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A Mathematical Model for Nonlinear Optimization Which Attempts Membership Functions to Address the Uncertainties

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Abstract: The problem of optimizing an objective function that exists within the constraints of equality and inequality is addressed by nonlinear programming (NLP). A linear program exists if all of the functions are linear; otherwise, the problem is referred to as a nonlinear program. The development of highly efficient and robust linear programming (LP) algorithms and software, the advent of high-speed computers, and practitioners' wider understanding and portability of mathematical modeling and analysis have all contributed to LP's importance in solving problems in a variety of fields. However, due to the nature of the nonlinearity of the objective functions and any of the constraints, several practical situations cannot be completely explained or predicted as a linear program. Efforts to overcome such nonlinear problems quickly and efficiently have made rapid progress in recent decades. The past century has seen rapid progress in the field of nonlinear modeling of real-world problems. Because of the uncertainty that exists in all aspects of nature and human life, these models must be viewed through a system known as a fuzzy system. In this article, a new fuzzy model is proposed to address the vagueness presented in the nonlinear programming problems (NLPPs). The proposed model is described; its mathematical formulation and detailed computational procedure are shown with numerical illustrations by employing trapezoidal fuzzy membership functions (TFMFs). Here, the computational procedure has an important role in acquiring the optimum result by utilizing the necessary and sufficient conditions of the Lagrangian multipliers method in terms of fuzziness. Additionally, the proposed model is based on the previous research in the literature, and the obtained optimal result is justified with TFMFs. A model performance evaluation was completed with different set of inputs, followed by a comparison analysis, results and discussion. Lastly, the performance evaluation states that the efficiency level of the proposed model is of high impact. The code to solve the model is implemented in LINGO, and it comes with a collection of built-in solvers for various problems.

Keywords: nonlinear optimization; fuzzy nonlinear programming problem; Lagrangian multiplier method in terms of fuzziness; fuzzy numbers; trapezoidal membership functions; ranking index

MSC: 90C05; 90C30; 90C70; 90C90

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1. Introduction

NLP typically describes rather more significant challenges than LP. The situation is perhaps always difficult if all of the constraints are linear and the objective function is nonlinear. For example, the feasible set may or may not be convex, and the optimum result could be placed within the feasible set, on its boundary, or at its vertex. For the most part, the scientific programming issue manages the ideal use or distribution of constrained assets to meet the ideal goal. The fuzzy NLP issue is valuable in taking care of issues due to the uncertain, emotional nature of the problematic definition, or due to its precise arrangement. In this case, an objective function must improve while working within certain constraints. Ref [1] introduced the theory of fuzzy and fuzzy decision-making, and the right decision used in decision problems to attain the optimum result [2]. In certain real-life situations, the decisions are ambiguous, and preliminary attempts at the choices are required to formulate a suitable model or cases to be analyzed. Similarly, we include fuzziness in our models of such situations in order to suggest approaches for dealing with fuzzy data [3]. An optimum solution in LP satisfies both the constraints and the objective function. Similarly, this problem has an objective function, and decision variables with fuzziness are presented in the constraints, including the coefficients in the form of fuzzy [3–5]. If the objective or limitations are nonlinear, at that point we think of it as an NLPP. In this model, to tackle such a nonlinear optimization problem, we begin with the necessary and sufficient conditions of the Lagrangian multipliers method with a fuzziness to obtain an optimum solution. However, this is carried out by employing TFMFs and their mathematical calculations. A fuzzy model is offered to the general NLPP, which helps to handle the vagueness and also justifies the obtained optimum solution within the description of MFs [5–7]. Furthermore, the proposed computational procedure was carried out in numerical illustration in two distinct ways: the first problem has been considered with fuzzy MFs, and the second one offered with a robust ranking index. Accordingly, the MF has been considered as a trapezoidal fuzzy number (TFN), which is linear and general, and preferably any kind of fuzzy number will handle the model effectively depending on its suitability. Finally, the evaluation of the optimal results for the mentioned two cases reveals the newness and cost-effectiveness of a fuzzy model, addressing the ambiguity and providing significantly more optimum values.

2. Literature Survey

This section highlights certain identified research work collections of existing fuzzy NLP, as shown below:

Tang and Wang [8] have suggested a nonsymmetric methodology for NLP with penalty coefficients in the form of fuzzy. This work also attempted to establish a structured model to solve the problem, and to classify the existing resources and limitations in the form of fuzzy, which is a type of nonlinear MF. Tang et al. [9] implemented an approach to the genetic algorithm for a penalty function and gradient search, and outlined a different hybrid genetic algorithmic approach to NLP. Fung et al. [10] extended the hybrid genetic algorithm and demonstrated essential strategies to apply to NLPPs involving all types of constraints. Sarimveis and Nikolakopoulos [11] offered an approach for constrained penalty weight optimization problems based on the LUDE algorithm. Syau and Lee [12] discussed the difficulties in fuzzy convex optimization as well as illustrations of multiobjective programming. Chen [13] preferred Yager's ranking index method to develop an optimization approach for constructing cost-based queueing decision problems by MFs with fuzzy numbers. Qin et al. [14] suggested an interval parameter NL model for managing stream water quality in a fuzzy situation. Fuzzy programming and interval procedures are combined in a common outline to report the fuzziness of nonlinear constraints on both sides. Kassem [15] devised a method for assessing the stability of optimal results to NL multiobjective optimization problems. Ravi Sankar et al. [16] suggested a method for optimizing an NL objective function that employs a genetic algorithm with coefficients and constraints that are under fuzziness. Abd-El-Wahed et al. [17] addressed a hybrid model which combines two heuristic optimization techniques and standard algorithms. Jameel and Sadeghi [18] discussed fuzzy NLPPs with some suitable numerical problems and evaluated them by comparing the crisp problem, attempting to show a more precise outcome.

