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Existence of Solutions for Coupled Higher-Order Fractional Integro-Differential Equations with Nonlocal Integral and Multi-Point Boundary Conditions Depending on Lower-Order Fractional Derivatives and Integrals

Muthaiah Subramanian [1,†], Jehad Alzabut [2,3,†], Mohamed I. Abbas [4,†], Chatthai Thaiprayoon [5,6,*,†] and Weerawat Sudsutad [7,†]

1 Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore 641407, India; subramanianmcbe@gmail.com
2 Deparment of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia; jalzabut@psu.edu.sa or jehad.alzabut@ostimteknik.edu.tr
3 Department of Industrial Engineering, OSTİM Technical University, 06374 Ankara, Turkey
4 Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Alexandria 21511, Egypt; miabbas@alexu.edu.eg
5 Research Group of Theoretical and Computation in Applied Science, Department of Mathematics, Faculty of Science, Burapha University, Chonburi 20131, Thailand
6 Center of Excellence in Mathematics, CHE, Sri Ayutthaya Road, Bangkok 10400, Thailand
7 Department of Statistics, Faculty of Science, Ramkhamhaeng University, Bangkok 10240, Thailand; weerawat.s@rumail.ru.ac.th
* Correspondence: chatthai@go.buu.ac.th
† These authors contributed equally to this work.

Abstract: In this article, we investigate the existence and uniqueness of solutions for a nonlinear coupled system of Liouville–Caputo type fractional integro-differential equations supplemented with non-local discrete and integral boundary conditions. The nonlinearity relies both on the unknown functions and their fractional derivatives and integrals in the lower order. The consequence of existence is obtained utilizing the alternative of Leray–Schauder, while the result of uniqueness is based on the concept of Banach contraction mapping. We introduced the concept of unification in the present work with varying parameters of the multi-point and classical integral boundary conditions. With the help of examples, the main results are well demonstrated.

Keywords: coupled system; integro-differential equations; Caputo derivatives; multi-point; integral boundary conditions; fixed point theorems

MSC: 26A33; 34A08; 34B15

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Citation: Subramanian, M.; Alzabut, J.; Abbas, M.I.; Thaiprayoon, C.; Sudsutad, W. Existence of Solutions for Coupled Higher-Order Fractional Integro-Differential Equations with Nonlocal Integral and Multi-Point Boundary Conditions Depending on Lower-Order Fractional Derivatives and Integrals. *Mathematics* **2022**, *10*, 1823. https://doi.org/10.3390/math10111823

Academic Editor: Christopher Goodrich

Received: 15 April 2022
Accepted: 20 May 2022
Published: 25 May 2022

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1. Introduction

In the mathematical modeling of many real-world problems, the study of coupled systems of fractional orders of differential equations (FDEs) has gained significant attention; for example, chaotic system synchronization [1,2], anomalous diffusion [3], ecological models [4], etc. We refer to some papers for some recent results on coupled systems with FDEs [5–13]. The use of fractional calculus methods is quite prominent in the mathematical modeling of various processes and phenomena. The main reason is that fractional operators, unlike integer operators, are non-local and able to trace the past effects of the phenomena involved; see [14–19] for examples and details. Some researchers have addressed the problem of fractional boundary value problems (BVPs), and a significant trend can be seen in the recent literature; for example, see [20–37] and the references cited therein. A few

authors have recently started investigating coupled fractional BVPs. Ahmad et al. [38] discussed the solvability of the following coupled FDEs with integral boundary conditions:

$$\begin{cases} {}^{C}\mathcal{D}^{q}x(t) = f(t, x(t), y(t)), \\ {}^{C}\mathcal{D}^{p}y(t) = h(t, x(t), y(t)), \\ x^{'}(0) = \alpha \int_0^{\xi} x^{'}(s)ds, \;\; x(1) = \beta \int_0^{1} g(x^{'}(s))ds, \\ y^{'}(0) = \alpha_1 \int_0^{\theta} y^{'}(s)ds, \;\; y(1) = \beta_1 \int_0^{1} g(y^{'}(s))ds, \\ t \in [0,1], \;\; 1 < q, p \leq 2, \;\; 0 \leq \xi, \theta \leq 1, \end{cases}$$

where ${}^{C}\mathcal{D}^{q}$, ${}^{C}\mathcal{D}^{p}$ denote the Caputo fractional derivatives (CFDs) of order q, p, f, h: $[0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and α, β, α_1, β_1 are real constants. The FDEs with integral and ordinary-fractional flux boundary conditions

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}x(t) = f(t, x(t), y(t)), \\ {}^{C}\mathcal{D}^{\beta}y(t) = h(t, x(t), y(t)), \\ x(0) + x(1) = a \int_0^{1} x(s)ds, \;\; x^{'}(0) = b\,{}^{C}\mathcal{D}^{\gamma}x(1), \\ y(0) + y(1) = a_1 \int_0^{1} y(s)ds, \;\; y^{'}(0) = b_1\,{}^{C}\mathcal{D}^{\delta}y(1), \\ t \in [0,1], \;\; 1 < \alpha, \beta \leq 2, \;\; 0 < \gamma, \delta \leq 1, \end{cases}$$

were discussed in [39], where ${}^{C}\mathcal{D}^{\alpha}$, ${}^{C}\mathcal{D}^{\beta}$, ${}^{C}\mathcal{D}^{\gamma}$, ${}^{C}\mathcal{D}^{\delta}$ denote the CFDs of order α, β, γ, δ, f, h: $[0,1] \times \mathbb{R}^2 \to \mathbb{R}$, are given continuous functions, and a, b, a_1, b_1 are real constants. Ahmad et al. analyzed in [40] the existence results for coupled system of FDEs:

$$\begin{cases} \mathcal{D}^{\alpha}u(t) = f(t, v(t), \mathcal{D}^{p}v(t)), \\ \mathcal{D}^{\beta}v(t) = g(t, u(t), \mathcal{D}^{q}u(t)), \\ u(0) = 0, \;\; u(1) = \gamma u(\eta), \\ v(0) = 0, \;\; v(1) = \gamma v(\eta), \\ 0 < t < 1, \;\; 1 < \alpha, \beta < 2, \;\; 0 < \eta < 1, \end{cases}$$

where $\mathcal{D}^{\alpha}$, $\mathcal{D}^{\beta}$, $\mathcal{D}^{p}$, $\mathcal{D}^{q}$ denote the Riemann–Liouville fractional derivatives of order α, β, p, q, f, $g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions, and γ is the real constant. Agarwal et al. [41] analyzed the results with discrete and integral boundary conditions of the existence of coupled fractional-order systems. In fractional BVP involving the Caputo derivatives, Subramanian et al. [42] studied coupled non-local slit-strip conditions.

In this article, we are investigating the existence of solutions for nonlinear coupled Caputo fractional integro-differential equations,

$$\begin{cases} {}^{C}\mathcal{D}^{\varrho}u(\tau) = f(\tau, u(\tau), v(\tau), {}^{C}\mathcal{D}^{\varsigma_1}v(\tau), \mathcal{I}^{\xi}v(\tau)), \;\; \tau \in [0,T] := \mathcal{U}, \\ {}^{C}\mathcal{D}^{\varsigma}v(\tau) = g(\tau, u(\tau), {}^{C}\mathcal{D}^{\varrho_1}u(\tau), \mathcal{I}^{\zeta}u(\tau), v(\tau)), \;\; \tau \in [0,T] := \mathcal{U}, \end{cases} \tag{1}$$

supplemented by nonlocal integral and multi-point boundary conditions,

$$\begin{cases} u(0) = \psi_1(v), \;\; u^{'}(0) = \epsilon_1 \int_0^{\nu_1} v^{'}(\theta)d\theta, \;\; u^{''}(0) = 0, \cdots, u^{n-2}(0) = 0, \\ u(T) = \lambda_1 \displaystyle\int_0^{\delta_1} v(\theta)d\theta + \mu_1 \sum_{j=1}^{k-2} \varpi_j v(\vartheta_j), \\ v(0) = \psi_2(u), \;\; v^{'}(0) = \epsilon_2 \int_0^{\nu_2} u^{'}(\theta)d\theta, \;\; v^{''}(0) = 0, \cdots, v^{n-2}(0) = 0, \\ v(T) = \lambda_2 \displaystyle\int_0^{\delta_2} u(\theta)d\theta + \mu_2 \sum_{j=1}^{k-2} \omega_j u(\varphi_j), \end{cases} \tag{2}$$

where ${}^{C}\mathcal{D}^{\varrho}$, ${}^{C}\mathcal{D}^{\varsigma}$, ${}^{C}\mathcal{D}^{\varrho_1}$, ${}^{C}\mathcal{D}^{\varsigma_1}$ are the Caputo fractional derivatives of order $n-1 < \varrho, \varsigma < n$, $0 < \varrho_1, \varsigma_1 < 1$, $\mathcal{I}^{\zeta}$, $\mathcal{I}^{\xi}$ are the Riemann–Liouville fractional integrals of order $\zeta, \xi > 0$, f, g:$\mathcal{U} \times \mathbb{R}^4 \to \mathbb{R}$, ψ_1, $\psi_2 : \mathcal{C}(\mathcal{U}, \mathbb{R}) \to \mathbb{R}$ are given continuous functions, $0 < \nu_1 < \nu_2 < \delta_1 <$

$\delta_2 < \vartheta_1 < \varphi_1 < \cdots < \vartheta_{n-2} < \varphi_{n-2} < T$, and ϵ_i, λ_i, μ_i $(i = 1, 2)$, ϖ_j, ω_j $(j = 1, 2, \ldots, k-2)$ are positive real constants.

The rest of the article is assembled appropriately. In Section 2, we retrieve those concepts for a good reference and prove an auxiliary lemma, which provides the basis for solving the problem. Section 3 presents the primary outcomes, while Sections 4–6 provide examples, some important observations, and closing remarks, respectively.

2. Preliminaries

Firstly, we remember some fundamental fractional calculus definitions.

Definition 1. *The fractional integral of order α with the lower limit zero for a function $\mathfrak{f}$ is defined as*

$$I^{\varrho}\mathfrak{f}(\tau) = \frac{1}{\Gamma(\varrho)}\int_0^{\tau}\frac{\mathfrak{f}(s)}{(\tau - s)^{1-\varrho}}ds, \tau > 0, \varrho > 0, \tag{3}$$

provided that the right-hand side is point-wise defined on [0.∞), where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\varrho) = \int_0^{\infty}\tau^{\varrho-1}e^{-\tau}d\tau$.

Definition 2. *The Riemann–Liouville fractional derivative of order $\varrho > 0$, $n - 1 < \varrho < n$, $n \in \mathbb{N}$ is defined as*

$$D_{0+}^{\varrho}\mathfrak{f}(\tau) = \frac{1}{\Gamma(n-\varrho)}\left(\frac{d}{d\tau}\right)^n\int_0^{\tau}(\tau - s)^{n-\varrho-1}\mathfrak{f}(s)ds, \tau > 0, \tag{4}$$

where the function $\mathfrak{f}$ has an absolutely continuous derivative up to order $(n-1)$.

Definition 3. *The Caputo derivative of order $\varrho \in [n-1, n)$ for a function $\mathfrak{f} : [0, \infty) \to (\mathbb{R})$ can be written as*

$${}^{C}\mathcal{D}_{0+}^{\varrho}\mathfrak{f}(\tau) = \mathcal{D}_{0+}^{\varrho}\left(\mathfrak{f}(\tau) - \sum_{k=0}^{n-1}\frac{\tau^k}{k!}\mathfrak{f}^{(k)}(0)\right), \tau > 0, n - 1 < \varrho < n. \tag{5}$$

Note that the Caputo fractional derivative of order $\varrho \in [n-1, n)$ exists almost everywhere on $[0, \infty)$ if $\mathfrak{f} \in \mathcal{AC}^n([0, \infty), (\mathbb{R}))$.

Remark 1. *If $\mathfrak{f} \in \mathcal{C}^n[0, \infty)$, then*

$${}^{C}D_{0+}^{\varrho}\hat{\mathfrak{f}}(\tau) = \frac{1}{\Gamma(n-\varrho)}\int_0^{\tau}\frac{\mathfrak{f}(n)(s)}{(\tau - s)^{\varrho+1-n}}ds = I^{n-\varrho}\mathfrak{f}(n)(\tau), \tau > 0, n - 1 < \varrho < n.$$

Lemma 1. *For any $\hat{f}, \hat{g} \in C[0, T]$, the solution of the linear system of FDEs*

$$\begin{cases} {}^{C}\mathcal{D}^{\varrho}u(\tau) = \hat{f}(\tau), & \tau \in \mathcal{U}, \\ {}^{C}\mathcal{D}^{\varsigma}v(\tau) = \hat{g}(\tau), & \tau \in \mathcal{U}, \end{cases} \tag{6}$$

supplemented with the boundary conditions (2) is equivalent to the system of integral equations

$$\begin{aligned} u(\tau) \;=\; & \frac{1}{\Gamma(\varrho)}\int_0^{\tau}(\tau-\theta)^{\varrho-1}\hat{f}(\theta)d\theta + \psi_1(v)[1 + \kappa_1\Lambda_4(\tau) - \Lambda_3(\tau)] + \psi_2(u)[\kappa_2\Lambda_3(\tau) - \Lambda_4(\tau)] \\ & + \frac{\Lambda_2(\tau)\epsilon_2}{\Gamma(\varrho-1)}\int_0^{\nu_2}\left(\int_0^{\theta}(\theta-\sigma)^{\varrho-2}\hat{f}(\sigma)d\sigma\right)d\theta + \frac{\Lambda_1(\tau)\epsilon_1}{\Gamma(\varsigma-1)}\int_0^{\nu_1}\left(\int_0^{\theta}(\theta-\sigma)^{\varsigma-2}\hat{g}(\sigma)d\sigma\right)d\theta \\ & + \Lambda_4(\tau)\left[\frac{\lambda_2}{\Gamma(\varrho)}\int_0^{\delta_2}\left(\int_0^{\theta}(\theta-\sigma)^{\varrho-1}\hat{f}(\sigma)d\sigma\right)d\theta + \mu_2\sum_{j=1}^{k-2}\frac{\omega_j}{\Gamma(\varrho)}\int_0^{\varphi_j}(\varphi_j-\theta)^{\varrho-1}\hat{f}(\theta)d\theta\right. \\ & \left. - \frac{1}{\Gamma(\varsigma)}\int_0^{T}(T-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta\right] + \Lambda_3(\tau)\left[\frac{\lambda_1}{\Gamma(\varsigma)}\int_0^{\delta_1}\left(\int_0^{\theta}(\theta-\sigma)^{\varsigma-1}\hat{g}(\sigma)d\sigma\right)d\theta\right. \end{aligned}$$

