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A Positivity-Preserving Improved Nonstandard Finite Difference Method to Solve the Black-Scholes Equation

Mohammad Mehdizadeh Khalsaraei [1,*], Ali Shokri [1], Higinio Ramos [2], Zahra Mohammadnia [1] and Pari Khakzad [1]

[1] Department of Mathematics, Faculty of Science, University of Maragheh, Maragheh 83111-55181, Iran; shokri@maragheh.ac.ir (A.S.); zahra.mohammadnia3@gmail.com (Z.M.); p.khakzad@yahoo.com (P.K.)
[2] Scientific Computing Group, Universidad de Salamanca, Plaza de la Merced, 37008 Salamanca, Spain; higra@usal.es
* Correspondence: mehdizadeh@maragheh.ac.ir

Abstract: In this paper, we evaluate and discuss different numerical methods to solve the Black–Scholes equation, including the θ-method, the mixed method, the Richardson method, the Du Fort and Frankel method, and the MADE (modified alternating directional explicit) method. These methods produce numerical drawbacks such as spurious oscillations and negative values in the solution when the volatility is much smaller than the interest rate. The MADE method sacrifices accuracy to obtain stability for the numerical solution of the Black–Scholes equation. In the present work, we improve the MADE scheme by using non-standard finite difference discretization techniques in which we use a non-local approximation for the reaction term (we call it the MMADE method). We will discuss the sufficient conditions to be positive of the new scheme. Also, we show that the proposed method is free of spurious oscillations even in the presence of discontinuous initial conditions. To demonstrate how efficient the new scheme is, some numerical experiments are performed at the end.

Keywords: Black–Scholes equation; MADE scheme; nonstandard finite differences; positivity-preserving scheme

MSC: 65M06; 65N06; 91B25

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1. Introduction

Financial mathematics is a field of applied mathematics that deals with financial markets. The field is of great help in the market of financial derivatives, futures, options and forward contracts, among others. If a holder has access to the simplest option, he or she will have the right but is not forced to buy or sell an underlying asset at a certain price or before a given date at a fixed strike price. Called European options, these options have two major brands, puts and calls; such an option is allowed to be exercised only at the expiration date T. American options are, however, allowed to be exercised at any time $t < T$. Option pricing is extremely important in such markets. Black and Scholes' seminal study is very important in this regard [1].

In [2], new singular perturbation techniques to price European, American, and barrier options are used. The use of these strategies has led to the simplification of problems by reducing the number of parameters. In [3], a numerical method is proposed by Xi et al., which uses the Crank–Nicolson approximation for the time derivative and a hybrid finite difference method in a uniform mesh for the spatial derivative. The introduced method is stable for the arbitrary volatility and arbitrary asset price and has a second-order accuracy for both spatial and temporal variables. In [4], Milev and Tagliani examined pricing the barrier options for the random asset price. The problem is postulated as the computation of a path integral, choosing an approach similar to the quadrature method. In [5], instead of the Crank–Nicolson method, a semi-implicit method is used that has no abnormal oscillations for discontinuous boundary conditions. Although this method has low accuracy, it has fewer restrictions on the length of the time step. The ADE method is proposed for pricing one-factor models in [6]. Furthermore, the accuracy, stability, and efficiency of the method are examined. Khalsaraei and Shokri [7] introduced an explicit method resting on a nonstandard discretization scheme to solve the option valuation problem using a double-barrier knock-out call option. The proposed nonstandard numerical scheme preserves the positivity as well as stability and consistency, and in [8], they proposed a new scheme based on a nonstandard discretization of the spatial derivatives. Along the unconditional positivity property, the scheme has an efficient performance in larger time steps. In [9], an explicit numerical method for solving the Black–Scholes equation without boundary conditions is presented by Jeong et al.. At each time step, they obtain a numerical solution at the new point by reducing one or two computational points. In [10], Anwar and Andallah introduced an explicit method for the numerical solution of the Black–Scholes equation in which the stability condition of the method is obtained by a convex combination. In addition to the above methods, there are other methods for numerically solving PDEs, ODEs, IEs, and Fractional equations [11–27].

Double-barrier options are becoming more and more popular. This article addresses the application of a nonstandard finite difference method to pricing the European call options using a discrete double-barrier. The positivity preserving and smoothing properties of the method are also discussed. Our emphasis is on the double-barrier knock-out option which uses discrete monitoring. This will satisfy the Black–Scholes pricing partial differential equation [5]:

$$-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \qquad (S,t) \in R^+ \times [0,T] \tag{1}$$

where S is the asset price, and $V(S,t)$ is the price of the option. This equation include initial and boundary conditions:

$$V(S,0) = \max(S-K,0)\mathbf{1}_{[L,U]}(S),$$

$$V(S,t) \to 0 \quad \text{as} \quad S \to 0 \quad \text{or} \quad S \to \infty,$$

with update of the initial condition at the monitoring dates t_i, $i = 1, \dots, F$:

$$V(S,t_i) = V(S,t_i^-)\mathbf{1}_{[L,U]}(S), \qquad 0 = t_0 < t_1 < \ldots < t_F = T,$$

where $\mathbf{1}_{[L,U]}(S)$ is the indicator function, i.e.,

$$\mathbf{1}_{[L,U]}(S) = \begin{cases} 1 & if \quad S \in [L,U] \\ 0 & if \quad S \notin [L,U]. \end{cases} \tag{2}$$

Here,

- K is the exercise price;
- T is the maturity;
- $r > 0$ is the interest rate;
- $\sigma > 0$ is the reference volatility.

Note that the option has a payoff condition of max $(S-K,0)$. The option is, nonetheless, rendered worthless if the maturity of the asset price S falls outside the corridor $[L,U]$ at the monitoring dates fixed before. Alternatively, the knock-out clause at the monitoring date will introduce a discontinuity at the barriers set at $S-L$ and $S-U$.

The organization of the remainder of the paper is as follows: In Section 2, we review different numerical methods that are used to solve the Black–Scholes equation. In Section 3, a summary of the NSFD strategy is provided. Section 4 examines how the nonstandard methods are examined in view of spatial nonlocalized discretization. In Section 5, we present the analysis of the method concerning the positivity preserving property, stability, and truncation error. The numerical results obtained from the new method are presented in Section 6.

