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On the Existence and Uniqueness of an R_ν-Generalized Solution to the Stokes Problem with Corner Singularity

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Abstract: We consider the Stokes problem with the homogeneous Dirichlet boundary condition in a polygonal domain with one re-entrant corner on its boundary. We define an R_ν-generalized solution of the problem in a nonsymmetric variational formulation. Such defined solution allows us to construct numerical methods for finding an approximate solution without loss of accuracy. In the paper, the existence and uniqueness of an R_ν-generalized solution in weighted sets is proved.

Keywords: R_ν-generalized solution; corner singularity; Stokes problem

MSC: 35Q30; 35A20

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Citation: Rukavishnikov, V.A.; Rukavishnikov, A.V. On the Existence and Uniqueness of an R_ν-Generalized Solution to the Stokes Problem with Corner Singularity. *Mathematics* **2022**, *10*, 1752. https://doi.org/10.3390/math10101752

Academic Editors: Hovik Matevossian, Maria Korovina and Sergey Lurie

Received: 27 April 2022
Accepted: 19 May 2022
Published: 20 May 2022

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1. Introduction

Boundary value problems with singularity play an important role in fracture mechanics [1,2]. The singularity can be caused both by the degeneracy of the coefficients and the right-hand sides of the equation (see [3–6]) and by the presence of re-entrant corners on the boundary of a polygonal domain (see, [7–10]). The singularity of the differential problem affects the accuracy of finding an approximate solution numerically, since the error depends on the regularity of the solution [11]. An efficient numerical method with a special mesh refinement to the boundary for the problems with degeneracy of the solution on the entire boundary of the domain has been developed. The method allows us to find a solution without a loss of accuracy at a rate of $\mathcal{O}(h)$ with respect to the grid step h in the norm of the weighted Sobolev space [12,13]. Several numerical methods for problems with a singularity caused by the presence of re-entrant corners at the boundary have been developed. The methods make it possible to reduce the influence of the singularity on the accuracy of finding an approximate solution. An overview of such approaches is given in [14]. Let us select from them the weighted finite element method (FEM), which is based on the definition of an R_ν-generalized solution that takes into account the asymptotics of the solution behavior [15–18] in the vicinity of the singularity point and introducing special weighted basis functions [19,20]. This allows us to find approximate solutions without a loss of accuracy.

The solution $(\mathbf{w}, p)$ of the Stokes problem in a domain with a re-entrant corner in polar coordinates (r, φ) is a linear combination of singular components and a regular one. The singular ones of the components of vector function $\mathbf{w}$ and function p have asymptotic behaviors r^{λ_i} and $r^{\lambda_{i-1}}$ respectively, where λ_i are eigenvalues of the Stokes operator satisfying the following equation in the case of homogeneous Dirichlet boundary conditions.

$$\lambda_i^2 \sin^2 \omega - \sin^2(\lambda_i \omega) = 0, \quad \lambda_i \in \mathbb{C}, \quad \lambda_i \neq 0.$$

In particular, if the re-entrant corner ω is equal to $\frac{3\pi}{2}$, then the smallest positive eigenvalue that characterizes the behavior of the solution in a neighborhood of the re-entrant corner is approximately equal to 0.544483. In this case, the solution (components of the velocity field) belongs to space $W_2^{1+0.544483-\varepsilon}(\Omega)$, where ε is an arbitrary positive number. According to the principle of consistent estimates [11], the classical FEM allows one to find an approximate solution at a rate not higher than $\mathcal{O}(h^{0.544})$. We have developed a weighted FEM for the Stokes problem [21,22], which allows us to find an approximate solution with a rate $\mathcal{O}(h)$ independently on the value of the re-entrant corner at the boundary. The approach is based on the introduction of the concept of an R_ν-generalized solution. The existence and uniqueness of which in weighted sets is studied in this paper. The solution proposed by us is determined in the nonsymmetric mixed variational formulation of the problem: find a pair of functions $(\mathbf{w}, p)$ such that the following identities:

$$a(\mathbf{w},\mathbf{v}) + b_1(\mathbf{v}, p) = l(\mathbf{v}), \tag{1}$$

$$b_2(\mathbf{w}, q) = 0 \tag{2}$$

hold, for arbitrary pairs of functions $(\mathbf{v}, q)$. Moreover, $a(\mathbf{w},\mathbf{v})$ is not in symmetric bilinear form, and forms $b_1(.,.)$ and $b_2(.,.)$ are not equal to each other and do not coincide with the standard form $b(.,.)$ (see [23]). The question of existence and uniqueness in the abstract mixed variational formulation (1), (2) was previously studied in [24,25] and generalized [26] for the presence of bilinear form $c(p,q) \neq 0$ in Equation (2). In the present study, the existence and uniqueness of the R_ν-generalized solution in weighted sets is proved based on auxiliary statements (see [27,28]).

The paper has the following structure. In Section 2, the Stokes problem is posed, and the necessary notation has been introduced. The concept of an R_ν-generalized solution is presented in a nonsymmetric mixed variational formulation of the problem. In Section 3, relying on the proved auxiliary assertions, we establish the existence and uniqueness of the R_ν-generalized solution in weighted sets. An estimate for the norms of the solution is obtained using the norm of the right-hand side function. Conclusions and useful remarks are made in Section 4.

2. Problem Statement

Let $\mathbf{R}^2$ be the two-dimensional Euclidean space a d $\mathbf{x} = (x_1, x_2)$ be its arbitrary element. Denote by $\Omega \subset \mathbf{R}^2$ a non-convex polygonal domain with one re-entrant corner $\omega \in (\pi, 2\pi)$ on boundary $\partial\Omega$ with a vertex at origin $\mathcal{O} = (0,0)$. Let $\bar{\Omega} = \Omega \cup \partial\Omega$ be its closure.

We consider the Stokes problem: find the velocity of the fluid $\mathbf{w} = \mathbf{w}(\mathbf{x}) = (w_1(\mathbf{x}), w_2(\mathbf{x}))$ and pressure $p = p(\mathbf{x})$, which satisfy the system of differential equations and boundary conditions.

$$-\triangle\mathbf{w} + \nabla p = \mathbf{f}, \text{ in } \quad \Omega, \tag{3}$$

$$\text{div } \mathbf{w} = 0, \text{ in } \quad \Omega, \tag{4}$$

$$\mathbf{w} = \mathbf{0}, \text{ on } \quad \partial\Omega. \tag{5}$$

Due to the fact that, on boundary $\partial\Omega$, there is a corner ω greater than π, although the solution $(\mathbf{w}, p)$ of the Stokes problem (3)–(5) is analytic in $\Omega\backslash(0,0)$, but $\nabla\mathbf{w}$ and p are singular at the origin. In particular, $\mathbf{w} \notin \mathbf{W}_2^2(\Omega)$ and $p \notin W_2^1(\Omega)$. As a result, it is necessary to determine the R_ν-generalized solution of problem (3)–(5) in special sets. The main idea behind this solution is based on the introduction of the weighted function $\rho(\mathbf{x})$ satisfying the following conditions in $\bar{\Omega}$: $\rho(\mathbf{x}) = (x_1^2 + x_2^2)^{1/2}$ if $\mathbf{x} \in \Omega_\delta$ and $\rho(\mathbf{x}) = \delta$; otherwise, we have the variational formulation of the problem to some degree ν. Here and later, let Ω_δ be the intersection of a circle with a radius $\delta, \delta > 0$, and center at origin $\mathcal{O} = (0,0)$ with closure $\bar{\Omega}$. The degree ν of the weighted function $\rho(\mathbf{x})$ depends on function $\mathbf{f} = \mathbf{f}(\mathbf{x})$ of the right-hand side of Equation (3) and the value of re-entrant corner ω on $\partial\Omega$. The presence of function $\rho^\nu(\mathbf{x})$ in the variational formulation suppresses the singularity and allows R_ν-generalized solution components $\mathbf{w}_\nu$ and p_ν to belong to sets of the weighted spaces $\mathbf{W}_{2,\nu}^2(\Omega)$ and $W_{2,\nu}^1(\Omega)$, respectively (see below for definitions). Let us introduce the necessary spaces and sets of functions. For more details, see [28].

