mathematics

Article

Quantum Weighted Fractional Fourier Transform

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Abstract: Quantum Fourier transform (QFT) is an important part of many quantum algorithms. However, there are few reports on quantum fractional Fourier transform (QFRFT). The main reason is that the definitions of fractional Fourier transform (FRFT) are diverse, while some definitions do not include unitarity, which leads to some studies pointing out that there is no QFRFT. In this paper, we first present a reformulation of the weighted fractional Fourier transform (WFRFT) and prove its unitarity, thereby proposing a quantum weighted fractional Fourier transform (QWFRFT). The proposal of QWFRFT provides the possibility for many quantum implementations of signal processing.

Keywords: quantum weighted fractional Fourier transform; quantum Fourier transform; quantum algorithm; quantum computing

MSC: 81-08

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1. Introduction

Feynman was the first to present the idea of quantum computing, that is, to directly use the state of microscopic particles to represent quantum information, which is considered to be the early prototype of the concept of quantum computing [1]. Subsequently, Deutsch formalized the concept of quantum computing, proposed the idea of a quantum Turing machine, and designed the first quantum parallel algorithm, which exhibited excellent performance beyond classical computing [2]. The proposal of Shor's algorithm caused researchers to realize that quantum computing had a natural parallel processing capability, which could introduce many disruptive technological innovations. Shor's algorithm states that a large number can be decomposed into the product of two prime factors in polynomial time. This greatly challenged the RSA (Rivest–Shamir–Adleman) encryption system, thus indicating that the RSA encryption system had been cracked in theory [3,4]. Grover's search algorithm convinced researchers of the power of quantum computing. Compared with the traditional search method, this algorithm can achieve the acceleration effect of square level [5]. Therefore, many improved Grover search algorithms have been proposed [6–10]. Meanwhile, quantum-inspired algorithms have also been proposed that can be simulated by classical computing [11–16]. Moreover, the quantum algorithm has been applied to solve linear systems of equations, which introduced new ideas for solving linear equations. This algorithm is also called the *HHL* algorithm [17]. The *HHL* algorithm has been widely used, and its improved algorithms have been continuously proposed [18–20]. Recently, quantum algorithms have been applied to solve differential equations [21–24]. A series of quantum computing technologies, such as quantum Fourier transform [25], quantum phase estimation [26], and the *HHL* algorithm, are called quantum basic linear algebra assembly [27]. At present, quantum computing has been widely used in cryptography, quantum simulation, machine learning, and other fields and shows a strong ability and great potential.

The Fourier transform plays an important role in the design of quantum algorithms, but little is known about the quantum algorithms of the fractional Fourier transform (*FRFT*). The initial definition of the *FRFT* was proposed in [28]. Its application provides a convenient technique for solving certain classes of ordinary and partial differential equations, which arise in quantum mechanics from classical quadratic Hamiltonians. The theoretical research of the *FRFT* has developed rapidly, and various definitions have been proposed, such as eigenvalue *FRFT* [29], weighted *FRFT* [30], and sampling *FRFT* [31]. These definitions are widely used in various fields of signal processing. So far, little is known about the reports and studies on the quantum fractional Fourier transform (*QFRFT*). The main reason is that the design of quantum algorithms should satisfy unitarity, and some FRFTs do not include unitarity. Thus, a quantum pseudo-fractional Fourier transform (*QPFRFT*) was proposed [32], and the authors showed that there was no *QFRFT*. However, we present a reformulation of the weighted fractional Fourier transform (*WFRFT*) and prove its unitarity, whereupon a quantum weighted fractional Fourier transform (*QWFRFT*) is proposed.

The remainder of this paper is organized as follows. The preliminary knowledge is described in Section 2. The unitarity of the WFRFT is proved in Section 3. Section 4 presents the *QWFRFT*. Finally, the conclusions are presented in Section 5.

2. Preparation

For a unitary matrix U, assuming that it has an eigenvector $|u\rangle$ and the corresponding eigenvalue $e^{2\pi i\varphi}$, $U|u\rangle = e^{2\pi i\varphi}|u\rangle$ is satisfied. Therefore, we can calculate φ through the phase estimation algorithm. The circuit of phase estimation is shown in Figure 1. It is not difficult to find that the quantum Fourier transform (*QFT*) is the key to phase estimation, and phase estimation is the key of many quantum algorithms.

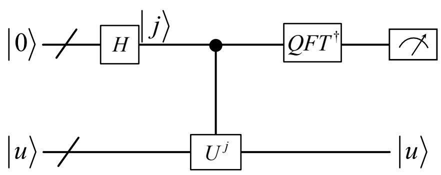

Figure 1. A circuit for phase estimation.

