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Classification of Complete Regular Minimal Surfaces in $\mathbb{R}^n$ with Total Curvature -6π

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Abstract: In this paper, we classify the complete regular orientable minimal surfaces in $\mathbb{R}^n$ with total curvature -6π and give a method to construct a series of complete non-holomorphic minimal surfaces with total curvature -6π. Specially, we give a simplified classification in another method if the surfaces lie in $\mathbb{R}^4$.

Keywords: total curvature; complete minimal surfaces; orientable surfaces

MSC: 53A10; 53C42; 49Q05; 53C45; 53A07

1. Introduction

The classification of complete regular minimal surfaces in $\mathbb{R}^n$ with finite total curvature is one of the most important problems in the theories of minimal surfaces. If M is a complete minimal surface in $\mathbb{R}^n$ with finite total curvature, by using Huber's theorem [1], Chern-Osserman [2] proved that M must be conformally equivalent to $\overline{M} \setminus \{p_1, \cdots, p_r\}$, where $\overline{M}$ is a compact Riemann surface, and $p_i (i = 1, \cdots, r)$ are ends of M. Furthermore, they [2] also proved that the the Gauss map of M can be extended to a meromorphic function on $\overline{M}$, and the total Gauss curvature of M is $-2\pi m$ for some positive integer $m \geq 0$. The classical Gauss–Bonnet theorem [3], Equation (12) proved by Chern-Osserman [2] and the Jorge–Meek formula [4] all give the relation between the total Gauss curvature and the topological properties of such surfaces. Consequently, it is natural to look for the classification of such minimal surfaces with prescribed total curvature or genus or number of ends.

For minimal surfaces with a finite total curvature in $\mathbb{R}^3$, Osserman [5] proved that the total curvature of minimal surfaces in $\mathbb{R}^3$ is equal to $-4\pi m$ for some integer $m \geq 0$, and that catenoid and Enneper surface are the only complete orientable minimal surfaces of total curvature -4π. Meeks [6] gave the classification of nonorientable complete minimal surfaces with total curvature greater than -8π. The orientable case can also be characterized by López [7] under the work of Chen–Gackstatter[8] and Jorge–Meeks [4]. For the total curvature -12π, Costa [9] obtained the classification of genus one embedded minimal surfaces in $\mathbb{R}^3$ with finite total curvature -12π. On the other hand, Chen and Gackstatter [8] constructed an example of genus two orientable minimal surface punctured in a point with total curvature -12π.

For minimal surfaces in $\mathbb{R}^n$, Chen [10] gave the characterization of surfaces with total curvature -2π. Then, Hoffman and Osserman gave a slightly different version of Chen's theorem (see Theorem 6.2 in [11]). They also gave a complete description of the complete minimal surfaces in $\mathbb{R}^n$ with total curvature -4π (see Theorem 6.7 in [11]). Chen [12] gave some properties of complete minimal surfaces in $\mathbb{R}^n$ with total curvature -8π.

In this paper, we give the classification of complete minimal surfaces in $\mathbb{R}^n$ with total curvature -6π. Proposition 4 states that if S is a complete regular minimal surface in $\mathbb{R}^n$ with total curvature -6π, there are three possibilities: S is simply connected with $n \leq 8$, or S is doubly connected with $n \leq 7$, or S is a holomorphic curve in $\mathbb{C}^2$ of genus 1. The last

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case can be excluded by Chen [12]; thus, we only have to consider the simply connected case and doubly connected case.

In Section 2, we give some basic results of minimal surfaces with finite total curvature. Propositions 2 and 3 give the representations of simply connectedness and k-connectedness, respectively, which are as devoted to the proof of our main theorems as what Hoffman and Osserman did in [11] for the total curvature -4π. However, distinguished with the -4π case, where a simple form of representation, especially in a doubly-connected case, is obtained by solving a system of equations, S may lie in an eight-dimensional affine subspace of $\mathbb{R}^n$ (see corollary in [11], p. 82). Thus, there may be more equations to be solved and more coefficients to be determined, resulting in a highly complicated form of representation.

In Section 3, we give the transformation

$$\phi_{2p-1} = \frac{\varphi_{2p-1} - i\varphi_{2p}}{2}, \quad \psi_{2p-1} = \frac{\varphi_{2p-1} + i\varphi_{2p}}{2}, \quad p = 1, 2, 3, 4.$$

Then

$$\varphi_{2p-1} = \phi_{2p-1} + \psi_{2p-1}, \quad \varphi_{2p} = i(\phi_{2p-1} - \psi_{2p-1}), \quad p = 1, 2, 3, 4. \tag{1}$$

With these forms, $\sum_{i=1}^{8} \varphi_i = 0$ if and only if $\sum_{p=1}^{4} \phi_{2p-1}\psi_{2p-1} = 0$. Thus, we get a system of Equation (22). By solving (22) and calculating the constraints condition in Propositions 2 and 3, we get the following theorems.

Theorem 1. *Suppose S is a complete regular simply connected minimal surface in $\mathbb{R}^n$ with a total curvature -6π; then, S lies in some affine eight-dimensional space, and in terms of suitable coordinates, up to a constant factor, it may be represented by*

$$X(S) = Re \int \Phi(\zeta) d\zeta, \tag{2}$$

where $\Phi = (\varphi_1, ..., \varphi_8)$ has the following form:

$$\begin{aligned} \Phi = \Big(&\phi_1(\zeta) + J\zeta + L, i\phi_1(\zeta) - i(J\zeta + L), \phi_3(\zeta) + R\zeta^2 + T\zeta + K, i\phi_3(\zeta) - i\Big(R\zeta^2 + T\zeta + K\Big), \\ &\phi_5(\zeta) + M\zeta + N, i\phi_5(\zeta) - i(M\zeta + N), \phi_7(\zeta) + P, i\phi_7(\zeta) - iP, \Big), \end{aligned} \tag{3}$$

where

$$\phi_1(\zeta) = a_1\zeta^3 + c_1\zeta + d_1, \phi_3(\zeta) = b_3\zeta^2 + c_3\zeta + d_3, \phi_5(\zeta) = c_5\zeta + d_5, \phi_7(\zeta) = d_7, a_1 \neq 0, \tag{4}$$

$$R = -\frac{Ja_1}{b_3}, \quad M = -\frac{Jc_1 + Kb_3 + Rd_3 + Tc_3}{c_5}, \quad P = -\frac{Ld_1 + Kd_3 + Nd_5}{d_7}, \tag{5}$$

$$T = -\frac{Rc_3 + La_1}{b_3}, \quad N = -\frac{Lc_1 + Jd_1 + Kc_3 + Td_3 + Md_5}{c_5}, \tag{6}$$

subject to the following constraints:

(i) If any denominator of fractions in (5) is 0, the corresponding numerator and left-hand side of the equality are both 0;
(ii) If any denominator of fractions in (6) is 0, the corresponding numerator is also 0;
(iii) $\{\phi_k\}, J\zeta + L, R\zeta^2 + T\zeta + K, M\zeta + N$ have no common factors.

