho}(\kappa)^{\ell_3}\star\Psi_{\rho\sigma}(\kappa)^{\ell_4};
\end{aligned} \tag{24}$$

$$\begin{aligned}
\Psi_{\sigma\tau}(\kappa\varpi) &= \Psi_{\Omega_{\sigma\tau\upsilon\rho}\Omega_{\tau\upsilon\rho\sigma}}(\kappa\varpi) \geq \Psi_{\daleth\sigma\daleth\tau}(\kappa)^{\ell_1}\star\Psi_{\daleth\tau\daleth\upsilon}(\kappa)^{\ell_2}\star\Psi_{\daleth\upsilon\daleth\rho}(\kappa)^{\ell_3}\star\Psi_{\daleth\rho\daleth\sigma}(\kappa)^{\ell_4} \\
&= \Psi_{\tau\upsilon}(\kappa)^{\ell_1}\star\Psi_{\upsilon\rho}(\kappa)^{\ell_2}\star\Psi_{\rho\sigma}(\kappa)^{\ell_3}\star\Psi_{\sigma\tau}(\kappa)^{\ell_4};
\end{aligned} \tag{25}$$

$$\begin{aligned}
\Psi_{\tau\upsilon}(\kappa\varpi) &= \Psi_{\Omega_{\tau\upsilon\rho\sigma}\Omega_{\upsilon\rho\sigma\tau}}(\kappa\varpi) \geq \Psi_{\daleth\tau\daleth\upsilon}(\kappa)^{\ell_1}\star\Psi_{\daleth\upsilon\daleth\rho}(\kappa)^{\ell_2}\star\Psi_{\daleth\rho\daleth\sigma}(\kappa)^{\ell_3}\star\Psi_{\daleth\sigma\daleth\tau}(\kappa)^{\ell_4} \\
&= \Psi_{\upsilon\rho}(\kappa)^{\ell_1}\star\Psi_{\rho\sigma}(\kappa)^{\ell_2}\star\Psi_{\sigma\tau}(\kappa)^{\ell_3}\star\Psi_{\tau\upsilon}(\kappa)^{\ell_4};
\end{aligned} \tag{26}$$

$$\begin{aligned}
\Psi_{\upsilon\rho}(\kappa\varpi) &= \Psi_{\Omega_{\upsilon\rho\sigma\tau}\Omega_{\rho\sigma\tau\upsilon}}(\kappa\varpi) \geq \Psi_{\daleth\upsilon\daleth\rho}(\kappa)^{\ell_1}\star\Psi_{\daleth\rho\daleth\sigma}(\kappa)^{\ell_2}\star\Psi_{\daleth\sigma\daleth\tau}(\kappa)^{\ell_3}\star\Psi_{\daleth\tau\daleth\upsilon}(\kappa)^{\ell_4} \\
&= \Psi_{\rho\sigma}(\kappa)^{\ell_1}\star\Psi_{\sigma\tau}(\kappa)^{\ell_2}\star\Psi_{\tau\upsilon}(\kappa)^{\ell_3}\star\Psi_{\upsilon\rho}(\kappa)^{\ell_4}.
\end{aligned} \tag{27}$$

Using the above four inequalities together, we have

$$
\begin{aligned}
\Pi(\kappa\varpi) &= \Psi_{\rho\sigma}(\kappa\varpi) \star \Psi_{\sigma\tau}(\kappa\varpi) \star \Psi_{\tau\upsilon}(\kappa\varpi) \star \Psi_{\upsilon\rho}(\kappa\varpi) \\
&\geq \left(\Psi_{\sigma\tau}(\kappa)^{\ell_1} \star \Psi_{\tau\upsilon}(\kappa)^{\ell_2} \star \Psi_{\upsilon\rho}(\kappa)^{\ell_3} \star \Psi_{\rho\sigma}(\kappa)^{\ell_4}\right) \\
&\quad \star \left(\Psi_{\tau\upsilon}(\kappa)^{\ell_1} \star \Psi_{\upsilon\rho}(\kappa)^{\ell_2} \star \Psi_{\rho\sigma}(\kappa)^{\ell_3} \star \Psi_{\sigma\tau}(\kappa)^{\ell_4}\right) \\
&\quad \star \left(\Psi_{\upsilon\rho}(\kappa)^{\ell_1} \star \Psi_{\rho\sigma}(\kappa)^{\ell_2} \star \Psi_{\sigma\tau}(\kappa)^{\ell_3} \star \Psi_{\tau\upsilon}(\kappa)^{\ell_4}\right) \\
&\quad \star \left(\Psi_{\rho\sigma}(\kappa)^{\ell_1} \star \Psi_{\sigma\tau}(\kappa)^{\ell_2} \star \Psi_{\tau\upsilon}(\kappa)^{\ell_3} \star \Psi_{\upsilon\rho}(\kappa)^{\ell_4}\right) \\
&= \left(\Psi_{\rho\sigma}(\kappa)^{\ell_4} \star \Psi_{\rho\sigma}(\kappa)^{\ell_3} \star \Psi_{\rho\sigma}(\kappa)^{\ell_2} \star \Psi_{\rho\sigma}(\kappa)^{\ell_1}\right) \\
&\quad \star \left(\Psi_{\sigma\tau}(\kappa)^{\ell_1} \star \Psi_{\sigma\tau}(\kappa)^{\ell_4} \star \Psi_{\sigma\tau}(\kappa)^{\ell_3} \star \Psi_{\sigma\tau}(\kappa)^{\ell_2}\right) \\
&\quad \star \left(\Psi_{\tau\upsilon}(\kappa)^{\ell_2} \star \Psi_{\tau\upsilon}(\kappa)^{\ell_1} \star \Psi_{\tau\upsilon}(\kappa)^{\ell_4} \star \Psi_{\tau\upsilon}(\kappa)^{\ell_3}\right) \\
&\quad \star \left(\Psi_{\upsilon\rho}(\kappa)^{\ell_3} \star \Psi_{\upsilon\rho}(\kappa)^{\ell_2} \star \Psi_{\upsilon\rho}(\kappa)^{\ell_1} \star \Psi_{\upsilon\rho}(\kappa)^{\ell_4}\right) \\
&\geq \left(\Psi_{\rho\sigma}(\kappa)^{\ell_4} . \Psi_{\rho\sigma}(\kappa)^{\ell_3} . \Psi_{\rho\sigma}(\kappa)^{\ell_2} . \Psi_{\rho\sigma}(\kappa)^{\ell_1}\right) \\
&\quad \star \left(\Psi_{\sigma\tau}(\kappa)^{\ell_1} . \Psi_{\sigma\tau}(\kappa)^{\ell_4} . \Psi_{\sigma\tau}(\kappa)^{\ell_3} . \Psi_{\sigma\tau}(\kappa)^{\ell_2}\right) \\
&\quad \star \left(\Psi_{\tau\upsilon}(\kappa)^{\ell_2} . \Psi_{\tau\upsilon}(\kappa)^{\ell_1} . \Psi_{\tau\upsilon}(\kappa)^{\ell_4} . \Psi_{\tau\upsilon}(\kappa)^{\ell_3}\right) \\
&\quad \star \left(\Psi_{\upsilon\rho}(\kappa)^{\ell_3} . \Psi_{\upsilon\rho}(\kappa)^{\ell_2} . \Psi_{\upsilon\rho}(\kappa)^{\ell_1} . \Psi_{\upsilon\rho}(\kappa)^{\ell_4}\right) \\
&= \Psi_{\rho\sigma}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \star \Psi_{\sigma\tau}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \\
&\quad \star \Psi_{\tau\upsilon}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \star \Psi_{\upsilon\rho}(\kappa)^{\ell_1+\ell_2+\ell_3+\ell_4} \\
&\geq \Psi_{\rho\sigma}(\kappa) \star \Psi_{\sigma\tau}(\kappa) \star \Psi_{\tau\upsilon}(\kappa) \star \Psi_{\upsilon\rho}(\kappa) = \Pi(\kappa).
\end{aligned}
$$