$$+\mu_1\sum_{j=1}^{k-2}\frac{\varpi_j}{\Gamma(\varsigma)}\int_0^{\vartheta_j}(\vartheta_j-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta-\frac{1}{\Gamma(\varrho)}\int_0^{T}(T-\theta)^{\varrho-1}\hat{f}(\theta)d\theta\Big], \tag{7}$$

and

$$\begin{aligned}v(\tau) &= \frac{1}{\Gamma(\varsigma)}\int_0^{\tau}(\tau-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta+\psi_2(u)[1+\kappa_2\Lambda_7(\tau)-\Lambda_8(\tau)]+\psi_1(v)[\kappa_1\Lambda_8(\tau)-\Lambda_7(\tau)]\\ &+\frac{\Lambda_5(\tau)\epsilon_1}{\Gamma(\varsigma-1)}\int_0^{\nu_1}\left(\int_0^{\theta}(\theta-\sigma)^{\varsigma-2}\hat{g}(\sigma)d\sigma\right)d\theta+\frac{\Lambda_6(\tau)\epsilon_2}{\Gamma(\varrho-1)}\int_0^{\nu_2}\left(\int_0^{\theta}(\theta-\sigma)^{\varrho-2}\hat{f}(\sigma)d\sigma\right)d\theta\\ &+\Lambda_7(\tau)\Big[\frac{\lambda_1}{\Gamma(\varsigma)}\int_0^{\delta_1}\left(\int_0^{\theta}(\theta-\sigma)^{\varsigma-1}\hat{g}(\sigma)d\sigma\right)d\theta+\mu_1\sum_{j=1}^{k-2}\frac{\varpi_j}{\Gamma(\varsigma)}\int_0^{\vartheta_j}(\vartheta_j-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta\\ &-\frac{1}{\Gamma(\varrho)}\int_0^{T}(T-\theta)^{\varrho-1}\hat{f}(\theta)d\theta\Big]+\Lambda_8(\tau)\Big[\frac{\lambda_2}{\Gamma(\varrho)}\int_0^{\delta_2}\left(\int_0^{\theta}(\theta-\sigma)^{\varrho-1}\hat{f}(\sigma)d\sigma\right)d\theta\\ &+\mu_2\sum_{j=1}^{k-2}\frac{\omega_j}{\Gamma(\varrho)}\int_0^{\varphi_j}(\varphi_j-\theta)^{\varrho-1}\hat{f}(\theta)d\theta-\frac{1}{\Gamma(\varsigma)}\int_0^{T}(T-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta\Big],\end{aligned} \tag{8}$$

where

$$\xi_1=\frac{\lambda_1\delta_1^2}{2}+\mu_1\sum_{j=1}^{k-2}\varpi_j\vartheta_j,\ \xi_2=\frac{\lambda_1\delta_1^n}{n}+\mu_1\sum_{j=1}^{k-2}\varpi_j\vartheta_j^{n-1},\ \xi_3=\frac{\lambda_2\delta_2^2}{2}+\mu_2\sum_{j=1}^{k-2}\omega_j\varphi_j,\ \xi_4=\frac{\lambda_2\delta_2^n}{n}+\mu_2\sum_{j=1}^{k-2}\omega_j\varphi_j^{n-1}, \tag{9}$$

$$\hat{\gamma}_1=1-\nu_1\nu_2\epsilon_1\epsilon_2,\hat{\gamma}_2=T^2-\xi_1\xi_3,\hat{\gamma}_3=T^n-\xi_1\xi_4,\hat{\gamma}_4=T^{n-1}\xi_3-\xi_4T,\hat{\gamma}_5=\xi_1T^{n-1}-T\xi_2,\hat{\gamma}_6=T^n-\xi_2\xi_3, \tag{10}$$

$$\begin{aligned}&\upsilon_1=\hat{\gamma}_2\nu_1\nu_2^{n-1}\epsilon_1\epsilon_2+\hat{\gamma}_1\hat{\gamma}_3,\ \upsilon_2=\hat{\gamma}_2\nu_2^{n-1}\epsilon_2+\hat{\gamma}_1\hat{\gamma}_4,\ \upsilon_3=\hat{\gamma}_2\nu_1^{n-1}\epsilon_1+\hat{\gamma}_1\hat{\gamma}_5,\ \upsilon_4=\hat{\gamma}_2\nu_1^{n-1}\nu_2\epsilon_1\epsilon_2+\hat{\gamma}_1\hat{\gamma}_6,\\ &\upsilon=\upsilon_2\upsilon_3-\upsilon_1\upsilon_4\neq 0,\end{aligned} \tag{11}$$

$$\begin{aligned}&\eta_1=1+\frac{(\nu_1\nu_2^{n-1}\epsilon_2\beta_1-\nu_1^{n-1}\beta_5)\epsilon_1}{\upsilon},\ \eta_2=\epsilon_1\nu_1+\frac{(\nu_1\nu_2^{n-1}\epsilon_2\beta_2-\nu_1^{n-1}\beta_6)\epsilon_1}{\upsilon},\ \eta_3=\frac{(\nu_1\nu_2^{n-1}\epsilon_2\beta_3-\nu_1^{n-1}\beta_7)\epsilon_1}{\upsilon},\\ &\eta_4=\frac{(\nu_1\nu_2^{n-1}\epsilon_2\beta_4-\nu_1^{n-1}\beta_8)\epsilon_1}{\upsilon},\ \eta_5=\epsilon_2\nu_2+\frac{(\nu_2^{n-1}\beta_1-\epsilon_1\nu_1^{n-1}\nu_2\beta_5)\epsilon_2}{\upsilon},\ \eta_6=1+\frac{(\nu_2^{n-1}\beta_2-\epsilon_1\nu_1^{n-1}\nu_2\beta_6)\epsilon_2}{\upsilon},\\ &\eta_7=\frac{(\nu_2^{n-1}\beta_3-\epsilon_1\nu_1^{n-1}\nu_2\beta_7)\epsilon_2}{\upsilon},\ \eta_8=\frac{(\nu_2^{n-1}\beta_4-\epsilon_1\nu_1^{n-1}\beta_8\nu_2)\epsilon_2}{\upsilon},\end{aligned} \tag{12}$$

$$\begin{aligned}&\beta_1=\hat{\gamma}_2(\upsilon_4-\upsilon_3\epsilon_2\nu_2),\ \beta_2=(\upsilon_4\epsilon_1\nu_1-\upsilon_3)\hat{\gamma}_2,\ \beta_3=(\upsilon_3\xi_3-\upsilon_4T)\hat{\gamma}_1,\ \beta_4=\hat{\gamma}_1(\upsilon_3T-\upsilon_4\xi_1),\\ &\beta_5=(\upsilon_2-\upsilon_1\epsilon_2\nu_2)\hat{\gamma}_2,\ \beta_6=(\upsilon_2\epsilon_1\nu_1-\upsilon_1)\hat{\gamma}_2,\ \beta_7=(\upsilon_1\xi_3-\upsilon_2T)\hat{\gamma}_1,\ \beta_8=\hat{\gamma}_1(\upsilon_1T-\upsilon_2\xi_1),\end{aligned} \tag{13}$$

$$\kappa_1=\lambda_2\delta_2+\mu_2\sum_{j=1}^{k-2}\omega_j,\ \kappa_2=\lambda_1\delta_1+\mu_1\sum_{j=1}^{k-2}\varpi_j, \tag{14}$$

$$\begin{aligned}&\Lambda_1(\tau)=\frac{\tau\eta_1}{\hat{\gamma}_1}+\frac{\tau^{n-1}\beta_1}{\upsilon},\ \Lambda_2(\tau)=\frac{\tau\eta_2}{\hat{\gamma}_1}+\frac{\tau^{n-1}\beta_2}{\upsilon},\ \Lambda_3(\tau)=\frac{\tau\eta_3}{\hat{\gamma}_1}+\frac{\tau^{n-1}\beta_3}{\upsilon},\ \Lambda_4(\tau)=\frac{\tau\eta_4}{\hat{\gamma}_1}+\frac{\tau^{n-1}\beta_4}{\upsilon},\\ &\Lambda_5(\tau)=\frac{\tau\eta_5}{\hat{\gamma}_1}-\frac{\tau^{n-1}\beta_5}{\upsilon},\ \Lambda_6(\tau)=\frac{\tau\eta_6}{\hat{\gamma}_1}-\frac{\tau^{n-1}\beta_6}{\upsilon},\ \Lambda_7(\tau)=\frac{\tau\eta_7}{\hat{\gamma}_1}-\frac{\tau^{n-1}\beta_7}{\upsilon},\ \Lambda_8(\tau)=\frac{\tau\eta_8}{\hat{\gamma}_1}-\frac{\tau^{n-1}\beta_8}{\upsilon}.\end{aligned} \tag{15}$$

Proof. Solving the FDEs (6) in a standard manner, we get

$$y(\tau)=\int_0^{\tau}\frac{(\tau-\theta)^{\varrho-1}}{\Gamma(\varrho)}\hat{f}(\theta)d\theta+a_0+a_1\tau+\cdots+a_{n-1}\tau^{n-1}, \tag{16}$$

$$z(\tau)=\int_0^{\tau}\frac{(\tau-\theta)^{\varsigma-1}}{\Gamma(\varsigma)}\hat{g}(\theta)d\theta+b_0+b_1\tau+\cdots+b_{n-1}\tau^{n-1}, \tag{17}$$

where $a_i,b_i\in\mathbb{R}$, $i=0,1,2,\cdots,n-1$, are arbitrary constants. Using the boundary conditions (2) in (16) and (17) together with notations (9)–(15), we obtain $a_0=\phi_1(z)$, $b_0=\phi_2(y)$, and

$$a_1-b_1\epsilon_1\nu_1-b_{n-1}\epsilon_1\nu_1^{n-1}=\frac{\epsilon_1}{\Gamma(\varsigma-1)}\int_0^{\nu_1}\left(\int_0^{\theta}(\theta-\sigma)^{\varsigma-2}\hat{g}(\sigma)d\sigma\right)d\theta, \tag{18}$$

$$b_1 - a_1\nu_2\epsilon_2 - a_{n-1}\epsilon_2\nu_2^{n-1} = \frac{\epsilon_2}{\Gamma(\varrho-1)}\int_0^{\nu_2}\left(\int_0^{\theta}(\theta-\sigma)^{\varrho-2}\hat{f}(\sigma)d\sigma\right)d\theta, \tag{19}$$

$$\begin{aligned} a_1T + a_{n-1}T^{n-1} - b_1\xi_1 - b_{n-1}\xi_2 = {} & \frac{\lambda_1}{\Gamma(\varsigma)}\int_0^{\delta_1}\left(\int_0^{\theta}(\theta-\sigma)^{\varsigma-1}\hat{g}(\sigma)d\sigma\right)d\theta \\ & + \mu_1\sum_{j=1}^{k-2}\frac{\varpi_j}{\Gamma(\varsigma)}\int_0^{\vartheta_j}(\vartheta_j-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta \\ & - \frac{1}{\Gamma(\varrho)}\int_0^{T}(T-\theta)^{\varrho-1}\hat{f}(\theta)d\theta \\ & + \kappa_2\phi_2(y) - \phi_1(z), \end{aligned} \tag{20}$$

$$\begin{aligned} b_1T + b_{n-1}T^{n-1} - a_1\xi_3 - a_{n-1}\xi_4 = {} & \frac{\lambda_2}{\Gamma(\varrho)}\int_0^{\delta_2}\left(\int_0^{\theta}(\theta-\sigma)^{\varrho-1}\hat{f}(\sigma)d\sigma\right)d\theta \\ & + \mu_2\sum_{j=1}^{k-2}\frac{\omega_j}{\Gamma(\varrho)}\int_0^{\varphi_j}(\varphi_j-\theta)^{\varrho-1}\hat{f}(\theta)d\theta \\ & - \frac{1}{\Gamma(\varsigma)}\int_0^{T}(T-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta \\ & + \kappa_1\phi_1(z) - \phi_2(y). \end{aligned} \tag{21}$$

Solving the system (18)–(21) for a_1, a_{n-1}, b_1 and b_{n-1}, we get

$$\begin{aligned} a_1 = \frac{1}{\hat{\gamma}_1}\Bigg[& \frac{\eta_1\epsilon_1}{\Gamma(\varsigma-1)}\int_0^{\nu_1}\left(\int_0^{\theta}(\theta-\sigma)^{\varsigma-2}\hat{g}(\sigma)d\sigma\right)d\theta \\ & + \frac{\eta_2\epsilon_2}{\Gamma(\varrho-1)}\int_0^{\nu_2}\left(\int_0^{\theta}(\theta-\sigma)^{\varrho-2}\hat{f}(\sigma)d\sigma\right)d\theta \\ & + \eta_3\Bigg(\frac{\lambda_1}{\Gamma(\varsigma)}\int_0^{\delta_1}\left(\int_0^{\theta}(\theta-\sigma)^{\varsigma-1}\hat{g}(\sigma)d\sigma\right)d\theta \\ & + \mu_1\sum_{j=1}^{k-2}\frac{\varpi_j}{\Gamma(\varsigma)}\int_0^{\vartheta_j}(\vartheta_j-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta - \frac{1}{\Gamma(\varrho)}\int_0^{T}(T-\theta)^{\varrho-1}\hat{f}(\theta)d\theta \\ & + \kappa_2\phi_2(y) - \phi_1(z)\Bigg) + \eta_4\Bigg(\frac{\lambda_2}{\Gamma(\varrho)}\int_0^{\delta_2}\left(\int_0^{\theta}(\theta-\sigma)^{\varrho-1}\hat{f}(\sigma)d\sigma\right)d\theta \\ & + \mu_2\sum_{j=1}^{k-2}\frac{\omega_j}{\Gamma(\varrho)}\int_0^{\varphi_j}(\varphi_j-\theta)^{\varrho-1}\hat{f}(\theta)d\theta - \frac{1}{\Gamma(\varsigma)}\int_0^{T}(T-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta \\ & + \kappa_1\phi_1(z) - \phi_2(y)\Bigg)\Bigg], \end{aligned}$$

$$\begin{aligned} b_1 = \frac{1}{\hat{\gamma}_1}\Bigg[& \frac{\eta_5\epsilon_1}{\Gamma(\varsigma-1)}\int_0^{\nu_1}\left(\int_0^{\theta}(\theta-\sigma)^{\varsigma-2}\hat{g}(\sigma)d\sigma\right)d\theta \\ & + \frac{\eta_6\epsilon_2}{\Gamma(\varrho-1)}\int_0^{\nu_2}\left(\int_0^{\theta}(\theta-\sigma)^{\varrho-2}\hat{f}(\sigma)d\sigma\right)d\theta \\ & + \eta_7\Bigg(\frac{\lambda_1}{\Gamma(\varsigma)}\int_0^{\delta_1}\left(\int_0^{\theta}(\theta-\sigma)^{\varsigma-1}\hat{g}(\sigma)d\sigma\right)d\theta \\ & + \mu_1\sum_{j=1}^{k-2}\frac{\varpi_j}{\Gamma(\varsigma)}\int_0^{\vartheta_j}(\vartheta_j-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta - \frac{1}{\Gamma(\varrho)}\int_0^{T}(T-\theta)^{\varrho-1}\hat{f}(\theta)d\theta \end{aligned}$$