Fuzzy programming techniques are likely to have a broader range of applications for nonlinear optimization and also stochastic optimization, specifically for allocation problems in supply chain management. A genetic algorithm technique has been used to illustrate the nonlinear transportation problem as an improved version of their previous findings for linear transportation problems, which obtained feasibility due to chromosome structures and genetic operators [19]. An innovative application for nonlinear network flow problems has been presented, which is strong enough to handle mixed-integer nonlinear optimization problems that incorporate a nonlinear transportation problem with the best solution [20].

An optimization algorithm for identifying the appropriate global solution for the problems with random noisy variables is described. Further MATLAB execution has been completed for nonlinear and stochastic problems with various test functions [21]. A suitable model has been developed, and the choice of the most appropriate optimization algorithms is essential in achieving cost-effective solutions to the nonlinear discrete transportation problem on mixed integer programming for both linear and nonlinear [22]. A rough interval approach was proposed for the solid transportation problem to maximize profit, with illustrative examples using LINGO with compromised solutions in various real-time scenarios [23]. A study has been proposed for a multiobjective allocation problem with a conveyance that follows the Weibull distribution using the stochastic approach with uncertainties. Furthermore, the study improves the efficiency of the computational results by demonstrating various optimization algorithms with sensitivity analysis [24]. The scheduling problem has been construed as a stochastic optimization problem, specifically in nonlinear programming, and then modeled in a multiobjective optimization problem that offers suitable scheduling with various models [25]. Bi-level preferential operation problems have been suggested, as well as a methodology for estimating the reliability parameters by employing the nonlinear optimization with the Kuhn Tucker approach [26]. An approach has been recommended for multiobjective optimization, mainly nonlinear supply chains under uncertainties, to obtain the lowest transportation cost while controlling all other deterioration factors. The LINGO is also used to enhance some optimization algorithms and data handling [27]. The outcomes are compared to a proposed approach for the design of the lowest cost canal sections in Newton's method, which is applied to KKT conditions for the constrained into unconstrained NL optimization problems with standard algorithms [28]. The fuzzy-based Lagrangian method can be described as the digital information mechanism to support vector machines for readily accessible biomedical data interpretation [29].

In recent decades, it has been desirable to solve optimization models such as energy, control systems, risk management, product inventory [30], and manufacturing processes [31]. Such specific problems are hazy and imprecise, but they can be addressed using fuzzy logic [32]. Several studies have recommended various methods and techniques for dealing with linear issues such as quadratic programming, solid transportation, as well as many industrial applications under uncertainties; this begins the best approach to addressing these types of NLPP in a cost-effective manner [33,34].

3. Preliminaries

In this section, some essential primary concepts and backgrounds are outlined in fuzzy mathematics [5,6]. Now it seems to address a few definitions which are most required:

Definition 1 [6]. *Let $T = [t_1, t_2, t_3, t_4]$ be a trapezoidal fuzzy number with the following MF,*

$$\mu_T(x) = \begin{cases} \frac{x-t_1}{t_2-t_1}, & t_1 \leq x \leq t_2 \\ 1, & t_2 \leq x \leq t_3 \\ \frac{x-t_4}{t_3-t_4}, & t_3 \leq x \leq t_4 \end{cases}$$

The MF $\mu_T(x)$ is illustrated in the Figure 1 below.

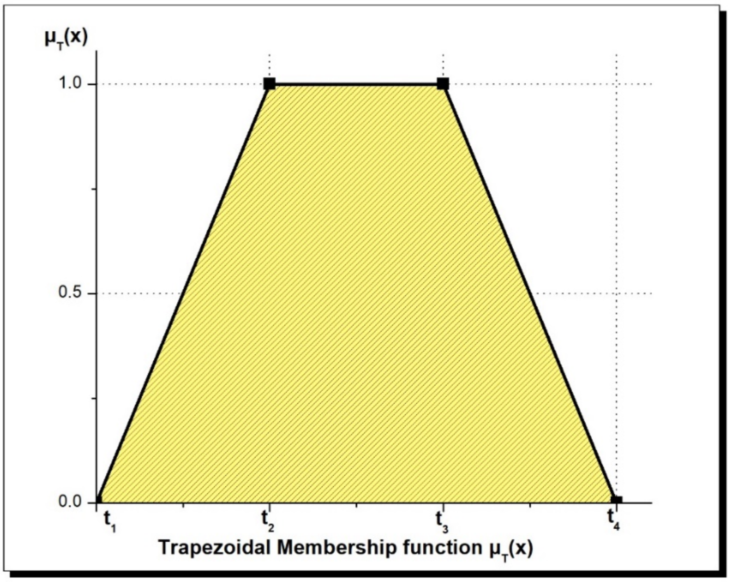

Figure 1. Trapezoidal Membership function $\mu_T(x)$.