Thus, $\Pi(\kappa\varpi) \geq \Pi(\kappa)$ leads to $\Pi(\kappa) \geq \Pi(\frac{\kappa}{\varpi}) \geq \Pi(\frac{\kappa}{\varpi^2}) \geq \ldots \geq \Pi(\frac{\kappa}{\varpi^m})$ for all $\kappa > 0$ and $m \geq 1$. Applying (24)–(27), we get

$$
\begin{aligned}
\Psi_{\rho\sigma}(\kappa\varpi) &\geq \Psi_{\sigma\tau}(\kappa)^{\ell_1} \star \Psi_{\tau\upsilon}(\kappa)^{\ell_2} \star \Psi_{\upsilon\rho}(\kappa)^{\ell_3} \star \Psi_{\rho\sigma}(\kappa)^{\ell_4} \\
&\geq \Psi_{\sigma\tau}(\kappa) \star \Psi_{\tau\upsilon}(\kappa) \star \Psi_{\upsilon\rho}(\kappa) \star \Psi_{\rho\sigma}(\kappa) = \Pi(\kappa) \geq \Pi(\frac{\kappa}{\varpi^m}), \\
\Psi_{\sigma\tau}(\kappa\varpi) &\geq \Psi_{\tau\upsilon}(\kappa)^{\ell_1} \star \Psi_{\upsilon\rho}(\kappa)^{\ell_2} \star \Psi_{\rho\sigma}(\kappa)^{\ell_3} \star \Psi_{\sigma\tau}(\kappa)^{\ell_4} \\
&\geq \Psi_{\tau\upsilon}(\kappa) \star \Psi_{\upsilon\rho}(\kappa) \star \Psi_{\rho\sigma}(\kappa) \star \Psi_{\sigma\tau}(\kappa) = \Pi(\kappa) \geq \Pi(\frac{\kappa}{\varpi^m}), \\
\Psi_{\tau\upsilon}(\kappa\varpi) &\geq \Psi_{\upsilon\rho}(\kappa)^{\ell_1} \star \Psi_{\rho\sigma}(\kappa)^{\ell_2} \star \Psi_{\sigma\tau}(\kappa)^{\ell_3} \star \Psi_{\tau\upsilon}(\kappa)^{\ell_4} \\
&\geq \Psi_{\upsilon\rho}(\kappa) \star \Psi_{\rho\sigma}(\kappa) \star \Psi_{\sigma\tau}(\kappa) \star \Psi_{\tau\upsilon}(\kappa) = \Pi(\kappa) \geq \Pi(\frac{\kappa}{\varpi^m}), \\
\Psi_{\upsilon\rho}(\kappa\varpi) &\geq \Psi_{\rho\sigma}(\kappa)^{\ell_1} \star \Psi_{\sigma\tau}(\kappa)^{\ell_2} \star \Psi_{\tau\upsilon}(\kappa)^{\ell_3} \star \Psi_{\upsilon\rho}(\kappa)^{\ell_4} \\
&\geq \Psi_{\rho\sigma}(\kappa) \star \Psi_{\sigma\tau}(\kappa) \star \Psi_{\tau\upsilon}(\kappa) \star \Psi_{\upsilon\rho}(\kappa) = \Pi(\kappa) \geq \Pi(\frac{\kappa}{\varpi^m}).
\end{aligned}
$$