The importance of the *QFT* goes without saying. However, little is known about the report of the *QFRFT*. In 2012, Parasa et al. proposed a *QPFRFT* using multiple-valued logic [32]. The reason why researchers call it "pseudo" is that the FRFT used did not include unitarity. The FRFT was proposed by Bailey et al. [33], and its definition is as follows:

$$F^{\alpha}[k] = \sum_{j=0}^{N-1} f[j]\cdot \exp\left(2\pi i\cdot\frac{kj}{N}\cdot\alpha\right). \tag{1}$$

Parasa et al. pointed out: "It must be noted that unlike the discrete Fourier transform, the *FRFT* is not a unitary operation. More formally, this means that there exists no unitary operator which can implement the following quantum computational operation".

$$\sum_{j=0}^{N-1} f(j)|j\rangle \overset{NOTPOSSLBLE}{\rightarrow} \sum_{k=0}^{N-1} F^{\alpha}(k)|k\rangle. \tag{2}$$

Therefore, Parasa et al. explicitly state that it is not possible to define the *QFRFT*. However, the definitions of the *FRFT* are diverse, and the definition of one class of *WFRFT* includes unitarity. Hence, Parasa et al.'s statement that there is no *QFRFT* is not rigorous.

In 1995, Shih proposed the definition of a *WFRFT* [30]. The alpha-order *FRFT* of the function $f(t)$ can be expressed as

$$F^{\alpha}[f(t)] = \sum_{l=0}^{3} A_l(\alpha) f_l(t). \quad (3)$$

Here,$f_0(t) = f(t)$, $f_1(t) = F[f_0(t)]$, $f_2(t) = F[f_1(t)]$, and $f_3(t) = F[f_2(t)]$ (F denotes Fourier transform). The weighting coefficient $A_l(\alpha)$ is expressed as

$$A_l(\alpha) = \cos\left(\frac{(\alpha - l)\pi}{4}\right) \cos\left(\frac{2(\alpha - l)\pi}{4}\right) \exp\left(\frac{3(\alpha - l)i\pi}{4}\right), \quad (4)$$

where $l = 0, 1, 2, 3$.

3. Unitarity of Weighted Fractional Fourier Transform

A complex matrix U satisfies

$$UU^H = U^H U = I, \quad (5)$$

where H denotes the conjugate transpose, and I is the identity matrix. Then, matrix U is called a unitary matrix.

The discrete form of the *WFRFT* (Equation (3)) can be expressed as

$$DWFRFT = A_0(\alpha) \cdot I + A_1(\alpha) \cdot DFT + A_2(\alpha) \cdot DFT^2 + A_3(\alpha) \cdot DFT^3, \quad (6)$$

where $A_l(\alpha)$ is Equation (4), and DFT is the discrete Fourier transform. It is not easy to prove the unitarity of Equation (6). Therefore, we present the reformulation of the *WFRFT* and prove its unitarity. First, Equation (4) can be written as

$$\begin{aligned}
A_l(\alpha) \quad &= \cos\left(\tfrac{(\alpha-l)\pi}{4}\right) \cos\left(\tfrac{2(\alpha-l)\pi}{4}\right) \exp\left(\tfrac{3(\alpha-l)i\pi}{4}\right) \\
&= \tfrac{1}{2} \times \left[\exp\left(\tfrac{(\alpha-l)\pi i}{4}\right) + \exp\left(\tfrac{-(\alpha-l)\pi i}{4}\right)\right] \\
&\times \tfrac{1}{2} \times \left[\exp\left(\tfrac{2(\alpha-l)\pi i}{4}\right) + \exp\left(\tfrac{-2(\alpha-l)\pi i}{4}\right)\right] \times \exp\left(\tfrac{3(\alpha-l)i\pi}{4}\right) \\
&= \tfrac{1}{4}\left(1 + \exp\left(\tfrac{2(\alpha-l)\pi i}{4}\right) + \exp\left(\tfrac{4(\alpha-l)\pi i}{4}\right) + \exp\left(\tfrac{6(\alpha-l)\pi i}{4}\right)\right) \\
&= \tfrac{1}{4}\sum_{k=0}^{3} \exp\left(\tfrac{2\pi i}{4}(\alpha - l)k\right) \\
&= \tfrac{1}{4}\sum_{k=0}^{3} \exp\left(\tfrac{2\pi i\alpha k}{4}\right) \exp\left(\tfrac{-2\pi i l k}{4}\right).
\end{aligned} \quad (7)$$

Let $B_k^{\alpha} = \exp\left(\frac{2\pi i k\alpha}{4}\right)$; $k = 0, 1, 2, 3$; then, Equation (7) can be expressed as

$$\begin{pmatrix} A_0^{\alpha} \\ A_1^{\alpha} \\ A_2^{\alpha} \\ A_3^{\alpha} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} B_0^{\alpha} \\ B_1^{\alpha} \\ B_2^{\alpha} \\ B_3^{\alpha} \end{pmatrix}. \quad (8)$$

We write Equation (6) as Equation (9).

$$DWFRFT = \left(I, DFT, DFT^2, DFT^3\right) \begin{pmatrix} A_0(\alpha) \\ A_1(\alpha) \\ A_2(\alpha) \\ A_3(\alpha) \end{pmatrix}. \quad (9)$$

Equation (8) is substituted into Equation (9), and we obtain

$$DWFRFT = \frac{1}{4}\left(I, DFT, DFT^2, DFT^3\right)\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -i & -1 & i\\ 1 & -1 & 1 & -1\\ 1 & i & -1 & -i\end{pmatrix}\begin{pmatrix}B_0^\alpha\\ B_1^\alpha\\ B_2^\alpha\\ B_3^\alpha\end{pmatrix}. \tag{10}$$