Theorem 2. *Let M be the plane minus a point 0, and up to a constant factor, let $X : M \to \mathbb{R}^8$ be defined by*

$$X = Re \int \Phi(\zeta) d\zeta. \tag{7}$$

Then, (7) defines a complete regular doubly connected minimal surface in $\mathbb{R}^8$ *with total curvature* -6π *of genus 0, and* Φ *has the following forms:*

$$\Phi = \frac{\tilde{\Phi}(\zeta)}{\zeta^3}, \tag{8}$$

where $\tilde{\Phi}$ *satisfies the same conditions as* Φ *in Theorem 1, in addition to*

$$R = \overline{b}_3. \tag{9}$$

Conversely, given a complete regular doubly connected minimal surface S in $\mathbb{R}^n$ *with total curvature* -6π *of genus 0, there exist functions and points satisfying conditions (8) and (9), such that S is given by (7).*

In Section 4, we give a universal method to construct a complete regular non-holomorphic minimal surface in $\mathbb{R}^n$ with total curvature -6π. In Equation (1), if all ψ_k vanish, the surface S determined by $\{\varphi_j\}$ is a complex analytic curve in $\mathbb{C}^4$. Thus, we can construct such minimal surfaces by deforming some specific complex analytic curve in $\mathbb{C}^4$.

Finally, in Section 5, a simplified classification of minimal surfaces in $\mathbb{R}^4$ with total curvature -6π is given from the viewpoint that the Gauss maps of such minimal surfaces have a fine representation about $\mathbb{CP}^1 \times \mathbb{CP}^1$ as follows:

Theorem 3. *Suppose S is a complete regular minimal surface in* $\mathbb{R}^4$ *with total curvature* -6π*. Then, in terms of suitable coordinates, it can be represented by one of the following forms:*

(a) *S is a regular complex analytic curve in* $\mathbb{C}^2$ *defined on* $\mathbb{C}$ *with the maximal degree* 4 *at the pole* ∞ *of its components;*

(b) *S is a regular complex analytic curve in* $\mathbb{C}^2$ *defined on* $\mathbb{C}\backslash\{0\}$ *with the maximal degree* 2, 1 *at the poles* $0, \infty$ *respectively of its components;*

(c) *Up to a constant factor,*

$$X(S) = Re \int \Phi(\zeta) d\zeta, \tag{10}$$

where Φ *has the form*

$$\Phi(\zeta) = (Q(\zeta) + \zeta P(\zeta), i(Q(\zeta) - \zeta P(\zeta)), \zeta Q(\zeta) - P(\zeta), -i(\zeta Q(\zeta) + P(\zeta))),$$

where $P(\zeta)$ *is polynomial of degree* 2, $Q(\zeta)$ *is polynomial of degree less than* 2, *and they have no common factor.*

(d) *Up to a constant factor,*

$$X(S) = Re \int \frac{\tilde{\Phi}(\zeta)}{\zeta^3} d\zeta, \tag{11}$$

where $\tilde{\Phi}$ *has the form*

$$\tilde{\Phi}(\zeta) = (\zeta^3 + (d_1 - 1)\zeta + d_3, -i(\zeta^3 + (d_1 + 1)\zeta - d_3), 2\zeta^2 - d_3\zeta + d_1, -i(d_3\zeta + d_1)),$$

where

$$d_3 = 0, \quad d_1 \neq 0,$$

or

$$d_3 \neq 0,\ d_1 = \xi_1 d_3, \quad d_3 \neq -\xi_1,\ \xi_1 \in \mathbb{R}.$$

Conversely, each form of (a)-(d) defines a complete regular minimal surface in $\mathbb{R}^4$ *with total curvature* -6π*.*

2. Preliminaries

Let S be an orientable surface in $\mathbb{R}^n$ defined by a conformal map

$$X : M \to \mathbb{R}^n,$$

where M is a Riemann surface. S is a minimal surface if and only if the mean curvature vector field vanishes everywhere. The generalized Gauss map of M is defined as

$$g : M \to G_{2,n}, \quad g(p) = T_pM,$$

where $G_{2,n}$ is the Grassmannian manifold consisting of all two-dimensional oriented planes in $\mathbb{R}^n$. Let v, w be an oriented pair of orthonormal vectors spanning T_pM; then, the complex vector $z = v + iw$ assigns a point of $\mathbb{C}^n$. A different choice of basis yields a point of the form $e^{i\theta}z$, and if we pass to the complex projective space $\mathbb{CP}^{n-1}$, the tangent plane T_pM corresponds to a unique point of $\mathbb{CP}^{n-1}$. The orthogonality of the pair v, w implies that the set of points obtained satisfies the equation

$$z_1^2 + \cdots + z_n^2 = 0,$$

which defines the quadric $Q_{n-2} \subset \mathbb{CP}^{n-1}$. We may identify the quadric Q_{n-2} with the Grassmannian manifold $G_{2,n}$. If $n = 4$, Q_2 may be holomorpfic to $\mathbb{S}^2 \times \mathbb{S}^2$ [11].

If (u_1, u_2) are locally isothermal parameters in a neighborhood of a point $p \in M$, then the pair of vectors

$$\frac{\partial X}{\partial u_1}, \quad \frac{\partial X}{\partial u_2}$$

are orthogonal and equal in length. Then, we may represent the generalized Gauss map by

$$g : M \to Q_{n-2}, \quad g(p) = \overline{\Phi(w)}.$$

where

$$\Phi(w) = (\varphi_1(w), \cdots, \varphi_n(w)),$$

and

$$\varphi_k(w) = 2\frac{\partial x_k}{\partial w} = \frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}, \quad w = u_1 + iu_2.$$

g is holomorphic if and only if $\Phi(w)$ is anti-holomorphic. From the lemma 4.3 of Osserman [5], if M is a minimal surface in $\mathbb{R}^n$, then g is an anti-analytic function.

Denote $\int_M KdA = -\pi C$, and χ the Euler characteristic of M; then, there are some properties of the corresponding minimal surface S.

Proposition 1 ([2]). *If the total curvature of S is finite, then*

(a). M is conformally to a compact Riemann surface $\overline{M}$ with a finite number, say r, points deleted;
(b). C is an even integer, and satisfies

$$C \geq 2(r - \chi) = 4g + 4r - 4, \tag{12}$$

where g is the genus of M (=genus of $\overline{M}$);
(c). The Gauss map of S extends to a map of $\overline{M}$ whose image $g(\overline{M})$ is an algebraic curve in $\mathbb{CP}^{n-1}$ lying in Q_{n-2}; the total curvature of S is equal in absolute value to the area of $g(M)$, counting multiplicities;
(d). $g(M)$ intersects a fixed finite number m times (counting multiplicity) every hyperplane in $\mathbb{CP}^{n-1}$ except for those hyperplanes containing one or more of the (finite number of) points of $g(\overline{M} \setminus M)$.