As $m \to \infty$, we have $\lim_{m\to\infty} \Pi(\frac{\kappa}{\varpi^m}) = 1$ for all $m \geq 1$. This means that $\Psi_{\rho\sigma}(\kappa\varpi) = \Psi_{\sigma\tau}(\kappa\varpi) = \Psi_{\tau\upsilon}(\kappa\varpi) = \Psi_{\upsilon\rho}(\kappa\varpi) = 1$ for all $\kappa > 0$, that is, $\rho = \sigma = \tau = \upsilon$. The uniqueness of ρ follows from (1). $\square$

Remark 2. *In Theorem 2, the continuity of $\star$ is only discussed at $(1,1)$, that is, if $\{\rho_m\}, \{\sigma_m\} \subset [0,1]$ are sequences such that $\{\rho_m\} \to 1$ and $\{\sigma_m\} \to 1$; therefore, $\{\rho_m \star \sigma_m\} \to 1$, which holds because $\{\rho_m \star \sigma_m\} \geq \{\rho_m . \sigma_m\} \to 1 \times 1 = 1$.*

Example 2. *Assume that $\zeta = \mathbb{R}$ and $(\mathbb{R}, \Psi^e)$ is defined as Example 1. Consider $\wp, \hbar > 0$ and $\varpi \in (0,1)$ are real numbers such that $8\wp \leq \hbar\varpi$, that is, $\frac{\wp}{\varpi} \leq \frac{\hbar}{8}$. For all $\rho, \sigma, \tau, \upsilon \in \mathbb{R}$, we define $\Omega : \mathbb{R}^4 \to \mathbb{R}$ and $\daleth : \mathbb{R} \to \mathbb{R}$ by $\Omega_{\rho\sigma\tau\upsilon} = \frac{\wp}{2}(\rho - \sigma)$ and $\daleth(\rho) = \frac{\hbar}{2}\rho$. It is clear that $\daleth$ is continuous, Ω and $\daleth$ are commuting, and $\Omega(\mathbb{R}^4) = \mathbb{R} = \daleth(\mathbb{R})$. Moreover, Ψ^e satisfies*

$$
\begin{aligned}
\Psi^e_{\Omega_{\rho\sigma\tau\upsilon}\Omega_{\widehat{\rho}\widehat{\sigma}\widehat{\tau}\widehat{\upsilon}}}(\varpi\kappa) &= \left(e^{|(\rho-\widehat{\rho})+(\sigma-\widehat{\sigma})|}\right)^{-\frac{\wp}{2\varpi\kappa}} \geq \left(e^{-\frac{2\max\{|(\rho-\widehat{\rho})|,|(\sigma-\widehat{\sigma})|\}}{2\kappa}}\right)^{\frac{\wp}{\varpi}} \\
&\geq \left(e^{-\frac{2\max\{|(\rho-\widehat{\rho})|,|(\sigma-\widehat{\sigma})|\}}{2\kappa}}\right)^{\frac{\hbar}{8}} = \left(e^{-\frac{\hbar}{8\kappa}}\right)^{\max\{|(\rho-\widehat{\rho})|,|(\sigma-\widehat{\sigma})|\}} \\
&= \min\left\{e^{-\frac{\hbar|(\rho-\widehat{\rho})|}{8\kappa}}, e^{-\frac{\hbar|(\sigma-\widehat{\sigma})|}{8\kappa}}\right\} \\
&\geq \min\left\{e^{-\frac{\hbar|(\rho-\widehat{\rho})|}{8\kappa}}, e^{-\frac{\hbar|(\sigma-\widehat{\sigma})|}{8\kappa}}, e^{-\frac{\hbar|(\tau-\widehat{\tau})|}{8\kappa}}, e^{-\frac{\hbar|(\upsilon-\widehat{\upsilon})|}{8\kappa}}\right\} \\
&= \min\left\{e^{-\frac{\hbar|(\rho-\widehat{\rho})|}{2(4\kappa)}}, e^{-\frac{\hbar|(\sigma-\widehat{\sigma})|}{2(4\kappa)}}, e^{-\frac{\hbar|(\tau-\widehat{\tau})|}{2(4\kappa)}}, e^{-\frac{\hbar|(\upsilon-\widehat{\upsilon})|}{2(4\kappa)}}\right\} \\
&= \min\left\{\left(\Psi^e_{\daleth_\rho\daleth_{\widehat{\rho}}}(\kappa)\right)^{\frac{1}{4}}, \left(\Psi^e_{\daleth_\sigma\daleth_{\widehat{\sigma}}}(\kappa)\right)^{\frac{1}{4}}, \left(\Psi^e_{\daleth_\tau\daleth_{\widehat{\tau}}}(\kappa)\right)^{\frac{1}{4}}, \left(\Psi^e_{\daleth_\upsilon\daleth_{\widehat{\upsilon}}}(\kappa)\right)^{\frac{1}{4}}\right\}.
\end{aligned}
$$

Thus, through Theorem 2, we deduce that Ω and $\daleth$ have a QCP.

4. Some Related Results

In this section, the view of (ζ, ω) as a friable FMS $(\zeta, \Psi^o, \min)$ is used. This tactic permits us to deduce some results involved in the metric space from the corresponding results in the fuzzy setting. Furthermore, without a partially ordered set, Theorem 3 is just a QCP result, similar to that of Karapınar and Luong ([28], Corollary 12).

Theorem 3. *Assume that (ζ, ω) is a CMS and that $\Omega : \zeta^4 \to \zeta$ and $\daleth : \zeta \to \zeta$ are two mappings such that:*

- $\Omega(\zeta^4) \subseteq \daleth(\zeta)$;
- $\daleth$ *is continuous;*
- $\daleth$ *is commuting with* Ω.