2. Finite Difference Approaches

In this section, we give a summary of different numerical methods to solve the Black–Scholes equation. Let us consider the computational domain $\Omega = [0, S_{\max}] \times [0, T]$ and discretize it in the following form. We introduce a grid of mesh points (S_j, t_n) where $S_j = j\Delta S, t_n = n\Delta t$, $j = 0, 1, \ldots, M; n = 0, 1, \ldots, X$, the spatial step size is given by $\Delta S = S_{\max}/M$, and the time step size is $\Delta t = T/X$. We denote the approximation of $V(S_j, t_n)$ by V_j^n.

2.1. The θ-Method

The θ-method is defined by replacing the partial derivatives with respect to S at the mesh point (S_j, t_n) as follows

$$\frac{\partial V}{\partial S} \approx \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S},$$

$$\frac{\partial^2 V}{\partial S^2} \approx (1-\theta)\frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta S^2} + \theta\frac{V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}}{\Delta S^2},$$

and the derivative with respect to t by

$$\frac{\partial V}{\partial t} \approx \frac{V_j^{n+1} - V_j^n}{\Delta t}.$$

Therefore, the family of the standard θ-method [1,28] for solving (1) leads to a difference equation

$$AV^{n+1} = BV^n, \tag{3}$$

where $V^n = (V_1^n, \ldots, V_{M-1}^n)^T$, $V^{n+1} = (V_1^{n+1}, \ldots, V_{M-1}^{n+1})^T$ and

$$A = tridiag\left\{-\frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2, \frac{1}{\Delta t} + \theta\left(\frac{\sigma S_j}{\Delta S}\right)^2, -\frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2\right\},$$
$$B = tridiag\left\{\frac{1-\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - \frac{rS_j}{2\Delta S}, \frac{1}{\Delta t} - (1-\theta)\left(\frac{\sigma S_j}{\Delta S}\right)^2 - r, \frac{1-\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 + \frac{rS_j}{2\Delta S}\right\}.$$

2.2. The Mixed Method

In this method, by using the Laplace Transform and taking the k-th derivative of (1), the following ordinary differential equation (ODE) is obtained [29]:

$$-\frac{1}{2}\sigma^2 S^2 \frac{d^2U^{(k)}}{dS^2} - rS\frac{dU^{(k)}}{dS} + (r+\lambda)U^{(k)} = \begin{cases} V(S,0), & k = 0, \\ -kU^{(k-1)}, & k = 1, 2, \cdots. \end{cases} \tag{4}$$

By using the centered differences for $\frac{dU^{(k)}}{dS}$ and $\frac{d^2U^{(k)}}{dS^2}$ we obtain

$$-\frac{1}{2}\sigma^2 S_j^2 \left(\frac{U_{j-1}^{(k)} - 2U_j^{(k)} + U_{j+1}^{(k)}}{\Delta S^2}\right) - rS_j\left(\frac{U_{j+1}^{(k)} - U_{j-1}^{(k)}}{2\Delta S}\right) + (r+\lambda)U_j^{(k)} = \begin{cases} V(S,0), & k = 0, \\ -kU^{(k-1)}, & k = 1, 2, \cdots, \end{cases} \tag{5}$$

which leads to:

$$\begin{cases} A_{pw}U^{(k)} = V(S,0), & k = 0, \\ A_{pw}U^{(k)} = -kU^{(k-1)}, & k = 1,2,\cdots, \end{cases} \tag{6}$$

with

$$A_{pw} = tridiag\left\{ -\frac{1}{2}\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 - r\frac{S_j}{\Delta S}\right]; (r+\lambda) + \left(\frac{\sigma S_j}{\Delta S}\right)^2; -\frac{1}{2}\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 + r\frac{S_j}{\Delta S}\right]\right\}. \tag{7}$$

If we take in (6) $k = 1,\ldots,X$ and $\lambda = \frac{X}{T}$, then the approximation $V_X(S_j,t)$ of $V(S_j,t)$ using the post-wider formula, is given by (see [30] for more details)

$$V_X(S_j,T) = \frac{(-1)^X}{X!}\left(\frac{X}{T}\right)^{X+1} U^{(M)}(S_j, \frac{X}{T}), \tag{8}$$

with $\lim_{X\to\infty} V_X(S_j,t) = V(S_j,t)$. An explicit form for $V_X(S_j,T)$ is obtained by combining (6) with (8)

$$V_X(S_j,T) = \left(\frac{X}{T}A_{pw}^{-1}\right)^{X+1} V(S_j,0), \tag{9}$$

so that $\frac{X}{T}A_{pw}^{-1}$ is the iteration matrix.

2.3. The Richardson Method

In this method, ref. [31] by replacing the derivative with respect to t by

$$\frac{\partial V}{\partial t} \approx \frac{V_j^{n+1} - V_j^{n-1}}{\Delta t},$$

and using the centered difference approximations for $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$, the discretization of (1) is given by the following difference equation

$$V^{n+1} = AV^n + V^{n-1}, \tag{10}$$

where $V^{n-1} = (V_1^{n-1},\ldots,V_{M-1}^{n-1})^T$, $V^n = (V_1^n,\ldots,V_{M-1}^n)^T$, $V^{n+1} = (V_1^{n+1},\ldots,V_{M-1}^{n+1})^T$, and A is the following tridiagonal matrix

$$A = tridiag\left\{\Delta t\left[-\frac{rS_j}{\Delta S} + \left(\frac{\sigma S_j}{\Delta S}\right)^2\right]; 2\Delta t\left[-\left(\frac{\sigma S_j}{\Delta S}\right)^2 - r\right]; \Delta t\left[\frac{rS_j}{\Delta S} + \left(\frac{\sigma S_j}{\Delta S}\right)^2\right]\right\}.$$

2.4. The Du Fort and Frankel Method

In this method [32], approximating the second derivative in the diffusion term by