We let

$$\begin{cases} Y_0 = I + DFT + DFT^2 + DFT^3\\ Y_1 = I - i\cdot DFT - DFT^2 + i\cdot DFT^3\\ Y_2 = I - DFT + DFT^2 - DFT^3\\ Y_3 = I + i\cdot DFT - DFT^2 - i\cdot DFT^3 \end{cases} \tag{11}$$

Definition 1. *A reformulation of the DWFRFT.*

$$\begin{aligned} DWFRFT &= \tfrac{1}{4}(Y_0, Y_1, Y_2, Y_3)\begin{pmatrix}B_0^\alpha\\ B_1^\alpha\\ B_2^\alpha\\ B_3^\alpha\end{pmatrix}\\ &= \tfrac{1}{4}\left(Y_0B_0^\alpha + Y_1B_1^\alpha + Y_2B_2^\alpha + Y_3B_3^\alpha\right)\\ &= \tfrac{1}{4}\sum_{k=0}^{3} Y_kB_k^\alpha. \end{aligned} \tag{12}$$

where $B_k^\alpha = \exp\left(\frac{2\pi ik\alpha}{4}\right); k = 0, 1, 2, 3.$

Proposition 1. Y_k *are real symmetric matrices.*

Proof of Proposition 1. In Equation (11), I is the identity matrix, and DFT can be expressed as

$$DFT = \frac{1}{\sqrt{N}}\cdot\begin{pmatrix} u^{0\times 0} & u^{0\times 1} & \dots & u^{0\times(n-1)}\\ u^{1\times 0} & u^{1\times 1} & \dots & u^{1\times(n-1)}\\ \vdots & \vdots & \ddots & \vdots\\ u^{(n-1)\times 0} & u^{(n-1)\times 1} & \dots & u^{(n-1)\times(n-1)} \end{pmatrix}, \tag{13}$$

where $u = \exp(-2\pi i/N)$. Here, DFT is a symmetric matrix, so that DFT^2, DFT^3, and DFT^4 are also symmetric matrices. We know that the result of adding symmetric matrices is still a symmetric matrix. Therefore, Y_k are symmetric matrices (Equation (11)).

Next, we prove that Y_k are real matrices. The integer powers of the Fourier transform are shown in Figure 2. Here, DFT^2 and DFT^4 are real matrices; the matrix of DFT^2 is shown in Equation (14), and DFT^4 is the identity matrix $DFT^4 = DFT^0 = I$.

$$DFT^2 = \begin{pmatrix} 1 & 0 & \dots & 0\\ 0 & 0 & \dots & 1\\ \vdots & \vdots & \iddots & \vdots\\ 0 & 1 & \dots & 0 \end{pmatrix}. \tag{14}$$

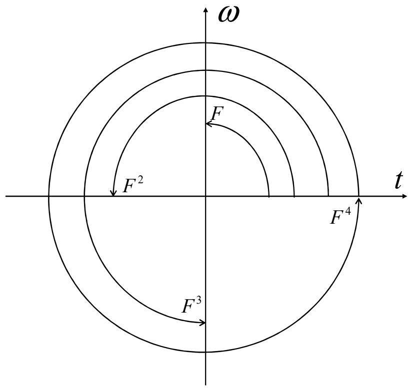

Figure 2. Time–frequency representation of Fourier transform.

Obviously, I and DFT^2 are real matrices. In Equation (13), each element of the DFT can be expressed as

$$u_{lk} = \exp(-2\pi ilk/N), \tag{15}$$

where $l = 0, 1, \ldots, n-1; k = 0, 1, \ldots, n-1$. Therefore, DFT^3 is an inverse Fourier transform, and each element of its matrix can be expressed as

$$w_{lk} = \exp(2\pi ilk/N), \tag{16}$$

where $l = 0, 1, \ldots, n-1$; $k = 0, 1, \ldots, n-1$. Thus, the result of $DFT + DFT^3$ is a real number,

$$\begin{aligned} w_{lk} + u_{lk} &= \exp(-2\pi ilk/N) + \exp(2\pi ilk/N) \\ &= \cos(2\pi lk/N) - i\sin(2\pi lk/N) + \cos(2\pi lk/N) + i\sin(2\pi lk/N) \\ &= 2\cos(2\pi lk/N). \end{aligned} \tag{17}$$

The result for $-iDFT + iDFT^3$ is

$$\begin{aligned} -iw_{lk} + iu_{lk} &= -i\exp(-2\pi ilk/N) + i\exp(2\pi ilk/N) \\ &= -i\cos(2\pi lk/N) - \sin(2\pi lk/N) + i\cos(2\pi lk/N) - \sin(2\pi lk/N) \\ &= -2\sin(2\pi lk/N). \end{aligned} \tag{18}$$

The result for $-DFT - DFT^3$ is

$$\begin{aligned} -w_{lk} - u_{lk} &= -\exp(-2\pi ilk/N) - \exp(2\pi ilk/N) \\ &= -\cos(2\pi lk/N) + i\sin(2\pi lk/N) - \cos(2\pi lk/N) - i\sin(2\pi lk/N) \\ &= -2\cos(2\pi lk/N). \end{aligned} \tag{19}$$