If Ω and $\daleth$ satisfy some of the conditions below for $\rho, \sigma, \tau, \upsilon, \widehat{\rho}, \widehat{\sigma}, \widehat{\tau}, \widehat{\upsilon} \in \zeta$:

(i) *for some $0 < \varpi < 1$,*

$$\omega_{\Omega_{\rho\sigma\tau\upsilon}\Omega_{\widehat{\rho}\widehat{\sigma}\widehat{\tau}\widehat{\upsilon}}} \leq \varpi \max\left\{\omega_{\daleth_\rho\daleth_{\widehat{\rho}}}, \omega_{\daleth_\sigma\daleth_{\widehat{\sigma}}}, \omega_{\daleth_\tau\daleth_{\widehat{\tau}}}, \omega_{\daleth_\upsilon\daleth_{\widehat{\upsilon}}}\right\}.$$

(ii) *for some $0 < \varpi < 1$ and some $\wp_1, \wp_2, \wp_3, \wp_4 \in [0, \frac{1}{3}]$,*

$$\omega_{\Omega_{\rho\sigma\tau\upsilon}\Omega_{\widehat{\rho}\widehat{\sigma}\widehat{\tau}\widehat{\upsilon}}} \leq \varpi\left(\wp_1\omega_{\daleth_\rho\daleth_{\widehat{\rho}}} + \wp_2\omega_{\daleth_\sigma\daleth_{\widehat{\sigma}}} + \wp_3\omega_{\daleth_\tau\daleth_{\widehat{\tau}}} + \wp_4\omega_{\daleth_\upsilon\daleth_{\widehat{\upsilon}}}\right).$$

(iii) *for some $\wp_1, \wp_2, \wp_3, \wp_4 \in [0,1)$ with $\wp_1 + \wp_2 + \wp_3 + \wp_4 < 1$,*

$$\omega_{\Omega_{\rho\sigma\tau\upsilon}\Omega_{\widehat{\rho}\widehat{\sigma}\widehat{\tau}\widehat{\upsilon}}} \leq \wp_1\omega_{\daleth_\rho\daleth_{\widehat{\rho}}} + \wp_2\omega_{\daleth_\sigma\daleth_{\widehat{\sigma}}} + \wp_3\omega_{\daleth_\tau\daleth_{\widehat{\tau}}} + \wp_4\omega_{\daleth_\upsilon\daleth_{\widehat{\upsilon}}}.$$

Then, there is a unique point $\rho \in \zeta$ such that $\rho = \daleth\rho = \Omega_{\rho\rho\rho\rho}$.

Proof. (i) Suppose that Ψ^o is defined as in Example 1. The completeness of (ζ, ω) leads to $(\zeta, \Psi^o, \min)$, which is a complete FMS. We fix $\rho, \sigma, \tau, \upsilon, \widehat{\rho}, \widehat{\sigma}, \widehat{\tau}, \widehat{\upsilon} \in \zeta$, and $\kappa > 0$, and we will achieve (1) by taking $\wp_1 = \wp_2 = \wp_3 = \wp_4 = \frac{1}{4}$ and $\star = \min$. If $\Psi^o_{\daleth_\rho\daleth_{\widehat{\rho}}}(\kappa) = 0$, $\Psi^o_{\daleth_\sigma\daleth_{\widehat{\sigma}}}(\kappa) = 0$, $\Psi^o_{\daleth_\tau\daleth_{\widehat{\tau}}}(\kappa) = 0$, or $\Psi^o_{\daleth_\upsilon\daleth_{\widehat{\upsilon}}}(\kappa) = 0$, then (1) is clear. Assume that $\Psi^o_{\daleth_\sigma\daleth_{\widehat{\sigma}}}(\kappa) = 1$,

$\Psi^{o}_{\daleth\rho\daleth\widehat{\rho}}(\kappa) = 1$, $\Psi^{o}_{\daleth\tau\daleth\widehat{\tau}}(\kappa) = 1$, and $\Psi^{o}_{\daleth\upsilon\daleth\widehat{\upsilon}}(\kappa) = 1$. This implies that $\omega_{\daleth\rho\daleth\widehat{\rho}} < \kappa$, $\omega_{\daleth\sigma\daleth\widehat{\sigma}} < \kappa$, $\omega_{\daleth\tau\daleth\widehat{\tau}} < \kappa$, and $\omega_{\daleth\upsilon\daleth\widehat{\upsilon}} < \kappa$. Therefore, $\kappa > \max\Big\{\omega_{\daleth\rho\daleth\widehat{\rho}}, \omega_{\daleth\sigma\daleth\widehat{\sigma}}, \omega_{\daleth\tau\daleth\widehat{\tau}}, \omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\Big\}$, and

$$\varpi\kappa > \varpi \max\Big\{\omega_{\daleth\rho\daleth\widehat{\rho}}, \omega_{\daleth\sigma\daleth\widehat{\sigma}}, \omega_{\daleth\tau\daleth\widehat{\tau}}, \omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\Big\} \geq \omega_{\Omega_{\rho\sigma\tau\upsilon}\Omega_{\widehat{\rho}\widehat{\sigma}\widehat{\tau}\widehat{\upsilon}}}.$$

Thus, $\Psi^{o}_{\Omega_{\rho\sigma\tau\upsilon}\Omega_{\widehat{\rho}\widehat{\sigma}\widehat{\tau}\widehat{\upsilon}}}(\kappa\varpi) = 1$ and (1) holds.