Definition 2 [6] (***α cut***)**.** *Let a fuzzy set T in X and any real number α in [0, 1], then the α -cut of T, denoted by* $^{\alpha}T$ *is the crisp set* $^{\alpha}T = \{x \in X : \mu_T(x) \geq \alpha\}$. *For illustration, let T be a fuzzy set whose membership function is given as above* $\mu_T(x)$. *To find the α-cut of T, where* $\alpha \in [0,1]$, *let us set the reference functions of T to each left and right.*

$$i.e., \alpha = \frac{x^{(1)} - t_1}{t_2 - t_1} \ \& \ \alpha = \frac{x^{(2)} - t_4}{t_3 - t_4}$$

Expressing x to α, where $x^{(1)} = (t_2 - t_1)\alpha + t_1$ *and* $x^{(2)} = t_4 + (t_3 - t_4)\alpha$ *which provides the* $\alpha-$*cut of T is* $^{\alpha}T = [x^{(1)},\ x^{(2)}] = [(t_2 - t_1)\alpha + t_1,\ t_4 + (t_3 - t_4)\alpha]$.

Definition 3 [6] (**Robust Ranking Index**). *The robust ranking index fulfills compensation, consistency, and additive properties, and produces outcomes that are controlled by human perception. If T is a fuzzy number, then the robust ranking index is measured as*

$$R(T) = \int_0^1 (0.5) * \left[T^L_\alpha, T^u_\alpha\right] d\alpha$$

where $\left[T^L_\alpha, T^u_\alpha\right] = [x^{(1)},\ x^{(2)}] = [(t_2 - t_1)\alpha + t_1,\ t_4 + (t_3 - t_4)\alpha]$ *is the α cut of the fuzzy number T. Here the robust ranking index R(T) offers the numerical significance of T.*

4. An Optimization Model for Fuzzy Nonlinear Programming

Research emphasis on fuzzy optimization issues in the area of NLP is mainly limited. However, little attention has focused on NLP, such as within quadratic programming, separable programming and search methods, and many others. However, apart from that, there are several numerous forms of fuzzy NLP addressed extensively in various significant issues, mostly in complex industrial systems. Research emphasis on problems of fuzzy optimization in the field of NLP is generally limited. Furthermore, there is little interest in NLP to address the vagueness of the issues. Besides this, in many real issues, many kinds of fuzzy NLPs occur, mainly in complex manufacturing systems. This cannot be signified and enlightened by traditional models.

Meanwhile, scientific studies on modeling techniques and enhancing approaches for NLP in fuzzy situations are important not only from the framework of fuzzy optimization but also in the application to the challenges. As a result, the fuzzy NL optimization model has been proposed in three steps: model formulation, computational procedure, and numerical illustrations, followed by a detailed comparative analysis with results and discussions.

4.1. Formulation of the Fuzzy NLP with Equality Constraints

Fuzzy NLPP has been well-defined as the problem of finding a fuzzy vector $\left[\left(x_1^{(k)}\right),\left(x_2^{(k)}\right),\cdots,\left(x_n^{(k)}\right)\right]$, where $\left(x_j^{(k)}\right), j = 1,2,3,\cdots,n$ & for all $k = 1,2,3,4$ is a TFMF, which optimizes the objective function Z, which is a real-valued function defined by 'n' fuzzy variables [7]

$$\left[Z^{(k)}\right] = f\left(\left[\left(x_1^{(k)}\right),\left(x_2^{(k)}\right),\cdots,\left(x_n^{(k)}\right)\right]\right), \text{for all } k = 1,2,3,4. \quad (1)$$

Under the constraints,

$$g^i\left(\left[\left(x_1^{(k)}\right),\left(x_2^{(k)}\right),\cdots,\left(x_n^{(k)}\right)\right]\right)\{\leq,\geq,\text{or} =\}\left(b_i^{(k)}\right), \text{for all } k = 1,2,3,4 \text{ \& } i = 1,2,\cdots,m.$$

where g^i's are 'm' real-valued function of 'n' fuzzy variables and b_i's are 'm' fuzzy constants, and

$$\left(x_j^{(k)}\right) \geq 0, i = 1,2,\cdots,m,\ j = 1,2,3,\cdots,n\ ,\ m < n \text{ \& for all } k = 1,2,3,4.$$

Moreover, as stated before, it can be restated as

$$\text{Maximize } \left[Z^{(k)}\right] = f\left(\left[\left(x_1^{(k)}\right),\left(x_2^{(k)}\right),\cdots,\left(x_n^{(k)}\right)\right]\right), \text{ for all } k = 1,2,3,4.$$

Under the constraints

$$g^i\left(\left[\left(x_1^{(k)}\right),\left(x_2^{(k)}\right),\cdots,\left(x_n^{(k)}\right)\right]\right) = \left(b_i^{(k)}\right); \text{ for all } k = 1,2,3,4 \text{ \& } i = 1,2,\cdots,m. \quad (2)$$

$$\left(x_j^{(k)}\right) \geq 0, i = 1,2,\cdots,m,\ j = 1,2,3,\cdots,n\ ,\ m < n \text{ \& for all } k = 1,2,3,4. \quad (3)$$

The fuzzy vector that satisfies conditions (2) and (3) is a feasible solution to the fuzzy NLP.

$$\left[X^{(k)}\right] = \left[\left(x_1^{(k)}\right),\left(x_2^{(k)}\right),\cdots,\left(x_n^{(k)}\right)\right], \text{ for all } k = 1,2,3,4.$$

4.2. Computational Procedure

This section describes the model's computational procedure, which uses Lagrangian necessary and sufficient conditions to find the optimum solution to a fuzzified NLPP.