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{j+1}^n - (V_j^{n+1} + V_j^{n-1}) + V_{j-1}^n}{\Delta S^2}$$

and replacing $\frac{\partial V}{\partial S}$ and $\frac{\partial V}{\partial t}$ by the centered difference approximations, we obtain the difference equation

$$V^{n+1} = AV^n + \alpha V^{n-1}, \tag{11}$$

where $V^{n-1} = (V_1^{n-1},\ldots,V_{M-1}^{n-1})^T$, $V^n = (V_1^n,\ldots,V_{M-1}^n)^T$, $V^{n+1} = (V_1^{n+1},\ldots,V_{M-1}^{n+1})^T$ and A is the following tridiagonal matrix

$$A = tridiag\left\{ \frac{-\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2}{\frac{1}{2\Delta t} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2}; \frac{-r}{\frac{1}{2\Delta t} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2}; \frac{\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2}{\frac{1}{2\Delta t} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2} \right\},$$

with

$$\alpha = \frac{\frac{1}{2\Delta t} - \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2}{\frac{1}{2\Delta t} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2}.$$

2.5. The MADE Method

The MADE scheme [31] approximates at each point (S_j, t_n) of the discretized domain the second derivative in Equation (1) by

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{1}{\Delta S^2}\left(V_{j+1}^n - 2V_j^{n+1} + V_{j-1}^n\right)$$

and the first partial derivatives by

$$\begin{aligned} \frac{\partial V}{\partial t} &\approx \frac{1}{\Delta t}\left(V_j^{n+1} - V_j^n\right), \\ \frac{\partial V}{\partial S} &\approx \frac{1}{2\Delta S}\left(V_{j+1}^n - V_{j-1}^n\right). \end{aligned}$$

The resulting difference equation to approximate (1) is given by

$$-\frac{1}{\Delta t}\left(V_j^{n+1} - V_j^n\right) + \frac{rS_j}{2\Delta S}\left(V_{j+1}^n - V_{j-1}^n\right) + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2\left(V_{j+1}^n - 2V_j^{n+1} + V_{j-1}^n\right) - rV_j^n = 0. \quad (12)$$

This scheme builds for each n a system

$$AV^{n+1} = BV^n, \quad (13)$$

where $V^n = (V_1^n, \ldots, V_{M-1}^n)^T$, $V^{n+1} = (V_1^{n+1}, \ldots, V_{M-1}^{n+1})^T$ and

$$\begin{aligned} A =& diag\left\{ \frac{1}{\Delta t} + \left(\frac{\sigma S_j}{\Delta S}\right)^2 \right\}, \\ B =& tridiag\left\{ -\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2; \frac{1}{\Delta t} - r; \frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 \right\}. \end{aligned} \quad (14)$$

According to Figures 1–7, the θ-method for different values of θ, the mixed method, the Richardson method, the Du Fort and Frankel method, and the MADE method display lower accuracy and often produce numerical drawbacks such as spurious oscillations and negative values in the respective solution every time the financial parameters of the Black–Scholes model σ and r satisfy the relationship $\sigma^2 \ll r$. This can be seen clearly in Figures 1–7 right), where the cross section at $t = T$ of the analytical solution and different numerical methods obtained are shown.

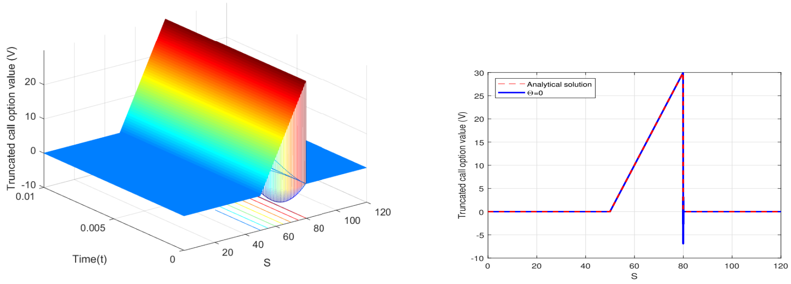

Figure 1. Spurious oscillations in the solution provided by the θ-method with $\Delta S = 0.01,\ \Delta t = 10^{-4}$, and $\theta = 0$. Parameter values: $r = 0.07,\ \sigma = 0.001,\ T = 0.01,\ U = 80,\ K = 50,\ L = 20,\ S_{max} = 120$.

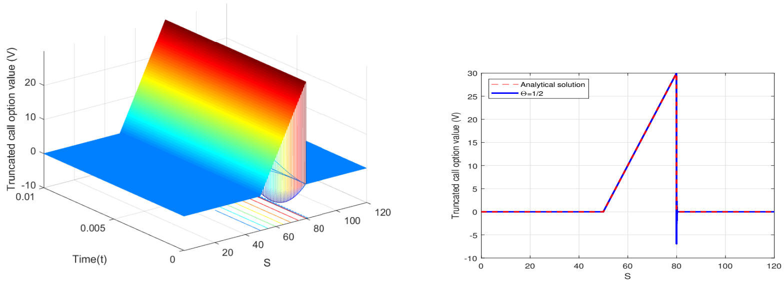

Figure 2. Spurious oscillations in the solution provided by the θ-method with $\Delta S = 0.01,\ \Delta t = 10^{-4}$, and $\theta = \frac{1}{2}$. Parameter values: $r = 0.07,\ \sigma = 0.001,\ T = 0.01,\ U = 80,\ K = 50,\ L = 20,\ S_{max} = 120$.

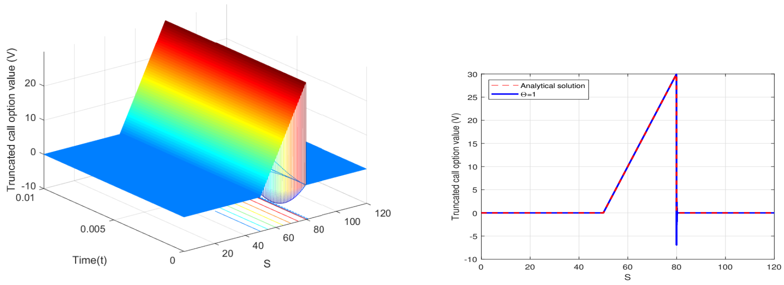

Figure 3. Spurious oscillations in the solution provided by the θ-method with $\Delta S = 0.01,\ \Delta t = 10^{-4}$, and $\theta = 1$. Parameter values: $r = 0.07,\ \sigma = 0.001,\ T = 0.01,\ U = 80,\ K = 50,\ L = 20,\ S_{max} = 120$.