The result for $iDFT - iDFT^3$ is

$$\begin{aligned} iw_{lk} - iu_{lk} &= i\exp(-2\pi ilk/N) - i\exp(2\pi ilk/N) \\ &= i\cos(2\pi lk/N) + \sin(2\pi lk/N) - i\cos(2\pi lk/N) + \sin(2\pi lk/N) \\ &= 2\sin(2\pi lk/N). \end{aligned} \tag{20}$$

Therefore, for Equation (11), Y_k are real symmetric matrices. □

Proposition 2. *The weighted fractional Fourier transform is unitary.*

Proof of Proposition 2. By the proof of Proposition 1, we know that Y_k are real symmetric matrices; that is, $(Y_k)^H = Y_k$. Therefore, the conjugate transpose of the $DWFRFT$ is

$$\begin{aligned} (DWFRFT)^H &= \tfrac{1}{4}\left(Y_0B_0^{\alpha} + Y_1B_1^{\alpha} + Y_2B_2^{\alpha} + Y_3B_3^{\alpha}\right)^H \\ &= \tfrac{1}{4}\left(Y_0B_0^{-\alpha} + Y_1B_1^{-\alpha} + Y_2B_2^{-\alpha} + Y_3B_3^{-\alpha}\right). \end{aligned} \tag{21}$$

Thus, we obtain

$$\begin{aligned} DWFRFT\cdot(DWFRFT)^H &= \tfrac{1}{16}\left(Y_0B_0^\alpha + Y_1B_1^\alpha + Y_2B_2^\alpha + Y_3B_3^\alpha\right)\left(Y_0B_0^{-\alpha} + Y_1B_1^{-\alpha} + Y_2B_2^{-\alpha} + Y_3B_3^{-\alpha}\right) \\ &= \tfrac{1}{16}\sum_{k=0}^{3}\sum_{l=0}^{3} Y_kY_lB_k^\alpha B_l^{-\alpha}. \end{aligned} \tag{22}$$

Here,

$$Y_kY_l = \begin{cases} 0, k \neq l \\ Y_k^2, k = l \end{cases} \tag{23}$$

Then, Equation (22) is written as

$$DWFRFT\cdot(DWFRFT)^H = \frac{1}{16}\sum_{k=0}^{3} Y_k^2. \tag{24}$$

After calculation, we know that $Y_k^2 = 4Y_k$. Equation (25) is obtained.

$$DWFRFT\cdot(DWFRFT)^H = \frac{1}{4}\sum_{k=0}^{3} Y_k = \frac{1}{4}(Y_0 + Y_1 + Y_2 + Y_3) = I. \tag{25}$$

Thus, the unitarity of the *WFRFT* is proved. □

We can also implement the new reformulation with the help of fast Fourier transform (*FFT*), and its implementation module is shown in Figure 3. The weighting coefficients are readjusted A_l^α in Figure 3; so, the computational complexity is $O(N \log N)$.