(ii) Here,

$$\begin{aligned}
\omega_{\Omega_{\rho\sigma\tau\upsilon}\Omega_{\widehat{\rho}\widehat{\sigma}\widehat{\tau}\widehat{\upsilon}}} &\leq \varpi\Big(\wp_1\omega_{\daleth\rho\daleth\widehat{\rho}} + \wp_2\omega_{\daleth\sigma\daleth\widehat{\sigma}} + \wp_3\omega_{\daleth\tau\daleth\widehat{\tau}} + \wp_4\omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\Big) \\
&\leq \varpi\left(\frac{1}{4}\omega_{\daleth\rho\daleth\widehat{\rho}} + \frac{1}{4}\omega_{\daleth\sigma\daleth\widehat{\sigma}} + \frac{1}{4}\omega_{\daleth\tau\daleth\widehat{\tau}} + \frac{1}{4}\omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\right) \\
&= \frac{\varpi}{4}\Big(\omega_{\daleth\rho\daleth\widehat{\rho}} + \omega_{\daleth\sigma\daleth\widehat{\sigma}} + \omega_{\daleth\tau\daleth\widehat{\tau}} + \omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\Big) \\
&\leq \frac{\varpi}{4} \times 4\max\Big\{\omega_{\daleth\rho\daleth\widehat{\rho}}, \omega_{\daleth\sigma\daleth\widehat{\sigma}}, \omega_{\daleth\tau\daleth\widehat{\tau}}, \omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\Big\} \\
&= \varpi\max\Big\{\omega_{\daleth\rho\daleth\widehat{\rho}}, \omega_{\daleth\sigma\daleth\widehat{\sigma}}, \omega_{\daleth\tau\daleth\widehat{\tau}}, \omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\Big\}.
\end{aligned}$$

(iii) If $\varpi = \wp_1 + \wp_2 + \wp_3 + \wp_4 < 1$,

$$\begin{aligned}
\omega_{\Omega_{\rho\sigma\tau\upsilon}\Omega_{\widehat{\rho}\widehat{\sigma}\widehat{\tau}\widehat{\upsilon}}} &\leq \wp_1\omega_{\daleth\rho\daleth\widehat{\rho}} + \wp_2\omega_{\daleth\sigma\daleth\widehat{\sigma}} + \wp_3\omega_{\daleth\tau\daleth\widehat{\tau}} + \wp_4\omega_{\daleth\upsilon\daleth\widehat{\upsilon}} \\
&\leq \wp_1\max\Big\{\omega_{\daleth\rho\daleth\widehat{\rho}}, \omega_{\daleth\sigma\daleth\widehat{\sigma}}, \omega_{\daleth\tau\daleth\widehat{\tau}}, \omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\Big\} \\
&\quad + \wp_2\max\Big\{\omega_{\daleth\rho\daleth\widehat{\rho}}, \omega_{\daleth\sigma\daleth\widehat{\sigma}}, \omega_{\daleth\tau\daleth\widehat{\tau}}, \omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\Big\} \\
&\quad + \wp_3\max\Big\{\omega_{\daleth\rho\daleth\widehat{\rho}}, \omega_{\daleth\sigma\daleth\widehat{\sigma}}, \omega_{\daleth\tau\daleth\widehat{\tau}}, \omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\Big\} \\
&\quad + \wp_4\max\Big\{\omega_{\daleth\rho\daleth\widehat{\rho}}, \omega_{\daleth\sigma\daleth\widehat{\sigma}}, \omega_{\daleth\tau\daleth\widehat{\tau}}, \omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\Big\} \\
&= (\wp_1 + \wp_2 + \wp_3 + \wp_4)\max\Big\{\omega_{\daleth\rho\daleth\widehat{\rho}}, \omega_{\daleth\sigma\daleth\widehat{\sigma}}, \omega_{\daleth\tau\daleth\widehat{\tau}}, \omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\Big\} \\
&= \varpi\max\Big\{\omega_{\daleth\rho\daleth\widehat{\rho}}, \omega_{\daleth\sigma\daleth\widehat{\sigma}}, \omega_{\daleth\tau\daleth\widehat{\tau}}, \omega_{\daleth\upsilon\daleth\widehat{\upsilon}}\Big\}.
\end{aligned}$$

□

Example 3. *Consider $\zeta = \mathbb{R}$, $\omega(\rho,\sigma) = |\rho - \sigma|$ for all $\rho,\sigma \in \mathbb{R}$ and for all $\ell_1, \ell_2, \ell_3, \ell_4, \xi, \Psi \in \mathbb{R}$ with $\Psi > |\ell_1| + |\ell_2| + |\ell_3| + |\ell_4|$. We define the mappings $\Omega : \mathbb{R}^4 \to \mathbb{R}$ and $\daleth : \mathbb{R} \to \mathbb{R}$ by $\Omega_{\rho\sigma\tau\upsilon} = \frac{(\ell_1\rho + \sigma\ell_2 + \tau\ell_3 + \upsilon\ell_4 + \xi)}{\Psi}$ and $\daleth\rho = \rho$ for all $\rho,\sigma,\tau,\upsilon \in \mathbb{R}$. It is easy to check that the two mappings verify the hypothesis (iii) of Theorem 3, and $(\rho_0, \rho_0, \rho_0, \rho_0)$ is a unique QCP of Ω and $\daleth$, where $\rho_0 = \frac{\xi}{\Psi - \ell_1 - \ell_2 - \ell_3 - \ell_4}$ and $\Omega_{\rho_0\rho_0\rho_0\rho_0} = \rho_0$.*

Now, we can generalize Theorem 1.7 [18] by obtaining a coupled coincidence point for $\Omega : \mathbb{R}^2 \to \mathbb{R}$ and $\daleth$. We only take $\ell_1 = \ell_2 = \frac{1}{2}$ as follows.