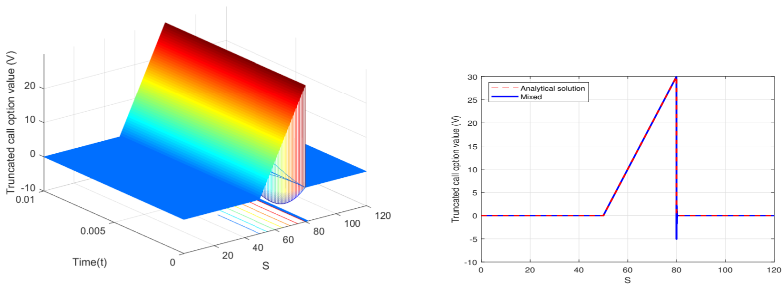

Figure 4. Spurious oscillations in the solution provided by the mixed method with $\Delta S = 0.01,\ \Delta t = 10^{-4}$. Parameter values: $r = 0.07,\ \sigma = 0.001,\ T = 0.01,\ U = 80,\ K = 50,\ L = 20,\ S_{max} = 120$.

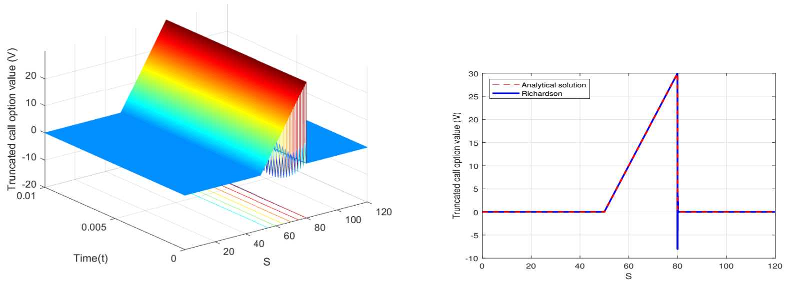

Figure 5. Spurious oscillations in the solution provided by the Richardson method with $\Delta S = 0.01,\ \Delta t = 10^{-4}$. Parameter values: $r = 0.07,\ \sigma = 0.001,\ T = 0.01,\ U = 80,\ K = 50,$ $L = 20,\ S_{max} = 120$.

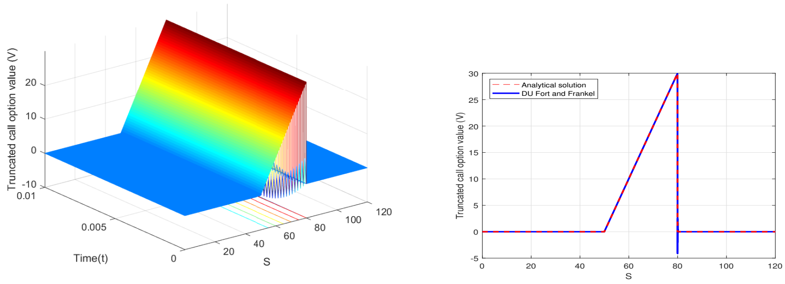

Figure 6. Spurious oscillations in the solution provided by the Du Fort and Frankel method with $\Delta S = 0.01,\ \Delta t = 10^{-4}$. Parameter values: $r = 0.07,\ \sigma = 0.001,\ T = 0.01,\ U = 80,\ K = 50,$ $L = 20,\ S_{max} = 120$.

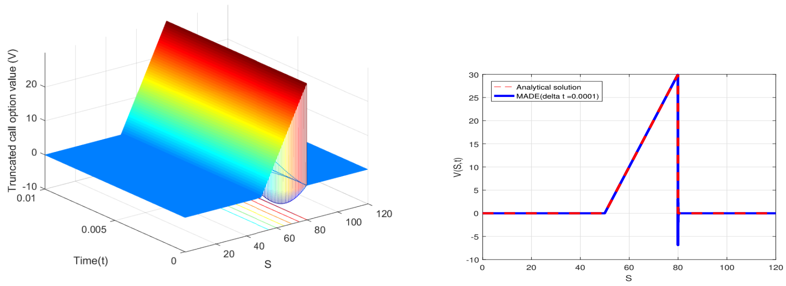

Figure 7. Spurious oscillations in the solution provided by the MADE scheme with $\Delta S = 0.01,$ $\Delta t = 10^{-4}$. Parameter values: $r = 0.07,\ \sigma = 0.001,\ T = 0.01,\ U = 80,\ K = 50,\ L = 20,\ S_{max} = 120$.

3. Nonstandard Finite-Difference Strategy

In this section, we will briefly introduce nonstandard finite difference methods (NSFDs) (for more details one can see [4,5,7,8,28,33–42]). Numerical methods based on naive finite difference approximations to solve ODEs and PDEs problems may not work well, and properties such as positivity of the solution cannot be transferred to the numerical solutions. As a result, there will be a need to devise and examine the numerical methods so that the problem may be solved. For this purpose, NSFD methods were introduced by Mickens [40]. The foundation for NSFD methods is a detailed study of finite difference schemes called exact finite differences. These results have also provided insight into the requisite structural properties of NSFD methods by expanding their application to special groups of differential equations for which exact schemes are not available. Constructing these methods is not always easy, and there is no general rule for doing so. However, Mikens presented a few rules for constructing nonstandard schemes [43]. These methods maintain ordinary properties such as stability, consistency, and convergence. In addition, these methods are designed to maintain the qualitative properties of the exact answer.