$$
\begin{aligned}
&+\kappa_2\phi_2(y)-\phi_1(z)\Big)+\eta_8\Big(\frac{\lambda_2}{\Gamma(\varrho)}\int_0^{\delta_2}\Big(\int_0^{\theta}(\theta-\sigma)^{\varrho-1}\hat{f}(\sigma)d\sigma\Big)d\theta\\
&+\mu_2\sum_{j=1}^{k-2}\frac{\omega_j}{\Gamma(\varrho)}\int_0^{\varphi_j}(\varphi_j-\theta)^{\varrho-1}\hat{f}(\theta)d\theta-\frac{1}{\Gamma(\varsigma)}\int_0^{T}(T-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta\\
&+\kappa_1\phi_1(z)-\phi_2(y)\Big)\Big],\\
a_{n-1}=\frac{1}{\upsilon}\Big[&\frac{\beta_1\epsilon_1}{\Gamma(\varsigma-1)}\int_0^{\nu_1}\Big(\int_0^{\theta}(\theta-\sigma)^{\varsigma-2}\hat{g}(\sigma)d\sigma\Big)d\theta\\
&+\frac{\beta_2\epsilon_2}{\Gamma(\varrho-1)}\int_0^{\nu_2}\Big(\int_0^{\theta}(\theta-\sigma)^{\varrho-2}\hat{f}(\sigma)d\sigma\Big)d\theta\\
&+\beta_3\Big(\frac{\lambda_1}{\Gamma(\varsigma)}\int_0^{\delta_1}\Big(\int_0^{\theta}(\theta-\sigma)^{\varsigma-1}\hat{g}(\sigma)d\sigma\Big)d\theta\\
&+\mu_1\sum_{j=1}^{k-2}\frac{\varpi_j}{\Gamma(\varsigma)}\int_0^{\vartheta_j}(\vartheta_j-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta-\frac{1}{\Gamma(\varrho)}\int_0^{T}(T-\theta)^{\varrho-1}\hat{f}(\theta)d\theta\\
&+\kappa_2\phi_2(y)-\phi_1(z)\Big)+\beta_4\Big(\frac{\lambda_2}{\Gamma(\varrho)}\int_0^{\delta_2}\Big(\int_0^{\theta}(\theta-\sigma)^{\varrho-1}\hat{f}(\sigma)d\sigma\Big)d\theta\\
&+\mu_2\sum_{j=1}^{k-2}\frac{\omega_j}{\Gamma(\varrho)}\int_0^{\varphi_j}(\varphi_j-\theta)^{\varrho-1}\hat{f}(\theta)d\theta-\frac{1}{\Gamma(\varsigma)}\int_0^{T}(T-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta\\
&+\kappa_1\phi_1(z)-\phi_2(y)\Big)\Big],\\
b_{n-1}=\frac{-1}{\upsilon}\Big[&\frac{\beta_5\epsilon_1}{\Gamma(\varsigma-1)}\int_0^{\nu_1}\Big(\int_0^{\theta}(\theta-\sigma)^{\varsigma-2}\hat{g}(\sigma)d\sigma\Big)d\theta\\
&+\frac{\beta_6\epsilon_2}{\Gamma(\varrho-1)}\int_0^{\nu_2}\Big(\int_0^{\theta}(\theta-\sigma)^{\varrho-2}\hat{f}(\sigma)d\sigma\Big)d\theta\\
&+\beta_7\Big(\frac{\lambda_1}{\Gamma(\varsigma)}\int_0^{\delta_1}\Big(\int_0^{\theta}(\theta-\sigma)^{\varsigma-1}\hat{g}(\sigma)d\sigma\Big)d\theta\\
&+\mu_1\sum_{j=1}^{k-2}\frac{\varpi_j}{\Gamma(\varsigma)}\int_0^{\vartheta_j}(\vartheta_j-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta-\frac{1}{\Gamma(\varrho)}\int_0^{T}(T-\theta)^{\varrho-1}\hat{f}(\theta)d\theta\\
&+\kappa_2\phi_2(y)-\phi_1(z)\Big)+\beta_8\Big(\frac{\lambda_2}{\Gamma(\varrho)}\int_0^{\delta_2}\Big(\int_0^{\theta}(\theta-\sigma)^{\varrho-1}\hat{f}(\sigma)d\sigma\Big)d\theta\\
&+\mu_2\sum_{j=1}^{k-2}\frac{\omega_j}{\Gamma(\varrho)}\int_0^{\varphi_j}(\varphi_j-\theta)^{\varrho-1}\hat{f}(\theta)d\theta-\frac{1}{\Gamma(\varsigma)}\int_0^{T}(T-\theta)^{\varsigma-1}\hat{g}(\theta)d\theta\\
&+\kappa_1\phi_1(z)-\phi_2(y)\Big)\Big],
\end{aligned}
$$

where $\hat{\gamma}_1$, υ, η_i and β_i $i=1,2,\cdots,8$, are given by (10)–(13) respectively. Substituting the values of a_0, a_1, a_{n-1}, b_0, b_1 and b_{n-1}, in (16) and (17), we obtain the solutions (7) and (8). □

3. Existence and Uniqueness Results

We define space $\mathcal{G} = \{u|u \in C(\mathcal{U},\mathbb{R}),\ {}^{C}\mathcal{D}^{\varrho_1}u \in C(\mathcal{U},\mathbb{R})\}$ equipped with norm $\|u\|_{\mathcal{G}} = \|u\| + \|{}^{C}\mathcal{D}^{\varrho_1}u\| = \sup_{\tau\in\mathcal{U}}|u(\tau)| + \sup_{\tau\in\mathcal{U}}|{}^{C}\mathcal{D}^{\varrho_1}u(\tau)|$. Furthermore, $\mathcal{H} = \{v|v \in C(\mathcal{U},\mathbb{R}),\ {}^{C}\mathcal{D}^{\varsigma_1}v \in C(\mathcal{U},\mathbb{R})\}$ equipped with norm $\|v\|_{\mathcal{H}} = \|v\| + \|{}^{C}\mathcal{D}^{\varsigma_1}v\| = \sup_{\tau\in\mathcal{U}}|v(\tau)| + \sup_{\tau\in\mathcal{U}}|{}^{C}\mathcal{D}^{\varsigma_1}v(\tau)|$. Obviously $(\mathcal{G},\|\cdot\|_{\mathcal{G}})$ and $(\mathcal{H},\|\cdot\|_{\mathcal{H}})$ are Banach spaces, and thus the product space $(\mathcal{G}\times\mathcal{H},\|\cdot\|_{\mathcal{G}\times\mathcal{H}})$ is a Banach space with norm $\|(u,v)\|_{\mathcal{G}\times\mathcal{H}} = \|u\|_{\mathcal{G}} + \|v\|_{\mathcal{H}}$ for $(u,v)\in\mathcal{G}\times\mathcal{H}$.

Using Lemma 1, we consider an operator $\Pi : \mathcal{G}\times\mathcal{H}\to\mathcal{G}\times\mathcal{H}$ as

$$\Pi(u,v)(\tau) = (\Pi_1(u,v)(\tau),\Pi_2(u,v)(\tau)), \tag{22}$$

where

$$\begin{aligned}
\Pi_1(u,v)(\tau) &= \frac{1}{\Gamma(\varrho)}\int_0^\tau(\tau-\theta)^{\varrho-1}\widehat{\mathcal{S}}_u(\theta)d\theta + \psi_1(v)[1+\kappa_1\Lambda_4(\tau)-\Lambda_3(\tau)] + \psi_2(u)[\kappa_2\Lambda_3(\tau)-\Lambda_4(\tau)]\\
&+\frac{\Lambda_2(\tau)\epsilon_2}{\Gamma(\varrho-1)}\int_0^{\nu_2}\left(\int_0^\theta(\theta-\sigma)^{\varrho-2}\widehat{\mathcal{S}}_u(\sigma)d\sigma\right)d\theta + \frac{\Lambda_1(\tau)\epsilon_1}{\Gamma(\varsigma-1)}\int_0^{\nu_1}\left(\int_0^\theta(\theta-\sigma)^{\varsigma-2}\tilde{\mathcal{S}}_v(\sigma)d\sigma\right)d\theta\\
&+\Lambda_4(\tau)\left[\frac{\lambda_2}{\Gamma(\varrho)}\int_0^{\delta_2}\left(\int_0^\theta(\theta-\sigma)^{\varrho-1}\widehat{\mathcal{S}}_u(\sigma)d\sigma\right)d\theta + \mu_2\sum_{j=1}^{k-2}\frac{\omega_j}{\Gamma(\varrho)}\int_0^{\varphi_j}(\varphi_j-\theta)^{\varrho-1}\widehat{\mathcal{S}}_u(\theta)d\theta\right.\\
&\left.-\frac{1}{\Gamma(\varsigma)}\int_0^T(T-\theta)^{\varsigma-1}\tilde{\mathcal{S}}_v(\theta)d\theta\right] + \Lambda_3(\tau)\left[\frac{\lambda_1}{\Gamma(\varsigma)}\int_0^{\delta_1}\left(\int_0^\theta(\theta-\sigma)^{\varsigma-1}\tilde{\mathcal{S}}_v(\sigma)d\sigma\right)d\theta\right.\\
&\left.+\mu_1\sum_{j=1}^{k-2}\frac{\omega_j}{\Gamma(\varsigma)}\int_0^{\vartheta_j}(\vartheta_j-\theta)^{\varsigma-1}\tilde{\mathcal{S}}_v(\theta)d\theta - \frac{1}{\Gamma(\varrho)}\int_0^T(T-\theta)^{\varrho-1}\widehat{\mathcal{S}}_u(\theta)d\theta\right],
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\Pi_2(u,v)(\tau) &= \frac{1}{\Gamma(\varsigma)}\int_0^\tau(\tau-\theta)^{\varsigma-1}\tilde{\mathcal{S}}_v(\theta)d\theta + \psi_2(u)[1+\kappa_2\Lambda_7(\tau)-\Lambda_8(\tau)] + \psi_1(v)[\kappa_1\Lambda_8(\tau)-\Lambda_7(\tau)]\\
&+\frac{\Lambda_5(\tau)\epsilon_1}{\Gamma(\varsigma-1)}\int_0^{\nu_1}\left(\int_0^\theta(\theta-\sigma)^{\varsigma-2}\tilde{\mathcal{S}}_v(\sigma)d\sigma\right)d\theta + \frac{\Lambda_6(\tau)\epsilon_2}{\Gamma(\varrho-1)}\int_0^{\nu_2}\left(\int_0^\theta(\theta-\sigma)^{\varrho-2}\widehat{\mathcal{S}}_u(\sigma)d\sigma\right)d\theta\\
&+\Lambda_7(\tau)\left[\frac{\lambda_1}{\Gamma(\varsigma)}\int_0^{\delta_1}\left(\int_0^\theta(\theta-\sigma)^{\varsigma-1}\tilde{\mathcal{S}}_v(\sigma)d\sigma\right)d\theta + \mu_1\sum_{j=1}^{k-2}\frac{\omega_j}{\Gamma(\varsigma)}\int_0^{\vartheta_j}(\vartheta_j-\theta)^{\varsigma-1}\tilde{\mathcal{S}}_v(\theta)d\theta\right.\\
&\left.-\frac{1}{\Gamma(\varrho)}\int_0^T(T-\theta)^{\varrho-1}\widehat{\mathcal{S}}_u(\theta)d\theta\right] + \Lambda_8(\tau)\left[\frac{\lambda_2}{\Gamma(\varrho)}\int_0^{\delta_2}\left(\int_0^\theta(\theta-\sigma)^{\varrho-1}\widehat{\mathcal{S}}_u(\sigma)d\sigma\right)d\theta\right.\\
&\left.+\mu_2\sum_{j=1}^{k-2}\frac{\omega_j}{\Gamma(\varrho)}\int_0^{\varphi_j}(\varphi_j-\theta)^{\varrho-1}\widehat{\mathcal{S}}_u(\theta)d\theta - \frac{1}{\Gamma(\varsigma)}\int_0^T(T-\theta)^{\varsigma-1}\tilde{\mathcal{S}}_v(\theta)d\theta\right].
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
\widehat{\mathcal{S}}_u(\tau) &= f(\tau,u(\tau),v(\tau),{}^{C}\mathcal{D}^{\varsigma_1}v(\tau),\mathcal{I}^{\xi}v(\tau)),\ \tau\in\mathcal{U},\\
\tilde{\mathcal{S}}_v(\tau) &= g(\tau,u(\tau),{}^{C}\mathcal{D}^{\varrho_1}u(\tau),\mathcal{I}^{\zeta}u(\tau),v(\tau)),\ \tau\in\mathcal{U},
\end{aligned}$$

and Λ_i $(i = 1,2,\cdots,8)$ are given by (15). Suitable for computation, we represent

$$\Delta_1 = \frac{1}{\Gamma(\varrho+1)}\left[T^{\varrho}(1+\overline{\Lambda}_3) + \overline{\Lambda}_2\epsilon_2\nu_2^{\varrho} + \overline{\Lambda}_4\left(\lambda_2\frac{\delta_2^{\varrho+1}}{(\varrho+1)} + \mu_2\sum_{j=1}^{k-2}\omega_j\varphi_j^{\varrho}\right)\right], \tag{25}$$

$$\Delta_2 = \frac{1}{\Gamma(\varsigma+1)}\left[T^{\varsigma}\overline{\Lambda}_4 + \overline{\Lambda}_1\epsilon_1\nu_1^{\varsigma} + \overline{\Lambda}_3\left(\lambda_1\frac{\delta_1^{\varsigma+1}}{(\varsigma+1)} + \mu_1\sum_{j=1}^{k-2}\omega_j\vartheta_j^{\varsigma}\right)\right], \tag{26}$$

$$\Delta_3 = \frac{1}{\Gamma(\varrho+1)}\left[T^{\varrho}\overline{\Lambda}_7 + \overline{\Lambda}_6\epsilon_2\nu_2^{\varrho} + \overline{\Lambda}_8\left(\lambda_2\frac{\delta_2^{\varrho+1}}{(\varrho+1)} + \mu_2\sum_{j=1}^{k-2}\omega_j\varphi_j^{\varrho}\right)\right], \tag{27}$$

$$\Delta_4 = \frac{1}{\Gamma(\varsigma+1)}\left[T^{\varsigma}(1+\overline{\Lambda}_8) + \overline{\Lambda}_5\epsilon_1\nu_1^{\varsigma} + \overline{\Lambda}_7\left(\lambda_1\frac{\delta_1^{\varsigma+1}}{(\varsigma+1)} + \mu_1\sum_{j=1}^{k-2}\varpi_j\vartheta_j^{\varsigma}\right)\right], \tag{28}$$