There are representations for complete regular minimal surfaces in $\mathbb{R}^n$ with finite total curvature:

Proposition 2 ([11]). *Let $X : \mathbb{C} \to \mathbb{R}^n$ be defined by*

$$X = Re \int \Phi(\zeta) d\zeta,$$

where $\Phi(\zeta) = (\varphi_1(\zeta), \cdots, \varphi_n(\zeta))$, and φ_j have the following properties:

(i) *Each φ_j is a polynomial;*
(ii) *The maximal degree of the φ_j is m;*
(iii) *The φ_j have no common factor;*
(iv) *$\sum_{i=1}^{n} \varphi_j^2 = 0$.*

Then, $X(\zeta)$ defines a complete regular simply connected minimal surface S in $\mathbb{R}^n$ with total curvature $-2\pi m$.

Conversely, every complete regular simply connected minimal surfaces in $\mathbb{R}^n$ with total curvature $-2\pi m$ is of the form given above.

Proposition 3 ([11]). *Let M denote the plane minus $k-1$ points, $\zeta_0, ..., \zeta_{k-2}$. Let $X : M \to \mathbb{R}^n$ be defined by*

$$X = Re \int \Phi(\zeta) d\zeta, \tag{13}$$

where the integral is taken from a fixed point to a variable point along any path in M. Assume that the complex vector $\Phi(\zeta)$ is of the form

$$\Phi(\zeta) = F(\zeta)(\tilde{\varphi}_1(\zeta), ..., \tilde{\varphi}_n(\zeta)), \tag{14}$$

where $F(\zeta)$ is a meromorphic function of the form

$$F(\zeta) = \frac{1}{\prod_{j=0}^{k-2} (\zeta - \zeta_j)^{\nu_j}}, \tag{15}$$

and the $\tilde{\varphi}_j$ satisfy

(i) *Each $\tilde{\varphi}_j$ is a polynomial;*
(ii) *The maximum degree of the $\tilde{\varphi}_j$ is m;*
(iii) *The $\tilde{\varphi}_j$ have no common factor;*
(iv) *$\sum_{j=1}^{n} \tilde{\varphi}_j^2 = 0$.*

Furthermore, the ν_j value satisfies

$$\begin{cases} \nu_j \geq 2, \quad j = 0, ..., k-2; \\ \sum\limits_{j=0}^{k-2} \nu_j \leq m+1. \end{cases} \tag{16}$$

Finally, given any closed curve $\gamma \subset M$,

$$Re \int_{\gamma} \Phi(\zeta) d\zeta = 0. \tag{17}$$

Then, (13) defines a complete regular minimal surface in $\mathbb{R}^n$ of genus 0, connectivity k and total curvature $-2\pi m$.

Conversely, given a complete regular surface S in $\mathbb{R}^n$ of genus 0, connectivity k and total curvature $-2\pi m$, there exist points $\zeta_0, ..., \zeta_{k-2}$, and functions $F, \tilde{\phi}_1(\zeta), ..., \tilde{\phi}_n(\zeta)$ satisfying (i)–(iv), (15)–(17) such that S is given by (13) and (14).

Remark 1. *Actually, the second condition of (16) in [11] can be stated as:*

$$\sum_{j=0}^{k-2} \nu_j \leq m.$$

Here, $\sum_{j=0}^{k-2} \nu_j = m+1$ could be excluded for the following reason. If it holds, similar to the proof of the simply connected case, we set

$$\tilde{\varphi}_1 = c\zeta^m + \sum_{j=0}^{m-1} C_j \zeta^j, \qquad \tilde{\varphi}_2 = ic\zeta^m + \sum_{j=0}^{m-1} C_j^{'} \zeta^j,$$

for some constant $c \neq 0, C_j, C_j^{'}$. Choose $\gamma = \partial D_R$ as the boundary of a sufficiently large disc so that $\zeta_j \in D_R$ for all j; then, the condition (17) implies

$$\begin{aligned} 0 &= Re \int_{\partial D_R} F(\zeta)\tilde{\varphi}_1(\zeta)d\zeta \\ &= Re \int_{\partial D_{1/R}} F(1/w)\tilde{\varphi}_1(1/w)d\frac{1}{w} \\ &= Re \int_{\partial D_{1/R}} -\frac{c + \sum_{j=0}^{m-1} C_j w^{m-j}}{\prod_{j=0}^{k-2}(1-\zeta_j w)^{\nu_j} w} dw \\ &= -Re(2\pi c i). \end{aligned}$$

For $\tilde{\varphi}_2$, by the similar calculation, we have

$$0 = Re \int_{\partial D_R} F(\zeta)\tilde{\varphi}_2(\zeta)d\zeta = Re(2\pi c).$$

Thus, $c = 0$ holds, which leads to a contradiction.

We say a minimal surface S lies fully in $\mathbb{R}^n$ if the image $X(M)$ does not lie in any proper affine subspace of $\mathbb{R}^n$. If S is a complete regular minimal surface in $\mathbb{R}^n$ with total curvature -6π, then S has the following properties:

Proposition 4 ([11]). *Let S be a complete oriented minimal surface of total curvature -6π lying fully in $\mathbb{R}^n$ with genus g and number of boundary components r. Then, there are three possibilities:*

1. *S is simply-connected with $n \leq 8$;*
2. *S is doubly-connected with $n \leq 7$;*
3. *$g = 1, r = 1$ and S is a holomorphic curve in $\mathbb{C}^2$.*

3. Proof of the Main Theorems

As noted in Proposition 4, if S is a complete regular minimal surface in $\mathbb{R}^n$ with finite total curvature -6π, there are three possibilities. From Chen [12], the case $g = 1$ can be ruled out. Then, we only consider the simply connected case and the doubly connected case of genus zreo. Thus, S must be conformally equivalent to the extended complex plane $\mathbb{C} \cup \{\infty\}$ punctured finite number of points by Huber's Theorem [1]. Combining the Propositions 2 and 4, S is given by

$$X = \mathrm{Re} \int \Phi(\zeta)d\zeta,$$

where $\Phi(\zeta) = (\varphi_1(\zeta), \cdots, \varphi_8(\zeta))$, and $\{\varphi_j, j = 1, \cdots 8\}$ are polynomials with maximal degree 3. Thus, we write

$$\varphi_j(\zeta) = a_j\zeta^3 + b_j\zeta^2 + c_j\zeta + d_j, \quad j = 1, \cdots, 8. \tag{18}$$

Note that the point $A = (a_1, \cdots, a_8) \in \mathbb{CP}^7$ is the image under the Gauss map of the point $\zeta = \infty$, i.e., it corresponds to the limit of the tangent plane to S as $\zeta \to \infty$. By a rotation in $\mathbb{R}^8$, we may assume that the vector A is a non-zero multiple of the vector $(1, i, 0, 0, 0, 0, 0, 0)$. The condition

$$\sum_{i=1}^{8} \varphi_j^2 = 0,$$

means

$$\sum_{i=1}^{8} a_j^2 \zeta^3 + \sum_{i=1}^{8} a_j b_j \zeta^2 + \cdots + \sum_{i=1}^{8} d_j^2 = 0.$$

