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Local Stability of Traveling Waves of a Model Describing Host Tissue Degradation by Bacteria

Xing He, Guansheng He and Chaohong Pan *

School of Mathematics and Physics, University of South China, Hengyang 421001, China; hexing20222022@163.com (X.H.); hgs@usc.edu.cn (G.H.)
* Correspondence: chaohongpan2020@163.com

Abstract: The focus of this paper is on the local stability of the traveling waves of reaction–diffusion systems that describe host-tissue degradation by bacteria. On the one hand, we discuss the asymptotic behavior of the solutions near the equilibrium points. On the other hand, the local stability of traveling waves is proved by the spectrum method based on the appropriate weighted functional space.

Keywords: local stability; spectrum method; weighted functional space; host-tissue degradation

MSC: 35K57; 35B35; 92D25

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1. Introduction

In this paper, we mainly focus on the model for host-tissue degradation by bacteria as follows

$$\begin{cases} u_t = u_{xx} - u + \omega - \gamma k u(1-\omega), \\ \omega_t = k u(1-\omega), \end{cases} \tag{1}$$

with the initial data

$$u(x,0) = u_0(x) \geq 0, \;\; \omega(x,0) = \omega_0(x) \geq 0, \;\; \forall x \in \mathbb{R},$$

where u describes the concentration of degradative enzymes produced by the bacteria, and $1-\omega$ corresponds to the volume fraction of the healthy tissue. In addition, it is generally believed that the population density of bacteria is proportional to ω. Both γ and k are positive constants, and k is the degradation rate of the tissue, which is usually very large in practical applications. Hence, k is a key parameter.

The earliest study of this model can be traced back to [1], which consists of a reaction–diffusion equation and an ordinary differential equation. This research predicted the wave velocity of the traveling wave of bacteria entering the tissue, and the experiment results in a matrix of proteins are consistent with this behavior. The background of bacterial infection is introduced in [2]. Hilhorst et al. [3] demonstrated the existence and uniqueness of solutions to this system and the convergence to a Stefan-like free boundary problem as the degradation rate tends to infinity. Furthermore, they [4] proved the existence of monotone traveling waves, established the conditions for nonlinear selection of the minimal wave speed of the system, and at all sufficiently large degradation rates the minimal speed is identical to the minimal speed of the limit problem. Recently, Zhang et al. [5] established the linear selection condition of the minimal wave speed of the system by ingeniously constructing upper and lower solutions, and verified some assumptions in the literature [4].

Through a simple calculation, it is easy to obtain that system (1) has two constant equilibrium points $e_0 = (0,0)$ and $e_1 = (1,1)$. In addition, we can easily get that e_0 is unstable and e_1 is stable. Therefore, system (1) is a monostable monotone system. In this paper, we are interested in the nonnegative traveling wave, connecting $(0,0)$ and $(1,1)$, which possesses the wave profile as

$$u(x,t) = \overline{U}(z),\ \omega(x,t) = \overline{W}(z),\ z = x - ct, \tag{2}$$

where the wave speed $c > 0$. Combining (1) and (2), we can obtain the system about $(\overline{U}, \overline{W})(z)$ as follows

$$\begin{cases} \overline{U}_{zz} + c\overline{U}_z - \overline{U} + \overline{W} - \gamma k \overline{U}(1 - \overline{W}) = 0, \\ c\overline{W}_z + k\overline{U}(1 - \overline{W}) = 0, \end{cases} \tag{3}$$

subject to the boundary conditions

$$(\overline{U}, \overline{W})(-\infty) = e_1,\ (\overline{U}, \overline{W})(+\infty) = e_0. \tag{4}$$

Further, using the transformation $(u,\omega)(x,t) = (U,W)(z,t)$, system (1) is transformed into the following system

$$\begin{cases} U_t = U_{zz} + cU_z - U + W - \gamma k U(1 - W), \\ W_t = cW_z + kU(1 - W), \end{cases} \tag{5}$$

subject to

$$U(z,0) = u_0(z),\ W(z,0) = \omega_0(z),\ \forall z \in \mathbb{R}.$$

$(\overline{U}, \overline{W})(z)$ is also the steady-state of system (5).