$$\widehat{\Delta}_1 = \frac{1}{\Gamma(\varrho+1)}\left[\varrho T^{\varrho-1} + \overline{\Lambda}_3^{'}T^{\varrho} + \overline{\Lambda}_2^{'}\epsilon_2\nu_2^{\varrho} + \overline{\Lambda}_4^{'}\left(\lambda_2\frac{\delta_2^{\varrho+1}}{(\varrho+1)} + \mu_2\sum_{j=1}^{k-2}\omega_j\varphi_j^{\varrho}\right)\right], \tag{29}$$

$$\widehat{\Delta}_2 = \frac{1}{\Gamma(\varsigma+1)}\left[T^{\varsigma}\overline{\Lambda}_4^{'} + \overline{\Lambda}_1^{'}\epsilon_1\nu_1^{\varsigma} + \overline{\Lambda}_3^{'}\left(\lambda_1\frac{\delta_1^{\varsigma+1}}{(\varsigma+1)} + \mu_1\sum_{j=1}^{k-2}\varpi_j\vartheta_j^{\varsigma}\right)\right], \tag{30}$$

$$\widehat{\Delta}_3 = \frac{1}{\Gamma(\varrho+1)}\left[T^{\varrho}\overline{\Lambda}_7^{'} + \overline{\Lambda}_6^{'}\epsilon_2\nu_2^{\varrho} + \overline{\Lambda}_8^{'}\left(\lambda_2\frac{\delta_2^{\varrho+1}}{(\varrho+1)} + \mu_2\sum_{j=1}^{k-2}\omega_j\varphi_j^{\varrho}\right)\right], \tag{31}$$

$$\widehat{\Delta}_4 = \frac{1}{\Gamma(\varsigma+1)}\left[\varsigma T^{\varsigma-1} + \overline{\Lambda}_8^{'}T^{\varsigma} + \overline{\Lambda}_5^{'}\epsilon_1\nu_1^{\varsigma} + \overline{\Lambda}_7^{'}\left(\lambda_1\frac{\delta_1^{\varsigma+1}}{(\varsigma+1)} + \mu_1\sum_{j=1}^{k-2}\varpi_j\vartheta_j^{\varsigma}\right)\right], \tag{32}$$

$$\Phi_1 = \left(\Delta_1 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\widehat{\Delta}_1 + \Delta_3 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\widehat{\Delta}_3\right)r_0 + \left(\Delta_2 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\widehat{\Delta}_2 + \Delta_4 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\widehat{\Delta}_4\right)s_0, \tag{33}$$

$$\begin{aligned}\Phi_2 = {} & \left(\Delta_1 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\widehat{\Delta}_1 + \Delta_3 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\widehat{\Delta}_3\right)r_1 \\ & + \left(\Delta_2 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\widehat{\Delta}_2 + \Delta_4 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\widehat{\Delta}_4\right)\left(\max\{s_1,s_2\} + \frac{s_3T^{\zeta}}{\Gamma(\zeta+1)}\right) \\ & + \left\{\kappa_2\overline{\Lambda}_3 + \overline{\Lambda}_4 + \frac{T^{1-\varrho_1}(\kappa_2\overline{\Lambda}_3^{'} + \overline{\Lambda}_4^{'})}{\Gamma(2-\varrho_1)} + 1 + \kappa_2\overline{\Lambda}_7 + \overline{\Lambda}_8 + \frac{T^{1-\varsigma_1}(\kappa_2\overline{\Lambda}_7^{'} + \overline{\Lambda}_8^{'})}{\Gamma(2-\varsigma_1)}\right\}\mathcal{W}_2,\end{aligned} \tag{34}$$

$$\begin{aligned}\Phi_3 = {} & \left(\Delta_1 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\widehat{\Delta}_1 + \Delta_3 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\widehat{\Delta}_3\right)\left(\max\{r_2,r_3\} + \frac{r_4T^{\xi}}{\Gamma(\xi+1)}\right) \\ & + \left(\Delta_2 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\widehat{\Delta}_2 + \Delta_4 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\widehat{\Delta}_4\right)s_4 \\ & + \left\{1 + \kappa_1\overline{\Lambda}_4 + \overline{\Lambda}_3 + \frac{T^{1-\varrho_1}(\kappa_1\overline{\Lambda}_4^{'} + \overline{\Lambda}_3^{'})}{\Gamma(2-\varrho_1)} + \kappa_1\overline{\Lambda}_8 + \overline{\Lambda}_7 + \frac{T^{1-\varsigma_1}(\kappa_1\overline{\Lambda}_8^{'} + \overline{\Lambda}_7^{'})}{\Gamma(2-\varsigma_1)}\right\}\mathcal{W}_1,\end{aligned} \tag{35}$$

$$\Psi_1 = \Delta_1\iota_1\mathcal{V}_1 + \Delta_2\iota_2\mathcal{V}_2 + (1 + \kappa_1\overline{\Lambda}_4 + \overline{\Lambda}_3)\mathcal{V}_1 + (\kappa_2\overline{\Lambda}_3 + \overline{\Lambda}_4)\mathcal{V}_2, \tag{36}$$

$$\Psi_2 = \widehat{\Delta}_1\iota_1\mathcal{V}_1 + \widehat{\Delta}_2\iota_2\mathcal{V}_2 + (\kappa_1\overline{\Lambda}_4^{'} + \overline{\Lambda}_3^{'})\mathcal{V}_1 + (\kappa_2\overline{\Lambda}_3^{'} + \overline{\Lambda}_4^{'})\mathcal{V}_2, \tag{37}$$

$$\Psi_3 = \Delta_4\iota_2\mathcal{V}_2 + \Delta_3\iota_1\mathcal{V}_1 + (1 + \kappa_2\overline{\Lambda}_7 + \overline{\Lambda}_8)\mathcal{V}_2 + (\kappa_1\overline{\Lambda}_8 + \overline{\Lambda}_7)\mathcal{V}_1, \tag{38}$$

$$\Psi_4 = \widehat{\Delta}_4\iota_2\mathcal{V}_2 + \widehat{\Delta}_3\iota_1\mathcal{V}_1 + (\kappa_2\overline{\Lambda}_7^{'} + \overline{\Lambda}_8^{'})\mathcal{V}_2 + (\kappa_1\overline{\Lambda}_8^{'} + \overline{\Lambda}_7^{'})\mathcal{V}_1, \tag{39}$$

$$\mathcal{P}_1 = \Delta_1\mathcal{T}_1 + \Delta_2\mathcal{T}_2, \quad \mathcal{P}_2 = \widehat{\Delta}_1\mathcal{T}_1 + \widehat{\Delta}_2\mathcal{T}_2, \quad \mathcal{P}_3 = \Delta_4\mathcal{T}_2 + \Delta_3\mathcal{T}_1, \quad \mathcal{P}_4 = \widehat{\Delta}_4\mathcal{T}_2 + \widehat{\Delta}_3\mathcal{T}_1, \tag{40}$$

$$\overline{\mathcal{P}} = \mathcal{P}_1 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\mathcal{P}_2, \quad \widehat{\mathcal{P}} = \mathcal{P}_3 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\mathcal{P}_4, \tag{41}$$

$$\Psi = \Psi_1 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\Psi_2, \quad \widehat{\Psi} = \Psi_3 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\Psi_4 \tag{42}$$

$$\mathcal{T}_1 = \sup_{\tau\in\mathcal{U}} f(\tau,0,0,0,0) < \infty, \quad \mathcal{T}_2 = \sup_{\tau\in\mathcal{U}} g(\tau,0,0,0,0) < \infty,$$

$$\iota_1 = 1 + \frac{T^{\xi}}{\Gamma(\xi+1)}, \quad \iota_2 = 1 + \frac{T^{\zeta}}{\Gamma(\zeta+1)}, \tag{43}$$

where $\overline{\Lambda_i} = \max_{\tau\in\mathcal{H}}|\Lambda_i(\tau)|$ and $\overline{\Lambda_i}^{'} = \max_{\tau\in\mathcal{H}}|\Lambda_i^{'}(\tau)|$ $i = 1,2,\cdots,8$. We need the following assumptions in the forthcoming analysis: $f, g : \mathcal{U}\times\mathbb{R}^4 \to \mathbb{R}$ and $\psi_1, \psi_2 : \mathcal{C}(\mathcal{U},\mathbb{R}) \to \mathbb{R}$ are continuous functions $\psi_1(0) = \psi_2(0) = 0$.

$(\mathcal{F}_1)$ There exist positive constants r_i and $s_i \geq 0$, $r_0 > 0$, $s_0 > 0$, $\forall w_i \in \mathbb{R}$, $i = 1,2,3,4$.

$$\begin{aligned} |f(\tau,w_1,w_2,w_3,w_4)| &\leq r_0 + r_1|w_1| + r_2|w_2| + r_3|w_3| + r_4|w_4|, \\ |g(\tau,w_1,w_2,w_3,w_4)| &\leq s_0 + s_1|w_1| + s_2|w_2| + s_3|w_3| + s_4|w_4|. \end{aligned}$$

$(\mathcal{F}_2)$ There exists positive constants $\mathcal{W}_1, \mathcal{W}_2 > 0$,

$$|\psi_1(v)| \leq \mathcal{W}_1\|v\|, \ \ |\psi_2(u)| \leq \mathcal{W}_2\|u\|, \forall\, u, v \in \ \mathcal{C}(\mathcal{U},\mathbb{R}).$$

$(\mathcal{F}_3)$ There exist positive constants $\mathcal{V}_i, i = 1,2$, $\forall\ \tau \in \mathcal{U}$ and $r_i, s_i \in \mathbb{R}(i = 1,2,3,4)$, we have

$$\begin{aligned} |f(\tau,r_1,r_2,r_3,r_4) - f(\tau,s_1,s_2,s_3,s_4)| &\leq \mathcal{V}_1\Big(|r_1 - s_1| + |r_2 - s_2| + |r_3 - s_3| + |r_4 - s_4|\Big), \\ |g(\tau,r_1,r_2,r_3,r_4) - g(\tau,s_1,s_2,s_3,s_4)| &\leq \mathcal{V}_2\Big(|r_1 - s_1| + |r_2 - s_2| + |r_3 - s_3| + |r_4 - s_4|\Big). \end{aligned}$$

$(\mathcal{F}_4)$ There exist positive constants $\mathcal{V}_i$ $(i = 1,2)$ such that

$$|\psi_1(r_1) - \psi_1(r_2)| \leq \mathcal{V}_1\|r_1 - r_2\|, \quad |\psi_2(r_1) - \psi_2(r_2)| \ \leq \ \mathcal{V}_2\|r_1 - r_2\|,$$

$\forall\ r_1, r_2 \in \mathbb{R}$.

Theorem 1. *Assume that* $(\mathcal{F}_1)$ *and* $(\mathcal{F}_2)$ *hold. Further, if* $\hat{\Phi} = \min\{\Phi_2, \Phi_3\} < 1$, *then the problem (1) and (2) has at least one solution on* $\mathcal{U}$.

Proof. In the first step, we show that operator $\Pi : \mathcal{G}\times\mathcal{H} \to \mathcal{G}\times\mathcal{H}$ is completely continuous. Operator Π is continuous by the continuity of f, g, ψ_1, ψ_2 functions. Let $\Omega \subset \mathcal{G}\times\mathcal{H}$ be bounded. Then, $\exists$ positive constants $\mathcal{L}_f$ and $\mathcal{L}_g$ such that

$$\begin{aligned} |\widehat{\mathcal{S}}_u(\tau)| &= |f(\tau,u(\tau),v(\tau),{}^{C}\mathcal{D}^{\varsigma_1}v(\tau),\mathcal{I}^{\xi}v(\tau))| \leq \mathcal{L}_f \\ |\tilde{\mathcal{S}}_v(\tau)| &= |g(\tau,u(\tau),{}^{C}\mathcal{D}^{\varrho_1}u(\tau),\mathcal{I}^{\zeta}u(\tau),v(\tau))| \leq \mathcal{L}_g, \end{aligned}$$

$\forall\ (u,v) \in \Omega$, and constants $\mathcal{L}_{\psi_1}$, $\mathcal{L}_{\psi_2} > 0$ such that $|\psi_1(v)| \leq \mathcal{L}_{\psi_1}$, $|\psi_2(u)| \leq \mathcal{L}_{\psi_2}$, $\forall u$, $v \in \mathcal{C}(\mathcal{U},\mathbb{R})$. Then, for any $(u,v) \in \Omega$, we have

$$\begin{aligned} |\Pi_1(u,v)(\tau)| \leq\ & \frac{1}{\Gamma(\varrho)}\int_0^{\tau}(\tau-\theta)^{\varrho-1}|\widehat{\mathcal{S}}_u(\theta)|d\theta + |\psi_1(v)|[1 + \kappa_1|\Lambda_4(\tau)| + |\Lambda_3(\tau)|] + |\psi_2(u)|[\kappa_2|\Lambda_3(\tau)| + |\Lambda_4(\tau)|] \\ & + \frac{|\Lambda_2(\tau)|\epsilon_2}{\Gamma(\varrho-1)}\int_0^{\nu_2}\left(\int_0^{\theta}(\theta-\sigma)^{\varrho-2}|\widehat{\mathcal{S}}_u(\sigma)d\sigma\right)d\theta + \frac{|\Lambda_1(\tau)|\epsilon_1}{\Gamma(\varsigma-1)}\int_0^{\nu_1}\left(\int_0^{\theta}(\theta-\sigma)^{\varsigma-2}|\tilde{\mathcal{S}}_v(\sigma)d\sigma\right)d\theta \\ & + |\Lambda_4(\tau)|\left[\frac{\lambda_2}{\Gamma(\varrho)}\int_0^{\delta_2}\left(\int_0^{\theta}(\theta-\sigma)^{\varrho-1}|\widehat{\mathcal{S}}_u(\sigma)d\sigma\right)d\theta + \mu_2\sum_{j=1}^{k-2}\frac{\omega_j}{\Gamma(\varrho)}\int_0^{\varphi_j}(\varphi_j-\theta)^{\varrho-1}|\widehat{\mathcal{S}}_u(\theta)|d\theta\right. \\ & \left. + \frac{1}{\Gamma(\varsigma)}\int_0^{T}(T-\theta)^{\varsigma-1}|\tilde{\mathcal{S}}_v(\theta)|d\theta\right] + |\Lambda_3(\tau)|\left[\frac{\lambda_1}{\Gamma(\varsigma)}\int_0^{\delta_1}\left(\int_0^{\theta}(\theta-\sigma)^{\varsigma-1}|\tilde{\mathcal{S}}_v(\sigma)d\sigma\right)d\theta\right. \\ & \left. + \mu_1\sum_{j=1}^{k-2}\frac{\varpi_j}{\Gamma(\varsigma)}\int_0^{\vartheta_j}(\vartheta_j-\theta)^{\varsigma-1}|\tilde{\mathcal{S}}_v(\theta)|d\theta + \frac{1}{\Gamma(\varrho)}\int_0^{T}(T-\theta)^{\varrho-1}|\widehat{\mathcal{S}}_u(\theta)|d\theta\right] \\ \leq\ & \Delta_1\mathcal{L}_f + \Delta_2\mathcal{L}_g + [1 + \kappa_1\overline{\Lambda_4} + \overline{\Lambda_3}]\mathcal{L}_{\psi_1} + [\kappa_2\overline{\Lambda_3} + \overline{\Lambda_4}]\mathcal{L}_{\psi_2}. \end{aligned}$$