Let $D^l z(\mathbf{x}) = \frac{\partial^{|l|} z(\mathbf{x})}{\partial x_1^{l_1} \partial x_2^{l_2}}$, where $l = (l_1, l_2)$, $|l| = l_1 + l_2$, l_i are non-negative integers, $i \in \{1, 2\}$ and $d\mathbf{x} = dx_1 dx_2$.

Denote by $L_{2,\beta}(\Omega)$ and $W^k_{2,\beta}(\Omega)$ the spaces of functions $z(\mathbf{x})$ with bounded norms:

$$\|z\|_{L_{2,\beta}(\Omega)} = \Big(\int_\Omega \rho^{2\beta}(\mathbf{x})\, z^2(\mathbf{x}) d\mathbf{x} \Big)^{1/2} \tag{6}$$

and

$$\|z\|_{W^k_{2,\beta}(\Omega)} = \Big(\|z\|^2_{L_{2,\beta}(\Omega)} + \sum_{1 \le |l| \le k} \int_\Omega \rho^{2\beta}(\mathbf{x}) |D^l z(\mathbf{x})|^2 d\mathbf{x} \Big)^{1/2} \tag{7}$$

respectively. Let $|z|_{W^k_{2,\beta}(\Omega)} = \Big(\sum\limits_{|l|=k} \int\limits_\Omega \rho^{2\beta}(\mathbf{x}) |D^l z(\mathbf{x})|^2 d\mathbf{x} \Big)^{1/2}$ be the seminorm of $W^k_{2,\beta}(\Omega)$. Denote by $W^{k,0}_{2,\beta}(\Omega)$ the closure with respect to the norm (7) of the set of infinitely differentiable compactly supported functions in Ω.

Next, we define the conditions where functions $z(\mathbf{x})$ obey the following:

$$0 < C_1 \le \|z\|_{L_{2,\beta}(\Omega \setminus \Omega_\delta)}, \tag{8}$$

$$|z(\mathbf{x})| \le C_2 \delta^{\beta - \tau} \rho^{\tau - \beta}(\mathbf{x}), \mathbf{x} \in \Omega_\delta, \tag{9}$$

$$|D^1 z(\mathbf{x})| \le C_2 \delta^{\beta - \tau} \rho^{\tau - \beta - 1}(\mathbf{x}), \mathbf{x} \in \Omega_\delta, \tag{10}$$

where C_2 is a positive constant, and τ is a small positive parameter independent from β, δ and $z(\mathbf{x})$.

Denote by $L_{2,\beta}(\Omega, \delta)$ the set of functions $z(\mathbf{x})$ from the space $L_{2,\beta}(\Omega)$, satisfying conditions (8) and (9) with bounded norm (6). Define its subset $L^0_{2,\beta}(\Omega, \delta) = \{z(\mathbf{x}) \in L_{2,\beta}(\Omega, \delta) : \|\rho^\beta z\|_{L_1(\Omega)} = 0\}$ with limited norm (6).

Let $W^1_{2,\beta}(\Omega, \delta)$ be the set of functions $z(\mathbf{x})$ from the space $W^1_{2,\beta}(\Omega)$ satisfying conditions (8)–(10) with limited norm (7). Denote by $W^{1,0}_{2,\beta}(\Omega, \delta)$ the set of functions from the space $W^{1,0}_{2,\beta}(\Omega)$ satisfying conditions (8)–(10) with bounded norm (7). Moreover, the linear combination of functions from $L_{2,\beta}(\Omega, \delta)$ ($W^{1,0}_{2,\beta}(\Omega, \delta)$) also belongs to set $L_{2,\beta}(\Omega, \delta)$ ($W^{1,0}_{2,\beta}(\Omega, \delta)$).

We will highlight in bold the sets of vector functions; that is, $\mathbf{L}_{2,\beta}(\Omega, \delta) = \{\mathbf{z}(\mathbf{x}) = (z_1(\mathbf{x}), z_2(\mathbf{x})) : z_i(\mathbf{x}) \in L_{2,\beta}(\Omega, \delta)\}$ with bounded vector norm $\|\mathbf{z}\|_{\mathbf{L}_{2,\beta}(\Omega)} = \Big(\|z_1\|^2_{L_{2,\beta}(\Omega)} + \|z_2\|^2_{L_{2,\beta}(\Omega)} \Big)^{1/2}$. A similar selection takes place for sets of vector functions $\mathbf{W}^1_{2,\beta}(\Omega, \delta)$ ($\mathbf{W}^{1,0}_{2,\beta}(\Omega, \delta)$) with vector norm (7).

Next, we define the bilinear and linear forms as follows.

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \Big[\nabla \mathbf{u} : \nabla(\rho^{2\nu} \mathbf{v}) \Big] d\mathbf{x}, \tag{11}$$

$$b_1(\mathbf{v}, s) = -\int_\Omega s \ \mathrm{div}\ (\rho^{2\nu} \mathbf{v}) d\mathbf{x}, \qquad b_2(\mathbf{u}, q) = -\int_\Omega (\rho^{2\nu} q)\ \mathrm{div}\ \mathbf{u}\, d\mathbf{x},$$

$$l(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot (\rho^{2\nu} \mathbf{v}) d\mathbf{x}.$$

Definition 1. *The pair* $(\mathbf{w}_\nu,\ p_\nu) \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta) \times L^0_{2,\nu}(\Omega,\delta)$ *is called an* R_ν*-generalized solution of problem (3)–(5) if for all pairs* $(\mathbf{z}, g) \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta) \times L^0_{2,\nu}(\Omega,\delta)$*, the integral identities of the following:*

$$a(\mathbf{w}_\nu, \mathbf{z}) + b_1(\mathbf{z}, p_\nu) = l(\mathbf{z}), \qquad b_2(\mathbf{w}_\nu, g) = 0 \tag{12}$$

hold, where $\mathbf{f} \in \mathbf{L}_{2,\gamma}(\Omega,\delta), \nu \geq \gamma \geq 0$.

In contrast to the classical variational formulation of the Stokes problem (3)–(5), bilinear forms $b_1(\cdot,\cdot)$ and $b_2(\cdot,\cdot)$ in Equation (12) are not equal to each other and standard bilinear form $b(\cdot,\cdot)$ (see [23]). Thus, the variational statement (12) is a nonsymmetric one. Guided by this fact, in order to prove the existence and uniqueness R_ν-generalized solution of the problem, we need to use the following sets of functions (see [28]).

$$K_1 = \{\mathbf{v} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta) : \forall s \in L^0_{2,\nu}(\Omega,\delta), b_1(\mathbf{v}, s) = -\int\limits_\Omega s \ \text{div}\ (\rho^{2\nu}\mathbf{v}) d\mathbf{x} = 0\}.$$

$$K_2 = \{\mathbf{u} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta) : \forall q \in L^0_{2,\nu}(\Omega,\delta), b_2(\mathbf{u}, q) = -\int\limits_\Omega (\rho^{2\nu}\, q)\ \text{div}\ \mathbf{u}\, d\mathbf{x} = 0\}.$$

In particular, the second one must contain the first component of the R_ν-generalized solution, namely $\mathbf{w}_\nu$.

3. Existence and Uniqueness of an R_ν-Generalized Solution

3.1. Supplementary Statements

Lemma 1. *(Friedrichs's inequality). For any* $z \in W^{1,0}_{2,0}(\Omega,\delta)$*, the inequality of the following:*

$$\|z\|_{L_{2,0}(\Omega)} \leq C_3 \|\nabla z\|_{L_{2,0}(\Omega)} \tag{13}$$

holds, where C_3 *is a positive constant that does not depend on* z.