If we choose

$$(a_1, \cdots, a_8) = (1, i, 0, 0, 0, 0, 0, 0),$$

then $\sum_{i=1}^{8} a_j b_j = 0$ yields $b_2 = i b_1$. φ_1, φ_2 have the forms:

$$\varphi_1 = \zeta^3 + b_1 \zeta^2 + c_1 \zeta + d_1 = (\zeta + \frac{b_1}{3})^3 + \tilde{c}_1 \zeta + \tilde{d}_1,$$

$$\varphi_2 = i\zeta^3 + i b_1 \zeta^2 + c_2 \zeta + d_2 = i(\zeta + \frac{b_1}{3})^3 + \tilde{c}_2 \zeta + \tilde{d}_2,$$

for some coefficients $\tilde{c}_1, \tilde{d}_1, \tilde{c}_2, \tilde{d}_2$. Now, we introduce a new variable $\tilde{\zeta} = \zeta + \frac{b_1}{3}$; then, φ_j also has the form (18), but with

$$b_1 = b_2 = 0.$$

Next, we may consider the real and imaginary parts of $(b_3, b_4, \cdots, b_8)$ as defining a pair of vectors in $\mathbb{R}^6$. Choosing an orthonormal pair of vectors in $\mathbb{R}^6$ orthogonal to both those vectors, we may make an orthogonal transformation of $\mathbb{R}^6$, leaving the first two coordinates unchanged, but with

$$b_5 = b_6 = b_7 = b_8 = 0.$$

For the vector (c_5, c_6, c_7, c_8), we repeat the process and obtain $c_7 = c_8 = 0$.

Thus, by the trick of resetting the coordinates above, we obtain the representation of Φ satisfying

$$\begin{aligned} &\deg \varphi_1 = \deg \varphi_2 = 3; \\ &\deg \varphi_k \leq 2, k = 3, 4; \\ &\deg \varphi_k \leq 1, k = 5, 6; \\ &\deg \varphi_k \leq 0, k = 7, 8; \\ &\deg \varphi_k \geq \deg \varphi_l, \forall\, k < l, \end{aligned} \tag{19}$$

where $\deg P(\zeta)$, for polynomial $P(\zeta)$, is the degree of the polynomial, and the coefficients of the square terms for φ_1, φ_2 vanish.

Set

$$\phi_{2p-1} = \frac{\varphi_{2p-1} - i\varphi_{2p}}{2}, \quad \psi_{2p-1} = \frac{\varphi_{2p-1} + i\varphi_{2p}}{2}, \quad p = 1, 2, 3, 4.$$

Then

$$\varphi_{2p-1} = \phi_{2p-1} + \psi_{2p-1}, \quad \varphi_{2p} = i(\phi_{2p-1} - \psi_{2p-1}), \quad p = 1, 2, 3, 4. \tag{20}$$

We can assume that $\deg \psi_{2p-1} \leq \deg \phi_{2p-1}$, otherwise changing the $2p$-th coordinate to the opposite. Then, together with condition (19), write ϕ_{2p-1} and ψ_{2p-1} as follows:

$$\begin{aligned} &\phi_1(\zeta) = a_1\zeta^3 + c_1\zeta + d_1, && \psi_1(\zeta) = G\zeta^3 + J\zeta + L, \\ &\phi_3(\zeta) = b_3\zeta^2 + c_3\zeta + d_3, && \psi_3(\zeta) = R\zeta^2 + T\zeta + K, \\ &\phi_5(\zeta) = c_5\zeta + d_5, && \psi_5(\zeta) = M\zeta + N, \\ &\phi_7(\zeta) = d_7, && \psi_7(\zeta) = P. \end{aligned} \tag{21}$$

where $a_1 \neq 0$; then, we have

$$\sum_{k=1}^{8} \varphi_k^2 = 0 \Leftrightarrow \sum_{p=1}^{4} \phi_{2p-1}\psi_{2p-1} = 0.$$

Since the functions above are all polynomials, and the degree is at most 6, we assume

$$P(\zeta) = \sum_{t=0}^{6} p_t\zeta^t := \sum_{p=1}^{4} \phi_{2p-1}\psi_{2p-1}.$$

Consider $p_6 = 0$, and directly, we get $G = 0$. Then, the following are left:

$$\begin{cases} 0 = p_4 = Ja_1 + Rb_3, \\ 0 = p_3 = La_1 + Tb_3 + Rc_3, \\ 0 = p_2 = Jc_1 + Kb_3 + Tc_3 + Rd_3 + Mc_5, \\ 0 = p_1 = Lc_1 + Jd_1 + Kc_3 + Td_3 + Nc_5 + Md_5, \\ 0 = p_0 = Ld_1 + Kd_3 + Nd_5 + Pd_7. \end{cases} \tag{22}$$

Given $\{a_k, b_k, c_k, d_k\}$, ref. (22) can be considered as a system of linear equations about $\{J, L, R, T, K, M, N, P\}$, which must have solutions, since it is homogeneous. By solving these equations, we get the identities (5) and (6).

If any denominator in (5) and (6) is zero, it is easy to see the fact that the corresponding numerator vanishes. If any denominator vanishes in (5), the principle $\deg \psi_{2p-1} \leq \phi_{2p-1}$ of resetting coordinates implies the fact that the left-hand side of the corresponding equality vanishes.

Notice that a common factor of $\{\varphi_k\}$ is also common factor of $\{\phi_k, \psi_k : k = 1, 3, 5, 7\}$. If such a common factor exists, ϕ_7 must vanish; thus, so does P, which leads to (iii). Actually, if $d_7 \neq 0$, then the constraint (iii) holds automatically.

So far, we give a complete proof of Theorem 1.

Now, we are going to consider the doubly connected case. From Proposition 3, it means $k = 2$, and $F(\zeta)$ has the form

$$F(\zeta) = \frac{1}{(\zeta - \zeta_0)^\nu}, \quad \nu = 2 \text{ or } 3. \tag{23}$$

Then, the condition (17) is non-trivial. We only need to consider the residue of $F(\zeta)\Phi(\zeta)$ at ζ_0.

Case 1: $\nu = 3$**.**

From some direct calculation, the residue of $F(\zeta)\varphi_k(\zeta)$ at ζ_0 is

$$(3a_1\zeta_0, 3ia_1\zeta_0, b_3 + R, ib_3 - iR, 0, 0, 0, 0).$$

Thus, (17) is equal to:

$$\begin{cases} \mathrm{Re}\,(2\pi i(3a_1\zeta_0)) = 0, \\ \mathrm{Re}\,(2\pi i(3ia_1\zeta_0)) = 0, \\ \mathrm{Re}\,(2\pi i(b_3 + R)) = 0, \\ \mathrm{Re}\,(2\pi i(ib_3 - iR)) = 0. \end{cases}$$

Then

$$\zeta_0 = 0, \quad R = \bar{b}_3. \tag{24}$$

Case 2: $\nu = 2$. Consider

$$\Phi(\zeta)d\zeta = \frac{\tilde{\Phi}(\zeta)}{(\zeta - \zeta_0)^2}d\zeta.$$

we can repeat the process similar to the case of $\nu = 3$ and then lead to some conditions to meet. However, there is another viewpoint to show these two cases are actually the same.