This class of schemes and their formulations pivot around two points. The first concerns how discrete representations should be formulated for derivatives. The second concerns the appropriate ways to approximate nonlinear terms. The forward finite difference formula is one of the simplest discretization schemes. In this formula, the derivative V_x is replaced by $\frac{V(x+h)-V(x)}{h}$, where h is the step size. However, in Mickens' schemes, $\frac{V(x+h)-V(x)}{\phi(h)}$ substitutes the term, where $\phi(h)$ is an increasing continuous function of h, which satisfies the following condition

$$\phi(h) = h + O(h^2), \quad 0 < \phi(h) < 1, \quad h \to 0. \tag{15}$$

Note that by taking the limit when $h \to 0$, we must obtain the first derivative whatever $\phi(h)$ is taken. It must be

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \lim_{h\to 0} \frac{V(x+\phi_1(h)) - V(x)}{\phi_2(h)}, \tag{16}$$

where $\phi_1(h)$ and $\phi_2(h)$ are continuous functions of the step size h verifying (15). We can call a scheme a nonstandard finite difference method only if at least one of the following is satisfied:

- The function in the denominator of the approximation of the discrete derivative must be expressed in terms of a function ϕ of the step size, provided that (16) holds. This rule allows the introduction of a complex analytic function of h in the denominator with the condition that

$$\phi(h) = h + O(h^2), \quad 0 < \phi(h) < 1, \quad h \to 0. \tag{17}$$

  Examples of functions $\phi(h)$ that satisfy this condition are [40]:

$$\tanh(h), \quad \frac{1-e^{-\lambda h}}{\lambda} \quad \text{or} \quad 2\sin\left(\frac{h}{2}\right).$$

- Generally, the nonlinear terms can be variously approximated non-locally on the computational grid. For example, reaction terms can be modeled as follows: (see [4,8,17,28,34–42])

$$\begin{aligned} &V \approx 3V_{j+1}^n - 2V_j^n, \\ &V^2 \approx a(V_j^n)^2 + bV_j^n V_{j+1}^n, \quad a+b=1, \quad a,b \in \mathbb{R}, \\ &V^3 \approx a(V_j^n)^3 + (1-a)(V_j^n)^2 V_{j+1}^n, \quad a \in \mathbb{R}. \end{aligned}$$

Although some special techniques can be found in [4,5,7,8,28,33–42], there is no proper general rule for choosing the function $\phi(h)$ or for deciding which nonlinear terms ought to be replaced.

4. Scheme Construction

In this section, to prevent spurious oscillations and negative values in the solution of the Black–Scholes equation in (1), we modify the MADE scheme by using nonstandard discretization techniques. By replacing

$$V(S_j, t_n) \approx a(V_{j-1}^n - 2V_j^n + V_{j+1}^n) + V_j^{n+1}, \tag{18}$$

in the reaction term of (1), we construct the MMADE method for (1) as:

$$\begin{aligned} &-\frac{1}{\Delta t}\left(V_j^{n+1} - V_j^n\right) + \frac{\sigma^2 S_j^2}{2\Delta S^2}\left(V_{j+1}^n - 2V_j^{n+1} + V_{j-1}^n\right) + \frac{rS_j}{2\Delta S}\left(V_{j+1}^n - V_{j-1}^n\right) \\ &- r(a(V_{j-1}^n - 2V_j^n + V_{j+1}^n) + V_j^{n+1}) = 0, \end{aligned} \tag{19}$$

The explicit form of (19) concerning the temporal variable is:

$$V_j^{n+1} = \frac{\left(-\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - ra\right)V_{j-1}^n + \left(\frac{1}{\Delta t} + 2ra\right)V_j^n + \left(\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - ra\right)V_{j+1}^n}{\frac{1}{\Delta t} + \left(\frac{\sigma S_j}{\Delta S}\right)^2 + r}, \tag{20}$$

where a is an arbitrary parameter to be determined by imposing positivity restrictions. The corresponding finite difference approximation provides for each $n = 0, 1, \ldots, X-1$, the difference equation

$$V^{n+1} = NV^n, \tag{21}$$

where N is the matrix:

$$N = tridiag\left\{\frac{-\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - ra}{\frac{1}{\Delta t} + \left(\frac{\sigma S_j}{\Delta S}\right)^2 + r}; \frac{\frac{1}{\Delta t} + 2ra}{\frac{1}{\Delta t} + \left(\frac{\sigma S_j}{\Delta S}\right)^2 + r}; \frac{\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - ra}{\frac{1}{\Delta t} + \left(\frac{\sigma S_j}{\Delta S}\right)^2 + r}\right\}, \tag{22}$$

and $V^n = (V_1^n, \ldots, V_{M-1}^n)^T$, $V^{n+1} = (V_1^{n+1}, \ldots, V_{M-1}^{n+1})^T$ are vectors containing approximations to V at the discrete points, at time levels t_n and t_{n+1}, respectively.

5. Analysis of the New Scheme

The following theorem gives sufficient conditions on the parameter a and the step size Δt so that (21) is a positivity-preserving scheme to solve the Black–Scholes problem in (1).

Theorem 1. *If $a \le -\frac{r}{8\sigma^2}$ and $\Delta t < \frac{1}{-2ra}$, then the scheme in (21) for solving the Black–Scholes equation in (1) is positivity-preserving.*

Proof. Since the denominators of the entries in the matrix N are positive, it suffices to prove that the numerators of the entries in N are non-negative. Thus, we impose that

$$\frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - \frac{rS_j}{2\Delta S} - ra \ge 0, \tag{23}$$

$$\frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 + \frac{rS_j}{2\Delta S} - ra \ge 0, \tag{24}$$

$$\frac{1}{\Delta t} + 2ra > 0. \tag{25}$$

From (23), we can write

$$\begin{aligned}
& ra \le \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - \frac{rS_j}{2\Delta S} \\
\Leftrightarrow\ & a \le \frac{1}{r}\left[\frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - \frac{rS_j}{2\Delta S}\right] \\
\Leftrightarrow\ & a \le \frac{\sigma^2}{2r}\left[\left(\frac{S_j}{\Delta S}\right)^2 - \frac{r}{\sigma^2}\left(\frac{S_j}{\Delta S}\right) + \frac{r^2}{4\sigma^4} - \frac{r^2}{4\sigma^4}\right] \\
\Leftrightarrow\ & a \le \frac{\sigma^2}{2r}\left[\left(\frac{S_j}{\Delta S} - \frac{r}{2\sigma^2}\right)^2 - \frac{r^2}{4\sigma^4}\right] \\
\Leftrightarrow\ & a \le \frac{\sigma^2}{2r}\left(\frac{S_j}{\Delta S} - \frac{r}{2\sigma^2}\right)^2 - \frac{r}{8\sigma^2}.
\end{aligned} \tag{26}$$