Lemma 2 ([28])**.** *For any* $z \in L_{2,\nu}(\Omega)$ *satisfying the conditions (8) and (9), the inequality of the following:*

$$\int\limits_{\Omega_\delta} \rho^{2(\nu-1)} z^2 d\mathbf{x} \leq C_4^2 \delta^{2\nu} \|z\|^2_{L_{2,\nu}(\Omega)} \tag{14}$$

holds, where C_4 *is a positive constant equal to* $\frac{C_2}{C_1}\sqrt{\frac{\varphi_1-\varphi_0}{2\tau}}$*,* $\varphi_1 - \varphi_0$ *is the value of the corner* ω *change in polar coordinates.*

Lemma 3 ([28])**.** *Function* $z \in W^{1,0}_{2,\nu}(\Omega,\delta)$ *if and only if* $\rho^\nu z \in W^{1,0}_{2,0}(\Omega,\delta)$ *and the following inequalities:*

$$\|\nabla(\rho^\nu z)\|^2_{L_{2,0}(\Omega)} \leq 2\|\nabla z\|^2_{L_{2,\nu}(\Omega)} + 2\nu^2 C_4^2 \delta^{2\nu} \|z\|^2_{L_{2,\nu}(\Omega)}, \tag{15}$$

$$|z|^2_{W^1_{2,\nu}(\Omega)} \leq 2\|\nabla(\rho^\nu z)\|^2_{L_{2,0}(\Omega)} + 2\nu^2 C_4^2 \delta^{2\nu} \|\rho^\nu z\|^2_{L_{2,0}(\Omega)} \tag{16}$$

hold.

Theorem 1 ([28])**.** *Let* $\nu > 0$*, then there exists* $\delta_0 = \delta_0(\nu) > 0$ *such that for any* $\delta \in (0, \delta_0)$*, any function* $\mathbf{u} \in K_2$*, represented as* $\left(\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}\right)$*, function* $\mathbf{v} = \left(\rho^{-2\nu}\frac{\partial(\rho^{2\nu}\psi)}{\partial x_2}, -\rho^{-2\nu}\frac{\partial(\rho^{2\nu}\psi)}{\partial x_1}\right)$ *belongs to the set* K_1*, and an estimate of the following:*

$$\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq C_5 \|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \tag{17}$$

holds, where $C_5 = \max\{\sqrt{2 + 16\nu^2 C_3^2 C_4^2 \delta^{2\nu+2}(1+\tau)^{-1}}, \sqrt{3 + 48\nu^2 C_3^2 C_4^2 \delta^{2\nu}(48C_3^2 + 1)}\}$.

Theorem 2 ([28]). *There exists $\varepsilon_1 > 0$, such that, for $\nu > 0$, there exists $\delta_1 = \delta_1(\varepsilon_1, \nu) > 0$, such that for any $\delta \in (0, \delta_1)$, any function $\mathbf{u} = \left(\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}\right) \in K_2$ and for $\mathbf{v} \in K_1$, represented as $\left(\rho^{-2\nu}\frac{\partial(\rho^{2\nu}\psi)}{\partial x_2}, -\rho^{-2\nu}\frac{\partial(\rho^{2\nu}\psi)}{\partial x_1}\right)$, the inequality of the following:*

$$\frac{1}{4}\|\nabla(\rho^\nu \mathbf{u})\|^2_{\mathbf{L}_{2,0}(\Omega)} \leq a(\mathbf{u}, \mathbf{v}) \tag{18}$$

holds, where $\delta_1 < \delta_0$.

Theorem 3 ([28]). *Let $\nu > 0$, then there exists $\delta_2 = \delta_2(\nu) = \min\{\delta_0(\nu), \sqrt{\frac{1+\tau}{8}}\} > 0$, such that for any $\delta \in (0, \delta_2)$, arbitrary function $\mathbf{v} \in K_1$, represented as $\left(\rho^{-2\nu}\frac{\partial(\rho^{2\nu}\psi)}{\partial x_2}, -\rho^{-2\nu}\frac{\partial(\rho^{2\nu}\psi)}{\partial x_1}\right)$, function $\mathbf{u} = \left(\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}\right)$ belongs to the set K_2, and an estimate of the following:*

$$\|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq C_6\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \tag{19}$$

holds, where a constant C_6 is equal to $\sqrt{4 + 48\nu^2 C_4^2 \delta^{2\nu}(48C_3^2 + 1)}$.

Theorem 4 ([28]). *There exists $\varepsilon_2 > 0$, such that for $\nu > 0$ there exists $\delta_3 = \delta_3(\varepsilon_2, \nu) > 0$, such that for arbitrary $\delta \in (0, \min\{\delta_2, \delta_3\})$, any function $\mathbf{v} = \left(\rho^{-2\nu}\frac{\partial(\rho^{2\nu}\psi)}{\partial x_2}, -\rho^{-2\nu}\frac{\partial(\rho^{2\nu}\psi)}{\partial x_1}\right) \in K_1$ and for $\mathbf{u} \in K_2$, represented as $\left(\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}\right)$, the inequality of the following:*

$$\frac{1}{4}\|\nabla(\rho^\nu \mathbf{v})\|^2_{\mathbf{L}_{2,0}(\Omega)} \leq a(\mathbf{u}, \mathbf{v}) \tag{20}$$

holds.

Let us formulate statements that define the LBB-conditions of the bilinear forms $b_1(\mathbf{v}, s)$ and $b_2(\mathbf{u}, q)$, $\mathbf{v}, \mathbf{u} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega, \delta)$ and $s, q \in L^0_{2,\nu}(\Omega, \delta)$.

Theorem 5. *Let $\nu > 0$; then there exists $\delta_4 = \delta_4(\nu) > 0$, such that for any $\delta \in (0, \delta_4)$, the inequalities of the following:*

$$\forall s \in L^0_{2,\nu}(\Omega, \delta): \quad \sup_{\mathbf{v} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)} \frac{b_1(\mathbf{v}, s)}{\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|s\|_{L_{2,\nu}(\Omega)}} \geq \beta_1 > 0, \tag{21}$$

$$\forall q \in L^0_{2,\nu}(\Omega, \delta): \quad \sup_{\mathbf{u} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)} \frac{b_2(\mathbf{u}, q)}{\|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|q\|_{L_{2,\nu}(\Omega)}} \geq \beta_2 > 0 \tag{22}$$

hold, where β_1 and β_2 constants that do not depend on $\mathbf{v}, s$ and $\mathbf{u}, q$, respectively.

Proof. Inequality (22) is proved in [27]. The inequality (21) is established in the same way. □

3.2. Relationship between Functions $\mathbf{v} \in K_1$ and $\mathbf{u} \in K_2$ and The Nonsymmetric Bilinear form $a(\mathbf{u}, \mathbf{v})$: Continuity of Bilinear and Linear Forms

Let us prove a theorem relating the norms of the functions $\mathbf{v} \in K_1$ and $\mathbf{u} \in K_2$ in space $\mathbf{W}^{1,0}_{2,\nu}(\Omega)$ with a nonsymmetric bilinear form $a(\mathbf{u}, \mathbf{v})$ (see Equation (11)).

Theorem 6. *There exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that, for $\nu > 0$, there exists $\delta_5 = \min\{\delta_0, \delta_1, \delta_2, \delta_3\} > 0$, such that for arbitrary $\delta \in (0, \delta_5)$, the following inequalities:*

$$\forall \mathbf{u} \in K_2: \quad \sup_{\mathbf{v} \in K_1} \frac{a(\mathbf{u}, \mathbf{v})}{\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}} \geq \alpha\|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}, \tag{23}$$

$$\forall \mathbf{v} \in K_1 : \qquad \sup_{\mathbf{u} \in K_2} \frac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}} \geq \alpha \|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \tag{24}$$

hold, where a constant $\alpha = \min\{\frac{1}{C_7}, \frac{1}{C_8}\}$ *and* $C_7 = 4C_5(2 + (1 + 2\nu^2 C_4^2 \delta^{2\nu})C_3^2)$, $C_8 = 4C_6(2 + (1 + 2\nu^2(1+2\nu)^2 C_4^2 \delta^{2\nu})C_3^2)$.