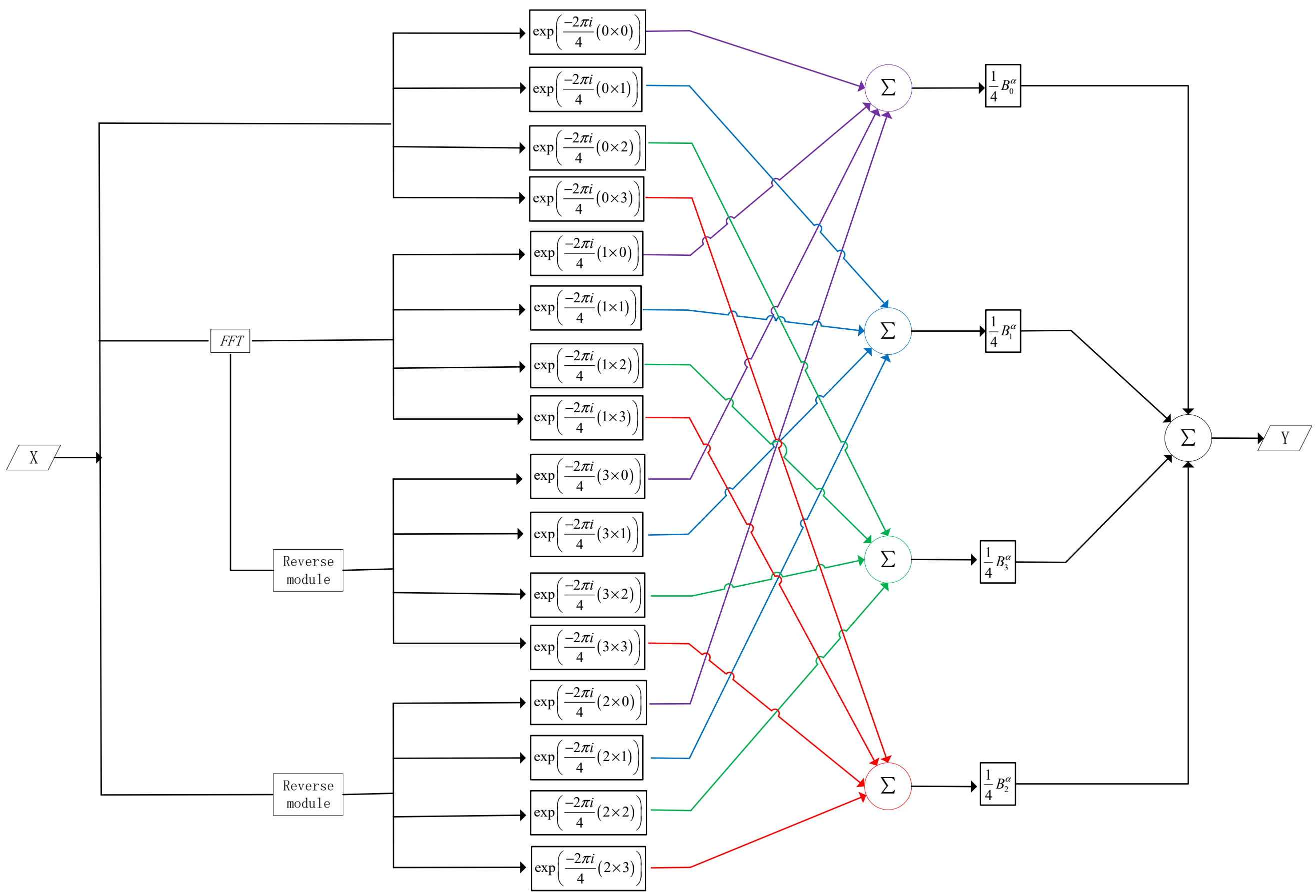

Figure 3. The reformulation of the *WFRFT* module.

4. Quantum Weighted Fractional Fourier Transform

In this section, we will present the *QWFRFT* with the help of the *QFT*. The *QFT* is an application of the classical Fourier transform to the amplitude of a quantum state. the vector $\boldsymbol{x}$ is transformed into the vector $\boldsymbol{y}$ by the classical Fourier transform,

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j u^{jk}; k = 0, 1, 2, \ldots, N-1 \tag{26}$$

where $u = e^{-2\pi i/N}$ and N is the signal length.

Similarly, *QFT* is applied to quantum state $|x\rangle = \sum_{j=0}^{N-1} x_j |j\rangle$ to obtain quantum state $|y\rangle = \sum_{k=0}^{N-1} y_k |k\rangle$, and we have

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j w_n^{jk}, \tag{27}$$

where $k = 0, 1, 2, \ldots, N-1$ and $w = e^{2\pi i/N}$. We note that Equation (27) is the inverse of the classical discrete Fourier transform; by convention, the *QFT* has the same effect as the inverse discrete Fourier transform.

In case that $|j\rangle$ is a basis state, the *QFT* can also be expressed as the map

$$QFT : |j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} w^{jk} |k\rangle. \tag{28}$$

Equivalently, the *QFT* can be viewed as a unitary matrix acting on quantum state vectors, where the unitary matrix F_N is given by

$$F_N = \frac{1}{\sqrt{N}} \begin{pmatrix} w^{0\times 0} & w^{0\times 1} & \ldots & w^{0\times(n-1)} \\ w^{1\times 0} & w^{1\times 1} & \ldots & w^{1\times(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ w^{(n-1)\times 0} & w^{(n-1)\times 1} & \ldots & w^{(n-1)\times(n-1)} \end{pmatrix}. \tag{29}$$

Sine $N = 2^n$ and $w = e^{2\pi i/2^n}$. The electronic circuit of the *QFT* is shown in Figure 4.

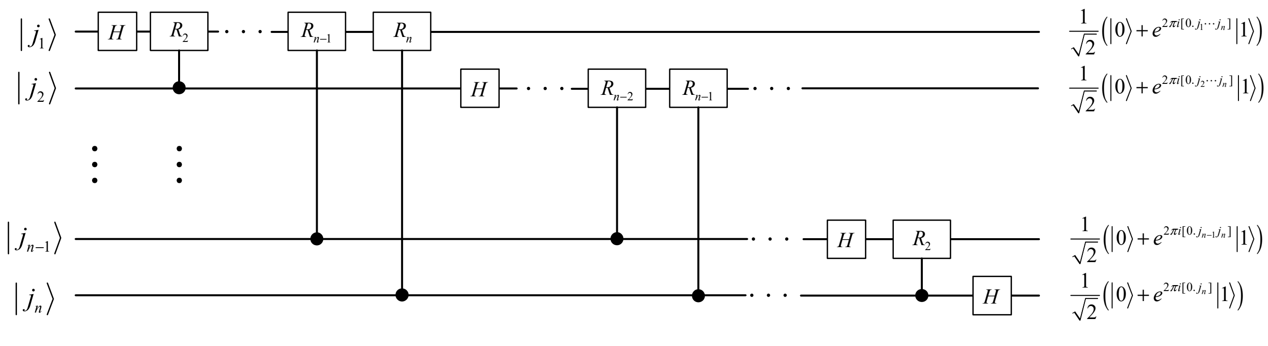

Figure 4. A circuit for the *QFT*.