Now, from the last inequality in (26), we have that if $a \le -\frac{r}{8\sigma^2}$, then (23) holds. In addition, (24) is a direct consequence of (23). Finally, from (25) we have that $\Delta t < \frac{1}{-2ra}$, which completes the proof. □

Theorem 2. *Under the hypothesis of Theorem 1, the MMADE scheme in (21) is conditionally stable and convergent with local truncation error $O(\Delta t, \Delta S^2)$.*

Proof. In view of the conditions in (23) and (24), we have that $\|N\|_\infty \leq \max_{j=1,\ldots,M-1}\left\{\dfrac{\frac{1}{\Delta t}+\frac{\sigma^2 S_j^2}{\Delta S^2}}{\frac{1}{\Delta t}+\frac{\sigma^2 S_j^2}{\Delta S^2}+r}\right\}$. Therefore, we have

$$\rho(N) \leq \|N\|_\infty \leq \max_{j=1,\ldots,M-1}\left\{\frac{\frac{1}{\Delta t}+\frac{\sigma^2 S_j^2}{\Delta S^2}}{\frac{1}{\Delta t}+\frac{\sigma^2 S_j^2}{\Delta S^2}+r}\right\} < 1, \tag{27}$$

where $\rho(N)$ is the spectral radius of the matrix N. By Lax's equivalence theorem [44], this implies that the scheme is conditionally stable and convergent.

The local truncation error is obtained from (19) considering the exact values, that is

$$\begin{aligned} T_j^n =& -\frac{V(S_j,t_{n+1})-V(S_j,t_n)}{\Delta t} + rS_j\frac{V(S_{j+1},t_n)-V(S_{j-1},t_n)}{2\Delta S} \\ &+\frac{1}{2}\sigma^2 S_j^2\frac{V(S_{j-1},t_n)-2V(S_j,t_{n+1})+V(S_{j+1},t_n)}{\Delta S^2} \\ &- r\big(a(V(S_{j-1},t_n)-2V(S_j,t_n)+V(S_{j+1},t_n))+V(S_j,t_{n+1})\big). \end{aligned}$$

Assuming that the previous values are exact, the Taylor's expansions of the above terms centered at (S_j,t_n) result in

$$\begin{aligned} V(S_j,t_{n+1}) =& V_j^n + \Delta t\left(\frac{\partial V}{\partial t}\right)_j^n + \frac{1}{2}\Delta t^2\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n + \frac{1}{6}\Delta t^3\left(\frac{\partial^3 V}{\partial t^3}\right)_j^n + \ldots, \\ V(S_{j+1},t_n) =& V_j^n + \Delta S\left(\frac{\partial V}{\partial S}\right)_j^n + \frac{1}{2}\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n + \frac{1}{6}\Delta S^3\left(\frac{\partial^3 V}{\partial S^3}\right)_j^n + \ldots, \\ V(S_{j-1},t_n) =& V_j^n - \Delta S\left(\frac{\partial V}{\partial S}\right)_j^n + \frac{1}{2}\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n - \frac{1}{6}\Delta S^3\left(\frac{\partial^3 V}{\partial S^3}\right)_j^n + \ldots. \end{aligned}$$

After substituting the above values into T_j^n, we obtain

$$\begin{aligned} T_j^n =& \left(-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV\right)_j^n - \frac{1}{2}\Delta t\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n - ra\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n \\ &- r\Delta t\left(\frac{\partial V}{\partial t}\right)_j^n + \ldots. \end{aligned}$$

Finally, since $V(S,t)$ is the solution of the Black–Scholes Equation (1), the local truncation error is

$$T_j^n = O(\Delta t) + O(\Delta S^2).$$

This concludes the proof. □

Now we want to investigate the accuracy of the approximate solution provided by (21). We consider the approximation given in (18) and substitute the terms V_{j-1}^n, V_{j+1}^n, and V_j^{n+1} by the Taylor's series expansions given in Theorem 2. This results in

$$\begin{aligned} V(S_j, t_n) \simeq & a\left(V_j^n - \Delta S\left(\frac{\partial V}{\partial S}\right)_j^n + \frac{1}{2}\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n + \ldots - 2V_j^n + V_j^n \right. \\ & \left. + \Delta S\left(\frac{\partial V}{\partial S}\right)_j^n + \frac{1}{2}\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n + \ldots \right) + \left(V_j^n + \Delta t\left(\frac{\partial V}{\partial t}\right)_j^n \right. \\ & \left. + \frac{1}{2}\Delta t^2\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n + \ldots \right) \\ \simeq & a\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n + V_j^n + \Delta t\left(\frac{\partial V}{\partial t}\right)_j^n + \ldots . \end{aligned}$$

Now, if we use this substitution in Equation (1) and take $a = -\frac{r}{8\sigma^2}$, we obtain

$$-\left[1 + r\Delta t\right]\left(\frac{\partial V}{\partial t}\right)_j^n + rS_j\left(\frac{\partial V}{\partial S}\right)_j^n + \left[\frac{1}{2}\sigma^2 S_j^2 + \frac{1}{8}\left(\frac{r\Delta S}{\sigma}\right)^2\right]\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n - rV_j^n \simeq 0, \tag{28}$$

which allows us to study how numerical diffusion and numerical dispersion are affected by the MMADE scheme. According to (28), if we take

$$r\Delta t \ll 1 \tag{29}$$

and

$$\frac{1}{8}\left(\frac{r\Delta S}{\sigma}\right)^2 \ll 1, \tag{30}$$

the proposed scheme guarantees an accurate solution for which positivity and stability are preserved. Otherwise, even though the conditions in Theorem 1 are satisfied, the terms $r\Delta t$ and $\frac{1}{8}\left(\frac{r\Delta S}{\sigma}\right)^2$ may produce a poor solution.