Proof. 1. For an arbitrary function $\mathbf{u} \in K_2$ defined in the formulations of Theorem 1, we apply estimate (16) of Lemma 3 combined with inequality (13) of Lemma 1, and we have the following.

$$\|\mathbf{u}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)} = \|\mathbf{u}\|^2_{\mathbf{L}_{2,\nu}(\Omega)} + |\mathbf{u}|^2_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq \|\rho^\nu \mathbf{u}\|^2_{\mathbf{L}_{2,0}(\Omega)} + 2\|\nabla(\rho^\nu \mathbf{u})\|^2_{\mathbf{L}_{2,0}(\Omega)} +$$

$$+ 2\nu^2 C_4^2 \delta^{2\nu} \|\rho^\nu \mathbf{u}\|^2_{\mathbf{L}_{2,0}(\Omega)} = 2\|\nabla(\rho^\nu \mathbf{u})\|^2_{\mathbf{L}_{2,0}(\Omega)} + (1 + 2\nu^2 C_4^2 \delta^{2\nu})\|\rho^\nu \mathbf{u}\|^2_{\mathbf{L}_{2,0}(\Omega)} \leq$$

$$\leq 2\|\nabla(\rho^\nu \mathbf{u})\|^2_{\mathbf{L}_{2,0}(\Omega)} + (1 + 2\nu^2 C_4^2 \delta^{2\nu})C_3^2 \|\nabla(\rho^\nu \mathbf{u})\|^2_{\mathbf{L}_{2,0}(\Omega)},$$

thus

$$\|\mathbf{u}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq (2 + (1 + 2\nu^2 C_4^2 \delta^{2\nu})C_3^2)\|\nabla(\rho^\nu \mathbf{u})\|^2_{\mathbf{L}_{2,0}(\Omega)}. \tag{25}$$

Applying Theorems 1 and 2 and their estimates (17) and (18), respectively, to Equation (25), for the function $\mathbf{v} \in K_1$, defined in the formulation of Theorem 1, we obtain a sequence of inequalities:

$$\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq C_5 \|\mathbf{u}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq$$

$$\leq 4C_5(2 + (1 + 2\nu^2 C_4^2 \delta^{2\nu})C_3^2)\|\nabla(\rho^\nu \mathbf{u})\|^2_{\mathbf{L}_{2,0}(\Omega)} \leq C_7 a(\mathbf{u},\mathbf{v}),$$

such that the following is the case:

$$\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq C_7 a(\mathbf{u},\mathbf{v}), \tag{26}$$

where C_7 is a constant equal to $4C_5(2 + (1 + 2\nu^2 C_4^2 \delta^{2\nu})C_3^2)$.

2. For an arbitrary function $\mathbf{v} \in K_1$, defined in the formulation of Theorem 3, by analogy with item 1, we conclude the following.

$$\|\mathbf{v}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq (2 + (1 + 2\nu^2(1+2\nu)^2 C_4^2 \delta^{2\nu})C_3^2)\|\nabla(\rho^\nu \mathbf{v})\|^2_{\mathbf{L}_{2,0}(\Omega)}. \tag{27}$$

Applying Theorems 3 and 4 and their estimates (19) and (20), respectively, to Equation (27), for the function $\mathbf{u} \in K_2$, defined in the formulation of Theorem 3, we obtain a sequence of inequalities:

$$\|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq C_6 \|\mathbf{v}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq$$

$$\leq 4C_6(2 + (1 + 2\nu^2(1+2\nu)^2 C_4^2 \delta^{2\nu})C_3^2)\|\nabla(\rho^\nu \mathbf{u})\|^2_{\mathbf{L}_{2,0}(\Omega)} \leq C_8 a(\mathbf{u},\mathbf{v}),$$

such that

$$\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq C_8 a(\mathbf{u},\mathbf{v}), \tag{28}$$

where C_8 is a constant equal to $4C_6(2 + (1 + 2\nu^2(1+2\nu)^2 C_4^2 \delta^{2\nu})C_3^2)$.

3. We perform the following sequence of reasoning. Divide both parts (26) and (28) into positive quantities $C_7\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}$ and $C_8\|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}$, respectively. We define the least upper bound over all functions $\mathbf{v} \in K_1$ and $\mathbf{u} \in K_2$ in the first and second inequalities, respectively. Let $\alpha = \min\{\frac{1}{C_7}, \frac{1}{C_8}\}$. Hence, we obtain estimates (23) and (24).

Theorem 6 is proved. □

Let us prove the continuity of the bilinear forms $a(\cdot,\cdot), b_1(\cdot,\cdot), b_2(\cdot,\cdot)$ and the linear form $l(\cdot)$.

Theorem 7. *For $\nu > 0$ the following statements hold:*

(1) For any functions $\mathbf{u}, \mathbf{v} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)$:

$$a(\mathbf{u},\mathbf{v}) \le C_9 \|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}, \tag{29}$$

where $C_9 = 8(1+\nu^2 C_3^2 C_4^2 \delta^{2\nu})(1+\nu^2 C_4^2 \delta^{2\nu})$;

(2) For arbitrary functions $s \in L^0_{2,\nu}(\Omega,\delta)$ and $\mathbf{v} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)$:

$$b_1(\mathbf{v},s) \le C_{10} \|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|s\|_{L_{2,\nu}(\Omega)}, \tag{30}$$

where $C_{10} = 2\sqrt{2+3\nu^2 C_4^2 \delta^{2\nu}}$;

(3) For any functions $q \in L^0_{2,\nu}(\Omega,\delta)$ and $\mathbf{u} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)$:

$$b_2(\mathbf{u},q) \le C_{11} \|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|q\|_{L_{2,\nu}(\Omega)}, \tag{31}$$

where $C_{11} = \sqrt{2}$;

(4) For arbitrary functions $\mathbf{v} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)$ and $\mathbf{f} \in \mathbf{L}_{2,\gamma}(\Omega,\delta)$, $\nu \ge \gamma \ge 0$:

$$l(\mathbf{v}) \le C_{12} \|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|\mathbf{f}\|_{\mathbf{L}_{2,\gamma}(\Omega)}, \tag{32}$$

where $C_{12} = \sqrt{2}C_{13}$, and C_{13} are constants in the estimate of norms for functions under embedding $L_{2,\gamma}(\Omega,\delta)$ to $L_{2,\nu}(\Omega,\delta)$.

Proof. (1) Consider arbitrary functions $\mathbf{u}, \mathbf{v} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)$ and the following:

$$\Big[a(\mathbf{u},\mathbf{v})\Big]^2 = \Big[\int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial(\rho^{2\nu} v_i)}{\partial x_j} d\mathbf{x}\Big]^2,$$

then

$$\Big[a(\mathbf{u},\mathbf{v})\Big]^2 \le 2\Big[\int_\Omega \sum_{j=1}^{2} \frac{\partial u_1}{\partial x_j} \frac{\partial(\rho^{2\nu} v_1)}{\partial x_j} d\mathbf{x}\Big]^2 + 2\Big[\int_\Omega \sum_{j=1}^{2} \frac{\partial u_2}{\partial x_j} \frac{\partial(\rho^{2\nu} v_2)}{\partial x_j} d\mathbf{x}\Big]^2 \le$$

$$\le 4\sum_{i,j=1}^{2} \Big[\int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial(\rho^{2\nu} v_i)}{\partial x_j} d\mathbf{x}\Big]^2 = 4\sum_{i,j=1}^{2} S_{i,j},$$

such that

$$\Big[a(\mathbf{u},\mathbf{v})\Big]^2 \le 4\sum_{i,j=1}^{2} S_{i,j}, \tag{33}$$

where $S_{i,j}$ has the following form.