Let us present the real-valued functions h^i's of n fuzzy variables such that

$$h^i\left(\left[\left(x_1^{(k)}\right),\left(x_2^{(k)}\right),\cdots,\left(x_n^{(k)}\right)\right]\right) = g^i\left(\left[\left(x_1^{(k)}\right),\left(x_2^{(k)}\right),\cdots,\left(x_n^{(k)}\right)\right]\right) - \left(b_i^{(k)}\right),$$
$$for\ all\ k = 1,2,3,4 \text{ \& } i = 1,2,\cdots,m.$$

Then, it becomes:

$$\text{Maximize } \left[Z^{(k)}\right] = f\left(\left[\left(x_1^{(k)}\right),\left(x_2^{(k)}\right),\cdots,\left(x_n^{(k)}\right)\right]\right), \text{for all } k = 1,2,3,4.$$

Subject to the constraints

$$h^i\left(\left[\left(x_1^{(k)}\right),\left(x_2^{(k)}\right),\cdots,\left(x_n^{(k)}\right)\right]\right) = 0, \text{for all } k = 1,2,3,4 \text{ \& } i = 1,2,\cdots,m.$$
$$\left(x_j^{(k)}\right) \geq 0, \text{where } i = 1,2,3,\cdots,m, j = 1,2,3,\cdots,n\ , m < n\ \&$$
$$\text{for all } k = 1,2,3,4 \text{ and } x_j\text{'}s \text{ are real} - \text{valued '}n\text{' fuzzy variables.}$$

Consequently, it can be restated as

$$\text{Maximize } \left[Z^{(k)}\right] = f\left(\left[\left(x_1^{(k)}\right), \left(x_2^{(k)}\right), \cdots, \left(x_n^{(k)}\right)\right]\right), \text{for all } k = 1,2,3,4.$$
$$h^i\left(\left[\left(x_1^{(k)}\right), \left(x_2^{(k)}\right), \cdots, \left(x_n^{(k)}\right)\right]\right) = 0, \text{for all } k = 1,2,3,4 \ \& \ i = 1,2,\cdots,m.$$
$$\left(x_j^{(k)}\right) \geq 0, \text{where } i = 1,2,3,\cdots,m,\ j = 1,2,3,\cdots,n \ \text{ \& for all } k = 1,2,3,4$$

Address the problem and represent the Lagrangian function defined to evaluate the stationary points using the Lagrangian multiplier method (LMM).

$$[L(X,\lambda)] = f(x_j) - \sum_{i=1}^{m} \lambda_i \left[h^i(x_j)\right],$$

where $i = 1,2,3,\cdots,m,\ j = 1,2,3,\cdots,n$ & $\lambda_i\prime s$ are 'm' Lagrange multiplier.

The LMM necessarily requires the following conditions:

$$\begin{aligned} \frac{\partial L}{\partial x_j} &\equiv \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^{(i)}}{\partial x_j} = 0 \\ \frac{\partial L}{\partial \lambda_i} &\equiv -\left[h^i(x_j)\right] = 0 \end{aligned}$$

for $i = 1,2,3,\cdots,m,\ j = 1,2,3,\cdots,n$.

The sufficiency conditions for the LMM can be stated as

$$H^B = \begin{vmatrix} 0 & M \\ M^T & N \end{vmatrix}_{(m+n)\times(m+n)}$$

where m & n denotes the number of constraints and the number of variables respectively, the matrix above is known as the Bordered Hessian matrix.

$$M = \begin{vmatrix} \frac{\partial h^{(i)}}{\partial x_j} & \frac{\partial h^{(i)}}{\partial x_{j+1}} & \cdots & \frac{\partial h^{(i)}}{\partial x_{j+n}} \\ \frac{\partial h^{(i+1)}}{\partial x_j} & \frac{\partial h^{(i+1)}}{\partial x_{j+1}} & \cdots & \frac{\partial h^{(i+1)}}{\partial x_{j+n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h^{(i+m)}}{\partial x_j} & \frac{\partial h^{(i+m)}}{\partial x_{j+1}} & \cdots & \frac{\partial h^{(i+m)}}{\partial x_{j+n}} \end{vmatrix}$$
$$N = \begin{vmatrix} \frac{\partial^2 L}{\partial x_j^2} & \frac{\partial^2 L}{\partial x_j \partial x_{j+1}} & \cdots & \frac{\partial^2 L}{\partial x_j \partial x_{j+n}} \\ \frac{\partial^2 L}{\partial x_{j+1} \partial x_j} & \frac{\partial^2 L}{\partial x_{j+1}^2} & \cdots & \frac{\partial^2 L}{\partial x_{j+1} \partial x_{j+n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 L}{\partial x_{j+n} \partial x_j} & \frac{\partial^2 L}{\partial x_{j+n} \partial x_{j+1}} & \cdots & \frac{\partial^2 L}{\partial x_{j+n}^2} \end{vmatrix}$$

Remark

In general, the Lagrangian function $L(x,\lambda)$ gives the extreme point (x_0,λ_0) which can be obtained at (x_0,λ_0) then the extreme point x_0 follows the following:

1. A maximum point if starting with the principal major determinant of order $(2m+1)$, then the last $(n-m)$ principal minor determinants of H^B form an alternating sign pattern starting with $(-1)^{m+1}$.
2. A minimum point if starting with the principal minor determinant of order $(2m+1)$, then the last $(n-m)$ principal minor determinants of H^B having the sign of $(-1)^m$.