The focus of our study is shifted to the local stability of the traveling wave. The stability of traveling waves to a scalar partial differential equation has been well-studied. For example, Gallay [6] proved nonlinear stability of the slowest monotonic frontier solution based on a renormalization group method for parabolic equations. Tsai and Sneyd [7] proved that the traveling wave front is stable, i.e., that any initial condition which vaguely resembles a traveling wave front evolves to the unique wave front in a buffered systems. By analyzing the position of the spectrum, the local stability of traveling waves of nonlinear reaction–diffusion equations in different weighted Banach spaces is proved in [8]. Using the upper and lower solutions method and a squeezing technique, Ma and Zhao [9] established a global asymptotic stability with phase shift of the minimal speed of traveling wave front for a class of monostable lattice equations. Wu and Xing [10] proved that the waves with critical speed are locally asymptotically stable in some polynomially-weighted spaces through Evans's function method, proper spatial decomposition and detailed semigroup decay estimation. The nonlinear stability of the traveling wave front of the time-delay reaction–diffusion equation is studied and the traveling wave front is exponentially stable to perturbations in some exponentially weighted L^∞ space [11]. In recent years, Alhasanat and Ou demonstrated the local stability of traveling waves of the Lotka–Volterra diffusion model by using the spectrum method in [12]. Very recently, Wang et al. [13], applying the squeeze theorem, proved the local stability of forced waves in a Lotka–Volterra competing system under shifting environments. We remark that more stable results are referred to [14–25].

Although there has been a success in studying the existence of traveling waves and the selection of linear and nonlinear the minimal wave speed for the model (1), less attention has been paid to the stability of traveling waves. In this paper, we will study the local stability of traveling waves of the model (1). Inspired by references [6–13], we choose to use the spectrum method to prove local stability of traveling waves, where the greatest difficulty lies in ensuring that the largest real part of spectrum λ is less than zero. In order to solve this problem, we discuss it through four steps. First, the generation of spectrum problem is mainly based on small perturbations to the traveling wave solutions. Second, we turn it into this problem of the essential spectrum under the weighted functional space. Third, according to the classical spectrum theory, it is transformed into the problem of the intersection of algebraic curves with a real axis. Finally, we take full advantage of the characteristics of the equation to analyze the size of eigenvalues and obtain the result of local stability.

The rest of this paper is organized as follows. In Section 2, we study the local asymptotic behavior of traveling waves. Then, the local stability of the traveling wave is shown in Section 3.

2. Local Asymptotic Behavior at Unstable Point

In this section, we mainly focus on the local asymptotic behavior of traveling wave $(\overline{U}, \overline{W})(z)$ near unstable point $(0,0)$.

We are interested in the asymptotic behavior of traveling waves for system (3) as $z \to +\infty$. Let

$$(\overline{U}, \overline{W})(z) \sim (\zeta_1 e^{-\mu z}, \zeta_2 e^{-\mu z}), \tag{6}$$

where ζ_1, ζ_2, μ are positive constants. By substituting (6) into (3), and then linearizing the system, we have

$$M(\mu) \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{7}$$

where $M(\mu)$ is a 2×2 matrix given by

$$M(\mu) = \begin{bmatrix} \mu^2 - c\mu - (1+\gamma k) & 1 \\ k & -c\mu \end{bmatrix}. \tag{8}$$

Then μ satisfies the characteristic equation

$$\mu^3 - c\mu^2 - \mu(1+\gamma k) + \frac{k}{c} = 0. \tag{9}$$

Assuming that μ_1, μ_2, μ_3 are three roots of Equation (9), respectively, according to the relationship between roots and coefficients, we get

$$\mu_1 + \mu_2 + \mu_3 = c > 0, \; \mu_1\mu_2\mu_3 = -\frac{k}{c} < 0. \tag{10}$$