6. Numerical Results with MMADE

Example 1. *Consider the Black–Scholes equation in* (1) *to price a truncated call option with parameters* $r = 0.07, \sigma = 0.001, a = -8800, T = 0.01, U = 80, K = 50, L = 20, S_{max} = 120, \Delta S = 0.01, \Delta t = 10^{-4}$.

Figure 8 illustrates the numerical solution provided by the MMADE scheme (left). We can see in Figure 8 (right) that the positivity and stability are preserved.

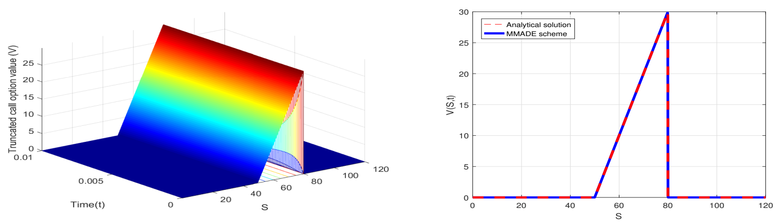

Figure 8. Numerical solution provided by the MMADE scheme. Parameters: $r = 0.07, \sigma = 0.001, a = -8800, T = 0.01, U = 80, K = 50, L = 20, S_{max} = 120, \Delta S = 0.01, \Delta t = 10^{-4}$.

Example 2. *Consider the Black–Scholes equation in* (1) *to price a truncated call option with different values of the interest rate r and parameters* $\sigma = 0.001, a = -1250, T = 1, U = 70, K = 50, L = 30, S_{\max} = 140, \Delta S = 0.05, and \Delta t = 10^{-3}$.

In Figure 9 the MMADE scheme preserves the positivity property, is stable, and the numerical results are in good agreement with the analytical solution for different values of the interest rate, $r = 0.01$ and $r = 0.001$.

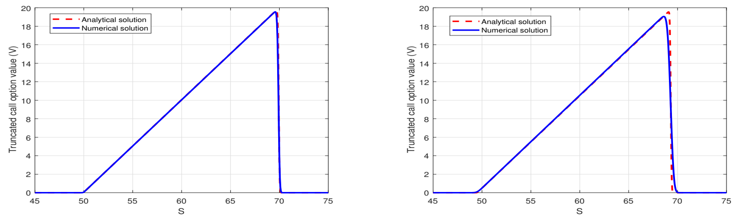

Figure 9. Numerical solution provided by the MMADE scheme with parameter values $r = 0.01$ (**right**) and parameter $r = 0.001$ (**left**). Other parameter values $\sigma = 0.001, a = -1250, T = 1$, $U = 70, K = 50, L = 30, S_{\max} = 140, \Delta S = 0.05, \Delta t = 10^{-3}$.

If one of conditions (29) or (30) is violated, then the numerical solution may have low accuracy (see Figure 10, for Example 1 (left) and Example 2 (right)). However, the MMADE scheme in both cases preserves the positivity of the solutions. On the other hand, if the conditions in Theorem 1 are violated, the MMADE scheme shows a poor performance (see Figure 11, for Example 1 (left) and Example 2 (right)).

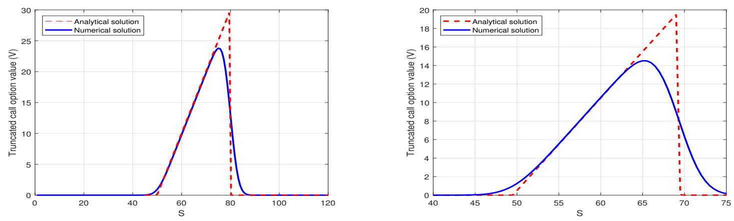

Figure 10. Poor performance of the MMADE scheme if one of the conditions (29) or (30) is violated. Example with parameter values $r = 0.07, \sigma = 0.001, a = -8800, T = 0.01, U = 80, K = 50$, $L = 20, S_{max} = 120, \Delta S = 0.75, \Delta t = 10^{-4}$ (**left**) and Example with parameter values $r = 0.01$, $\sigma = 0.001, a = -1250, T = 1, U = 70, K = 50, L = 30, S_{\max} = 140, \Delta S = 0.5, \Delta t = 10^{-3}$ (**right**).

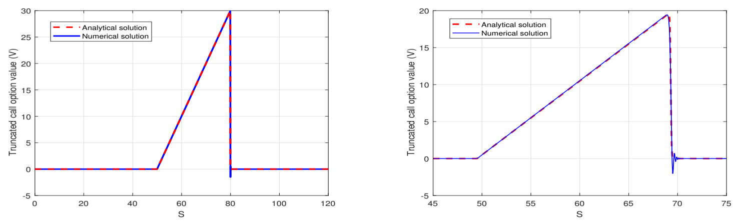

Figure 11. Poor performance of the MMADE scheme if the conditions in Theorem 1 are not satisfied. Example with parameter values $r = 0.07, \sigma = 0.001, a = -1300, T = 0.01, U = 80, K = 50$, $L = 20, S_{max} = 120, \Delta S = 0.01, \Delta t = 10^{-4}$ (**left**) and Example with parameter values $r = 0.01$, $\sigma = 0.001, a = -50, T = 1, U = 70, K = 50, L = 30, S_{\max} = 140, \Delta S = 0.05, \Delta t = 10^{-3}$ (**right**).

When conducting financial analysis, sensitivity analysis is one of the most important financial methods for assessing the risk of investment and examining its financial indicators under conditions of uncertainty. Variables such as the remaining time to the expiry of the option, the strike price, and the volatility or the interest rate, affect the price of the options. A brief review of these changes and their implications follows, namely, the calculation of sensitivities or Greeks. The most important Greeks are:

$$\text{Delta} \qquad \Delta = \frac{\partial V}{\partial S},$$

$$\text{Gamma} \qquad \Gamma = \frac{\partial^2 V}{\partial S^2}.$$

The variation of the price option relative to the increase or decrease in stock prices is measured by Delta. Delta determines the sensitivity of the price option to change the stock price. The buyer will have the discretion to determine the amount of change in the profit or loss of his investment and the maximum loss caused by the failure to enter into an option contract.