Let M be the complex plane minus a point ζ_0, which means indeed that M is a Riemann surface $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ minus two points $p_1 = \infty$, and $p_2 = \zeta_0$, and define a Möbius transformation

$$\sigma(w) = \frac{\zeta_0 w + 1}{w},$$

with $\sigma(\infty) = \zeta_0$, and $\sigma(0) = \infty$. Let $\zeta = \sigma(w)$; then, we have

$$\Phi(\zeta)d\zeta = \frac{\tilde{\Phi}(\frac{\zeta_0 w+1}{w})}{\frac{1}{w^2}}d\frac{1}{w} = -\frac{w^3\tilde{\Phi}(\frac{\zeta_0 w+1}{w})}{w^3}dw,$$

where the components of $w^3\tilde{\Phi}(\frac{\zeta_0 w+1}{w})$ are polynomials of degree at most 3. Thus, it is the same as **Case 1** if there exists some component of $w^3\tilde{\Phi}(\frac{\zeta_0 w+1}{w})$ of degree exactly 3. If not, we write

$$\tilde{\varphi}_k(\zeta) = a_k\zeta^3 + b_k\zeta^2 + c_k\zeta + d_k,$$

then

$$w^3\tilde{\varphi}_k(\frac{\zeta_0 w + 1}{w}) = a_k(\zeta_0 w + 1)^3 + b_k w(\zeta_0 w + 1)^2 + c_k w^2(\zeta_0 w + 1) + d_k w^3,$$

with coefficient $a_k\zeta_0^3 + b_k\zeta_0^2 + c_k\zeta_0 + d_k$ of degree 3. If all the coefficients of degree 3 vanish, ζ_0 becomes a common root of all $\tilde{\varphi}_k$, which is a contradiction! Thus, we get the Theorem 2.

It seems that the description of the doubly connected case just adds one condition (9) compared to the simply connected case. However, we will see later that the condition (9) may play an important role in the doubly connected surfaces.

Remark 2. *From Proposition 4, we know that such surfaces in Theorem 2 are in some affine seven-dimensional space. In fact, restriction (9) implies this result. Since $R = \bar{b}_3$ means*

$$\begin{cases} \varphi_3(\zeta) = 2Re\, b_3\zeta^2 + (c_3 + T)\zeta + (d_3 + K), \\ \varphi_4(\zeta) = -2Im\, b_3\zeta^2 + i(c_3 - T)\zeta + i(d_3 - K). \end{cases}$$

Here, notice that the coefficients of degree 2 are both real, so by the trick of resetting coordinates, we can set deg $\varphi_4 \leq 1$. Continue the process on $\varphi_k, k = 4, 5, ..., 8$, and then, we can get $\varphi_8 = 0$.

4. A Method to Give Some Families of Examples

In (22), there are two series of coefficients $\{a_k, b_k, c_k, d_k\}$ and $\{J, L, R, T, K, M, N, P\}$. Notice that if the second ones all vanish, the surface corresponding to Φ is a complex analytic curve. So, we can regard the second series of coefficients as the deformation upon some complex analytic curve. From this viewpoint and Theorem 1, there is a method to construct a series of complete non-holomorphic simply connected minimal surfaces with total curvature -6π as follows:

- **Step 1.** Fix a series of data $\{a_k, b_k, c_k, d_k\}$ with $a_1 \neq 0$. To simplify the next step, we could just choose $b_3c_5d_7 \neq 0$ and others arbitrarily.
- **Step 2.** Equation (22) consists of homogeneous linear equations for J, L, R, T, K, M, N, and P. By the theory of linear equations, it is always solvable. In the simplified case with $b_3c_5d_7 \neq 0$, given $J, K, L \in \mathbb{C}$ arbitrarily, we can determine R, T, M, N, P by using (5) and (6) successively.
- **Step 3.** Define ϕ_k, ψ_k as (21), φ as (20). If $b_3c_5d_7 \neq 0$, then

$$X(S) = \mathrm{Re} \int \Phi(\zeta) d\zeta$$

defines a complete regular simply connected minimal surface in $\mathbb{R}^8$ with total curvature -6π. Moreover, if J, K,and$L$ given in **Step 2** are not all zero, the surface is non-holomorphic. If $b_3c_5d_7 = 0$, the constraint (iii) in Theorem 1 needs to be checked.

In this method, there is a gap that the constraint (iii) holds naturally if choosing $b_3c_5d_7 \neq 0$. This is for the reasons that $\phi_7(\zeta) = d_7$ has no factor; therefore, the terms in constraint (iii) have no common factor.

Example 1. *We can give a detailed example by using the method given above. Choose* $a_1 = b_3 = c_5 = d_7 = 1$, *other* b_k, c_k, d_k *vanishing, and* $J = 2, K = L = 0$; *then, the Gauss map is*

$$\Phi = (\zeta^3 + 2\zeta, i(\zeta^3 - 2\zeta), -\zeta^2, 3i\zeta^2, \zeta, i\zeta, 1, i),$$

thus, $X(S)$ *is defined by*

$$\begin{aligned}
x_1 &= \frac{1}{4}x^4 - \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 + x^2 - y^2, & x_2 &= -x^3y + xy^3 + 2xy,\\
x_3 &= -\frac{1}{3}x^3 + xy^2, & x_4 &= y^3 - 3x^2y,\\
x_5 &= \frac{1}{2}x^2 - \frac{1}{2}y^2, & x_6 &= -xy,\\
x_7 &= x, & x_8 &= -y.
\end{aligned}$$

We can check easily that S *lies fully in* $\mathbb{R}^8$.

Similar to the simply connected case, Theorem 2 also gives a method to construct a series of complete non-holomorphic doubly connected minimal surfaces with total curvature -6π as follows:

- **Step 1.** Fix a series of data $\{a_k, b_k, c_k, d_k\}$ with $a_1 \neq 0$, $b_3c_5d_7 \neq 0$, others arbitrarily.
- **Step 2.** Given $K, L \in \mathbb{C}$ arbitrarily, then R and J come from (9) and (5), respectively. T, M, N, and P are determined by (5) and (6) successively.
- **Step 3.** Define ϕ_k, ψ_k as (21), $\tilde{\varphi}_k$ as (20), Φ as (8). Then, (7) defines a complete doubly connected minimal surface with total curvature -6π. Moreover, it cannot be holomorphic.

Here, in **Step 3**, we say the surface is non-holomorphic since there is only exactly one component of $\tilde{\Phi}$ of degree 2 from Remark 2, which cannot happen if S is holomorphic.