The necessary conditions for the LMM in fuzzified form may be expressed as

$$[L(X,\lambda)] = f\left(x_j{}^{(k)}\right) - \sum_{i=1}^{m} \lambda_i{}^{(k)}\left[h^i(x_j)^{(k)}\right] = 0,$$
$$\left[h^i(x_j)^{(k)}\right] = 0$$

In a fuzzified manner, the sufficiency conditions for the LMM are as follows:

$$N = f\left(x_j{}^{(k)}\right) - \sum_{i=1}^{m} \lambda_i{}^{(k)}\left[h^i(x_j)^{(k)}\right] = 0,$$
$$M = \left[h^i(x_j)^{(k)}\right] = 0$$

where $i = 1, 2, 3, \cdots, m,\ j = 1, 2, 3, \cdots, n, k = 1, 2, 3, 4$ & $\lambda_i{\prime}s$ are 'm' Lagrange multiplier.

Next, we determine the fuzzy MF of the above conditions.

Let $A = \left(f_j{}^{(k)}\right),\ B = \left(\lambda_i{}^{(k)}\right),\ C = \left(h^i{}_j{}^{(k)}\right), k = 1, 2, 3, 4$

$$\mu_A(x) = \begin{cases} \frac{x - f_j^{(1)}}{f_j^{(2)} - f_j^{(1)}} & \text{for } f_j^{(1)} \le x \le f_j^{(2)} \\ 1 & \text{for } f_j^{(2)} \le x \le f_j^{(3)} \\ \frac{x - f_j^{(4)}}{f_j^{(3)} - f_j^{(4)}} & \text{for } f_j^{(3)} \le x \le f_j^{(4)} \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_B(x) = \begin{cases} \frac{x - \lambda_i^{(1)}}{\lambda_i^{(2)} - \lambda_i^{(1)}} & \text{for } \lambda_i^{(1)} \le x \le \lambda_i^{(2)} \\ 1 & \text{for } \lambda_i^{(2)} \le x \le \lambda_i^{(3)} \\ \frac{x - \lambda_i^{(4)}}{\lambda_i^{(3)} - \lambda_i^{(4)}} & \text{for } \lambda_i^{(3)} \le x \le \lambda_i^{(4)} \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_C(x) = \begin{cases} \frac{x - h_j^{i(1)}}{h_j^{i(2)} - h_j^{i(1)}} & \text{for } h_j^{i(1)} \le x \le h_j^{i(2)} \\ 1 & \text{for } h_j^{i(2)} \le x \le h_j^{i(3)} \\ \frac{x - h_j^{i(4)}}{h_j^{i(3)} - h_j^{i(4)}} & \text{for } h_j^{i(3)} \le x \le h_j^{i(4)} \\ 0 & \text{otherwise} \end{cases}$$

To estimate $B * C{\prime}s$ fuzzy MF, start with the computations of α the level confidence interval for B and C MFs and then proceed to $B * C{\prime}s$ MFs computation. Substitute $a_1^{(\alpha)}$ and $a_2^{(\alpha)}$ in the place of x for the membership function B (i.e., $\mu_B(x)$) in the first and second equations.

$$\alpha = \frac{x_1^{(\alpha)} - \lambda_i^{(1)}}{\lambda_i^{(2)} - \lambda_i^{(1)}} \Rightarrow x_1^{(\alpha)} = \left(\lambda_i^{(2)} - \lambda_i^{(1)}\right)\alpha + \lambda_i^{(1)};$$
$$\alpha = \frac{x_2^{(\alpha)} - \lambda_i^{(4)}}{\lambda_i^{(3)} - \lambda_i^{(4)}} \Rightarrow x_2^{(\alpha)} = \left(\lambda_i^{(3)} - \lambda_i^{(4)}\right)\alpha + \lambda_i^{(4)}$$

Therefore, B_α it is symbolized as the interval of confidence of B,

$$B_\alpha = \left[\left(\lambda_i^{(2)} - \lambda_i^{(1)}\right)\alpha + \lambda_i^{(1)},\ \left(\lambda_i^{(3)} - \lambda_i^{(4)}\right)\alpha + \lambda_i^{(4)}\right]$$

Similarly, the interval of confidence of C, is symbolized as C_α is

$$C_\alpha = \left[\left(h_j^{i(2)} - h_j^{i(1)}\right)\alpha + h_j^{i(1)},\ \left(h_j^{i(3)} - h_j^{i(4)}\right)\alpha + h_j^{i(4)}\right]$$