$$S_{i,j} = \Big[\int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial(\rho^{2\nu} v_i)}{\partial x_j} d\mathbf{x}\Big]^2. \tag{34}$$

Insofar as we have the following:

$$\frac{\partial(\rho^\nu u_i)}{\partial x_j} = \rho^\nu\Big(\frac{\partial u_i}{\partial x_j}\Big) + \Big(\frac{\partial \rho^\nu}{\partial x_j}\Big) u_i,$$

then

$$\frac{\partial u_i}{\partial x_j} = \rho^{-\nu}\frac{\partial(\rho^\nu u_i)}{\partial x_j} - \rho^{-\nu}\Big(\frac{\partial \rho^\nu}{\partial x_j}\Big)u_i, \tag{35}$$

and

$$\frac{\partial(\rho^{2\nu} v_i)}{\partial x_j} = \Big(\frac{\partial \rho^\nu}{\partial x_j}\Big)(\rho^\nu v_i) + \rho^\nu\frac{\partial(\rho^\nu v_i)}{\partial x_j}. \tag{36}$$

Let us substitute the right-hand parts (35) and (36) instead of their left-hand parts to Equation (34); then, by applying the Cauchy–Schwarz inequality, we have the following.

$$S_{i,j} = \Big[\int\limits_\Omega \Big(\rho^{-\nu}\frac{\partial(\rho^\nu u_i)}{\partial x_j} - \rho^{-\nu}\Big(\frac{\partial \rho^\nu}{\partial x_j}\Big)u_i\Big)\Big(\Big(\frac{\partial \rho^\nu}{\partial x_j}\Big)(\rho^\nu v_i) + \rho^\nu\frac{\partial(\rho^\nu v_i)}{\partial x_j}\Big)d\mathbf{x}\Big]^2 =$$

$$= \Big[\int\limits_\Omega \Big(\frac{\partial(\rho^\nu u_i)}{\partial x_j}\Big)\Big(\frac{\partial \rho^\nu}{\partial x_j}v_i\Big)d\mathbf{x} + \int\limits_\Omega \Big(\frac{\partial(\rho^\nu u_i)}{\partial x_j}\Big)\Big(\frac{\partial(\rho^\nu v_i)}{\partial x_j}\Big)d\mathbf{x} - \int\limits_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_j}u_i\Big)\Big(\frac{\partial \rho^\nu}{\partial x_j}v_i\Big)d\mathbf{x} -$$

$$- \int\limits_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_j}u_i\Big)\Big(\frac{\partial(\rho^\nu v_i)}{\partial x_j}\Big)d\mathbf{x}\Big]^2 \le 4\Big[\Big(\int\limits_\Omega \Big(\frac{\partial(\rho^\nu u_i)}{\partial x_j}\Big)\Big(\frac{\partial \rho^\nu}{\partial x_j}v_i\Big)d\mathbf{x}\Big)^2 +$$

$$+\Big(\int\limits_\Omega \Big(\frac{\partial(\rho^\nu u_i)}{\partial x_j}\Big)\Big(\frac{\partial(\rho^\nu v_i)}{\partial x_j}\Big)d\mathbf{x}\Big)^2 +$$

$$+\Big(\int\limits_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_j}u_i\Big)\Big(\frac{\partial \rho^\nu}{\partial x_j}v_i\Big)d\mathbf{x}\Big)^2 + \Big(\int\limits_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_j}u_i\Big)\Big(\frac{\partial(\rho^\nu v_i)}{\partial x_j}\Big)d\mathbf{x}\Big)^2\Big] \le$$

$$\le 4\Big[\Big(\int\limits_\Omega \Big(\frac{\partial(\rho^\nu u_i)}{\partial x_j}\Big)^2 d\mathbf{x}\Big)\Big(\int\limits_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_j}\Big)^2 v_i^2 d\mathbf{x}\Big) + \Big(\int\limits_\Omega \Big(\frac{\partial(\rho^\nu u_i)}{\partial x_j}\Big)^2 d\mathbf{x}\Big)\Big(\int\limits_\Omega \Big(\frac{\partial(\rho^\nu v_i)}{\partial x_j}\Big)^2 d\mathbf{x}\Big) +$$

$$+\Big(\int\limits_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_j}\Big)^2 u_i^2 d\mathbf{x}\Big)\Big(\int\limits_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_j}\Big)^2 v_i^2 d\mathbf{x}\Big) + \Big(\int\limits_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_j}\Big)^2 u_i^2 d\mathbf{x}\Big)\Big(\int\limits_\Omega \Big(\frac{\partial(\rho^\nu v_i)}{\partial x_j}\Big)^2 d\mathbf{x}\Big)\Big].$$

Let us summarize $S_{i,1}$ and $S_{i,2}$ using inequality $a_1b_1 + a_2b_2 \le (a_1+a_2)(b_1+b_2)$, $a_k, b_k \ge 0$; we conclude the following.

$$\sum_{j=1}^{2} S_{i,j} \le 4\Big[\Big(\int\limits_\Omega \Big[\Big(\frac{\partial \rho^\nu}{\partial x_1}\Big)^2 + \Big(\frac{\partial \rho^\nu}{\partial x_2}\Big)^2\Big]v_i^2 d\mathbf{x}\Big) \cdot \Big(\int\limits_\Omega \Big[\Big(\frac{\partial(\rho^\nu u_i)}{\partial x_1}\Big)^2 + \Big(\frac{\partial(\rho^\nu u_i)}{\partial x_2}\Big)^2\Big]d\mathbf{x}\Big) +$$

$$+\Big(\int\limits_\Omega \Big[\Big(\frac{\partial(\rho^\nu u_i)}{\partial x_1}\Big)^2 + \Big(\frac{\partial(\rho^\nu u_i)}{\partial x_2}\Big)^2\Big]d\mathbf{x}\Big) \cdot \Big(\int\limits_\Omega \Big[\Big(\frac{\partial(\rho^\nu v_i)}{\partial x_1}\Big)^2 + \Big(\frac{\partial(\rho^\nu v_i)}{\partial x_2}\Big)^2\Big]d\mathbf{x}\Big) +$$

$$+\Big(\int\limits_\Omega \Big[\Big(\frac{\partial \rho^\nu}{\partial x_1}\Big)^2 + \Big(\frac{\partial \rho^\nu}{\partial x_2}\Big)^2\Big]u_i^2 d\mathbf{x}\Big) \cdot \Big(\int\limits_\Omega \Big[\Big(\frac{\partial \rho^\nu}{\partial x_1}\Big)^2 + \Big(\frac{\partial \rho^\nu}{\partial x_2}\Big)^2\Big]v_i^2 d\mathbf{x}\Big) +$$

$$+\Big(\int\limits_\Omega \Big[\Big(\frac{\partial \rho^\nu}{\partial x_1}\Big)^2 + \Big(\frac{\partial \rho^\nu}{\partial x_2}\Big)^2\Big]u_i^2 d\mathbf{x}\Big) \cdot \Big(\int\limits_\Omega \Big[\Big(\frac{\partial(\rho^\nu v_i)}{\partial x_1}\Big)^2 + \Big(\frac{\partial(\rho^\nu v_i)}{\partial x_2}\Big)^2\Big]d\mathbf{x}\Big)\Big].$$

For an arbitrary β, we have the following.

$$\frac{\partial \rho^\beta}{\partial x_i} = \begin{cases} \beta\rho^{\beta-2}x_i, \mathbf{x} \in \Omega_\delta, \\ 0, \mathbf{x} \in \bar{\Omega} \setminus \Omega_\delta, \end{cases} \tag{37}$$

Apply (37) with $\beta = \nu$; then, we have the following.

$$\left(\frac{\partial \rho^{\nu}}{\partial x_1}\right)^2 + \left(\frac{\partial \rho^{\nu}}{\partial x_2}\right)^2 = \begin{cases} \nu^2 \rho^{2\nu-2}, \mathbf{x} \in \Omega_\delta, \\ 0, \mathbf{x} \in \bar{\Omega} \setminus \Omega_\delta, \end{cases} \tag{38}$$