Likewise, we get $|\Pi_1'(u,v)(\tau)| \leq \widehat{\Delta}_1\mathcal{L}_f + \widehat{\Delta}_2\mathcal{L}_g + [\kappa_1\overline{\Lambda}_4' + \overline{\Lambda}_3']\mathcal{L}_{\psi_1} + [\kappa_2\overline{\Lambda}_3' + \overline{\Lambda}_4']\mathcal{L}_{\psi_2}$, which implies that $|{}^C\mathcal{D}^{\varrho_1}\Pi_1(u,v)(\tau)| \leq \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}(\widehat{\Delta}_1\mathcal{L}_f + \widehat{\Delta}_2\mathcal{L}_g + [\kappa_1\overline{\Lambda}_4' + \overline{\Lambda}_3']\mathcal{L}_{\psi_1} + [\kappa_2\overline{\Lambda}_3' + \overline{\Lambda}_4']\mathcal{L}_{\psi_2})$. Thus, we have

$$\begin{aligned}\|\Pi_1(u,v)\|_{\mathcal{G}} \leq\ & \Delta_1\mathcal{L}_f + \Delta_2\mathcal{L}_g + [1 + \kappa_1\overline{\Lambda_4} + \overline{\Lambda_3}]\mathcal{L}_{\psi_1} + [\kappa_2\overline{\Lambda_3} + \overline{\Lambda_4}]\mathcal{L}_{\psi_2} \\ & + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}(\widehat{\Delta}_1\mathcal{L}_f + \widehat{\Delta}_2\mathcal{L}_g + [\kappa_1\overline{\Lambda}_4' + \overline{\Lambda}_3']\mathcal{L}_{\psi_1} + [\kappa_2\overline{\Lambda}_3' + \overline{\Lambda}_4']\mathcal{L}_{\psi_2}). \end{aligned} \tag{44}$$

We obtain that, equivalently, $|\Pi_2(u,v)(\tau)| \leq \Delta_4\mathcal{L}_g + \Delta_3\mathcal{L}_f + [1 + \kappa_2\overline{\Lambda}_7 + \overline{\Lambda}_8]\mathcal{L}_{\psi_2} + [\kappa_1\overline{\Lambda}_8 + \overline{\Lambda}_7]\mathcal{L}_{\psi_1}$, and $|\Pi_2'(u,v)(\tau)| \leq \widehat{\Delta}_4\mathcal{L}_g + \widehat{\Delta}_3\mathcal{L}_f + [\kappa_2\overline{\Lambda}_7' + \overline{\Lambda}_8']\mathcal{L}_{\psi_2} + [\kappa_1\overline{\Lambda}_8' + \overline{\Lambda}_7']\mathcal{L}_{\psi_1}$, which implies that $|{}^C\mathcal{D}^{\varsigma_1}\Pi_2(u,v)(\tau)| \leq \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}(\widehat{\Delta}_4\mathcal{L}_g + \widehat{\Delta}_3\mathcal{L}_f + [\kappa_2\overline{\Lambda}_7' + \overline{\Lambda}_8']\mathcal{L}_{\psi_2} + [\kappa_1\overline{\Lambda}_8' + \overline{\Lambda}_7']\mathcal{L}_{\psi_1})$. Thus, we have

$$\begin{aligned}\|\Pi_2(u,v)\|_{\mathcal{H}} \leq\ & \Delta_4\mathcal{L}_g + \Delta_3\mathcal{L}_f + [1 + \kappa_2\overline{\Lambda}_7 + \overline{\Lambda}_8]\mathcal{L}_{\psi_2} + [\kappa_1\overline{\Lambda}_8 + \overline{\Lambda}_7]\mathcal{L}_{\psi_1} \\ & + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}(\widehat{\Delta}_4\mathcal{L}_g + \widehat{\Delta}_3\mathcal{L}_f + [\kappa_2\overline{\Lambda}_7' + \overline{\Lambda}_8']\mathcal{L}_{\psi_2} + [\kappa_1\overline{\Lambda}_8' + \overline{\Lambda}_7']\mathcal{L}_{\psi_1}). \end{aligned} \tag{45}$$

Thus, Π is uniformly bounded by (44) and (45). Operator Π must be shown to be equicontinuous.

For $\tau_1, \tau_2 \in \mathcal{U}$ with $\tau_1 < \tau_2$, we have

$$\begin{aligned}& |\Pi_1(u,v)(\tau_2) - \Pi_1(u,v)(\tau_1)| \\ \leq\ & \frac{\mathcal{L}_f}{\Gamma(\varrho+1)}[(\tau_2-\tau_1)^\varrho + (\tau_2^\varrho - \tau_1^\varrho)] + |\psi_1(v)|\Big[\kappa_1(|\Lambda_4(\tau_2) - \Lambda_4(\tau_1)|) + (|\Lambda_3(\tau_2) - \Lambda_3(\tau_1)|)\Big] \\ & + |\psi_2(u)|\Big[\kappa_2(|\Lambda_3(\tau_2) - \Lambda_3(\tau_1)|) + (|\Lambda_4(\tau_2) - \Lambda_4(\tau_1)|)\Big] + \frac{(|\Lambda_2(\tau_2) - \Lambda_2(\tau_1)|)\epsilon_2\nu_2^\varrho\mathcal{L}_f}{\Gamma(\varrho+1)} \\ & + \frac{(|\Lambda_1(\tau_2) - \Lambda_1(\tau_1)|)\epsilon_1\nu_1^\varsigma\mathcal{L}_g}{\Gamma(\varsigma+1)} + (|\Lambda_4(\tau_2) - \Lambda_4(\tau_1)|)\left[\frac{\lambda_2\delta_2^{\varrho+1}\mathcal{L}_f}{\Gamma(\varrho+2)} + \mu_2\sum_{j=1}^{k-2}\frac{\omega_j(\varphi_j)^\varrho\mathcal{L}_f}{\Gamma(\varrho+1)} + \frac{T^\varsigma\mathcal{L}_g}{\Gamma(\varsigma+1)}\right] \\ & + (\Lambda_3(\tau_2) - \Lambda_3(\tau_1))\left[\frac{\lambda_1\delta_1^{\varsigma+1}\mathcal{L}_g}{\Gamma(\varsigma+2)} + \mu_1\sum_{j=1}^{k-2}\frac{\varpi_j\vartheta_j^\varsigma\mathcal{L}_g}{\Gamma(\varsigma+1)} + \frac{T^\varrho\mathcal{L}_f}{\Gamma(\varrho+1)}\right], \end{aligned}$$

and

$$\begin{aligned}& |\Pi_1'(u,v)(\tau_2) - \Pi_1'(u,v)(\tau_1)| \\ \leq\ & \left|\int_0^{\tau_1}\frac{[(\tau_2-\theta)^{\varrho-2} - (\tau_1-\theta)^{\varrho-2}]}{\Gamma(\varrho-1)} \times f(\theta, u(\theta), v(\theta), {}^C\mathcal{D}^{\varsigma_1}v(\theta), \mathcal{I}^\xi v(\theta))d\theta\right| \\ & + \left|\int_{\tau_1}^{\tau_2}\frac{(\tau_2-\theta)^{\varrho-2}}{\Gamma(\varrho-1)}f(\theta, u(\theta), v(\theta), {}^C\mathcal{D}^{\varsigma_1}v(\theta), \mathcal{I}^\xi v(\theta))d\theta\right| + |\psi_1(v)|\Big[\kappa_1(|\Lambda_4'(\tau_2) - \Lambda_4'(\tau_1)|) + (|\Lambda_3'(\tau_2) - \Lambda_3'(\tau_1)|)\Big] \\ & + |\psi_2(u)|\Big[\kappa_2(|\Lambda_3'(\tau_2) - \Lambda_3'(\tau_1)|) + (|\Lambda_4'(\tau_2) - \Lambda_4'(\tau_1)|)\Big] + \frac{(|\Lambda_2'(\tau_2) - \Lambda_2'(\tau_1)|)\epsilon_2\nu_2^\varrho\mathcal{L}_f}{\Gamma(\varrho+1)} \\ & + \frac{(|\Lambda_1'(\tau_2) - \Lambda_1'(\tau_1)|)\epsilon_1\nu_1^\varsigma\mathcal{L}_g}{\Gamma(\varsigma+1)} + (|\Lambda_4'(\tau_2) - \Lambda_4'(\tau_1)|)\left[\frac{\lambda_2\delta_2^{\varrho+1}\mathcal{L}_f}{\Gamma(\varrho+2)} + \mu_2\sum_{j=1}^{k-2}\frac{\omega_j(\varphi_j)^\varrho\mathcal{L}_f}{\Gamma(\varrho+1)} + \frac{T^\varsigma\mathcal{L}_g}{\Gamma(\varsigma+1)}\right] \\ & + (|\Lambda_3'(\tau_2) - \Lambda_3'(\tau_1)|)\left[\frac{\lambda_1\delta_1^{\varsigma+1}\mathcal{L}_g}{\Gamma(\varsigma+2)} + \mu_1\sum_{j=1}^{k-2}\frac{\varpi_j\vartheta_j^\varsigma\mathcal{L}_g}{\Gamma(\varsigma+1)} + \frac{T^\varrho\mathcal{L}_f}{\Gamma(\varrho+1)}\right]. \end{aligned}$$

Thus, we have

$$
\begin{aligned}
&|{}^{C}\mathcal{D}^{\varrho_1}\Pi_1(u,v)(\tau_2) - {}^{C}\mathcal{D}^{\varrho_1}\Pi_1(u,v)(\tau_1)| \\
\leq\ & \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\left\{\frac{\mathcal{L}_f}{\Gamma(\varrho)}[(\tau_2-\tau_1)^{\varrho-1}+(\tau_2^{\varrho-1}-\tau_1^{\varrho-1})]+|\psi_1(v)|\Big[\kappa_1(|\Lambda_4^{'}(\tau_2)-\Lambda_4^{'}(\tau_1)|)+(|\Lambda_3^{'}(\tau_2)-\Lambda_3^{'}(\tau_1)|)\Big]\right. \\
&+|\psi_2(u)|\Big[\kappa_2(|\Lambda_3^{'}(\tau_2)-\Lambda_3^{'}(\tau_1)|)+(|\Lambda_4^{'}(\tau_2)-\Lambda_4^{'}(\tau_1)|)\Big]+\frac{(|\Lambda_2^{'}(\tau_2)-\Lambda_2^{'}(\tau_1)|)\epsilon_2\nu_2^{\varrho}\mathcal{L}_f}{\Gamma(\varrho+1)} \\
&+\frac{(|\Lambda_1^{'}(\tau_2)-\Lambda_1^{'}(\tau_1)|)\epsilon_1\nu_1^{\varsigma}\mathcal{L}_g}{\Gamma(\varsigma+1)}+(|\Lambda_4^{'}(\tau_2)-\Lambda_4^{'}(\tau_1)|)\left[\frac{\lambda_2\delta_2^{\varrho+1}\mathcal{L}_f}{\Gamma(\varrho+2)}+\mu_2\sum_{j=1}^{k-2}\frac{\omega_j(\varphi_j)^{\varrho}\mathcal{L}_f}{\Gamma(\varrho+1)}+\frac{T^{\varsigma}\mathcal{L}_g}{\Gamma(\varsigma+1)}\right] \\
&\left.+(|\Lambda_3^{'}(\tau_2)-\Lambda_3^{'}(\tau_1)|)\left[\frac{\lambda_1\delta_1^{\varsigma+1}\mathcal{L}_g}{\Gamma(\varsigma+2)}+\mu_1\sum_{j=1}^{k-2}\frac{\omega_j\vartheta_j^{\varsigma}\mathcal{L}_g}{\Gamma(\varsigma+1)}+\frac{T^{\varrho}\mathcal{L}_f}{\Gamma(\varrho+1)}\right]\right\}.
\end{aligned}
$$

Thus, we obtain $\|\Pi_1(u,v)(\tau_2)-\Pi_1(u,v)(\tau_1)\|_{\mathcal{G}} \to 0$ independent of u and v as $\tau_2 \to \tau_1$. According to the above, we get