Thus $\{B_\alpha * C_\alpha\}$ is referred by $\{\text{Term I}, \text{Term II}\}$,where Term I & II are given below: Term I is

$$= (\lambda_i^{(2)} h_j^{i(2)} - \lambda_i^{(2)} h_j^{i(1)} - \lambda_i^{(1)} h_j^{i(2)} + \lambda_i^{(1)} h_j^{i(1)})\alpha^2 + (\lambda_i^{(2)} h_j^{i(1)} - \lambda_i^{(1)} h_j^{i(1)} + \lambda_i^{(1)} h_j^{i(2)} - \lambda_i^{(1)} h_j^{i(1)})\alpha + \lambda_i^{(1)} h_j^{i(1)}$$

and Term II is

$$= (\lambda_i^{(3)} h_j^{i(3)} - \lambda_i^{(3)} h_j^{i(4)} - \lambda_i^{(4)} h_j^{i(3)} + \lambda_i^{(4)} h_j^{i(4)})\alpha^2 + (\lambda_i^{(3)} h_j^{i(4)} - \lambda_i^{(4)} h_j^{i(4)} + \lambda_i^{(4)} h_j^{i(3)} - \lambda_i^{(4)} h_j^{i(4)})\alpha + \lambda_i^{(4)} h_j^{i(4)}$$

Let

$$\mathrm{X} = (\lambda_i^{(2)} h_j^{i(2)} - \lambda_i^{(2)} h_j^{i(1)} - \lambda_i^{(1)} h_j^{i(2)} + \lambda_i^{(1)} h_j^{i(1)})\alpha^2 + (\lambda_i^{(2)} h_j^{i(1)} - \lambda_i^{(1)} h_j^{i(1)} + \lambda_i^{(1)} h_j^{i(2)} - \lambda_i^{(1)} h_j^{i(1)})\alpha + \lambda_i^{(1)} h_j^{i(1)}$$

Solving for $\propto$ and after simplification results in

$$\alpha = \frac{-\left(\lambda_i^{(1)} h_j^{i(2)} + \lambda_i^{(2)} h_j^{i(1)} - 2\lambda_i^{(1)} h_j^{i(1)}\right) \pm \begin{pmatrix} \left(\lambda_i^{(1)} h_j^{i(2)} + \lambda_i^{(2)} h_j^{i(1)} - 2\lambda_i^{(1)} h_j^{i(1)}\right)^2 - 4\lambda_i^{(2)} h_j^{i(2)} \lambda_i^{(1)} h_j^{i(1)} \\ +4\lambda_i^{(2)} h_j^{i(1)} \lambda_i^{(1)} h_j^{i(1)} + 4\lambda_i^{(1)} h_j^{i(2)} \lambda_i^{(1)} h_j^{i(1)} - 4\lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(1)} h_j^{i(1)} \end{pmatrix}^{\frac{1}{2}}}{2\left(\lambda_i^{(2)} h_j^{i(2)} - \lambda_i^{(2)} h_j^{i(1)} - \lambda_i^{(1)} h_j^{i(2)} + \lambda_i^{(1)} h_j^{i(1)}\right)}$$

Let

$$\mathrm{X} = (\lambda_i^{(3)} h_j^{i(3)} - \lambda_i^{(3)} h_j^{i(4)} - \lambda_i^{(4)} h_j^{i(3)} + \lambda_i^{(4)} h_j^{i(4)})\alpha^2 + (\lambda_i^{(3)} h_j^{i(4)} - \lambda_i^{(4)} h_j^{i(4)} + \lambda_i^{(4)} h_j^{i(3)} - \lambda_i^{(4)} h_j^{i(4)})\alpha + \lambda_i^{(4)} h_j^{i(4)}$$

Solving for $\propto$ and after simplification results in

$$\alpha = \frac{-\left(\lambda_i^{(3)} h_j^{i(4)} + \lambda_i^{(4)} h_j^{i(3)} - 2\lambda_i^{(4)} h_j^{i(4)}\right) \pm \begin{pmatrix} \left(\lambda_i^{(3)} h_j^{i(4)} + \lambda_i^{(4)} h_j^{i(3)} - 2\lambda_i^{(4)} h_j^{i(4)}\right)^2 - 4\lambda_i^{(3)} h_j^{i(3)} \lambda_i^{(4)} h_j^{i(4)} \\ +4\lambda_i^{(3)} h_j^{i(4)} \lambda_i^{(4)} h_j^{i(4)} + 4\lambda_i^{(4)} h_j^{i(3)} \lambda_i^{(4)} h_j^{i(4)} - 4\lambda_i^{(4)} h_j^{i(4)} \lambda_i^{(4)} h_j^{i(4)} \end{pmatrix}^{\frac{1}{2}}}{2\left(\lambda_i^{(3)} h_j^{i(3)} - \lambda_i^{(3)} h_j^{i(4)} - \lambda_i^{(4)} h_j^{i(3)} + \lambda_i^{(4)} h_j^{i(4)}\right)}$$