Using the inequality (14) of Lemma 2 for u_i and v_i and the definition of the $L_{2,0}(\Omega)$ norm for $\nabla(\rho^\nu u_i)$ and $\nabla(\rho^\nu v_i)$, we have the following:

$$\sum_{j=1}^{2} S_{i,j} \leq 4\nu^2 C_4^2 \delta^{2\nu} \|v_i\|^2_{L_{2,\nu}(\Omega)} \cdot \|\nabla(\rho^\nu u_i)\|^2_{L_{2,0}(\Omega)} + 4\|\nabla(\rho^\nu u_i)\|^2_{L_{2,0}(\Omega)} \cdot \|\nabla(\rho^\nu v_i)\|^2_{L_{2,0}(\Omega)} +$$

$$+4\left[\nu^2 C_4^2 \delta^{2\nu}\right]^2 \|u_i\|^2_{L_{2,\nu}(\Omega)} \cdot \|v_i\|^2_{L_{2,\nu}(\Omega)} + 4\nu^2 C_4^2 \delta^{2\nu} \|u_i\|^2_{L_{2,\nu}(\Omega)} \cdot \|\nabla(\rho^\nu v_i)\|^2_{L_{2,0}(\Omega)} =$$

$$= 4\left[\nu^2 C_4^2 \delta^{2\nu} \|u_i\|^2_{L_{2,\nu}(\Omega)} + \|\nabla(\rho^\nu u_i)\|^2_{L_{2,0}(\Omega)}\right] \cdot \left[\nu^2 C_4^2 \delta^{2\nu} \|v_i\|^2_{L_{2,\nu}(\Omega)} + \|\nabla(\rho^\nu v_i)\|^2_{L_{2,0}(\Omega)}\right],$$

such that the following is the case.

$$\sum_{j=1}^{2} S_{i,j} \leq 4\left[\nu^2 C_4^2 \delta^{2\nu} \|u_i\|^2_{L_{2,\nu}(\Omega)} + \|\nabla(\rho^\nu u_i)\|^2_{L_{2,0}(\Omega)}\right] \times$$

$$\times \left[\nu^2 C_4^2 \delta^{2\nu} \|v_i\|^2_{L_{2,\nu}(\Omega)} + \|\nabla(\rho^\nu v_i)\|^2_{L_{2,0}(\Omega)}\right]. \tag{39}$$

Applying equalities $\|u_i\|_{L_{2,\nu}(\Omega)} = \|\rho^\nu u_i\|_{L_{2,0}(\Omega)}$, $\|v_i\|_{L_{2,\nu}(\Omega)} = \|\rho^\nu v_i\|_{L_{2,0}(\Omega)}$ and inequality (13) of Lemma 1 for evaluating the right-hand side (39), we derive the following.

$$\sum_{j=1}^{2} S_{i,j} \leq 4\left(1 + \nu^2 C_3^2 C_4^2 \delta^{2\nu}\right)^2 \|\nabla(\rho^\nu u_i)\|^2_{L_{2,0}(\Omega)} \cdot \|\nabla(\rho^\nu v_i)\|^2_{L_{2,0}(\Omega)}. \tag{40}$$

Let us sum the inequalities (40) over variable i, $i = 1, 2$, and substitute it into Equation (33). Using inequality $a_1 b_1 + a_2 b_2 \leq (a_1 + a_2)(b_1 + b_2), a_k, b_k \geq 0$, we have a sequence of inequalities:

$$\left[a(\mathbf{u}, \mathbf{v})\right]^2 \leq 16\left(1 + \nu^2 C_3^2 C_4^2 \delta^{2\nu}\right)^2 \sum_{i=1}^{2} \left[\|\nabla(\rho^\nu u_i)\|^2_{L_{2,0}(\Omega)} \cdot \|\nabla(\rho^\nu v_i)\|^2_{L_{2,0}(\Omega)}\right] \leq$$

$$\leq 16\left(1 + \nu^2 C_3^2 C_4^2 \delta^{2\nu}\right)^2 \left[\sum_{i=1}^{2} \|\nabla(\rho^\nu u_i)\|^2_{L_{2,0}(\Omega)}\right] \cdot \left[\sum_{i=1}^{2} \|\nabla(\rho^\nu v_i)\|^2_{L_{2,0}(\Omega)}\right],$$

so that the following is the case.

$$\left[a(\mathbf{u}, \mathbf{v})\right]^2 \leq 16\left(1 + \nu^2 C_3^2 C_4^2 \delta^{2\nu}\right)^2 \left[\sum_{i=1}^{2} \|\nabla(\rho^\nu u_i)\|^2_{L_{2,0}(\Omega)}\right] \cdot \left[\sum_{i=1}^{2} \|\nabla(\rho^\nu v_i)\|^2_{L_{2,0}(\Omega)}\right]. \tag{41}$$

Due to inequality (15) of Lemma 3 for the factors of the right-hand side (41), we derive the following.

$$\left[a(\mathbf{u}, \mathbf{v})\right]^2 \leq 64\left(1 + \nu^2 C_3^2 C_4^2 \delta^{2\nu}\right)^2 \left(1 + \nu^2 C_4^2 \delta^{2\nu}\right)^2 \|\mathbf{u}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|\mathbf{v}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)}.$$

It remains to take the square root of both sides of the last inequality. The estimate (29) is proven.

(2) Consider arbitrary functions $s \in L^0_{2,\nu}(\Omega,\delta)$, $\mathbf{v} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)$ and the following.

$$\Big[b_1(\mathbf{v},s)\Big]^2 = \Big[\int\limits_\Omega s\Big(\frac{\partial(\rho^{2\nu}v_1)}{\partial x_1} + \frac{\partial(\rho^{2\nu}v_2)}{\partial x_2}\Big)d\mathbf{x}\Big]^2. \tag{42}$$

Insofar as the following is the case:

$$\frac{\partial(\rho^{2\nu}v_i)}{\partial x_i} = \rho^\nu\Big(\frac{\partial\rho^\nu}{\partial x_i}\Big)v_i + \rho^\nu\frac{\partial(\rho^\nu v_i)}{\partial x_i}, \qquad i=1,2,$$

then, using this and the Cauchy–Schwarz inequality, we have a sequence of estimates:

$$\Big(\int\limits_\Omega s\frac{\partial(\rho^{2\nu}v_i)}{\partial x_i}d\mathbf{x}\Big)^2 \le 2\Big(\int\limits_\Omega \rho^{2\nu}s^2 d\mathbf{x}\Big)\cdot\Big(\int\limits_\Omega \Big(\frac{\partial\rho^\nu}{\partial x_i}\Big)^2 v_i^2 d\mathbf{x}\Big) + 2\Big(\int\limits_\Omega \rho^{2\nu}s^2 d\mathbf{x}\Big)\times$$

$$\times\Big(\int\limits_\Omega \Big(\frac{\partial(\rho^\nu v_i)}{\partial x_i}\Big)^2 d\mathbf{x}\Big) \le 2\|s\|^2_{L_{2,\nu}(\Omega)}\cdot\Big(\int\limits_\Omega\Big[\sum_{i=1}^2\Big(\frac{\partial\rho^\nu}{\partial x_i}\Big)^2\Big]v_i^2 d\mathbf{x} + \int\limits_\Omega\sum_{i=1}^2\Big(\frac{\partial(\rho^\nu v_i)}{\partial x_i}\Big)^2 d\mathbf{x}\Big),$$

such that the following is the case.

$$\Big(\int\limits_\Omega s\frac{\partial(\rho^{2\nu}v_i)}{\partial x_i}d\mathbf{x}\Big)^2 \le 2\|s\|^2_{L_{2,\nu}(\Omega)}\Big(\int\limits_\Omega\Big[\sum_{i=1}^2\Big(\frac{\partial\rho^\nu}{\partial x_i}\Big)^2\Big]v_i^2 d\mathbf{x} + \int\limits_\Omega\sum_{i=1}^2\Big(\frac{\partial(\rho^\nu v_i)}{\partial x_i}\Big)^2 d\mathbf{x}\Big). \tag{43}$$