Therefore, the *QFT* of the quantum state $|j\rangle = |j_1 j_2 \ldots j_n\rangle$ can be expressed as

$$QFT(|j_1 j_2 \ldots j_n\rangle) = \frac{1}{2^{n/2}} \left(|0\rangle + e^{2\pi i[0.j_n]}|1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i[0.j_{n-1}j_n]}|1\rangle\right) \otimes \ldots \otimes \left(|0\rangle + e^{2\pi i[0.j_1 j_2 \ldots j_n]}|1\rangle\right), \tag{30}$$

where the binary of decimals can be expressed as

$$[0.j_1 j_2 \dots j_m] = \sum_{k=1}^{m} j_k 2^{-k}. \tag{31}$$

For instance, $[0.j_1] = j_1/2$ and $[0.j_1 j_2] = j_1/2 + j_2/2^2$. Then, the QFT can be further expressed as

$$QFT(|j_1 j_2 \dots j_n\rangle) = \frac{1}{2^{n/2}} \left(|0\rangle + w_1^{[j_n]}|1\rangle\right) \otimes \left(|0\rangle + w_2^{[j_{n-1}j_n]}|1\rangle\right) \otimes \dots \otimes \left(|0\rangle + w_n^{[j_1 j_2 \dots j_n]}|1\rangle\right). \tag{32}$$

Here, we use $[0.j_1 j_2 \dots j_m] = [j_1 j_2 \dots j_n]/2^m$, and $w_m = w^{-2m} = e^{2\pi i/2^m}$.

To implement the $QWFRFT$, we first present the integer powers ($QFT^0, QFT^1, QFT^2, QFT^3$) of the QFT.

1. We know that $QFT^0 = I$, and I is the identity matrix; obviously, this is a unitary operator. Then, its operation can be expressed as

$$|\alpha\rangle^{-} I^{-} |\beta_0\rangle$$

2. The QFT is a unitary operator. The Fourier transform of a quantum state $|\alpha\rangle$ can be expressed as

$$|\alpha\rangle^{-} QFT^{-} |\beta_1\rangle$$

3. The quadratic power of the QFT can be expressed as

$$QFT^2 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \dots & 0 \end{pmatrix}$$

For the vector $(\alpha_0, \alpha_1, \dots, \alpha_{n-1})$, the transformation can be expressed as

$$(\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \dots & 0 \end{pmatrix} = (\alpha_0, \alpha_{n-1}, \dots, \alpha_1)$$

In order to realize the quantum circuit of the above matrix, multiple swap gates are required. The swap gate of two quanta is shown in Figure 5.

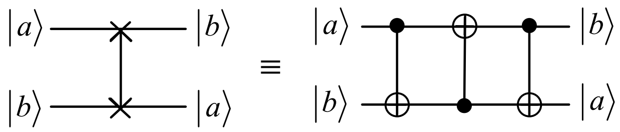

Figure 5. Swap gate.

Thus, for QFT^2, we provide quantum circuits of eight quantum states, as shown in Figure 6.

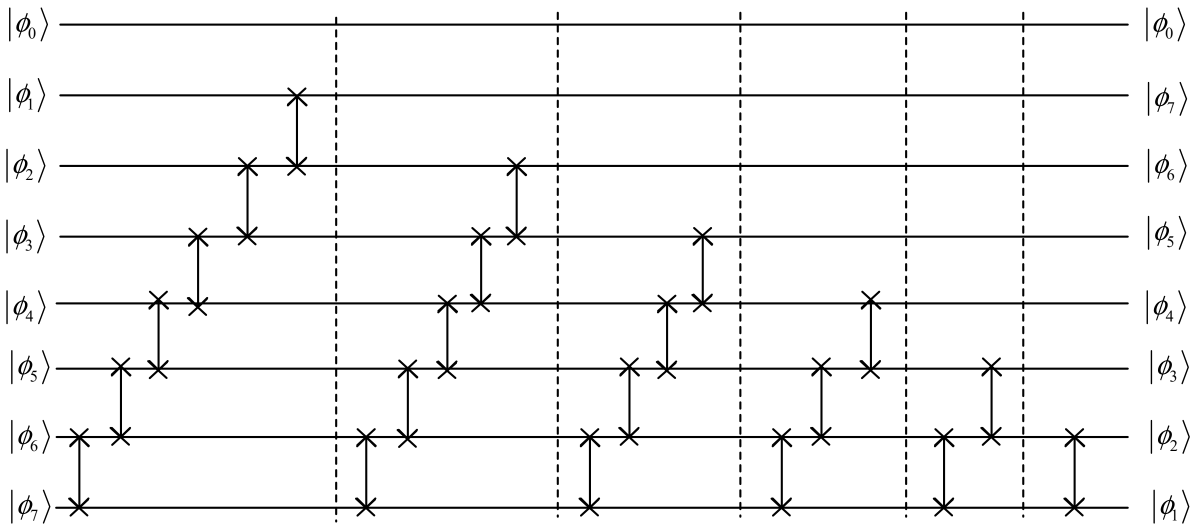

Figure 6. A circuit forthe QFT^2.

For a $2^n \times 2^n$ dimensional identity matrix, we can obtain the QFT^2 by row transformation, as shown in Figure 7.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{2^n \times 2^n} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{2^n \times 2^n}$$

Figure 7. Matrix of the QFT^2.

Therefore, the quantum circuit of Figure 6 can be simplified as Figure 8.