Letting $g(\mu) = \mu^3 - c\mu^2 - \mu(1+\gamma k) + \frac{k}{c}$, then we get $g(0) = \frac{k}{c} > 0$ and $g(-\infty) < 0$. Therefore, there exists a negative root $\mu_3 < 0$ for Equation (9), and the other two roots satisfy the following equation

$$\mu^2 + (\mu_3 - c)\mu - \frac{k}{c\mu_3} = 0. \tag{11}$$

Further, we can get the two roots as

$$\mu_1 = \frac{(c-\mu_3) - \sqrt{(c-\mu_3)^2 + \frac{4k}{c\mu_3}}}{2}, \; \mu_2 = \frac{(c-\mu_3) + \sqrt{(c-\mu_3)^2 + \frac{4k}{c\mu_3}}}{2}. \tag{12}$$

By [4], for all γ, k system (3) possesses monotone traveling waves if and only if $c \geq c_{min} \geq c_0$, where c_{min} is the minimal wave speed and c_0 is the linear speed. According to Lemma 2.2 in [5], for $\mu_0 = \mu_1 = \mu_2$, μ_0 and c_0 must satisfy the following equations

$$\begin{cases} c_0\mu_0 = -(1+\gamma k) + \sqrt{(1+\gamma k)^2 + 3k}, \\ 3\mu_0^2 = -(1+\gamma k) + 2\sqrt{(1+\gamma k)^2 + 3k}. \end{cases} \tag{13}$$

When $c \geq c_0$, μ_1 and μ_2 are the real numbers, then we have $\mu_3 < 0 < \mu_1 < \mu_2$. When $z \to +\infty$, the asymptotic behavior of the traveling wave $(\overline{U}, \overline{W})(z)$ can be expressed as:

$$\begin{pmatrix} \overline{U}(z) \\ \overline{W}(z) \end{pmatrix} \sim C_1 \begin{pmatrix} \zeta_1(\mu_1) \\ \zeta_2(\mu_1) \end{pmatrix} e^{-\mu_1 z} + C_2 \begin{pmatrix} \zeta_1(\mu_2) \\ \zeta_2(\mu_2) \end{pmatrix} e^{-\mu_2 z}, \tag{14}$$

for constants $C_1 > 0$, or $C_1 = 0$ with $C_2 > 0$. In particular, the eigenvectors corresponding to eigenvalues $\mu_i (i = 1, 2)$ can be expressed as

$$\begin{pmatrix} \zeta_1(\mu_i) \\ \zeta_2(\mu_i) \end{pmatrix} = \begin{pmatrix} c\mu_i \\ k \end{pmatrix}. \tag{15}$$

Theorem 1. *For any $c > c_0$, when $z \to +\infty$, the wavefront $\overline{U}(z)$ has the following asymptotic behavior:*

$$\overline{U}(z) \sim C_1 e^{-\mu_1 z}, C_1 > 0. \tag{16}$$

Before stating the results of local stability, we need to make a notation.

Notation: $L^p(\mathbb{R})$ is the integrable functions of Lebesgue space defined on the real number field. The asymptotic behavior of traveling wave $(U, W)(z)$ is exponentially decaying. To judge the rate of exponential decay, we need to introduce a weighted functional spaces L^p_w, which is defined as

$$L^p_w = \{f(z) : w(z)^{-1} f(z) \in L^p(\mathbb{R}), p \geq 1\}. \tag{17}$$

In addition, we also require its norm to satisfy condition

$$\|f(z)\|_{L^p_w} = \left\{ \int_{-\infty}^{+\infty} w(z)^{-1} |f(z)|^p dz \right\}^{\frac{1}{p}}, \tag{18}$$

where $\frac{1}{w(z)}$ is the weight function, and its expression is

$$w(z) = \begin{cases} 1, & z \leq z_0, \\ e^{-\alpha(z - z_0)}, & z > z_0, \end{cases} \tag{19}$$

where α and z_0 are constants and α is related to the rate of exponential decay.

3. Main Results

In this section, we will state our main results and give the proof of the local stability.