Using representation (38) and inequality (14) of Lemma 2, we conclude the following.

$$\int\limits_\Omega\Big[\sum_{i=1}^2\Big(\frac{\partial\rho^\nu}{\partial x_i}\Big)^2\Big]v_i^2 d\mathbf{x} = \nu^2\int\limits_{\Omega_\delta}\rho^{2(\nu-1)}v_i^2 d\mathbf{x} \le \nu^2 C_4^2\delta^{2\nu}\|v_i\|^2_{L_{2,\nu}(\Omega)}.$$

Applying this to estimate the first term of the second factor on the right-hand side (43), we have the following.

$$\Big(\int\limits_\Omega s\frac{\partial(\rho^{2\nu}v_i)}{\partial x_i}d\mathbf{x}\Big)^2 \le 2\|s\|^2_{L_{2,\nu}(\Omega)}\cdot\Big(\nu^2C_4^2\delta^{2\nu}\|v_i\|^2_{L_{2,\nu}(\Omega)} + \|\nabla(\rho^\nu v_i)\|^2_{L_{2,0}(\Omega)}\Big). \tag{44}$$

Let us use inequality (15) of Lemma 3 for the last term of the second factor on the right-hand side (44); then, we estimate value $\Big[b_1(\mathbf{v},s)\Big]^2$ in Equation (42):

$$\Big[b_1(\mathbf{v},s)\Big]^2 \le 2\Big[\Big(\int\limits_\Omega s\frac{\partial(\rho^{2\nu}v_1)}{\partial x_1}d\mathbf{x}\Big)^2 + \Big(\int\limits_\Omega s\frac{\partial(\rho^{2\nu}v_2)}{\partial x_2}d\mathbf{x}\Big)^2\Big] \le$$

$$\le 4\|s\|^2_{L_{2,\nu}(\Omega)}\cdot\Big(\nu^2C_4^2\delta^{2\nu}\|\mathbf{v}\|^2_{\mathbf{L}_{2,\nu}(\Omega)} + 2|\mathbf{v}|^2_{\mathbf{W}^1_{2,\nu}(\Omega)} + 2\nu^2C_4^2\delta^{2\nu}\|\mathbf{v}\|^2_{\mathbf{L}_{2,\nu}(\Omega)}\Big) \le$$

$$\le 4\Big(2+3\nu^2C_4^2\delta^{2\nu}\Big)\|s\|^2_{L_{2,\nu}(\Omega)}\cdot\|\mathbf{v}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)},$$

such that the following is the case.

$$\Big[b_1(\mathbf{v},s)\Big]^2 \le 4\Big(2+3\nu^2C_4^2\delta^{2\nu}\Big)\|s\|^2_{L_{2,\nu}(\Omega)}\cdot\|\mathbf{v}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)}.$$

It remains to take the square root of both parts of the last inequality. The estimate (30) is set.

(3) Consider arbitrary functions $q \in L^0_{2,\nu}(\Omega,\delta)$, $\mathbf{u} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)$ and the following.

$$\Big[b_2(\mathbf{u},q)\Big]^2 = \Big[\int\limits_{\Omega} \rho^{2\nu} q\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big) d\mathbf{x}\Big]^2. \tag{45}$$

Using the Cauchy–Schwarz inequality, we have the following.

$$\Big(\int\limits_{\Omega} \Big(\rho^{\nu} q\Big)\Big(\rho^{\nu}\Big(\frac{\partial u_i}{\partial x_i}\Big)\Big) d\mathbf{x}\Big)^2 \le \Big(\int\limits_{\Omega} \rho^{2\nu} q^2 d\mathbf{x}\Big) \cdot \Big(\int\limits_{\Omega} \rho^{2\nu}\Big(\frac{\partial u_i}{\partial x_i}\Big)^2 d\mathbf{x}\Big) \le$$

$$\le \|q\|^2_{L_{2,\nu}(\Omega)} \cdot \Big(\int\limits_{\Omega} \rho^{2\nu}\Big[\Big(\frac{\partial u_i}{\partial x_1}\Big)^2 + \Big(\frac{\partial u_i}{\partial x_2}\Big)^2\Big] d\mathbf{x}\Big) \le \|q\|^2_{L_{2,\nu}(\Omega)} \cdot \|u_i\|^2_{W^1_{2,\nu}(\Omega)},$$

Applying this, to evaluate the right-hand side (45), we conclude the following:

$$\Big[b_2(\mathbf{u},q)\Big]^2 \le 2\sum_{i=1}^{2}\Big[\int\limits_{\Omega} \rho^{2\nu} q\Big(\frac{\partial u_i}{\partial x_i}\Big) d\mathbf{x}\Big]^2 \le 2\|q\|^2_{L_{2,\nu}(\Omega)} \cdot \|\mathbf{u}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)},$$

then the following is obtained.

$$\Big[b_2(\mathbf{u},q)\Big]^2 \le 2\|q\|^2_{L_{2,\nu}(\Omega)} \cdot \|\mathbf{u}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)}.$$

It remains to take the square root of both parts of the last estimate. Inequality (31) is established.

(4) Consider arbitrary functions $\mathbf{v} \in \mathbf{W}^1_{2,\nu}(\Omega,\delta)$, $\mathbf{f} \in \mathbf{L}_{2,\gamma}(\Omega,\delta)$, $\nu \ge \gamma \ge 0$ and the following.

$$\Big[l(\mathbf{v})\Big]^2 = \Big[\sum_{i=1}^{2}\int\limits_{\Omega} \rho^{2\nu} f_i v_i d\mathbf{x}\Big]^2 \le 2\sum_{i=1}^{2}\Big[\int\limits_{\Omega} \rho^{2\nu} f_i v_i d\mathbf{x}\Big]^2. \tag{46}$$

For $i = 1,2$, we have the following:

$$\Big[\int\limits_{\Omega} \rho^{2\nu} f_i v_i d\mathbf{x}\Big]^2 \le \|f_i\|^2_{L_{2,\nu}(\Omega)} \cdot \|v_i\|^2_{L_{2,\nu}(\Omega)} \le C^2_{13}\|f_i\|^2_{L_{2,\gamma}(\Omega)} \cdot \|v_i\|^2_{W^1_{2,\nu}(\Omega)},$$

then

$$\Big[l(\mathbf{v})\Big]^2 \le 2C^2_{13}\sum_{i=1}^{2}\Big[\|f_i\|^2_{L_{2,\gamma}(\Omega)} \cdot \|v_i\|^2_{W^1_{2,\nu}(\Omega)}\Big] \le$$

$$\le 2C^2_{13}\Big[\sum_{i=1}^{2}\|f_i\|^2_{L_{2,\gamma}(\Omega)}\Big] \cdot \Big[\sum_{i=1}^{2}\|v_i\|^2_{W^1_{2,\nu}(\Omega)}\Big] = 2C^2_{13}\|\mathbf{f}\|^2_{\mathbf{L}_{2,\gamma}(\Omega)} \cdot \|\mathbf{v}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)},$$

such that the following is obtained.

$$\Big[l(\mathbf{v})\Big]^2 \le 2C^2_{13}\|\mathbf{f}\|^2_{\mathbf{L}_{2,\gamma}(\Omega)} \cdot \|\mathbf{v}\|^2_{\mathbf{W}^1_{2,\nu}(\Omega)}.$$

It remains to take the square root of both parts of the last inequality. The estimate (32) is set.

Theorem 7 is proved. □

3.3. Existence and Uniqueness Theorem for R_ν-Generalized Solution

Let us prove a theorem on the existence and uniqueness of an R_ν-generalized solution $(\mathbf{w}_\nu, p_\nu) \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta) \times L^0_{2,\nu}(\Omega,\delta)$ of the Stokes problem (3)–(5) in a nonsymmetric variational formulation (12).