Corollary 1. *Assume that $\star$ is a κ-norm of the Y-type such that $\mu \star \kappa \geq \mu\kappa$ for all $\mu, \kappa \in [0,1]$. Suppose that $(\zeta, \Psi, \star)$ is a complete FMS and $\Omega : \zeta^2 \to \zeta$, $\daleth : \zeta \to \zeta$ are two mappings such that*

- $\Omega(\zeta^2) \subseteq \daleth(\zeta)$,
- *$\daleth$ is continuous,*
- *$\daleth$ is commuting with Ω,*
- *for all $\rho, \sigma, \widehat{\rho}, \widehat{\sigma} \in \zeta$,*

$$\Psi_{\Omega_{\rho\sigma}\Omega_{\widehat{\rho}\widehat{\sigma}}}(\kappa\varpi) \geq \Psi_{\daleth\rho\daleth\widehat{\rho}}(\kappa)^{\ell_1} \star \Psi_{\daleth\sigma\daleth\widehat{\sigma}}(\kappa)^{\ell_2},$$

where $\varpi \in (0,1)$ and ℓ_1, ℓ_2 are real numbers in $[0,1]$ such that $\ell_1 + \ell_2 \leq 1$.

Then, there exists a unique $\rho \in \zeta$ *such that* $\rho = \daleth\rho = \Omega_{\rho\rho}$.

Proof. Define $\ell_3 = \ell_4 = 0$ and $\Omega^* : \zeta^4 \to \zeta$ as $\Omega^*_{\rho\sigma\tau\upsilon} = \Omega_{\rho\sigma}$ for all $\rho, \sigma, \tau, \upsilon \in \zeta$. Then, $\Omega^*(\zeta^4) = \Omega(\zeta^2) \subseteq \daleth(\zeta)$ and Ω^* is commuting with $\daleth$, that is, $\daleth\Omega^*_{\rho\sigma\tau\upsilon} = \daleth\Omega_{\rho\sigma} = \Omega_{\daleth\rho\daleth\sigma} = \Omega^*_{\daleth\rho\daleth\sigma\daleth\tau\daleth\upsilon}$. In addition, one can write

$$\begin{aligned} \Psi_{\Omega^*_{\rho\sigma\tau\upsilon}\Omega^*_{\widehat{\rho}\widehat{\sigma}\widehat{\tau}\widehat{\upsilon}}}(\kappa\varpi) &= \Psi_{\Omega_{\rho\sigma}\Omega_{\widehat{\rho}\widehat{\sigma}}}(\kappa\varpi) \geq \Psi_{\daleth\rho\daleth\widehat{\rho}}(\kappa)^{\ell_1} \star \Psi_{\daleth\sigma\daleth\widehat{\sigma}}(\kappa)^{\ell_2} \\ &= \Psi_{\daleth\rho\daleth\widehat{\rho}}(\kappa)^{\ell_1} \star \Psi_{\daleth\sigma\daleth\widehat{\sigma}}(\kappa)^{\ell_2} \star 1 \star 1 \\ &\geq \Psi_{\daleth\rho\daleth\widehat{\rho}}(\kappa)^{\ell_1} \star \Psi_{\daleth\sigma\daleth\widehat{\sigma}}(\kappa)^{\ell_2} \star \Psi_{\daleth\tau\daleth\widehat{\tau}}(\kappa)^{\ell_3} \star \Psi_{\daleth\upsilon\daleth\widehat{\upsilon}}(\kappa)^{\ell_4}. \end{aligned}$$

Hence, by Theorem 2, there is $\rho \in \zeta$ such that $\daleth\rho = \Omega^*_{\rho\sigma\tau\upsilon}$. If $\sigma \in \zeta$ satisfies $\Omega_{\sigma\sigma} = \daleth\sigma$, then $\daleth\sigma = \Omega_{\sigma\sigma} = \Omega^*_{\sigma\sigma\sigma\sigma}$. Thus, $\sigma = \rho$. □

The proof of the corollary below follows immediately from Theorem 3.

Corollary 2 ([18]). *Assume that* (ζ, ω) *is a CMS and* $\Omega : \zeta^2 \to \zeta$, $\daleth : \zeta \to \zeta$ *are two mappings such that:*

- $\Omega(\zeta^2) \subseteq \daleth(\zeta)$;
- $\daleth$ *is continuous;*
- $\daleth$ *is commuting with* Ω.

If Ω *and* $\daleth$ *satisfy some of the conditions below for* $\rho, \sigma, \widehat{\rho}, \widehat{\sigma} \in \zeta$:

(i) *for some* $0 < \varpi < 1$,
$$\omega_{\Omega_{\rho\sigma}\Omega_{\widehat{\rho}\widehat{\sigma}}} \leq \varpi \max\Big\{\omega_{\daleth\rho\daleth\widehat{\rho}}, \omega_{\daleth\sigma\daleth\widehat{\sigma}}\Big\}.$$

(ii) *for some* $0 < \varpi < 1$ *and some* $\wp_1, \wp_2 \in [0, \frac{1}{2}]$,
$$\omega_{\Omega_{\rho\sigma}\Omega_{\widehat{\rho}\widehat{\sigma}}} \leq \varpi\Big(\wp_1\omega_{\daleth\rho\daleth\widehat{\rho}} + \wp_2\omega_{\daleth\sigma\daleth\widehat{\sigma}}\Big).$$

(iii) *for some* $\wp_1, \wp_2 \in [0, 1)$ *with* $\wp_1 + \wp_2 + \wp_3 + \wp_4 < 1$,
$$\omega_{\Omega_{\rho\sigma}\Omega_{\widehat{\rho}\widehat{\sigma}}} \leq \wp_1\omega_{\daleth\rho\daleth\widehat{\rho}} + \wp_2\omega_{\daleth\sigma\daleth\widehat{\sigma}}.$$

Then, there is a unique point $\rho \in \zeta$ *such that* $\rho = \daleth\rho = \Omega_{\rho\rho}$.

5. Supportive Applications

This section was specially prepared to highlight the importance of the theoretical results and how to use them to find the existence of the solution to a Lipschitzian and integral quadruple system.