$$
\begin{aligned}
&|\Pi_2(u,v)(\tau_2)-\Pi_2(u,v)(\tau_1)| \\
\leq\ & \frac{\mathcal{L}_g}{\Gamma(\varsigma+1)}[(\tau_2-\tau_1)^{\varsigma}+(\tau_2^{\varsigma}-\tau_1^{\varsigma})]+|\psi_2(u)|\Big[\kappa_2(|\Lambda_7(\tau_2)-\Lambda_7(\tau_1)|)+(|\Lambda_8(\tau_2)-\Lambda_8(\tau_1)|)\Big] \\
&+|\psi_1(v)|\Big[\kappa_1(|\Lambda_8(\tau_2)-\Lambda_8(\tau_1)|)+(|\Lambda_7(\tau_2)-\Lambda_7(\tau_1)|)\Big]+\frac{(|\Lambda_5(\tau_2)-\Lambda_5(\tau_1)|)\epsilon_1\nu_1^{\varsigma}\mathcal{L}_g}{\Gamma(\varsigma+1)} \\
&+\frac{(|\Lambda_6(\tau_2)-\Lambda_6(\tau_1)|)\epsilon_2\nu_2^{\varrho}\mathcal{L}_f}{\Gamma(\varrho+1)}+(|\Lambda_7(\tau_2)-\Lambda_7(\tau_1)|)\left[\frac{\lambda_1\delta_1^{\varsigma+1}\mathcal{L}_g}{\Gamma(\varsigma+2)}+\mu_1\sum_{j=1}^{k-2}\frac{\omega_j\vartheta_j^{\varsigma}\mathcal{L}_g}{\Gamma(\varsigma+1)}+\frac{T^{\varrho}\mathcal{L}_f}{\Gamma(\varrho+1)}\right] \\
&+(|\Lambda_8(\tau_2)-\Lambda_8(\tau_1)|)\left[\frac{\lambda_2\delta_2^{\varrho+1}\mathcal{L}_f}{\Gamma(\varrho+2)}+\mu_2\sum_{j=1}^{k-2}\frac{\omega_j(\varphi_j)^{\varrho}\mathcal{L}_f}{\Gamma(\varrho+1)}+\frac{T^{\varsigma}\mathcal{L}_g}{\Gamma(\varsigma+1)}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&|\Pi_2^{'}(u,v)(\tau_2)-\Pi_2^{'}(u,v)(\tau_1)| \\
\leq\ & \left|\int_0^{\tau_1}\frac{[(\tau_2-\theta)^{\varsigma-2}-(\tau_1-\theta)^{\varsigma-2}]}{\Gamma(\varsigma-1)}\times g(\theta,u(\theta),{}^{C}\mathcal{D}^{\varrho_1}u(\theta),\mathcal{I}^{\zeta}u(\theta),v(\theta))d\theta\right| \\
&+\left|\int_{\tau_1}^{\tau_2}\frac{(\tau_2-\theta)^{\varsigma-2}}{\Gamma(\varsigma-1)}g(\theta,u(\theta),{}^{C}\mathcal{D}^{\varrho_1}u(\theta),\mathcal{I}^{\zeta}u(\theta),v(\theta))d\theta\right|+|\psi_2(u)|\Big[\kappa_2(|\Lambda_7^{'}(\tau_2)-\Lambda_7^{'}(\tau_1)|)+(|\Lambda_8^{'}(\tau_2)-\Lambda_8^{'}(\tau_1)|)\Big] \\
&+|\psi_1(v)|\Big[\kappa_1(|\Lambda_8^{'}(\tau_2)-\Lambda_8^{'}(\tau_1)|)+(|\Lambda_7^{'}(\tau_2)-\Lambda_7^{'}(\tau_1)|)\Big]+\frac{(|\Lambda_5^{'}(\tau_2)-\Lambda_5^{'}(\tau_1)|)\epsilon_1\nu_1^{\varsigma}\mathcal{L}_g}{\Gamma(\varsigma+1)} \\
&+\frac{(|\Lambda_6^{'}(\tau_2)-\Lambda_6^{'}(\tau_1)|)\epsilon_2\nu_2^{\varrho}\mathcal{L}_f}{\Gamma(\varrho+1)}+(|\Lambda_7^{'}(\tau_2)-\Lambda_7^{'}(\tau_1)|)\left[\frac{\lambda_1\delta_1^{\varsigma+1}\mathcal{L}_g}{\Gamma(\varsigma+2)}+\mu_1\sum_{j=1}^{k-2}\frac{\omega_j\vartheta_j^{\varsigma}\mathcal{L}_g}{\Gamma(\varsigma+1)}+\frac{T^{\varrho}\mathcal{L}_f}{\Gamma(\varrho+1)}\right] \\
&+(|\Lambda_8^{'}(\tau_2)-\Lambda_8^{'}(\tau_1)|)\left[\frac{\lambda_2\delta_2^{\varrho+1}\mathcal{L}_f}{\Gamma(\varrho+2)}+\mu_2\sum_{j=1}^{k-2}\frac{\omega_j(\varphi_j)^{\varrho}\mathcal{L}_f}{\Gamma(\varrho+1)}+\frac{T^{\varsigma}\mathcal{L}_g}{\Gamma(\varsigma+1)}\right].
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&|{}^{C}\mathcal{D}^{\varsigma_1}\Pi_2(u,v)(\tau_2) - {}^{C}\mathcal{D}^{\varsigma_1}\Pi_2(u,v)(\tau_1)| \\
\leq\ & \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\Bigg\{\frac{\mathcal{L}_g}{\Gamma(\varsigma+1)}[(\tau_2-\tau_1)^{\varsigma}+(\tau_2^{\varsigma}-\tau_1^{\varsigma})]+|\psi_2(u)|\Big[\kappa_2(|\Lambda_7^{'}(\tau_2)-\Lambda_7^{'}(\tau_1)|)+(|\Lambda_8^{'}(\tau_2)-\Lambda_8^{'}(\tau_1)|)\Big] \\
&+|\psi_1(v)|\Big[\kappa_1(|\Lambda_8^{'}(\tau_2)-\Lambda_8^{'}(\tau_1)|)+(|\Lambda_7^{'}(\tau_2)-\Lambda_7^{'}(\tau_1)|)\Big]+\frac{(|\Lambda_5^{'}(\tau_2)-\Lambda_5^{'}(\tau_1)|)\epsilon_1\nu_1^{\varsigma}\mathcal{L}_g}{\Gamma(\varsigma+1)} \\
&+\frac{(|\Lambda_6^{'}(\tau_2)-\Lambda_6^{'}(\tau_1)|)\epsilon_2\nu_2^{\varrho}\mathcal{L}_f}{\Gamma(\varrho+1)}+(|\Lambda_7^{'}(\tau_2)-\Lambda_7^{'}(\tau_1)|)\Bigg[\frac{\lambda_1\delta_1^{\varsigma+1}\mathcal{L}_g}{\Gamma(\varsigma+2)}+\mu_1\sum_{j=1}^{k-2}\frac{\omega_j\vartheta_j^{\varsigma}\mathcal{L}_g}{\Gamma(\varsigma+1)}+\frac{T^{\varrho}\mathcal{L}_f}{\Gamma(\varrho+1)}\Bigg] \\
&+(|\Lambda_8^{'}(\tau_2)-\Lambda_8^{'}(\tau_1)|)\Bigg[\frac{\lambda_2\delta_2^{\varrho+1}\mathcal{L}_f}{\Gamma(\varrho+2)}+\mu_2\sum_{j=1}^{k-2}\frac{\omega_j(\varphi_j)^{\varrho}\mathcal{L}_f}{\Gamma(\varrho+1)}+\frac{T^{\varsigma}\mathcal{L}_g}{\Gamma(\varsigma+1)}\Bigg]\Bigg\},
\end{aligned}
$$

which means that $\|\Pi_2(u,v)(\tau_2)-\Pi_2(u,v)(\tau_1)\|_{\mathcal{H}}\to 0$ independent of u and v as $\tau_2\to\tau_1$. Hence, the operator $\Pi(u,v)$ is equicontinuous, and thus it is completely continuous by Lemma (see Lemma 1.2 [15]). Next, we demonstrate that the set $\Phi=\{(u,v)\in\mathcal{G}\times\mathcal{H}|(u,v)=\eta\Pi(u,v),0<\eta<1\}$ is bounded. Let $(u,v)\in\Phi$; then, $(u,v)=\eta\Pi(u,v)$, and for any $\tau\in\mathcal{U}$, we have $u(\tau)=\eta\Pi_1(u,v)(\tau)$, $v(\tau)=\eta\Pi_2(u,v)(\tau)$. Thus,

$$
\begin{aligned}
|u(\tau)|_{\mathcal{G}} \leq\ & \Delta_1\Big(r_0+r_1\|u\|_{\mathcal{G}}+\Big(\max\{r_2,r_3\}+\frac{r_4T^{\xi}}{\Gamma(\xi+1)}\Big)\|v\|_{\mathcal{H}}\Big) \\
&+\Delta_2\Big(s_0+\Big(\max\{s_1,s_2\}+\frac{s_3T^{\zeta}}{\Gamma(\zeta+1)}\Big)\|u\|_{\mathcal{G}}+s_4\|v\|_{\mathcal{H}}\Big) \\
&+\Big(1+\kappa_1|\overline{\Lambda}_4|+|\overline{\Lambda}_3|\Big)\mathcal{W}_1\|v\|_{\mathcal{H}}+\Big(\kappa_2|\overline{\Lambda}_3|+|\overline{\Lambda}_4|\Big)\mathcal{W}_2\|u\|_{\mathcal{G}}, \\
|u^{'}(\tau)| \leq\ & \widehat{\Delta}_1\Big(r_0+r_1\|u\|_{\mathcal{G}}+\Big(\max\{r_2,r_3\}+\frac{r_4T^{\xi}}{\Gamma(\xi+1)}\Big)\|v\|_{\mathcal{H}}\Big) \\
&+\widehat{\Delta}_2\Big(s_0+\Big(\max\{s_1,s_2\}+\frac{s_3T^{\zeta}}{\Gamma(\zeta+1)}\Big)\|u\|_{\mathcal{G}}+s_4\|v\|_{\mathcal{H}}\Big) \\
&+\Big(\kappa_1|\overline{\Lambda}_4^{'}|+|\overline{\Lambda}_3^{'}|\Big)\mathcal{W}_1\|v\|_{\mathcal{H}}+\Big(\kappa_2|\overline{\Lambda}_3^{'}|+|\overline{\Lambda}_4^{'}|\Big)\mathcal{W}_2\|u\|_{\mathcal{G}}, \\
|{}^{C}\mathcal{D}^{\varrho_1}u(\tau)| \leq\ & \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\Bigg\{\widehat{\Delta}_1\Big(r_0+r_1\|u\|_{\mathcal{G}}+\Big(\max\{r_2,r_3\}+\frac{r_4T^{\xi}}{\Gamma(\xi+1)}\Big)\|v\|_{\mathcal{H}}\Big) \\
&+\widehat{\Delta}_2\Big(s_0+\Big(\max\{s_1,s_2\}+\frac{s_3T^{\zeta}}{\Gamma(\zeta+1)}\Big)\|u\|_{\mathcal{G}}+s_4\|v\|_{\mathcal{H}}\Big) \\
&+\Big(\kappa_1|\overline{\Lambda}_4^{'}|+|\overline{\Lambda}_3^{'}|\Big)\mathcal{W}_1\|v\|_{\mathcal{H}}+\Big(\kappa_2|\overline{\Lambda}_3^{'}|+|\overline{\Lambda}_4^{'}|\Big)\mathcal{W}_2\|u\|_{\mathcal{G}}\Bigg\}.
\end{aligned}
$$

Hence, we have

$$\begin{aligned}
\|u\| \leq\ & \Delta_1\Big(r_0 + r_1\|u\|_{\mathcal{G}} + \Big(\max\{r_2, r_3\} + \frac{r_4 T^{\xi}}{\Gamma(\xi+1)}\Big)\|v\|_{\mathcal{H}}\Big) \\
& + \Delta_2\Big(s_0 + \Big(\max\{s_1, s_2\} + \frac{s_3 T^{\zeta}}{\Gamma(\zeta+1)}\Big)\|u\|_{\mathcal{G}} + s_4\|v\|_{\mathcal{H}}\Big) \\
& + \Big(1 + \kappa_1|\overline{\Lambda}_4| + |\overline{\Lambda}_3|\Big)\mathcal{W}_1\|v\|_{\mathcal{H}} + \Big(\kappa_2|\overline{\Lambda}_3| + |\overline{\Lambda}_4|\Big)\mathcal{W}_2\|u\|_{\mathcal{G}} \\
& + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\Bigg\{\widehat{\Delta}_1\Big(r_0 + r_1\|u\|_{\mathcal{G}} + \Big(\max\{r_2, r_3\} + \frac{r_4 T^{\xi}}{\Gamma(\xi+1)}\Big)\|v\|_{\mathcal{H}}\Big) \\
& + \widehat{\Delta}_2\Big(s_0 + \Big(\max\{s_1, s_2\} + \frac{s_3 T^{\zeta}}{\Gamma(\zeta+1)}\Big)\|u\|_{\mathcal{G}} + s_4\|v\|_{\mathcal{H}}\Big) \\
& + \Big(\kappa_1|\overline{\Lambda}_4^{'}| + |\overline{\Lambda}_3^{'}|\Big)\mathcal{W}_1\|v\|_{\mathcal{H}} + \Big(\kappa_2|\overline{\Lambda}_3^{'}| + |\overline{\Lambda}_4^{'}|\Big)\mathcal{W}_2\|u\|_{\mathcal{G}}\Bigg\}.
\end{aligned} \tag{46}$$

According to the above, we get

$$\begin{aligned}
\|v\| \leq\ & \Delta_4\Big(s_0 + \Big(\max\{s_1, s_2\} + \frac{s_3 T^{\zeta}}{\Gamma(\zeta+1)}\Big)\|u\|_{\mathcal{G}} + s_4\|v\|_{\mathcal{H}}\Big) \\
& + \Delta_3\Big(r_0 + r_1\|u\|_{\mathcal{G}} + \Big(\max\{r_2, r_3\} + \frac{r_4 T^{\xi}}{\Gamma(\xi+1)}\Big)\|v\|_{\mathcal{H}}\Big) \\
& + \Big(1 + \kappa_2|\overline{\Lambda}_7| + |\overline{\Lambda}_8|\Big)\mathcal{W}_2\|u\|_{\mathcal{G}} + \Big(\kappa_1|\overline{\Lambda}_8| + |\overline{\Lambda}_7|\Big)\mathcal{W}_1\|v\|_{\mathcal{H}} \\
& + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\Bigg\{\widehat{\Delta}_4\Big(s_0 + \Big(\max\{s_1, s_2\} + \frac{s_3 T^{\zeta}}{\Gamma(\zeta+1)}\Big)\|u\|_{\mathcal{G}} + s_4\|v\|_{\mathcal{H}}\Big) \\
& + \widehat{\Delta}_3\Big(r_0 + r_1\|u\|_{\mathcal{G}} + \Big(\max\{r_2, r_3\} + \frac{r_4 T^{\xi}}{\Gamma(\xi+1)}\Big)\|v\|_{\mathcal{H}}\Big) \\
& + \Big(\kappa_2|\overline{\Lambda}_7^{'}| + |\overline{\Lambda}_8^{'}|\Big)\mathcal{W}_2\|u\|_{\mathcal{G}} + \Big(\kappa_1|\overline{\Lambda}_8^{'}| + |\overline{\Lambda}_7^{'}|\Big)\mathcal{W}_1\|v\|_{\mathcal{H}}\Bigg\}.
\end{aligned} \tag{47}$$

Using the above inequalities in combination with the notations (46) and (47), we deduce the result below. $\|u\| + \|v\| \leq \Phi_1 + \min\{\Phi_2, \Phi_3\}\|(u,v)\|_{\mathcal{G}\times\mathcal{H}}$, which leads to $\|(u,v)\|_{\mathcal{G}\times\mathcal{H}} \leq \dfrac{\Phi_1}{1 - \min\{\Phi_2, \Phi_3\}}$. This concludes that the set $\min\{\Phi_2, \Phi_3\}$ is bounded. Therefore, operator Π has at least one fixed point by Theorem (see Theorem 1.9 [15]), which means the system (1)–(2) has at least one solutions on $\mathcal{U}$. □

Theorem 2. *Assume that* $(\mathcal{F}_3)$ *and* $(\mathcal{F}_4)$ *holds. Additionally, if*

$$\Psi + \widehat{\Psi} < 1, \tag{48}$$

where $\Psi, \widehat{\Psi}$ *are defined by (42), then on* $\mathcal{U}$ *there is a unique solution to the problems (1) and (2).*