Putting $\alpha = 0$ and $\alpha = 1$, then the domain of x and hence the MF of $\{B_\alpha * C_\alpha\}$ is as follows

$$\mu_{\lambda_i h_j}(x) = \begin{cases} \dfrac{-\left(\lambda_i^{(1)} h_j^{i(2)} + \lambda_i^{(2)} h_j^{i(1)} - 2\lambda_i^{(1)} h_j^{i(1)}\right) \pm \begin{pmatrix} \left(\lambda_i^{(1)} h_j^{i(2)} + \lambda_i^{(2)} h_j^{i(1)} - 2\lambda_i^{(1)} h_j^{i(1)}\right)^2 - 4\lambda_i^{(2)} h_j^{i(2)} \lambda_i^{(1)} h_j^{i(1)} \\ +4\lambda_i^{(2)} h_j^{i(1)} \lambda_i^{(1)} h_j^{i(1)} + 4\lambda_i^{(1)} h_j^{i(2)} \lambda_i^{(1)} h_j^{i(1)} - 4\lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(1)} h_j^{i(1)} \end{pmatrix}^{\frac{1}{2}}}{2\left(\lambda_i^{(2)} h_j^{i(2)} - \lambda_i^{(2)} h_j^{i(1)} - \lambda_i^{(1)} h_j^{i(2)} + \lambda_i^{(1)} h_j^{i(1)}\right)} \\ \dfrac{-\left(\lambda_i^{(3)} h_j^{i(4)} + \lambda_i^{(4)} h_j^{i(3)} - 2\lambda_i^{(4)} h_j^{i(4)}\right) \pm \begin{pmatrix} \left(\lambda_i^{(3)} h_j^{i(4)} + \lambda_i^{(4)} h_j^{i(3)} - 2\lambda_i^{(4)} h_j^{i(4)}\right)^2 - 4\lambda_i^{(3)} h_j^{i(3)} \lambda_i^{(4)} h_j^{i(4)} \\ +4\lambda_i^{(3)} h_j^{i(4)} \lambda_i^{(4)} h_j^{i(4)} + 4\lambda_i^{(4)} h_j^{i(3)} \lambda_i^{(4)} h_j^{i(4)} - 4\lambda_i^{(4)} h_j^{i(4)} \lambda_i^{(4)} h_j^{i(4)} \end{pmatrix}^{\frac{1}{2}}}{2\left(\lambda_i^{(3)} h_j^{i(3)} - \lambda_i^{(3)} h_j^{i(4)} - \lambda_i^{(4)} h_j^{i(3)} + \lambda_i^{(4)} h_j^{i(4)}\right)} \\ 0 \quad \text{otherwise} \end{cases}$$

Next evaluation of the FMF with the Lagrangian function of necessary condition, which is

$$\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i}{\partial x_j} = 0 \Rightarrow f' - \sum_{i=1}^{m} \lambda_i h_i' = 0$$

In a fuzzified manner, the necessary conditions for the LMM are as follows:

$$f\left(x_j{}^{(k)}\right) - \sum_{i=1}^{m} \lambda_i{}^{(k)} \left[h^i\left(x_j\right)^{(k)\prime}\right] = 0,$$

where $i = 1, 2, 3, \cdots, m,\ j = 1, 2, 3, \cdots, n\ , k = 1, 2, 3, 4$ & $\lambda_i\prime s$ are 'm' Lagrange multiplier.

Next, we determine the fuzzy MF of the above conditions.

$$\text{Let } A = \left(f_j{}^{(k)}\right),\ B = \left(\lambda_i{}^{(k)}\right),\ C = \left(h^i{}_j{}^{(k)}\right),$$

where i = 1,2,3,$\cdots$,m, j = 1,2,3,$\cdots$, n & k = 1,2,3,4

$$\mu_A(x) \ = \begin{cases} \frac{x-f_j^{(1)}}{f_j^{(2)}-f_j^{(1)}} & \text{for} \quad f_j^{(1)} \le x \le f_j^{(2)} \\ 1 & \text{for} \quad f_j^{(2)} \le x \le f_j^{(3)} \\ \frac{x-f_j^{(4)}}{f_j^{(3)}-f_j^{(4)}} & \text{for} \quad f_j^{(3)} \le x \le f_j^{(4)} \\ 0 & \text{otherwise} \end{cases}$$

Now compute the confidence interval for each degree α for the MF A. By replacing the α values in both the equations of the MF of A,

$$\begin{aligned} \alpha \ &= \ \frac{x_1^{(\alpha)}-f'^{(1)}}{f'^{(2)}-f'^{(1)}} \ \Rightarrow \ a_1^{(\alpha)} \ = \ \left(f'^{(2)} - f'^{(1)}\right)\alpha + f'^{(1)} \\ \alpha \ &= \ \frac{x_2^{(\alpha)}-f'^{(4)}}{f'^{(3)}-f'^{(4)}} \ \Rightarrow \ a_2^{(\alpha)} \ = \ \left(f'^{(3)} - f'^{(4)}\right)\alpha + f'^{(4)} \\ \therefore\ \{A\}_\alpha \ &= \ \left[\left\{\left(f'^{(2)} - f'^{(1)}\right)\alpha + f'^{(1)}\right\}, \left\{\left(f'^{(3)} - f'^{(4)}\right)\alpha + f'^{(4)}\right\}\right] \end{aligned}$$