5.1. Lipschitzian Quadruple Systems

Assume that $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 : \mathbb{R} \to \mathbb{R}$ are LMs and $\wp_1, \wp_2, \wp_3, \wp_4 \in \mathbb{R}$ are real numbers. Let $\mho : \mathbb{R} \to \mathbb{R}$ be defined by $\mho(\rho) = \sum\limits_{i=1}^{4} \wp_i\Gamma_i(\rho)$ for all $\rho \in \mathbb{R}$; then, $\mho$ is also an LM and $\varpi_\mho \leq \sum\limits_{i=1}^{4} |\wp_i|\varpi_{\Gamma_i}$. It is easy to see that if $\Lambda = \sum\limits_{i=1}^{4} |\wp_i|\varpi_{\Gamma_i} < 1$, then $\mho$ is a contraction; thus, there is a unique $\rho_0 \in \mathbb{R}$ such that $\mho_{\rho_0} = \rho_0$. Now, for all $\rho, \sigma, \tau, \upsilon \in \mathbb{R}$, define $\Omega : \zeta^4 \to \zeta$ as

$$\Omega_{\rho\sigma\tau\upsilon} = \wp_1\Gamma_1(\rho) + \wp_2\Gamma_2(\sigma) + \wp_3\Gamma_3(\tau) + \wp_4\Gamma_4(\upsilon).$$

It is obvious that for all $\rho \in \mathbb{R}$, $\Omega_{\rho\rho\rho\rho} = \mho_{\rho}$. In addition, we have

$$\begin{aligned} \omega(\Omega_{\rho_1\rho_2\rho_3\rho_4}, \Omega_{\sigma_1\sigma_2\sigma_3\sigma_4}) &= \sum_{i=1}^{4} |\wp_i||\Gamma_i(\rho_i) - \Gamma_i(\sigma_i)| \\ &\leq \sum_{i=1}^{4} |\wp_i| \varpi_{\Gamma_i} |\rho_i - \sigma_i| \leq \Lambda \max_{1\leq j\leq 4} \omega(\rho_j, \sigma_j). \end{aligned}$$

If $\Lambda < 1$, then Ω satisfies (1) with $\daleth\rho = \rho$ for all $\rho \in \mathbb{R}$.
According to the above results, we can state the corollary below.

Corollary 3. *Assume that $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 : \mathbb{R} \to \mathbb{R}$ are LMs and $\wp_1, \wp_2, \wp_3, \wp_4 \in \mathbb{R}$ such that $\sum\limits_{i=1}^{4} |\wp_i| \varpi_{\Gamma_i} < 1$; then, the system*

$$\begin{cases} \rho = \wp_1\Gamma_1(\rho) + \wp_2\Gamma_2(\sigma) + \wp_3\Gamma_3(\tau) + \wp_4\Gamma_4(\upsilon), \\ \sigma = \wp_1\Gamma_1(\sigma) + \wp_2\Gamma_2(\tau) + \wp_3\Gamma_3(\upsilon) + \wp_4\Gamma_4(\rho), \\ \tau = \wp_1\Gamma_1(\tau) + \wp_2\Gamma_2(\upsilon) + \wp_3\Gamma_3(\rho) + \wp_4\Gamma_4(\sigma), \\ \upsilon = \wp_1\Gamma_1(\upsilon) + \wp_2\Gamma_2(\rho) + \wp_3\Gamma_3(\sigma) + \wp_4\Gamma_4(\tau). \end{cases} \tag{28}$$

has a unique solution $(\rho_0, \rho_0, \rho_0, \rho_0)$, where ρ_0 is the only real solution of $\rho = \sum\limits_{i=1}^{4} \wp_i\Gamma_i(\rho)$.

Example 4. *Consider the system*

$$\begin{cases} 24\cos\rho - \frac{18}{1+\sigma^2} + 144 = 120\rho + \left(\frac{4}{1+\tau^2}\right)^2 - 15\arcsin\upsilon, \\ 24\cos\sigma - \frac{18}{1+\tau^2} + 144 = 120\sigma + \left(\frac{4}{1+\upsilon^2}\right)^2 - 15\arcsin\rho, \\ 24\cos\tau - \frac{18}{1+\upsilon^2} + 144 = 120\tau + \left(\frac{4}{1+\rho^2}\right)^2 - 15\arcsin\sigma, \\ 24\cos\upsilon - \frac{18}{1+\rho^2} + 144 = 120\rho + \left(\frac{4}{1+\sigma^2}\right)^2 - 15\arcsin\tau, \end{cases} \tag{29}$$

If we select $\Gamma_1(\rho) = 6 + \cos\rho$, $\Gamma_2(\rho) = \frac{1}{1+\rho^2}$, $\Gamma_3(\rho) = \left(\frac{1}{1+\rho^2}\right)^2$, and $\Gamma_4(\rho) = \arcsin\upsilon$, then Γ_1, Γ_2, Γ_3, and Γ_4 are LMs, and $\varpi_{\Gamma_1} = \varpi_{\Gamma_4} = 1$, $\varpi_{\Gamma_2} = \frac{3\sqrt{3}}{8}$, and $\varpi_{\Gamma_3} = \frac{27}{64}$. Let $\wp_1 = \frac{1}{5}$, $\wp_2 = -\frac{3}{20}$, $\wp_3 = \frac{2}{15}$, and $\wp_4 = \frac{1}{8}$. Then, $\sum\limits_{i=1}^{4} |\wp_i| \varpi_{\Gamma_i} = 0.479 < 1$ because system (29) is a special case of system (28). So, the problem (29) has a unique solution $(\rho_0, \rho_0, \rho_0, \rho_0)$, where ρ_0 represents a unique solution of

$$24\cos\rho - \frac{18}{1+\rho^2} + 144 = 120\rho + \left(\frac{4}{1+\rho^2}\right)^2 - 15\arcsin\rho.$$

By programming in Matlab or Mathematica or by using the bisection method, we can approximate the value $\rho_0 = 1.26624$.