Theorem 2. *(Local Stability) For any $c > c_0$, if parameter $\alpha \in (\frac{\sqrt{k}}{c}, \sqrt{1 + \gamma k})$ is true, the wavefront $(\overline{U}, \overline{W})(z)$ is locally stable in the weighted functional space L^p_w.*

The detailed proof is mainly divided into the following four steps:
Step 1: Added a small perturbation to the traveling wave. We set

$$\begin{cases} U(z, t) = \overline{U}(z) + \delta \phi_1(z) e^{\lambda t}, \\ W(z, t) = \overline{W}(z) + \delta \phi_2(z) e^{\lambda t}, \end{cases} \tag{20}$$

where ϕ_1 and ϕ_2 are real functions, $\delta \ll 1$, λ is a parameter. Substituting (20) into (5), and then linearizing the system, we get the following spectrum problem:

$$\lambda \Phi = \Theta \Phi := N(z) \Phi'' + c \Phi' + J(z) \Phi, \tag{21}$$

where

$$\Phi = (\phi_1, \phi_2)^T, N(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, J(z) = \begin{bmatrix} -1 + (\overline{W} - 1)\gamma k & 1 + \gamma k \overline{U} \\ -k(\overline{W} - 1) & -k\overline{U} \end{bmatrix}. \tag{22}$$

Next, we need to know the sign of the maximum real part of the spectrum λ with operator Θ. At this point, the local stability of traveling waves can be transformed to investigate the spectrum problem(λ) in L^p_w which is defined in Section 2.

Step 2: Spectrum problem(λ) in the weighted functional space L^p_w. Next, we set

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} w\Psi_1 \\ w\Psi_2 \end{pmatrix}. \tag{23}$$

Substituting (23) into (21), a new spectrum problem in the weighted functional space L^p_w is obtained as

$$\lambda \Psi = \Theta_w \Psi := N(z)\Psi'' + O(z)\Psi' + P(z)\Psi, \tag{24}$$

where $N(z)$ is defined in (22), and other matrices are shown as follows

$$\begin{aligned} &\Psi = (\Psi_1, \Psi_2)^T,\ O(z) = \begin{bmatrix} c + 2\frac{w'}{w} & 0 \\ 0 & c \end{bmatrix}, \\ &P(z) = \begin{bmatrix} \frac{w''}{w} + c\frac{w'}{w} - 1 + \gamma k(\overline{W} - 1) & 1 + \gamma k \overline{U} \\ -k(\overline{W} - 1) & c\frac{w'}{w} - k\overline{U} \end{bmatrix}. \end{aligned} \tag{25}$$

In order to apply the spectrum method, we need to select an appropriate constant α such that the matrices $O(z)$ and $P(z)$ are bounded. Clearly, we can get

$$\begin{aligned} &\lim_{z \to +\infty} (1 + \gamma k \overline{U}) = 1,\ \lim_{z \to +\infty} -k(\overline{W} - 1) = k, \\ &\lim_{z \to -\infty} (1 + \gamma k \overline{U}) = 1 + \gamma k,\ \lim_{z \to -\infty} -k(\overline{W} - 1) = 0. \end{aligned} \tag{26}$$

It is easy to check that the matrices $O(z)$ and $P(z)$ are bounded. In the weighted functional space L^p_w, if the maximum real part of the eigenvalue of the essential spectrum of operator Θ_w is less than zero, we can obtain the main result of the local stability of the traveling wave.

Step 3: The essential spectrum of the operator Θ_w.

Lemma 1. *Define the algebraic curves*

$$S_\pm := \{\lambda \mid det(-\tau^2 N + i\tau O_\pm + P_\pm - \lambda I) = 0, -\infty < \tau < +\infty\}, \tag{27}$$

where $O_\pm$ and $P_\pm$ are the limits of $O(z)$ and $P(z)$ at $z \to \pm\infty$. If parameter $\alpha \in (\frac{\sqrt{k}}{c}, \sqrt{1 + \gamma k})$ is true, the essential spectrum of operator Θ_w is contained in the union of regions inside or on the curves S_+ and S_- which are on the left-half complex plane.

Proof. Next, we will adopt the idea of classification to prove the above lemma in two cases.