Comparing with the method to construct simply connected minimal surfaces, where we choose $b_3c_5d_7 \neq 0$ to simplify the process, here, $b_3c_5d_7 \neq 0$ works. In other words, if the condition $b_3c_5d_7 \neq 0$ in **Step 1** is removed, the construction might be invalid. For example, we give $\{a_k, b_k, c_k, d_k\}$ as follows:

$$\phi_1(\zeta) = \zeta^3 + \zeta - 1, \quad \phi_3(\zeta) = \zeta^2 - \zeta - 1, \quad \phi_5(\zeta) = \phi_7(\zeta) = 0. \tag{25}$$

Then, combining (9), Equation (22) can be rewritten as:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} J \\ L \\ K \\ T \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

which is overdetermined by some simple calculation. It means that the method given before cannot continue if such bad $\{a_k, b_k, c_k, d_k\}$ are given.

Example 2. *We also give a detailed example. Let $a_1 = b_3 = c_5 = d_7 = 1$, other b_k, c_k, d_k vanish and $K = L = 0$; then, the Gauss map is*

$$\Phi = (1 - \frac{1}{\zeta^2}, i + \frac{i}{\zeta^2}, \frac{2}{\zeta}, 0, \frac{1}{\zeta^2}, \frac{i}{\zeta^2}, \frac{1}{\zeta^3}, \frac{i}{\zeta^3}),$$

thus, $X(S)$ is defined by

$$\begin{aligned} x_1 &= x + \frac{x}{x^2+y^2}, & x_2 &= -y - \frac{y}{x^2+y^2}, \\ x_3 &= ln(x^2+y^2), & x_4 &= 0, \\ x_5 &= \frac{-x}{x^2+y^2}, & x_6 &= \frac{-y}{x^2+y^2}, \\ x_7 &= \frac{-x^2+y^2}{2(x^2+y^2)^2}, & x_8 &= \frac{-xy}{(x^2+y^2)^2}, \end{aligned}$$

where $(x, y) \neq (0, 0)$. In addition, it lies fully in $\mathbb{R}^7$.

Remark 3. *In our methods, to construct specific minimal surfaces, after giving a vector in*

$$\{(a_1, c_1, d_1, b_3, c_3, d_3, c_5, d_5, d_7) \in \mathbb{C}_* \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}_* \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}_* \times \mathbb{C} \times \mathbb{C}_*\},$$

where $\mathbb{C}_* = \mathbb{C}\backslash\{0\}$, any $(J, K, L) \in \mathbb{C}^3$ decides such a simply connected minimal surface. In other words, any vector in $\mathbb{C}_*^3 \times \mathbb{C}^8$ decides such a simply connected minimal surface. In the other case, replacing $(J, K, L) \in \mathbb{C}^3$ by $(K, L) \in \mathbb{C}^2$, any vector in $\mathbb{C}_*^3 \times \mathbb{C}^7$ decides such a doubly connected minimal surface.

There is a natural problem to check out which ones indeed result in the same surfaces up to an isometry. If solved, the expressions of such surfaces might be further simplified.

5. Complete Minimal Surfaces with Total Curvature -6π in $\mathbb{R}^4$

Now, we consider the minimal surfaces with total curvature -6π in lower dimensional Euclidean space. As seen in Section 1, complete minimal surfaces in $\mathbb{R}^3$ have total curvature $-4m\pi$ for some non-negative integer m. Thus, the surfaces discussed must lie in at least *four*-dimensional Euclidean space. If it is exactly 4, the minimal surfaces need to be much more restricted. Therefore, we simplify the result of the classification in $\mathbb{R}^4$.

Naturally, we could set some coefficients given before vanishing to force the surface to lie in $\mathbb{R}^4$. However, in $\mathbb{R}^4$, there is another viewpoint on the Gauss maps of minimal surfaces given by Hoffman-Osserman [11]: Q_2 may be biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. The inverse $F : \mathbb{CP}^1 \times \mathbb{CP}^1 \to Q_2$ is defined as

$$F(w_1, w_2) = (1 + w_1 w_2, i(1 - w_1 w_2), w_1 - w_2, -i(w_1 + w_2)), \tag{26}$$

where regarding $\mathbb{CP}^1$ as $\mathbb{C} \cup \{\infty\}$ and

$$F(\infty, w_2) = (w_2, -iw_2, 1, -i), \tag{27}$$
$$F(w_1, \infty) = (w_1, -iw_1, -1, -i), \tag{28}$$
$$F(\infty, \infty) = (1, -i, 0, 0). \tag{29}$$

5.1. Further Structure of Q_2

We know that an $O(4, \mathbb{R})$-action on Q_2 gives a reselection of coordinates of the related minimal surfaces; thus, the orbit of the actions determines a unique surface. The following proposition describes the induced action on $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Proposition 5. *There is a corresponding pair of actions $\sigma \in O(4, \mathbb{R})$ on Q_2 and $\tilde{\sigma}$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ so that the following diagram is commutated:*

$$\begin{array}{ccc} \mathbb{CP}^1 \times \mathbb{CP}^1 & \xrightarrow{\tilde{\sigma}} & \mathbb{CP}^1 \times \mathbb{CP}^1 \\ \Big\downarrow F & & \Big\downarrow F \\ Q_2 & \xrightarrow{\sigma \in O(4,\mathbb{R})} & Q_2, \end{array}$$

where $\tilde{\sigma}$ has the form

$$\begin{aligned} \tilde{\sigma}(w_1, w_2) &= \left(\frac{a_1 w_1 + b_1}{c_1 w_1 + d_1}, \frac{a_2 w_2 + b_2}{c_2 w_2 + d_2}\right), \quad \text{if } \sigma \in SO(4, \mathbb{R}), \\ \tilde{\sigma}(w_1, w_2) &= \left(\frac{a_2 w_2 + b_2}{c_2 w_2 + d_2}, \frac{a_1 w_1 + b_1}{c_1 w_1 + d_1}\right), \quad \text{if } \sigma \in O(4, \mathbb{R}) \setminus SO(4, \mathbb{R}), \end{aligned} \tag{30}$$

with $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SU(2), \quad i = 1, 2.$

Proof. The inverse of F is given by

$$(w_1, w_2) = F^{-1}([z_1, z_2, z_3, z_4]) = \left(\frac{z_3 + iz_4}{z_1 - iz_2}, \frac{z_3 - iz_4}{z_1 - iz_2}\right) = \left(-\frac{z_1 + iz_2}{z_3 - iz_4}, -\frac{z_1 + iz_2}{z_3 + iz_4}\right).$$

Then, it is easy to see that if

$$\sigma_0 = \mathrm{diag}(1, 1, 1, -1) \in O(4, \mathbb{R}) \backslash SO(4, \mathbb{R}),$$

the induced action $\tilde{\sigma}$ gives an exchange of w_1 and w_2. Therefore, we only need to consider the $SO(4, \mathbb{R})$-actions.