5.2. An Integral Quadruple System

Assume that $\ell_1, \ell_2 \in \mathbb{R}$ with $\ell_1 < \ell_2$ and set $\varphi = [\ell_1, \ell_2]$. Let $\zeta = L^1(\varphi)$ be equipped with $\omega_1(\Gamma, \aleph) = \int_{\varphi} |\Gamma(\kappa), \aleph(\kappa)| \omega\kappa$, where $\int$ is the Lebesgue integral. It is clear that $(L^1(\varphi), \omega_1)$ is a CMS. Suppose that $\varpi, \wp_1, \wp_2, \wp_3, \wp_4 \in \mathbb{R}$ are real numbers and $\beth : \mathbb{R}^4 \to \mathbb{R}$ is a mapping satisfying $\beth(0,0,0,0) = 0$ and

$$|\beth_{\rho_1\rho_2\rho_3\rho_4} - \beth_{\sigma_1\sigma_2\sigma_3\sigma_4}| \leq \varpi \sum_{i=1}^{4} \wp_i |\rho_i - \sigma_i|, \ \forall (\rho_1, \rho_2, \rho_3, \rho_4,), (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathbb{R}^4.$$

If $B \in \mathbb{R}$, we want to find the functions $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \in L^1(\varphi)$ such that

$$\Gamma_i(\rho) = B + \int_{[\ell_1,\rho]} \beth(\Gamma_i(\kappa), \Gamma_{i+1}(\kappa), \Gamma_{i+2}(\kappa), \Gamma_{i+3}(\kappa))\omega\kappa, \quad (30)$$

is fulfilled for all $\rho \in \varphi$, $i = 1, 2, 3, 4$.

For $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \in L^1(\varphi)$, and all $\rho \in \varphi$, define the mapping Ω by

$$\Omega_{\Gamma_1\Gamma_2\Gamma_3\Gamma_4}(\rho) = B + \int_{[\ell_1,\rho]} \beth(\Gamma_1(\kappa), \Gamma_2(\kappa), \Gamma_3(\kappa), \Gamma_4(\kappa))\omega\kappa. \quad (31)$$

According to (30) and (31), we see that $\Omega_{\Gamma_1\Gamma_2\Gamma_3\Gamma_4} \in L^1(\varphi)$; hence, $\Omega : L^1(\varphi)^4 \to L^1(\varphi)$ is well defined.

In addition,

$$\begin{aligned}
& \omega_1\left(\Omega_{\Gamma_1\Gamma_2\Gamma_3\Gamma_4}, \Omega_{\aleph_1\aleph_2\aleph_3\aleph_4}\right) \\
= & \int_\varphi |\Omega_{\Gamma_1\Gamma_2\Gamma_3\Gamma_4}(\rho) - \Omega_{\aleph_1\aleph_2\aleph_3\aleph_4}(\rho)|\omega\rho \\
= & \int_\varphi \left(\int_{[\ell_1,\rho]} |\beth(\Gamma_1(\kappa), \Gamma_2(\kappa), \Gamma_3(\kappa), \Gamma_4(\kappa)) - \beth(\aleph_1(\kappa), \aleph_2(\kappa), \aleph_3(\kappa), \aleph_4(\kappa))|\omega\kappa \right)\omega\rho \\
\leq & \int_\varphi \left(\int_{[\ell_1,\rho]} \varpi \sum_{i=1}^{4} \wp_i |\Gamma_i - \aleph_i|\omega\kappa \right)\omega\rho \\
\leq & \varpi \sum_{i=1}^{4} \wp_i \int_\varphi \left(\int_\varphi |\Gamma_i - \aleph_i|\omega\kappa \right)\omega\rho \\
= & \varpi \sum_{i=1}^{4} \wp_i \int_\varphi \omega_1(\Gamma_i, \aleph_i)\omega\rho = \varpi(\ell_2 - \ell_1) \sum_{i=1}^{4} \wp_i \omega_1(\Gamma_i, \aleph_i).
\end{aligned}$$

If we take $\varpi(\ell_2 - \ell_1) \sum_{i=1}^{4} \wp_i = \Lambda < 1$, then Ω justifies (1) with $\daleth(\Gamma) = \Gamma$ for all $\Gamma \in L^1(\varphi)$. We conclude from the above results that system (30) has a unique solution $(\Gamma_0, \Gamma_0, \Gamma_0, \Gamma_0)$, where Γ_0 is a unique solution of the equation

$$\Gamma_0(\rho) = B + \int_{[\ell_1,\rho]} \beth(\Gamma_0(\kappa), \Gamma_0(\kappa), \Gamma_0(\kappa), \Gamma_0(\kappa))\omega\kappa,$$

for $\Gamma_0 \in L^1(\varphi)$ and all $\rho \in \varphi$.

6. Conclusions

The study of fuzzy sets led to the fuzzification of a number of mathematical notions, and it has applications in a variety of fields, including neural networking theory, image processing, control theory, modeling theory, and many more. In fixed-point theory, contraction-type mappings in FMSs are extremely important. So, in this manuscript, we investigated QCP results for commuting mappings without assuming a partially ordered set in the setting of FMSs. Furthermore, some new results are presented to generalize some of the previous results on this topic. In addition, non-trivial examples are given. Moreover, some applications for finding a unique solution for Lipschitzian and integral quadruple systems are provided to support and strengthen our study. In our future paper, we intend to establish a fixed-point theorem for cyclic $\phi-$contractive mappings in an $M-$complete FMS.

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