Theorem 8. *Let the following conditions be satisfied for $\nu > 0$:*

(1) The bilinear forms $a(\mathbf{u},\mathbf{v}), b_1(\mathbf{v},s), b_2(\mathbf{u},q)$ and the linear form $l(\mathbf{v})$ are continuous for arbitrary functions $\mathbf{u},\mathbf{v} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)$, $s,q \in L^0_{2,\nu}(\Omega,\delta)$ and $\mathbf{f} \in \mathbf{L}_{2,\gamma}(\Omega,\delta), \nu \geq \gamma \geq 0$, and we have inequalities (29)–(32);

(2) There exists $\delta_6 = \min\{\delta_4,\delta_5\} > 0$, such that for any $\delta \in (0,\delta_6)$, the estimates (21)–(24) of the following:

$$\forall s \in L^0_{2,\nu}(\Omega,\delta): \sup_{\mathbf{v}\in\mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)} \frac{b_1(\mathbf{v},s)}{\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|s\|_{L_{2,\nu}(\Omega)}} \geq \beta_1 > 0,$$

$$\forall q \in L^0_{2,\nu}(\Omega,\delta): \sup_{\mathbf{u}\in\mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)} \frac{b_2(\mathbf{u},q)}{\|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|q\|_{L_{2,\nu}(\Omega)}} \geq \beta_2 > 0$$

$$\forall \mathbf{u} \in K_2: \quad \sup_{\mathbf{v}\in K_1} \frac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}} \geq \alpha\|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)},$$

$$\forall \mathbf{v} \in K_1: \quad \sup_{\mathbf{u}\in K_2} \frac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}} \geq \alpha\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}$$

hold; then, for an arbitrary function $\mathbf{f} \in \mathbf{L}_{2,\gamma}(\Omega,\delta), \gamma \leq \nu$, there is a unique R_ν-generalized solution $(\mathbf{w}_\nu, p_\nu) \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta) \times L^0_{2,\nu}(\Omega,\delta)$ of the Stokes problem (3)–(5) in statement (12). There is a positive constant $C_{14} = \frac{C_{12}}{\alpha}\left(1 + \frac{\alpha + C_9}{\beta_1}\right)$ that satisfies the following inequality.

$$\|\mathbf{w}_\nu\|_{\mathbf{W}^1_{2,\nu}(\Omega)} + \|p_\nu\|_{L_{2,\nu}(\Omega)} \leq C_{14}\|\mathbf{f}\|_{\mathbf{L}_{2,\gamma}(\Omega)}. \tag{47}$$

Proof. Due to the fulfillment of inequalities (21)–(24), we can apply Theorem 2.1 in combination with Remark 2.1 (see [24]), which was proved for an abstract mixed problem in a nonsymmetric formulation, to our variational formulation (12). Then, there is a unique R_ν-generalized solution $(\mathbf{w}_\nu, p_\nu) \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta) \times L^0_{2,\nu}(\Omega,\delta)$.

Let us prove estimate (47).

1. Consider an arbitrary function $\mathbf{v} \in K_1$ and then $b_1(\mathbf{v}, p_\nu) = 0$ and $a(\mathbf{w}_\nu,\mathbf{v}) = l(\mathbf{v})$. Using sequentially estimates (23) ($\mathbf{w}_\nu \in K_2$) and (32), we conclude the following:

$$\alpha\|\mathbf{w}_\nu\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq \sup_{\mathbf{v}\in K_1} \frac{a(\mathbf{w}_\nu,\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}} = \sup_{\mathbf{v}\in K_1} \frac{l(\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}} \leq$$

$$\leq \sup_{\mathbf{v}\in K_1} \frac{C_{12}\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \cdot \|\mathbf{f}\|_{\mathbf{L}_{2,\gamma}(\Omega)}}{\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}} = C_{12}\|\mathbf{f}\|_{\mathbf{L}_{2,\gamma}(\Omega)},$$

such that the following is the case.

$$\|\mathbf{w}_\nu\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq \frac{C_{12}}{\alpha}\|\mathbf{f}\|_{\mathbf{L}_{2,\gamma}(\Omega)}. \tag{48}$$

2. Consider an arbitrary function $\mathbf{v} \in \mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)$. Using sequentially the estimates (21), (29), (32) and (48), we derive the following:

$$\beta_1\|p_\nu\|_{L_{2,\nu}(\Omega)} \leq \sup_{\mathbf{v}\in\mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)} \frac{b_1(\mathbf{v},p_\nu)}{\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}} = \sup_{\mathbf{v}\in\mathbf{W}^{1,0}_{2,\nu}(\Omega,\delta)} \frac{l(\mathbf{v}) - a(\mathbf{w}_\nu,\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}} \leq$$

$$\leq C_{12}\|\mathbf{f}\|_{\mathbf{L}_{2,\gamma}(\Omega)} + C_9\|\mathbf{w}_\nu\|_{\mathbf{W}^1_{2,\nu}(\Omega)} \leq C_{12}\left(1 + \frac{C_9}{\alpha}\right)\|\mathbf{f}\|_{\mathbf{L}_{2,\gamma}(\Omega)},$$

then we have the following.

$$\|p_\nu\|_{L_{2,\nu}(\Omega)} \le \frac{C_{12}}{\beta_1}\left(1+\frac{C_9}{\alpha}\right)\|\mathbf{f}\|_{\mathbf{L}_{2,\gamma}(\Omega)}. \tag{49}$$

3. Summarize the right-hand and left-hand parts of inequalities (48) and (49), respectively; we have

$$\|\mathbf{w}_\nu\|_{\mathbf{W}^1_{2,\nu}(\Omega)} + \|p_\nu\|_{L_{2,\nu}(\Omega)} \le \frac{C_{12}}{\alpha}\left(1+\frac{\alpha + C_9}{\beta_1}\right)\|\mathbf{f}\|_{\mathbf{L}_{2,\gamma}(\Omega)}.$$

Theorem 8 is proved. □

4. Conclusions

In the presented paper, the concept of an R_ν-generalized solution of the Stokes problem with a homogeneous Dirichlet boundary condition in a polygonal domain with one re-entrant corner on its boundary is introduced. The variational formulation of the problem is nonsymmetric. The existence and uniqueness of the R_ν-generalized solution of the Stokes problem in weighted sets is established. An estimate of its norm in terms of the norm of the right-hand side function is obtained.

The results of the paper can be extended to the case of a polygonal domain with several re-entrant corners on its boundary. Apply the approach to solve other fluid dynamics problems with corner singularity. In particular, for linearized Navier–Stokes equations in rotation and convective forms, see [29,30]. To perform this, we need to prove estimates (23) and (24) of Theorem 6 and (29) of Theorem 7 that correspond to these problems. Consider cases of other boundary conditions that are different from the homogeneous Dirichlet condition, which is a no-slip behavior of the fluid on the fixed walls (for example, Neumann, Robin and periodic or mixed boundary conditions, including inhomogeneous ones). An established fact of the existence and uniqueness of an R_ν-generalized solution in weighted sets is that, firstly, it will make it possible to create efficient numerical methods without a loss of accuracy for hydrodynamic problems with a corner singularity. Secondly, it will help us in obtaining a priori estimates for the convergence rate of an approximate R_ν-generalized solution to an exact one in the norms of weighted spaces. Thirdly, it will promote the determination of the optimal sets of parameters ν and δ to achieve the required order of accuracy, as well as finding the body of optimal parameters depending on the value of the re-entrant corner. In particular, this was established for the Lame system in a domain with a re-entrant corner equal to 2π, which is a mathematical model for the crack problem (see [14,20]).

Author Contributions: V.A.R. and A.V.R. contributed equally and significantly in writing this article. All authors have read and agreed to the published version of the manuscript.

Funding: The reported study was supported by Russian Science Foundation, project No. 21-11-00039, https://rscf.ru/en/project/21-11-00039/ (accessed on 16 May 2022).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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