Case 1: When $z \to +\infty$, substituting (19) and (26) into (25), it is easy to get

$$O_+ = \begin{bmatrix} c - 2\alpha & 0 \\ 0 & c \end{bmatrix}, P_+ = \begin{bmatrix} \alpha^2 - c\alpha - 1 - \gamma k & 1 \\ k & -c\alpha \end{bmatrix}. \tag{28}$$

Solving the equation $det(-\tau^2 N + i\tau O_+ + P_+ - \lambda I) = 0$, we obtain

$$\begin{vmatrix} -\tau^2 + (c - 2\alpha)i\tau + \alpha^2 - c\alpha - 1 - \gamma k - \lambda & 1 \\ k & ci\tau - c\alpha - \lambda \end{vmatrix} = 0. \tag{29}$$

Define the functions as

$$\begin{cases} \Gamma_1 := -\tau^2 + \alpha^2 - c\alpha - 1 - \gamma k + (c - 2\alpha)\tau i = A + Bi, \\ \Gamma_2 := -c\alpha + c\tau i = C + Di, \end{cases} \tag{30}$$

then we can get the two roots of Equation (29) as

$$\lambda_{1,2} = \frac{1}{2}\left[(\Gamma_1 + \Gamma_2) \pm \sqrt{(\Gamma_1 - \Gamma_2)^2 + 4k}\right]. \tag{31}$$

By applying the formula

$$Re(\sqrt{a + bi}) = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}},\ (b \neq 0), \tag{32}$$

the expressions of $Re(\lambda_1)$ and $Re(\lambda_2)$ are as follows

$$\begin{cases} Re(\lambda_1) = \dfrac{A + C}{2} + \dfrac{1}{2}Re\left\{\sqrt{[(A - C)^2 - (B - D)^2 + 4k] + 2(A - C)(B - D)i}\right\}, \\ Re(\lambda_2) = \dfrac{A + C}{2} - \dfrac{1}{2}Re\left\{\sqrt{[(A - C)^2 - (B - D)^2 + 4k] + 2(A - C)(B - D)i}\right\}. \end{cases} \tag{33}$$

To prove the local stability, we require that the real parts of the eigenvalues λ_1 and λ_2 are less than zero. Therefore, we can assume that $Re(\lambda_{max}) = max\{Re(\lambda_1), Re(\lambda_2)\}$. Combining (32) and (33) to get the expression of $Re(\lambda_{max})$ as follows

$$Re(\lambda_{max}) = \frac{A + C}{2} + \frac{\sqrt{2}}{4}\sqrt{F^2 - G^2 + 4k + \sqrt{[F^2 + G^2 + 4k]^2 - 16kG^2}}, \tag{34}$$

where $F = A - C$ and $G = B - D$. Because of $-16kG^2 < 0$ and $\sqrt{a^2 + b^2} \leq |a| + |b|$, a simple calculation leads to

$$Re(\lambda_{max}) < \frac{A + C}{2} + \frac{|A - C|}{2} + \sqrt{k}. \tag{35}$$

Further, A is a quadratic function with respect to τ, and C is a negative constant. In order for the real part of eigenvalue λ_{max} to be less than zero, (35) needs to satisfy the following inequalities

$$\begin{cases} A < C, \\ Re(\lambda_{max}) < 0. \end{cases} \tag{36}$$

Thus, by (30) and (36), it follows that

$$\begin{cases} -\tau^2 + \alpha^2 - c\alpha - 1 - \gamma k < -c\alpha, \\ \sqrt{k} - c\alpha < 0, \end{cases} \tag{37}$$

where $-\infty < \tau < +\infty$. For the first inequality of (37), we need $\alpha^2 - c\alpha - 1 - \gamma k < -c\alpha$ to hold true. By a simple calculation, we have

$$\frac{\sqrt{k}}{c} < \alpha < \sqrt{1 + \gamma k}. \tag{38}$$

Hence, the algebraic curve $S_+ = \{\lambda_1 \mid -\infty < \tau < +\infty\} \cup \{\lambda_2 \mid -\infty < \tau < +\infty\}$ is on the left-half complex plane.