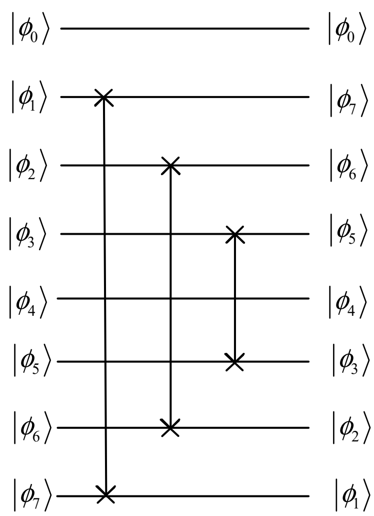

Figure 8. A circuit for the QFT^2.

Thus, the QFT^2 for quantum state $|\alpha\rangle$ can be expressed as

$$|\alpha\rangle \bar{\ } QFT^{2-}|\beta_2\rangle$$

1. The third power of the QFT, which is equivalent to the inverse operation of the QFT, is also a unitary operator.

$$|\alpha\rangle \bar{\ } QFT^{3-}|\beta_3\rangle$$

Therefore, the $QWFRFT$ of the quantum state by Equation (10) can be expressed as

$$\begin{aligned} QWFRFT(|\alpha\rangle) &= \frac{1}{4}\Big(I(|\alpha\rangle), QFT(|\alpha\rangle), QFT^2(|\alpha\rangle), QFT^3(|\alpha\rangle)\Big)\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{pmatrix} B_0^\alpha \\ B_1^\alpha \\ B_2^\alpha \\ B_3^\alpha \end{pmatrix} \\ &= \frac{1}{4}(|\beta_0\rangle, |\beta_1\rangle, |\beta_2\rangle, |\beta_3\rangle)\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{pmatrix} B_0^\alpha \\ B_1^\alpha \\ B_2^\alpha \\ B_3^\alpha \end{pmatrix}. \end{aligned} \tag{33}$$

Equation (33) can be further written as

$$\begin{aligned} QWFRFT(|\alpha\rangle) &= \frac{1}{4}(|\beta_0\rangle, |\beta_1\rangle, |\beta_2\rangle, |\beta_3\rangle)\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{pmatrix} B_0^\alpha \\ B_1^\alpha \\ B_2^\alpha \\ B_3^\alpha \end{pmatrix} \\ &= \frac{1}{4}(|\beta_0\rangle, |\beta_1\rangle, |\beta_2\rangle, |\beta_3\rangle)\begin{pmatrix} \exp\left(\frac{-2\pi i 0\times 0}{4}\right) & \exp\left(\frac{-2\pi i 0\times 1}{4}\right) & \exp\left(\frac{-2\pi i 0\times 2}{4}\right) & \exp\left(\frac{-2\pi i 0\times 3}{4}\right) \\ \exp\left(\frac{-2\pi i 1\times 0}{4}\right) & \exp\left(\frac{-2\pi i 1\times 1}{4}\right) & \exp\left(\frac{-2\pi i 1\times 2}{4}\right) & \exp\left(\frac{-2\pi i 1\times 3}{4}\right) \\ \exp\left(\frac{-2\pi i 2\times 0}{4}\right) & \exp\left(\frac{-2\pi i 2\times 1}{4}\right) & \exp\left(\frac{-2\pi i 2\times 2}{4}\right) & \exp\left(\frac{-2\pi i 2\times 3}{4}\right) \\ \exp\left(\frac{-2\pi i 3\times 0}{4}\right) & \exp\left(\frac{-2\pi i 3\times 1}{4}\right) & \exp\left(\frac{-2\pi i 3\times 2}{4}\right) & \exp\left(\frac{-2\pi i 3\times 3}{4}\right) \end{pmatrix}\begin{pmatrix} B_0^\alpha \\ B_1^\alpha \\ B_2^\alpha \\ B_3^\alpha \end{pmatrix} \end{aligned} \tag{34}$$

where $B_k^\alpha = \exp\left(\frac{2\pi i k\alpha}{4}\right)$; $k = 0, 1, 2, 3$. Then, Equation (34) can be written again as

$$\begin{aligned} QWFRFT(|\alpha\rangle) &= \frac{1}{4}\sum_{l=0}^{3}\sum_{k=0}^{3}|\beta_l\rangle \exp\left(\frac{-2\pi i l k}{4}\right)B_k^\alpha \\ &= \frac{1}{4}\sum_{l=0}^{3}\sum_{k=0}^{3}|\beta_l\rangle \exp\left(\frac{-2\pi i l k}{4}\right)\exp\left(\frac{2\pi i k\alpha}{4}\right) \\ &= \frac{1}{4}\sum_{l=0}^{3}\sum_{k=0}^{3}|\beta_l\rangle \exp\left(\frac{2\pi i k(\alpha - l)}{4}\right). \end{aligned} \tag{35}$$

With the help of the quantum artificial neural network ($QANN$), we are inspired to design a $QWFRFT$. Here, we first introduce the $QANN$ [34,35]. If we use $\{|e_1\rangle, |e_2\rangle, \ldots, |e_M\rangle\}$ to denote the canonical basis for $\mathbb{C}^M$, then the quantum artificial neural network above can be rewritten as

$$Q(|x\rangle) = \sum_{k=1}^{M}\sum_{j=1}^{N}\left(\alpha_{j,k}^{(1)}\sigma_k\left(\left\langle w_{j,k}^{(1)}\middle|T|x\right\rangle + \theta_{j,k}^{(1)}\right) + i\alpha_{j,k}^{(2)}\sigma_k\left(\left\langle w_{j,k}^{(2)}\middle|T|x\right\rangle + \theta_{j,k}^{(2)}\right)\right)|e_k\rangle. \tag{36}$$