Proof. Let us fix this $\widehat{\varepsilon} \geq \max\left\{\frac{\overline{\mathcal{P}}+\widehat{\mathcal{P}}}{1-(\Psi+\widehat{\Psi})}\right\}$, where Ψ, $\widehat{\Psi}$ and $\overline{\mathcal{P}}$, $\widehat{\mathcal{P}}$ are respectively given by (41) and (42), and show that $\Pi\mathcal{B}_{\widehat{\varepsilon}} \subset \mathcal{B}_{\widehat{\varepsilon}}$, where the operator Π is given by (22) and $\mathcal{B}_{\widehat{\varepsilon}} = \{(u,v) \in \mathcal{G}\times\mathcal{H} : \|(u,v)\| \leq \widehat{\varepsilon}\}$. For $(u,v) \in \mathcal{B}_{\widehat{\varepsilon}}$, $\tau \in \mathcal{H}$, we have

$$\begin{aligned}
|\widehat{\mathcal{S}}_u(\tau)| &= |f(\tau, u(\tau), v(\tau), {}^{C}\mathcal{D}^{\varsigma_1}v(\tau), \mathcal{I}^{\xi}v(\tau))| \leq \mathcal{V}_1\iota_1\Big(\|(u,v)\|_{\mathcal{G}\times\mathcal{H}}\Big) + \mathcal{T}_1 \leq \mathcal{V}_1\iota_1\widehat{\varepsilon} + \mathcal{T}_1, \\
|\tilde{\mathcal{S}}_v(\tau)| &= |g(\tau, u(\tau), {}^{C}\mathcal{D}^{\varrho_1}u(\tau), \mathcal{I}^{\zeta}u(\tau), v(\tau))| \leq \mathcal{V}_2\iota_2\Big(\|(u,v)\|_{\mathcal{G}\times\mathcal{H}}\Big) + \mathcal{T}_2 \leq \mathcal{V}_2\iota_2\widehat{\varepsilon} + \mathcal{T}_2, \\
|\psi_1(v)| &\leq \mathcal{V}_1\widehat{\varepsilon},\ |\psi_2(u)| \leq \mathcal{V}_2\widehat{\varepsilon},
\end{aligned}$$

which lead to

$$|\Pi_1(u,v)(\tau)| \leq \Psi_1\widehat{\varepsilon} + \mathcal{P}_1,$$

where Ψ_1 and $\mathcal{P}_1$ are given by (36) and (40). With the above notes, we get

$$|\Pi_1'(u,v)(\tau)| \leq \Psi_2\widehat{\varepsilon} + \mathcal{P}_2,$$

which means that

$$|{}^C\mathcal{D}^{\varrho_1}\Pi_1(u,v)(\tau)| \leq \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\Big(\Psi_2\widehat{\varepsilon} + \mathcal{P}_2\Big).$$

Thus, we get

$$\|\Pi_1(u,v)\|_{\mathcal{G}} \leq \Big(\Psi_1 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\Psi_2\Big)\widehat{\varepsilon} + \Big(\mathcal{P}_1 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\mathcal{P}_2\Big). \tag{49}$$

Similarly, we get

$$|\Pi_2(u,v)(\tau)| \leq \Psi_3\widehat{\varepsilon} + \mathcal{P}_3,$$

where Ψ_3 and $\mathcal{P}_3$ are given by (38) and (40). With the above notes, we get

$$|\Pi_2'(u,v)(\tau)| \leq \Psi_4\widehat{\varepsilon} + \mathcal{P}_4,$$

which means that

$$|{}^C\mathcal{D}^{\varsigma_1}\Pi_2(u,v)(\tau)| \leq \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\Big(\Psi_4\widehat{\varepsilon} + \mathcal{P}_4\Big).$$

Hence, we have

$$\|\Pi_2(u,v)\|_{\mathcal{H}} \leq \Big(\Psi_3 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\Psi_4\Big)\widehat{\varepsilon} + \Big(\mathcal{P}_3 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\mathcal{P}_4\Big). \tag{50}$$

So, (49) and (50) follow $\|\Pi(u,v)\|_{\mathcal{G}\times\mathcal{H}} \leq \widehat{\varepsilon}$, and thus, $\Pi\mathcal{B}_{\widehat{\varepsilon}} \subset \mathcal{B}_{\widehat{\varepsilon}}$.

Now, for (u_1,v_1), $(u_2,v_2) \in \mathcal{G}\times\mathcal{H}$ and any $\tau \in \mathcal{U}$, we have

$$|\Pi_1(u_1,v_1)(\tau) - \Pi_1(u_2,v_2)(\tau)| \leq \Psi_1(\|u_1-u_2\|_{\mathcal{G}} + \|v_1-v_2\|_{\mathcal{H}}).$$

Next, we find that

$$|\Pi_1'(u_1,v_1)(\tau) - \Pi_1'(u_2,v_2)(\tau)| \leq \Psi_2(\|u_1-u_2\|_{\mathcal{G}} + \|v_1-v_2\|_{\mathcal{H}}).$$

Thus, we have

$$|{}^C\mathcal{D}^{\varrho_1}\Pi_1(u_1,v_1)(\tau) - {}^C\mathcal{D}^{\varrho_1}\Pi_1(u_2,v_2)(\tau)| \leq \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\Big(\Psi_2(\|u_1-u_2\|_{\mathcal{G}} + \|v_1-v_2\|_{\mathcal{H}})\Big),$$

Hence, we get

$$\|\Pi_1(u_1,v_1) - \Pi_1(u_2,v_2)\|_{\mathcal{G}} \leq \Big(\Psi_1 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\Psi_2\Big)(\|u_1-u_2\|_{\mathcal{G}} + \|v_1-v_2\|_{\mathcal{H}}). \tag{51}$$

Similarly, we have

$$\|\Pi_2(u_1,v_1) - \Pi_2(u_2,v_2)\|_{\mathcal{H}} \leq \Big(\Psi_3 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\Psi_4\Big)(\|u_1-u_2\|_{\mathcal{G}} + \|v_1-v_2\|_{\mathcal{H}}). \tag{52}$$

So, (51) and (52) follow

$$\begin{aligned} &\|\Pi(u_1,v_1) - \Pi(u_2,v_2)\|_{\mathcal{G}\times\mathcal{H}} \\ \leq\ &\Big(\Psi_1 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\Psi_2 + \Psi_3 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\Psi_4\Big)(\|u_1-u_2\|_{\mathcal{G}} + \|v_1-v_2\|_{\mathcal{H}}). \end{aligned}$$

It follows that, in view of the condition (48), the operator Π is a contraction. Thus, by Theorem (see Theorem 1.2.2 [14]), the system (1) and (2) has a unique solution on $\mathcal{U}$. □

4. Examples

Example 1. *Consider the system of Caputo type FIDEs given by*

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{68}{25}}u(\tau) = f(\tau, u(\tau), v(\tau), {}^{C}\mathcal{D}^{\frac{44}{25}}v(\tau), \mathcal{I}^{\frac{46}{25}}v(\tau)), \\ {}^{C}\mathcal{D}^{\frac{62}{25}}v(\tau) = g(\tau, u(\tau), {}^{C}\mathcal{D}^{\frac{73}{50}}u(\tau), \mathcal{I}^{\frac{39}{25}}u(\tau), v(\tau)), \end{cases} \tag{53}$$

subject to the boundary conditions

$$\begin{cases} u(0) = \psi_1(v), u'(0) = \epsilon_1 \int_0^{\nu_1} v'(\theta)d\theta, \ u(T) = \lambda_1 \int_0^{\delta_1} v(\theta)d\theta + \mu_1 \sum_{j=1}^{k-2} \varpi_j v(\vartheta_j), \\ v(0) = \psi_2(u), v'(0) = \epsilon_2 \int_0^{\nu_2} u'(\theta)d\theta, \ v(T) = \lambda_2 \int_0^{\delta_2} u(\theta)d\theta + \mu_2 \sum_{j=1}^{k-2} \omega_j u(\varphi_j). \end{cases} \tag{54}$$

Here, $\varrho = \frac{68}{25}$, $\varsigma = \frac{62}{25}$, $\varrho_1 = \frac{73}{50}$, $\varsigma_1 = \frac{44}{25}$, $\xi = \frac{46}{25}$, $\zeta = \frac{39}{25}$, $T = 1$, $\delta_1 = \frac{1}{8}$, $\delta_2 = \frac{37}{200}$, $\vartheta_1 = \frac{37}{250}$, $\vartheta_2 = \frac{79}{500}$, $\vartheta_3 = \frac{47}{250}$, $\vartheta_4 = \frac{11}{50}$, $\varphi_1 = \frac{113}{500}$, $\varphi_2 = \frac{59}{250}$, $\varphi_3 = \frac{6}{25}$, $\varphi_4 = \frac{49}{200}$, $\varpi_1 = \frac{1}{160}$, $\varpi_2 = \frac{58}{625}$, $\varpi_3 = \frac{29}{400}$, $\varpi_4 = \frac{21}{250}$, $\omega_1 = \frac{17}{200}$, $\omega_2 = \frac{12}{125}$, $\omega_3 = \frac{19}{250}$, $\omega_4 = \frac{7}{125}$, $\lambda_1 = \frac{1}{80}$, $\lambda_2 = \frac{7}{200}$, $\mu_1 = \frac{9}{400}$, $\mu_2 = \frac{23}{500}$,$\epsilon_1 = \frac{11}{400}$, $\epsilon_2 = \frac{19}{500}$, $\nu_1 = \frac{53}{200}$, $\nu_2 = \frac{133}{500}$. *We consider the functions,*

$$\begin{aligned} |f(\tau, u_1, u_2, u_3, u_4)| &\leq \frac{1}{20(\tau^2+1)} + \frac{1}{70(2+\tau)^2}\left(2u_2 + \frac{|u_1|}{1+|u_1|}\right) + \frac{\sin u_3}{700} + \frac{\arctan u_4}{140(3+\tau)}, \\ |g(\tau, u_1, u_2, u_3, u_4)| &\leq \frac{1}{(\tau^4+1)2} + \frac{1}{150(1+\tau^2)}\left(\frac{u_2}{3} + 3u_1\right) + \frac{\cos u_3}{800} + \frac{|u_4|}{400(1+|u_4|)}, \\ |\psi_1(v)| &\leq \frac{1}{110}\|v\|, \qquad |\psi_2(u)| \leq \frac{1}{600}\|u\|. \end{aligned}$$

Clearly

$$\begin{aligned} |f(\tau, u_1, u_2, u_3, u_4)| &\leq \frac{1}{20} + \frac{1}{140}|u_1| + \frac{1}{70}|u_2| + \frac{1}{700}|u_3| + \frac{1}{420}|u_4|, \\ |g(\tau, u_1, u_2, u_3, u_4)| &\leq \frac{1}{2} + \frac{1}{50}|u_1| + \frac{1}{450}|u_2| + \frac{1}{800}|u_3| + \frac{1}{400}|u_4|, \\ |\psi_1(v)| &\leq \frac{1}{110}\|v\|, \qquad |\psi_2(u)| \leq \frac{1}{600}\|u\|. \end{aligned}$$

With the given data, we find that $r_0 = \frac{1}{20}$, $r_1 = \frac{1}{140}$, $r_2 = \frac{1}{70}$, $r_3 = \frac{1}{700}$, $r_4 = \frac{1}{420}$, $s_0 = \frac{1}{2}$, $s_1 = \frac{1}{50}$, $s_2 = \frac{1}{450}$, $s_3 = \frac{1}{800}$, $s_4 = \frac{1}{400}$, $\Delta_1 = 0.46845200402823617$, $\Delta_2 = 0.0007227881649342847$, $\Delta_3 = 0.0011251030301272463$, $\Delta_4 = 0.6151822290610394$, $\widehat{\Delta}_1 = 0.8713184743902467$, $\widehat{\Delta}_2 = 0.0014641295646385377$, $\widehat{\Delta}_3 = 0.00186307333192$, $\widehat{\Delta}_4 = 1.0704190784503973$, *we find that* $\widehat{\Phi} = min\{\Phi_1, \Phi_2\} < 1$. *Thus, the assumption of Theorem 1 holds and the problem (53) and (54) has at least one solution on* $[0, 1]$.

Example 2. *We consider the functions*

$$\begin{aligned} |f(\tau, u_1, u_2, u_3, u_4)| &\leq \frac{1}{9\sqrt{\tau^2+64}}\left(\frac{3}{\tau+1} + \cos(u_1+u_2) + \frac{|u_3|}{|u_3|+1} + \arctan u_4\right), \\ |g(\tau, u_1, u_2, u_3, u_4)| &\leq \frac{1}{24\sqrt{36+\tau^2}}\left(\frac{\tau^2}{\tau+2} + \left(u_2 + \frac{|u_1|}{1+|u_1|}\right) + \sin u_3 + |u_4|\right), \\ |\psi_1(v)| &\leq \frac{1}{110}\|v\|, \qquad |\psi_2(u)| \leq \frac{1}{60}\|u\|. \end{aligned}$$

Using the given data, it is found that $\mathcal{V}_1 = \frac{1}{72}$, $\mathcal{V}_2 = \frac{1}{144}$, $\mathcal{V}_1 = \frac{1}{110}$, $\mathcal{V}_2 = \frac{1}{60}$, $\Delta_1 = 0.46845200402823617$, $\Delta_2 = 0.0007227881649342847$, $\Delta_3 = 0.0011251030301272463$, $\Delta_4 = 0.6151822290610394$, $\widehat{\Delta}_1 = 0.8713184743902467$, $\widehat{\Delta}_2 = 0.0014641295646385377$,

$\widehat{\Delta}_3 = 0.00186307333192$, $\widehat{\Delta}_4 = 1.0704190784503973$, *with* $\Psi_1 + \frac{T^{1-\varrho_1}}{\Gamma(2-\varrho_1)}\,\Psi_2 \approx 0.0514018187$ 3251667, *and* $\Psi_3 + \frac{T^{1-\varsigma_1}}{\Gamma(2-\varsigma_1)}\Psi_4 \approx 0.05324949686716335$; *hence, the Theorem 2 is satisfied, and here the problem (53)–(54) has a unique solution on* $[0,1]$.