To consider the tangent map of $\sigma \in SO(4, \mathbb{R})$, we choose a series of fine vectors in Lie algebra $so(4)$ of $SO(4, \mathbb{R})$ as follows:

$$e_1 = \varepsilon_{13} - \varepsilon_{31} + \varepsilon_{42} - \varepsilon_{24},\ e_2 = \varepsilon_{14} - \varepsilon_{41} + \varepsilon_{23} - \varepsilon_{32},\ e_3 = \varepsilon_{21} - \varepsilon_{12} + \varepsilon_{34} - \varepsilon_{43},$$
$$e_4 = \varepsilon_{13} - \varepsilon_{31} - \varepsilon_{42} + \varepsilon_{24},\ e_5 = \varepsilon_{14} - \varepsilon_{41} - \varepsilon_{23} + \varepsilon_{32},\ e_6 = \varepsilon_{21} - \varepsilon_{12} - \varepsilon_{34} + \varepsilon_{43},$$

where ε_{ij} is the 4×4 matrix whose element of the i-th row and j-th column is 1 and elements in other positions are 0. Then, $\{e_i\}_{i=1}^6$ are linearly independent and $\mathrm{Span}\{e_1, ..., e_6\} = so(4)$. By calculating, the corresponding one-parameter transformations on $\mathbb{CP}^1 \times \mathbb{CP}^1$ are given respectively as:

$$\begin{cases} W_1(t) = w_1, \\ W_2(t) = \dfrac{w_2 \cos t - \sin t}{\cos t + w_2 \sin t}; \end{cases} \quad \begin{cases} W_1(t) = w_1, \\ W_2(t) = \dfrac{w_2 \cos t + i \sin t}{\cos t + i\, w_2 \sin t}; \end{cases} \quad \begin{cases} W_1(t) = w_1, \\ W_2(t) = e^{2it} w_2; \end{cases}$$

$$\begin{cases} W_1(t) = \dfrac{w_1 \cos t - \sin t}{\cos t + w_1 \sin t}, \\ W_2(t) = w_2; \end{cases} \quad \begin{cases} W_1(t) = \dfrac{w_1 \cos t - i \sin t}{\cos t - i\, w_1 \sin t}, \\ W_2(t) = w_2; \end{cases} \quad \begin{cases} W_1(t) = e^{2it} w_1, \\ W_2(t) = w_2. \end{cases}$$

Thus, it is obversed that any action $\tilde{\sigma}$ for $\sigma \in SO(4,\mathbb{R})$ can be decomposed onto the two projections $\mathbb{CP}^1$ independently. Onto each one, the actions have forms

$$W_i = \frac{a_i w_i + b_i}{c_i w_i + d_i}, \quad \text{with} \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SU(2,\mathbb{C}), \quad i = 1,2. \tag{31}$$

Thus, any $SO(4,\mathbb{R})$ action on Q_2 induces an $SU(2,\mathbb{C}) \times SU(2,\mathbb{C})$ action on $\mathbb{CP}^1 \times \mathbb{CP}^1$ by F with (31).

Conversely, according to the well-known structure of $SU(2,\mathbb{C})$,

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix}, \quad \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

($t \in \mathbb{R}$) generate $SU(2,\mathbb{C})$. Thus, any $SU(2,\mathbb{C}) \times SU(2,\mathbb{C})$ acting on $\mathbb{CP}^1 \times \mathbb{CP}^1$ as (31) is related to an $SO(4,\mathbb{R})$ action on Q_2 induced by F. □

5.2. Proof of Theorem 3

Since the induced metric on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is

$$ds^2 = \frac{2|dw_1|^2}{(1+|w_1|^2)^2} + \frac{2|dw_2|^2}{(1+|w_2|^2)^2},$$

which is just the product metric of the standard metric defined on a sphere of radius $1/\sqrt{2}$. Each projection to $\mathbb{CP}^1$ defines holomorphic functions from $\overline{M}$ to $\mathbb{CP}^1$, so it will be a finite-sheeted covering of $\mathbb{CP}^1$. Denote $n_i = \deg w_i$, $(i = 1,2)$, then

$$\int_M K dA = -Area(g(\bar{M})) = -2\pi(n_1 + n_2)$$

The surfaces with total curvature -6π have only two cases ([11], Sec. 2):

(i) One of the projections is constant, and the other is a three-sheeted covering. That is, $g(\overline{M})$ is a three-sheeted covering of $\mathbb{CP}^1$;

(ii) One of the projections is single sheeted and the other is double sheeted. Thus, $g(\overline{M})$ has genus 0 and is given by (w_1, w_2) as before with one of w_1, w_2 expressed as a rational function of degree 2 in the other variable.

Then, write $M = \overline{M} \backslash \{p_0, ..., p_{k-1}\}$. For surfaces considered, we have $k = 1$ (simply connected) or 2 (doubly connected) and $\overline{M} \simeq \mathbb{S}^2 \simeq \mathbb{CP}^1$.

Case 1. k = 1. Without loss of generality, assume $p_0 = \infty$ and denote $F^{-1}(g(p_0)) = (\xi_1, \xi_2)$. We can choose an $SU(2,\mathbb{C}) \times SU(2,\mathbb{C})$ action to transform (ξ_1, ξ_2) into (∞, ∞). Up to a reselection of coordinates, the following assumption makes sense:

$$p_0 = \infty, \quad F^{-1}(g(p_0)) = (\infty, \infty). \tag{32}$$

Case 1.1. $g(\overline{M})$ is a three-sheeted covering of $\mathbb{CP}^1$. Then, without loss of generality, we assume $g(\zeta) = F(\mu, w_2(\zeta))$ where $\mu \in \mathbb{CP}^1$ is a constant and $w_2(\zeta)$ is a meromorphic function of degree 3. Write $w_2(\zeta) = P(\zeta)/Q(\zeta)$ where $P(\zeta)$ and $Q(\zeta)$ are polynomials with no common factor and $\max\{\deg P, \deg Q\} = 3$. The assumption (32) implies $\mu = \infty$ and $3 = \deg P > \deg Q$. Hence, the Gauss image can be represented as:

$$\Phi_1(\zeta) = \eta_1(\zeta)(w_2(\zeta), -iw_2(\zeta), 1, -i)$$

where $\eta_1(\zeta)$ is a polynomial with zero at some point ζ_0 if and only if ζ_0 is a pole of w_2 with the same order. Hence, $\eta_1(\zeta) = \alpha_1 Q(\zeta)$ with $\alpha_1 \neq 0$. Thus,

$$\Phi_1(\zeta) = \alpha_1(P(\zeta), -iP(\zeta), Q(\zeta), -iQ(\zeta)).$$

Then, up to conjugate, the determined surface M is complex analytic and expressed by polynomials of maximal degree 4, where the condition that P and Q have no common factor is equal to the regularization. It is form (a) in Theorem 3.