Put $y_{j,k}^{(i)} = \sigma_k\left(\sum_{t=1}^{n}\left\langle w_{j,k}^{(i)}(t)\middle|T|x_t\right\rangle + \theta_{j,k}^{(i)}\right)$ and $\left|\alpha_k^{(i)}\right\rangle = \sum_{j=1}^{N}\alpha_{j,k}^{(i)}y_{j,k}^{(i)}|e_k\rangle$. Then, a $QANN$ can be illustrated by Figures 9 and 10 below.

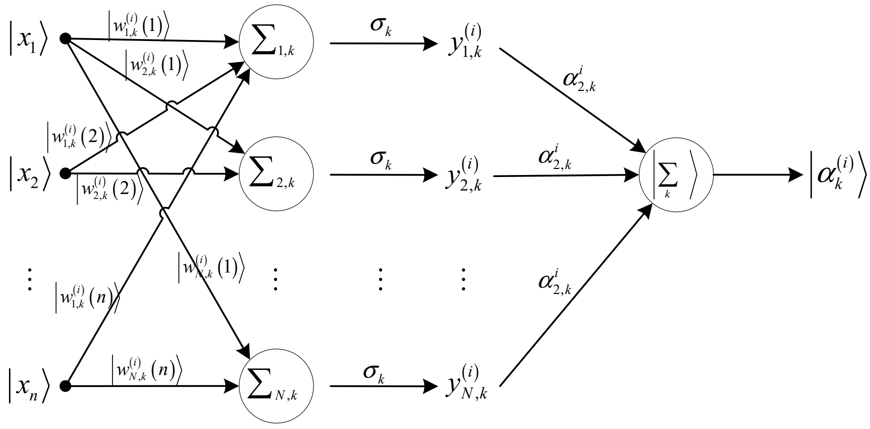

Figure 9. The output $\left|\alpha_k^{(i)}\right\rangle$ of a *QANN*, where $i = 1, 2$; $k = 1, 2, \ldots, M$.

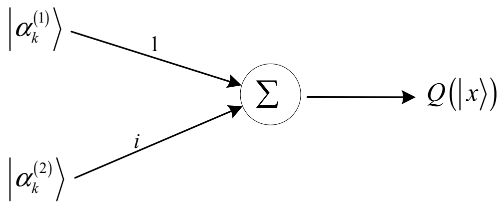

Figure 10. The output $Q(|x\rangle)$ of a *QANN*.

Thus, we can present the circuit of the *QWFRFT*, as shown in Figure 11.

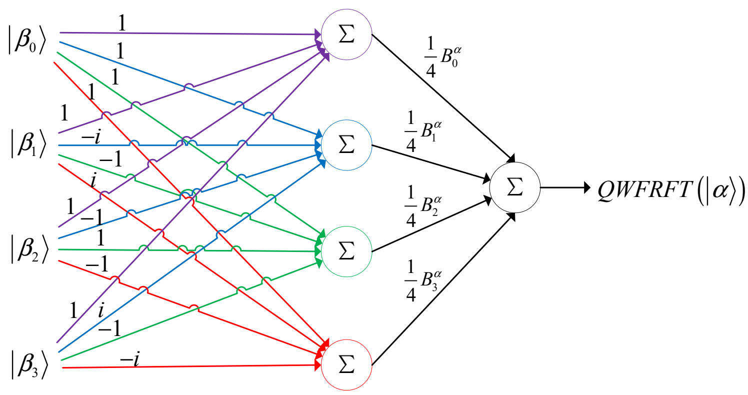

Figure 11. A circuit for the *QWFRFT*.

So far, we have completed the *QWFRFT* and circuit implementation. The work of this paper is a supplement to the work of Parasa et al. At one point, researchers pointed out that there is no quantum-weighted fractional Fourier transform [32]. However, our study illustrates the diversity of *FRFT* and proposes *QWFRFT*. Due to the characteristics of quantum parallelism, we believe that the *QWFRFT* has a wider application space.

At present, our method is only applicable to closed systems. The standard quantum theory has shown its limit to describe successfully experimental results. Counterintuitive results are obtained in different experiments [36,37]. The open system effects need to be further analyzed.

5. Conclusions

Unitarity is a prerequisite for the realization of quantum algorithms. In this paper, we proposed the reformulation of the *WFRFT*. The unitarity of the *WFRFT* was proved by means of the proposed reformulation. The *QFT* is an important part of the *QWFRFT*. Furthermore, we presented the integer power operation and quantum circuit of the *QFT*, which lays the foundation for the *QWFRFT*. Finally, we designed the circuit of the *QWFRFT* with the help of a quantum artificial neural network and proposed the electronic circuit of the *QWFRFT*. The results of this paper show that there is a *QFRFT* algorithm, which lays the foundation for further research.

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