Case 2: When $z \to -\infty$, substituting (19) and (26) into (25), it is easy to get

$$O_- = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}, P_- = \begin{bmatrix} -1 & 1+\gamma k \\ 0 & -k \end{bmatrix}. \tag{39}$$

Similarly, solving the equation $det(-\tau^2 N + i\tau O_- + P_- - \lambda I) = 0$, we have

$$\lambda_3 = -\tau^2 - 1 + c\tau i, \ \lambda_4 = -k + c\tau i. \tag{40}$$

To make the real part of the two eigenvalues λ_3 and λ_4 less than zero, $k > 0$ needs to be satisfied, and it is a natural fact. Therefore, the algebraic curve $S_- = \{\lambda_3 \mid -\infty < \tau < +\infty\} \cup \{\lambda_4 \mid -\infty < \tau < +\infty\}$ is also on the left-half complex plane.

From the above analysis, we know that the essential spectrum of operator Θ_w is on the left-half complex plane. □

Finally, we will need to check the sign of the principal eigenvalue in the point spectrum (21) to guarantee locally stable of the traveling waves.

Step 4: Sign of the principal eigenvalue in the point spectrum.

Next, we will discuss the asymptotic behavior of traveling waves of system (3) as $z \to -\infty$. Let

$$(\overline{U}, \overline{W})(z) \sim (1 - \zeta_3 e^{\mu z}, 1 - \zeta_4 e^{\mu z}), \tag{41}$$

where ζ_3, ζ_4, μ are positive constants. By substituting (41) into (3), and then linearizing the system, we have

$$E(\mu) \begin{bmatrix} \zeta_3 \\ \zeta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{42}$$

where $E(\mu)$ is a 2×2 matrix given by

$$E(\mu) = \begin{bmatrix} \mu^2 + c\mu - 1 & 1+\gamma k \\ 0 & c\mu - k \end{bmatrix}. \tag{43}$$

Suppose $\mu_i (i = 4,5,6)$ is the eigenvalues of matrix (43). According to Vieta Theorem, it is easy to know that these three eigenvalues are two positive numbers and one negative number. Further, by a simple calculation, we have

$$\mu_4 = \frac{-c + \sqrt{c^2+4}}{2}, \ \mu_5 = \frac{-c - \sqrt{c^2+4}}{2}, \ \mu_6 = \frac{k}{c}. \tag{44}$$

Without loss of generality, we assume that $\mu_5 < 0 < \mu_4 < \mu_6$, then the asymptotic behavior of traveling waves is as follows:

$$\begin{pmatrix} \overline{U}(z) \\ \overline{W}(z) \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} - C_3 \begin{pmatrix} \zeta_3(\mu_6) \\ \zeta_4(\mu_6) \end{pmatrix} e^{\mu_6 z} - C_4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\mu_4 z}, \tag{45}$$

for constants $C_3 > 0$, or $C_3 = 0$ with $C_4 > 0$. In particular, the eigenvector corresponding to the eigenvalue μ_6 can be expressed as

$$\begin{pmatrix} \zeta_3(\mu_6) \\ \zeta_4(\mu_6) \end{pmatrix} = \begin{pmatrix} -(1+\gamma k) \\ \mu_6^2 + c\mu_6 - 1 \end{pmatrix}. \tag{46}$$

To check the sign of the principal eigenvalue in the point spectrum (21), we need to consider the related linear partial differential system

$$u_t = N(z) u_{zz} + c u_z + J(z) u, \tag{47}$$

where $u(z,t) = (u_1(z,t), u_2(z,t))^T$. For the solution semiflow $P_t = u(z,t,\phi)$ with any given initial data $\phi \in L^p$, we denote the solution of (47) by $e^{\lambda t}\Phi$. Therefore, we can easily verify that P_t is compact and strongly positive. From the Krein–Rutman theorem in [26], we know that P_t has a simple principal eigenvalue λ_{MAX} with a strongly positive eigenvector, and all other eigenvalues must satisfy $|e^{\lambda t}| < e^{\lambda_{MAX} t}$.

Next, we will discuss the range of eigenvalue λ. On the contrary, if neither $\lambda = 0$ nor $\lambda > 0$ holds, then the results of locally stable can be obtained.

Case 1. When $\lambda = 0$.