5. A Variant of a Problem

Note that the boundary conditions (1) include the strips of different lengths when modifying the strips in boundary conditions to the same lengths (1); then, the problem reduces to the form

$$\begin{cases} u(0) = \psi_1(v),\ \ u^{'}(0) = \epsilon_1 \int_0^{\nu} v^{'}(\theta)d\theta,\ \ u^{''}(0) = 0, \cdots, u^{n-2}(0) = 0, \\ u(T) = \lambda_1 \int_0^{\delta} v(\theta)d\theta + \mu_1 \sum_{j=1}^{k-2} \varpi_j v(\vartheta_j), \\ v(0) = \psi_2(u),\ \ v^{'}(0) = \epsilon_2 \int_0^{\nu} u^{'}(\theta)d\theta,\ \ v^{''}(0) = 0, \cdots, v^{n-2}(0) = 0, \\ v(T) = \lambda_2 \int_0^{\delta} u(\theta)d\theta + \mu_2 \sum_{j=1}^{k-2} \omega_j u(\vartheta_j). \end{cases} \tag{55}$$

Concerning the problem (1) with (55) instead of (2), we obtained the operator $\widehat{\Pi} : \mathcal{G} \times \mathcal{H} \to \mathcal{G} \times \mathcal{H}$ defined by

$$\widehat{\Pi}(u,v)(\tau) = (\widehat{\Pi}_1(u,v)(\tau), \widehat{\Pi}_2(u,v)(\tau)),$$

where

$$\begin{aligned} \widehat{\Pi}_1(u,v)(\tau) &= \frac{1}{\Gamma(\varrho)} \int_0^{\tau} (\tau-\theta)^{\varrho-1} \widehat{\mathcal{Q}}_u(\theta)d\theta + \psi_1(v)[1 + \kappa_1\Lambda_4(\tau) - \Lambda_3(\tau)] + \psi_2(u)[\kappa_2\Lambda_3(\tau) - \Lambda_4(\tau)] \\ &+ \frac{\Lambda_2(\tau)\epsilon_2}{\Gamma(\varrho-1)} \int_0^{\nu} \left(\int_0^{\theta} (\theta-\sigma)^{\varrho-2} \widehat{\mathcal{Q}}_u(\sigma)d\sigma \right) d\theta + \frac{\Lambda_1(\tau)\epsilon_1}{\Gamma(\varsigma-1)} \int_0^{\nu} \left(\int_0^{\theta} (\theta-\sigma)^{\varsigma-2} \tilde{\mathcal{Q}}_v(\sigma)d\sigma \right) d\theta \\ &+ \Lambda_4(\tau) \left[\frac{\lambda_2}{\Gamma(\varrho)} \int_0^{\delta} \left(\int_0^{\theta} (\theta-\sigma)^{\varrho-1} \widehat{\mathcal{Q}}_u(\sigma)d\sigma \right) d\theta + \mu_2 \sum_{j=1}^{k-2} \frac{\omega_j}{\Gamma(\varrho)} \int_0^{\vartheta_j} (\vartheta_j-\theta)^{\varrho-1} \widehat{\mathcal{Q}}_u(\theta)d\theta \right. \\ &- \left. \frac{1}{\Gamma(\varsigma)} \int_0^{T} (T-\theta)^{\varsigma-1} \tilde{\mathcal{Q}}_v(\theta)d\theta \right] + \Lambda_3(\tau) \left[\frac{\lambda_1}{\Gamma(\varsigma)} \int_0^{\delta} \left(\int_0^{\theta} (\theta-\sigma)^{\varsigma-1} \tilde{\mathcal{Q}}_v(\sigma)d\sigma \right) d\theta \right. \\ &+ \left. \mu_1 \sum_{j=1}^{k-2} \frac{\varpi_j}{\Gamma(\varsigma)} \int_0^{\vartheta_j} (\vartheta_j-\theta)^{\varsigma-1} \tilde{\mathcal{Q}}_v(\theta)d\theta - \frac{1}{\Gamma(\varrho)} \int_0^{T} (T-\theta)^{\varrho-1} \widehat{\mathcal{Q}}_u(\theta)d\theta \right], \end{aligned} \tag{56}$$

and

$$\begin{aligned} \widehat{\Pi}_2(u,v)(\tau) &= \frac{1}{\Gamma(\varsigma)} \int_0^{\tau} (\tau-\theta)^{\varsigma-1} \tilde{\mathcal{Q}}_v(\theta)d\theta + \psi_2(u)[1 + \kappa_2\Lambda_7(\tau) - \Lambda_8(\tau)] + \psi_1(v)[\kappa_1\Lambda_8(\tau) - \Lambda_7(\tau)] \\ &+ \frac{\Lambda_5(\tau)\epsilon_1}{\Gamma(\varsigma-1)} \int_0^{\nu} \left(\int_0^{\theta} (\theta-\sigma)^{\varsigma-2} \tilde{\mathcal{Q}}_v(\sigma)d\sigma \right) d\theta + \frac{\Lambda_6(\tau)\epsilon_2}{\Gamma(\varrho-1)} \int_0^{\nu} \left(\int_0^{\theta} (\theta-\sigma)^{\varrho-2} \widehat{\mathcal{Q}}_u(\sigma)d\sigma \right) d\theta \\ &+ \Lambda_7(\tau) \left[\frac{\lambda_1}{\Gamma(\varsigma)} \int_0^{\delta} \left(\int_0^{\theta} (\theta-\sigma)^{\varsigma-1} \tilde{\mathcal{Q}}_v(\sigma)d\sigma \right) d\theta + \mu_1 \sum_{j=1}^{k-2} \frac{\varpi_j}{\Gamma(\varsigma)} \int_0^{\vartheta_j} (\vartheta_j-\theta)^{\varsigma-1} \tilde{\mathcal{Q}}_v(\theta)d\theta \right. \\ &- \left. \frac{1}{\Gamma(\varrho)} \int_0^{T} (T-\theta)^{\varrho-1} \widehat{\mathcal{Q}}_u(\theta)d\theta \right] + \Lambda_8(\tau) \left[\frac{\lambda_2}{\Gamma(\varrho)} \int_0^{\delta} \left(\int_0^{\theta} (\theta-\sigma)^{\varrho-1} \widehat{\mathcal{Q}}_u(\sigma)d\sigma \right) d\theta \right. \\ &+ \left. \mu_2 \sum_{j=1}^{k-2} \frac{\omega_j}{\Gamma(\varrho)} \int_0^{\vartheta_j} (\vartheta_j-\theta)^{\varrho-1} \widehat{\mathcal{Q}}_u(\theta)d\theta - \frac{1}{\Gamma(\varsigma)} \int_0^{T} (T-\theta)^{\varsigma-1} \tilde{\mathcal{Q}}_v(\theta)d\theta \right]. \end{aligned} \tag{57}$$

where

$$\begin{aligned} \widehat{\mathcal{Q}}_u(\tau) &= f(\tau, u(\tau), v(\tau), {}^{C}\mathcal{D}^{\varsigma_1} v(\tau), \mathcal{I}^{\xi} v(\tau)),\ \tau \in \mathcal{U}, \\ \tilde{\mathcal{Q}}_v(\tau) &= g(\tau, u(\tau), {}^{C}\mathcal{D}^{\varrho_1} u(\tau), \mathcal{I}^{\zeta} u(\tau), v(\tau)),\ \tau \in \mathcal{U}. \end{aligned}$$

and

$$\xi_1 = \frac{\lambda_1\delta^2}{2} + \mu_1\sum_{j=1}^{k-2}\varpi_j\vartheta_j,\ \ \xi_2 = \frac{\lambda_1\delta^n}{n} + \mu_1\sum_{j=1}^{k-2}\varpi_j\vartheta_j^{n-1},\ \ \xi_3 = \frac{\lambda_2\delta^2}{2} + \mu_2\sum_{j=1}^{k-2}\omega_j\vartheta_j,\ \ \xi_4 = \frac{\lambda_2\delta^n}{n} + \mu_2\sum_{j=1}^{k-2}\omega_j\vartheta_j^{n-1},$$

$$\hat{\gamma}_1 = 1-\nu^2\epsilon_1\epsilon_2,\ \ \hat{\gamma}_2 = T^2-\xi_1\xi_3,\ \ \hat{\gamma}_3 = T^n-\xi_1\xi_4,\ \ \hat{\gamma}_4 = T^{n-1}\xi_3-\xi_4T,\ \ \hat{\gamma}_5 = \xi_1T^{n-1}-T\xi_2,\ \ \hat{\gamma}_6 = T^n-\xi_2\xi_3,$$

$$\upsilon_1 = \hat{\gamma}_2\nu^n\epsilon_1\epsilon_2+\hat{\gamma}_1\hat{\gamma}_3,\ \ \upsilon_2 = \hat{\gamma}_2\nu^{n-1}\epsilon_2+\hat{\gamma}_1\hat{\gamma}_4,\ \ \upsilon_3 = \hat{\gamma}_2\nu^{n-1}\epsilon_1+\hat{\gamma}_1\hat{\gamma}_5,\ \ \upsilon_4 = \hat{\gamma}_2\nu^n\epsilon_1\epsilon_2+\hat{\gamma}_1\hat{\gamma}_6,\ \ \upsilon = \upsilon_2\upsilon_3-\upsilon_1\upsilon_4\neq 0,$$

$$\eta_1 = 1+\frac{(\nu^n\epsilon_2\beta_1-\nu^{n-1}\beta_5)\epsilon_1}{\upsilon},\ \ \eta_2 = \epsilon_1\nu+\frac{(\nu^n\epsilon_2\beta_2-\nu^{n-1}\beta_6)\epsilon_1}{\upsilon},\ \ \eta_3 = \frac{(\nu^n\epsilon_2\beta_3-\nu^{n-1}\beta_7)\epsilon_1}{\upsilon},$$

$$\eta_4 = \frac{(\nu^n\epsilon_2\beta_4-\nu^{n-1}\beta_8)\epsilon_1}{\upsilon},\ \ \eta_5 = \epsilon_2\nu+\frac{(\nu^{n-1}\beta_1-\epsilon_1\nu^n\beta_5)\epsilon_2}{\upsilon},\ \ \eta_6 = 1+\frac{(\nu^{n-1}\beta_2-\epsilon_1\nu^n\beta_6)\epsilon_2}{\upsilon},$$

$$\eta_7 = \frac{(\nu^{n-1}\beta_3-\epsilon_1\nu^n\beta_7)\epsilon_2}{\upsilon},\ \ \eta_8 = \frac{(\nu^{n-1}\beta_4-\epsilon_1\nu^n\beta_8)\epsilon_2}{\upsilon},$$

$$\beta_1 = \hat{\gamma}_2(\upsilon_4-\upsilon_3\epsilon_2\nu),\ \ \beta_2 = (\upsilon_4\epsilon_1\nu-\upsilon_3)\hat{\gamma}_2,\ \ \beta_3 = (\upsilon_3\xi_3-\upsilon_4T)\hat{\gamma}_1,\ \ \beta_4 = \hat{\gamma}_1(\upsilon_3T-\upsilon_4\xi_1),$$

$$\beta_5 = (\upsilon_2-\upsilon_1\epsilon_2\nu)\hat{\gamma}_2,\ \ \beta_6 = (\upsilon_2\epsilon_1\nu-\upsilon_1)\hat{\gamma}_2,\ \ \beta_7 = (\upsilon_1\xi_3-\upsilon_2T)\hat{\gamma}_1,\ \ \beta_8 = \hat{\gamma}_1(\upsilon_1T-\upsilon_2\xi_1),$$

$$\kappa_1 = \lambda_2\delta+\mu_2\sum_{j=1}^{k-2}\omega_j,\ \ \kappa_2 = \lambda_1\delta+\mu_1\sum_{j=1}^{k-2}\varpi_j,$$

$$\Lambda_1(\tau) = \frac{\tau\eta_1}{\hat{\gamma}_1}+\frac{\tau^{n-1}\beta_1}{\upsilon},\ \ \Lambda_2(\tau) = \frac{\tau\eta_2}{\hat{\gamma}_1}+\frac{\tau^{n-1}\beta_2}{\upsilon},\ \ \Lambda_3(\tau) = \frac{\tau\eta_3}{\hat{\gamma}_1}+\frac{\tau^{n-1}\beta_3}{\upsilon},\ \ \Lambda_4(\tau) = \frac{\tau\eta_4}{\hat{\gamma}_1}+\frac{\tau^{n-1}\beta_4}{\upsilon},$$

$$\Lambda_5(\tau) = \frac{\tau\eta_5}{\hat{\gamma}_1}-\frac{\tau^{n-1}\beta_5}{\upsilon},\ \ \Lambda_6(\tau) = \frac{\tau\eta_6}{\hat{\gamma}_1}-\frac{\tau^{n-1}\beta_6}{\upsilon},\ \ \Lambda_7(\tau) = \frac{\tau\eta_7}{\hat{\gamma}_1}-\frac{\tau^{n-1}\beta_7}{\upsilon},\ \ \Lambda_8(\tau) = \frac{\tau\eta_8}{\hat{\gamma}_1}-\frac{\tau^{n-1}\beta_8}{\upsilon}.$$

6. Discussion

For Caputo form FIDEs, we examined the consequences of existence and uniqueness supplemented by non-local multi-point and integral boundary conditions by Leray–Schauder's alternative and Banach's fixed-point theorem. By fixing the parameters (ϵ_1, ϵ_2, λ_1, λ_2, μ_1, μ_2) involved in the problem (1) and (2), our results correspond to certain specific problems. Suppose that taking $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$ in the results provided, we are given the problems (1) with the form

$$\begin{cases} u(0) = \psi_1(v),\ u^{'}(0) = \epsilon_1\int_0^{\nu_1} v^{'}(\theta)d\theta,\ u^{''}(0) = 0,\cdots,u^{n-2}(0) = 0,\ u(T) = 0, \\ v(0) = \psi_2(u),\ v^{'}(0) = \epsilon_2\int_0^{\nu_2} u^{'}(\theta)d\theta,\ v^{''}(0) = 0,\cdots,v^{n-2}(0) = 0,\ v(T) = 0, \end{cases}$$

while the results are

$$\begin{cases} u(0) = \psi_1(v),\ u^{'}(0) = 0,\cdots,u^{n-2}(0) = 0,\ u(T) = \lambda_1\int_0^{\delta_1} v(\theta)d\theta+\mu_1\sum_{j=1}^{k-2}\varpi_j v(\vartheta_j), \\ v(0) = \psi_2(u),\ v^{'}(0) = 0,\cdots,v^{n-2}(0) = 0\ v(T) = \lambda_2\int_0^{\delta_2} u(\theta)d\theta+\mu_2\sum_{j=1}^{k-2}\omega_j u(\varphi_j), \end{cases}$$

followed by $\epsilon_1 = \epsilon_2 = 0$. Using the methods used in the previous section, we can solve the above-related problems (1) and (2).

Author Contributions: Conceptualization, M.S., J.A. and M.I.A.; methodology and validation, M.S., J.A. and M.I.A.; investigation and formal analysis, C.T., J.A. and W.S.; resources, M.S.; data curation, M.I.A.; writing—original draft preparation, M.S. and J.A.; writing—review and editing, C.T. and W.S.; funding acquisition, C.T. and J.A. All authors have read and agreed to the published version of the manuscript.

Funding: This work was financially supported by the Faculty of Science, Burapha University, Thailand (Grant no. *SC*06/2564).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Alzabut is thankful and grateful to Prince Sultan University and OSTİM Technical University for their endless support. C. Thaiprayoon would like to gratefully acknowledge Burapha University and the Center of Excellence in Mathematics (CEM), CHE, Sri Ayutthaya Rd., Bangkok, 10400, Thailand, for supporting this research.

Conflicts of Interest: The authors have stated that they have no competing interest.

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