Gamma measures the ratio of the introduced Delta change relative to the stock price changes. By measuring Gamma, investors and traders can determine the amount of need for change and adjustment in their portfolio composition, which consists of shares and stock options. The lower the Gamma value, the investor realizes that the number of changes in the size and composition of the portfolio is less than required, and the need for this change becomes more intense as Gamma increases.

We compare the values obtained for Delta and Gamma in Example 1 using the MADE and MMADE schemes. Results for Delta are shown in Figure 12. Delta is correct when the asset price exceeds the upper barrier. When the asset price is lower than the upper barrier, Delta is negative due to the risk of hitting the barrier. Furthermore, the Delta value obtained with the MADE scheme shows spurious oscillations near the upper barrier, while using the MMADE scheme eliminates these oscillations. The results for Gamma are shown in Figure 13. Gamma is correct when the asset price is higher than the upper barrier. It is negative when the asset price is lower than the upper barrier due to the risk of hitting the barrier. The Gamma value obtained using the MADE scheme shows spurious oscillations, while these oscillations are eliminated by using the MMADE scheme.

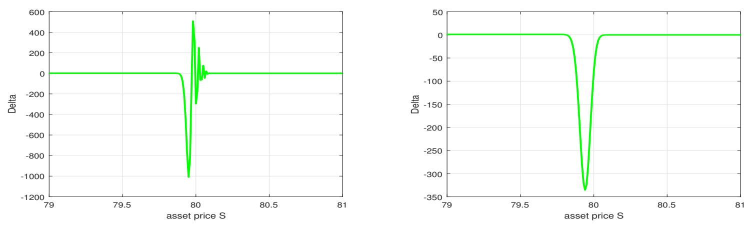

Figure 12. Plots of Delta obtained with the MADE scheme (**left**) and the MMADE scheme (**right**), with parameter values $r = 0.07$, $\sigma = 0.001$, $a = -8800$, $K = 50$, $T = 0.01$, $U = 80$, $L = 20$, $S_{max} = 120$, $\Delta S = 0.01$ and $\Delta t = 10^{-4}$.

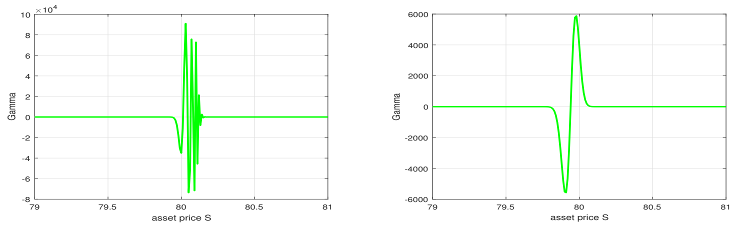

Figure 13. Plots of Gamma obtained with the MADE scheme (**left**) and the MMADE scheme (**right**), with parameter values $r = 0.07$, $\sigma = 0.001$, $a = -8800$, $K = 50$, $T = 0.01$, $U = 80$, $L = 20$, $S_{max} = 120$, $\Delta S = 0.01$ and $\Delta t = 10^{-4}$.

Now, we compare the values obtained for Delta and Gamma in Example 2 using the MADE and the MMADE schemes. The results for Delta are shown in Figure 14, and the results for Gamma are shown in Figure 15. Similar comments as those in Example 1 are valid for this example. Again, spurious oscillations can be shown with the use of the MADE scheme.

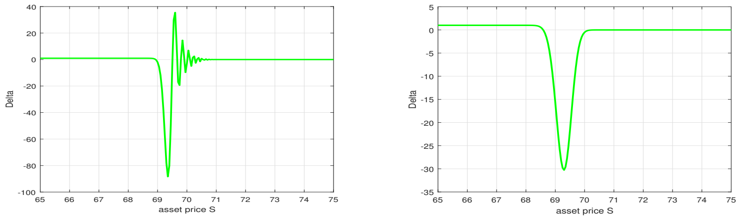

Figure 14. Plots of Delta obtained with the MADE scheme (**left**) and the MMADE scheme (**right**), with parameter values $r = 0.01$, $\sigma = 0.001, a = -1250, T = 1, U = 70, K = 50, L = 30$, $S_{\max} = 140, \Delta S = 0.05, \Delta t = 10^{-3}$.

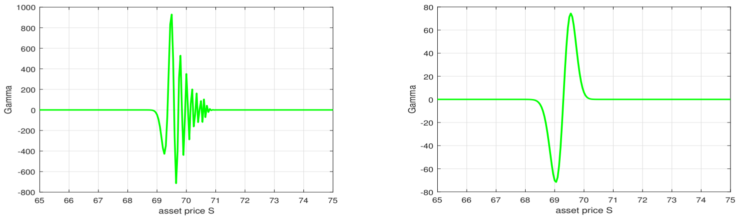

Figure 15. Plots of Gamma obtained with the MADE scheme (**left**) and the MMADE scheme (**right**), with parameter values $r = 0.01$, $\sigma = 0.001, a = -1250, T = 1, U = 70, K = 50, L = 30$, $S_{\max} = 140, \Delta S = 0.05, \Delta t = 10^{-3}$.

7. Conclusions and Discussion

The article introduces a novel numerical method for a Black–Scholes partial differential equation arising in European option pricing. In the proposed method, a nonstandard discretization technique is used to improve the MADE method. To this end, a non-local expression to approximate the reaction sentence of the Black–Scholes equation in the MADE method is used. One of its advantages over the MADE method and other numerical methods such as the θ-method, the mixed method, the Richardson method, and the Du Fort and Frankel method, is that it eliminates spurious oscillations and preserves the positivity property. Furthermore, it is stable and the numerical results agree with the analytical solution. Comparisons of the Delta and Gamma parameters obtained with the MADE and MMADE schemes show that the MMADE scheme eliminates the spurious oscillations near the upper barrier. The results of the study suggest that nonstandard difference schemes might be effective at solving the nonlinear Black–Scholes equation. As part of our subsequent investigation, we will employ nonstandard methods to address these problems.

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