By using Theorem 1 for any $c > c_0$ as $z \to +\infty$, we have $(\overline{U}, \overline{W})(z) \sim (C_1 e^{-\mu_1 z}, C_1 e^{-\mu_1 z})$. Therefore, it is easy to verify that $\lambda = 0$ is the eigenvalue of (21), and the corresponding strong positive eigenvector is $(-\overline{U}'(z), -\overline{W}'(z))$. Moreover, if (38) holds, we can infer that $\mu_1 < \alpha < \mu_2$. In this case, we can easily verify the strong positive eigenvector $(-\overline{U}'(z), -\overline{W}'(z))$ is not in the weighted functional space L^p_w.

Case 2. When $\lambda > 0$.

To verify whether this situation exists, we first consider the asymptotic behavior of $\Phi(z)$ as $z \to +\infty$ and $z \to -\infty$. Assuming $\Phi(z) \in L^p_w$, this yields the smallest positive eigenvalue of $\Phi(z)$ greater than α.

For $z \to +\infty$, $\overline{\Phi}(z) = (-\overline{U}'(z), -\overline{W}'(z))$ is the positive solution of (47) and we have $\overline{\Phi}(z) > \Phi(z)$.

For $z \to -\infty$, suppose that $\Phi(z)$ has asymptotic behavior similar to $ke^{\mu z}$ for some positive numbers k and μ. By substituting it into the spectrum problem (24), we obtain the characteristic equation in eigenvalue λ as follows

$$\begin{vmatrix} \mu^2 + c\mu - 1 - \lambda & 1 + \gamma k \\ 0 & c\mu - k - \lambda \end{vmatrix} = 0. \tag{48}$$

If we use $\widetilde{\mu}_i (i = 4, 5, 6)$ to denote the three roots of (48), then their expressions are as follows

$$\widetilde{\mu}_4 = \frac{-c + \sqrt{c^2 + 4(1+\lambda)}}{2},\ \widetilde{\mu}_5 = \frac{-c - \sqrt{c^2 + 4(1+\lambda)}}{2},\ \widetilde{\mu}_6 = \frac{k + \lambda}{c}. \tag{49}$$

When $\lambda > 0$, we can compare the relationship between $\mu_i (i = 4, 6)$ and $\widetilde{\mu}_i (i = 4, 6)$, and it is easy to obtain their relations as $\mu_4 < \widetilde{\mu}_4$, $\mu_6 < \widetilde{\mu}_6$. In other words, we can know that the positive root of $\widetilde{\mu}$ gets bigger as λ increases.

Assuming $\widetilde{\mu}_6 > \widetilde{\mu}_4 > 0$ holds, when $z \to -\infty$, from (45), we can get $\overline{\Phi}(z) \sim K_1 e^{\mu_6 z}$ and $\Phi(z) \sim K_1 e^{\widetilde{\mu}_6 z}$. Therefore, we have $\overline{\Phi}(z) > \Phi(z)$. Similarly, if the assumption $\widetilde{\mu}_4 > \widetilde{\mu}_6 > 0$ holds, we can also obtain $\overline{\Phi}(z) > \Phi(z)$.

By choosing $\widetilde{k}$ large enough, we have $\widetilde{k}\overline{\Phi}(z) \geq |\Phi(z)|$. Further, applying the comparison principle to system (47), we can obtain $\widetilde{k}\overline{\Phi}(z) \geq |\Phi(z)| e^{\lambda t}$, which contradicts $\lambda > 0$.

The above analysis implies that the real parts of all eigenvalues λ of (21) should be negative for $\Phi(z) \in L^p_w$. Summarizing the above four steps, we can get the result of the local stability of the traveling waves. The proof is complete.

4. Conclusions

To reveal the dynamical behaviors of traveling waves in models of host-tissue degradation by bacteria, we investigate the local stability of traveling wave solutions by choosing appropriate weighted functional space and applying spectrum methods. For one thing, our results are novel for the local stability of the traveling wave solution of this model. For another, from an application point of view, our research methods can be generalized to some models related to Aedes aegypti (see, e.g., [27,28]). Next, we will further consider other types stabilization of traveling waves of this model or generalize this approach to other models.

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