Case 1.2. $g(\overline{M})$ is given by (w_1, w_2) and, without loss of generality, $w_2 = G(w_1)$ with meromorphic G of degree 2. Then, the Gauss image can be represented as: $g(\zeta) = F(w_1(\zeta), G(w_1(\zeta)))$, where $w_1(\zeta)$ is a holomorphic bijection on $\mathbb{CP}^1$ with $w_1(\infty) = \infty$. So, we can just say $w_1 = \zeta$ up to a constant factor. Then, $g(\zeta) = F(\zeta, G(\zeta))$. Write $G(\zeta) = P(\zeta)/Q(\zeta)$ with $P(\zeta)$ and $Q(\zeta)$ are polynomials with no common factor and $\max\{\deg P, \deg Q\} = 2$. The assumption (32) implies $G(\infty) = \infty$, that is $2 = \deg P > \deg Q$. Then

$$\Phi_2(\zeta) = \eta_2(\zeta)(1 + \zeta G(\zeta), i(1 - \zeta G(\zeta)), \zeta - G(\zeta), -i(\zeta + G(\zeta))).$$

Similarly, we can assume $\eta_2(\zeta) = \alpha_2 Q(\zeta)$ with $\alpha_2 \neq 0$; thus

$$\Phi_2(\zeta) = \alpha_2(Q(\zeta) + \zeta P(\zeta), i(Q(\zeta) - \zeta P(\zeta)), \zeta Q(\zeta) - P(\zeta), -i(\zeta Q(\zeta) + P(\zeta))),$$

which is form (c) in Theorem 3.

Case 2. k = 2. In addition, without loss of generality, assume $p_0 = \infty,\ p_1 = 0$. Similar with **Case 1**, assume further

$$\begin{cases} p_0 = \infty, & F^{-1}(g(p_0)) = (\infty, \infty), \\ p_1 = 0, & F^{-1}(g(p_1)) = (\xi_1, \xi_2). \end{cases} \tag{33}$$

When choosing action in $SU(2,\mathbb{C}) \times SU(2,\mathbb{C})$, we can let $\xi_i \in \mathbb{R} \cup \{\infty\}, i = 1, 2$. In addition, ξ_1, ξ_2 cannot be both ∞, so we further assume $\xi_2 \neq \infty$.

Case 2.1. $g(\overline{M})$ is a three-sheeted covering of $\mathbb{CP}^1$. Then, $g(\zeta) = F(C, w_2(\zeta))$ where $w_2(\zeta)$ is a meromorphic function of degree 3, where C is a constant in $\mathbb{CP}^1$, which forces $\xi_1 = C = \infty$. Write $w_2(\zeta) = P(\zeta)/Q(\zeta)$ like **Case 1.1.** Thus, we have $3 = \deg P > \deg Q$ and $w_2(0) = \xi_2$. Then, the Gauss image can be represented as:

$$\Phi_3(\zeta) = \eta_3(\zeta)(w_2(\zeta), -iw_2(\zeta), 1, -i), \tag{34}$$

where $\eta_3(\zeta)$ is an analytic function in $\mathbb{C} \setminus \{0\}$ with zero at some point ζ_0 if and only if ζ_0 is a pole of w_2 with the same order. Combining the discussion in the proof of Theorem 2, $\zeta = 0$ is the pole of $\eta_3(\zeta)$ with order three. Hence, we can assume $\eta_3(\zeta) = \beta_1 Q(\zeta)/\zeta^3$ with $\beta_1 \neq 0$. Thus,

$$\Phi_3(\zeta) = \frac{\beta_1}{\zeta^3}(P(\zeta), -iP(\zeta), Q(\zeta), -iQ(\zeta)). \tag{35}$$

and assume detailedly

$$P(\zeta) = \zeta^3 + b_1\zeta^2 + c_1\zeta + d_3\,\xi_2, \quad Q(\zeta) = b_3\zeta^2 + c_3\zeta + d_3,$$

where b_1, c_1, b_3, c_3, and d_3 are complex constants with $d_3 \neq 0$. The residue condition

$$Re \int_\gamma \Phi_3(\zeta) d\zeta = 0$$

implies

$$b_1 = b_3 = 0.$$

Then, $w_2(\zeta) = \dfrac{\zeta^3 + c_1\zeta + d_3\,\xi_2}{c_3\zeta + d_3}$, and combining the condition that $P(\zeta)$ and $Q(\zeta)$ have no common factor, we obtain

$$d_3 \neq 0,\ d_3^2 + c_1c_3^2 + \xi_2c_3^3 \neq 0. \tag{36}$$

Then, up to conjugate and a constant factor, the determined surface M is expressed by

$$\left(z - \frac{c_1}{z} - \frac{d_3\,\xi_2}{2z^2}, -\frac{c_3}{z} - \frac{d_3}{2z^2}\right) \in \mathbb{C}^2, \quad z \in \mathbb{C}\backslash\{0\}.$$

where additional condition (36) guarantees the order at 0 and regularization. It is form (b) in Theorem 3.

Case 2.2. $g(\overline{M})$ is given by (w_1, w_2) and $w_1 = G(w_2)$ (or $w_2 = G(w_1)$ similarly) with meromorphic G of degree 2. Then, $g(\zeta) = F(G(\zeta), \zeta)$. Write $G(\zeta) = P(\zeta)/Q(\zeta)$ like **Case 1.2**. Thus, we have $2 = \deg P > \deg Q$ and $G(0) = \xi_1$. Then, assume detailedly

$$P(\zeta) = \zeta^2 + c_1\zeta + d_1, \quad Q(\zeta) = c_3\zeta + d_3,$$

then, the Gauss image can be represented as:

$$\Phi_4(\zeta) = \eta_4(\zeta)(1 + \zeta G(\zeta), i(1 - \zeta G(\zeta)), G(\zeta) - \zeta, -i(G(\zeta) + \zeta)).$$

Similarly, we can assume $\eta_4(\zeta) = \frac{\beta_2 Q(\zeta)}{\zeta^3}$, thus

$$\begin{aligned}\Phi_4(\zeta) = \beta_2\Big(&\zeta^3 + c_1\zeta^2 + (d_1 + c_3)\zeta + d_3, -i(\zeta^3 + c_1\zeta^2 + (d_1 - c_3)\zeta - d_3),\\ &(1 - c_3)\zeta^2 + (c_1 - d_3)\zeta + d_1, -i((1 + c_3)\zeta^2 + (c_1 + d_3)\zeta + d_1)\Big)\end{aligned}$$

In addition, considering the residue condition (17) leads to $c_1 = 0, c_3 = -1$. Then, $G(\zeta) = \dfrac{\zeta^2 + d_1}{-\zeta + d_3}$.

Case 2.2.1. $\xi_1 = \infty$. $G(0) = \xi_1 = \infty$ implies $d_3 = 0$. No common factor of $P(\zeta)$ and $Q(\zeta)$ gives $d_1 \neq 0$.

Case 2.2.2. $\xi_1 \neq \infty$. $d_1 = \xi_1 d_3,\ d_3 \neq 0, d_3 \neq -\xi_1$ hold similarly.

Combining **Case 2.2.1** and **Case 2.2.2** results in form (d) in Theorem 3.

So far, we complete the proof of Theorem 3.

Remark 4. *In accord with Theorems 1 and 2, the surfaces of forms (a) and (c) in Theorem 3 are related to simply connected cases, while (b) and (d) are doubly connected.*

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