As the MF of $B * C$ is denoted by $\mu_{B*C}(x)$ and has been calculated previously. Next, determine the α level intervals for the fuzzy MF $*C$, by substituting $b_1^\alpha\ and\ b_2^\alpha$ in both the equation of the MF of $B * C$. Then MF $B * C$ is as follows

$$\begin{gathered}
\{B(*)C\}_\alpha \ = \\
\Big[\Big\{2\alpha^2 \Big(\lambda_i^{(2)} h_j^{i(2)}\lambda_i^{(3)}h_i'^{(3)} - \lambda_i^{(2)}h_j^{i(2)}\lambda_i^{(3)}h_i'^{(4)} - \lambda_i^{(2)}h_j^{i(2)}\lambda_i^{(4)}h_i'^{(3)} + \lambda_i^{(2)}h_j^{i(2)}\lambda_i^{(4)}h_i'^{(4)} - \lambda_i^{(2)}h_j^{i(1)}\lambda_i^{(3)}h_i'^{(3)} \\
+\lambda_i^{(2)}h_j^{i(1)}\lambda_i^{(3)}h_i'^{(4)} + \lambda_i^{(2)}h_j^{i(1)}\lambda_i^{(4)}h_i'^{(3)} - \lambda_i^{(2)}h_j^{i(1)}\lambda_i^{(4)}h_i'^{(4)} - \lambda_i^{(1)}h_j^{i(2)}\lambda_i^{(3)}h_i'^{(3)} + \lambda_i^{(1)}h_j^{i(2)}\lambda_i^{(3)}h_i'^{(4)} + \\
\lambda_i^{(1)}h_j^{i(2)}\lambda_i^{(4)}h_i'^{(3)} - \lambda_i^{(1)}h_j^{i(2)}\lambda_i^{(4)}h_i'^{(4)} + \lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(3)}h_i'^{(3)} - \lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(3)}h_i'^{(4)} - \lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(4)}h_i'^{(3)} + \\
\lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(4)}\, h_i'^{(4)}\Big) + 2\alpha\Big(\lambda_i^{(2)}\, h_j^{i(1)}\lambda_i^{(3)}h_i'^{(4)} - \lambda_i^{(2)}h_j^{i(1)}\lambda_i^{(4)}h_i'^{(4)} + \lambda_i^{(2)}h_j^{i(1)}\lambda_i^{(4)}h_i'^{(3)} + \lambda_i^{(2)}h_j^{i(1)}\lambda_i^{(4)}h_i'^{(4)} \\
-\lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(3)}h_i'^{(4)} + \lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(4)}h_i'^{(4)} - \lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(4)}h_i'^{(3)} + \lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(4)}h_i'^{(4)} + \lambda_i^{(1)}h_j^{i(2)}\lambda_i^{(3)}h_i'^{(4)} - \\
\lambda_i^{(1)}h_j^{i(2)}\lambda_i^{(4)}h_i'^{(4)} + \lambda_i^{(1)}h_j^{i(2)}\lambda_i^{(4)}h_i'^{(3)} - \lambda_i^{(1)}h_j^{i(2)}\lambda_i^{(4)}h_i'^{(4)} - \lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(3)}h_i'^{(4)} + \lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(4)}h_i'^{(4)} - \\
\lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(4)}h_i'^{(3)} + \lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(4)}\, h_i'^{(4)}\Big) + \lambda_i^{(1)}h_j^{i(1)}\lambda_i^{(4)}\, h_i'^{(4)}\Big\},
\end{gathered}$$

$$\begin{gathered}
\Big\{2\alpha^2 \Big(\lambda_i^{(2)}\, h_i'^{(2)}\lambda_i^{(3)}h_j^{i(3)} - \lambda_i^{(2)}h_i'^{(2)}\lambda_i^{(3)}h_j^{i(4)} - \lambda_i^{(2)}h_i'^{(2)}\lambda_i^{(4)}h_j^{i(3)} + \lambda_i^{(2)}h_i'^{(2)}\lambda_i^{(4)}h_j^{i(4)} - \lambda_i^{(2)}h_i'^{(1)}\lambda_i^{(3)}h_j^{i(3)} \\
+\lambda_i^{(2)}h_i'^{(1)}\lambda_i^{(3)}h_j^{i(4)} + \lambda_i^{(2)}h_i'^{(1)}\lambda_i^{(4)}h_j^{i(3)} - \lambda_i^{(2)}h_i'^{(1)}\lambda_i^{(4)}h_j^{i(4)} - \lambda_i^{(1)}h_i'^{(2)}\lambda_i^{(3)}h_j^{i(3)} + \lambda_i^{(1)}h_i'^{(2)}\lambda_i^{(3)}h_j^{i(4)} + \\
\lambda_i^{(1)}h_i'^{(2)}\lambda_i^{(4)}h_j^{i(3)} - \lambda_i^{(1)}h_i'^{(2)}\lambda_i^{(4)}h_j^{i(4)} + \lambda_i^{(1)}h_i'^{(1)}\lambda_i^{(3)}h_j^{i(3)} - \lambda_i^{(1)}h_i'^{(1)}\lambda_i^{(3)}h_j^{i(4)} - \lambda_i^{(1)}h_i'^{(1)}\lambda_i^{(4)}h_j^{i(3)} + \\
\lambda_i^{(1)}h_i'^{(1)}\lambda_i^{(4)}\, h_j^{i(4)}\Big) + 2\alpha\Big(\lambda_i^{(2)}\, h_i'^{(1)}\lambda_i^{(3)}h_j^{i(4)} - \lambda_i^{(2)}h_i'^{(1)}\lambda_i^{(4)}h_j^{i(4)} + \lambda_i^{(2)}h_i'^{(1)}\lambda_i^{(4)}h_j^{i(3)} - \lambda_i^{(2)}h_i'^{(1)}\lambda_i^{(4)}h_j^{i(4)} \\
-\lambda_i^{(1)}h_i'^{(1)}\lambda_i^{(3)}h_j^{i(4)} + \lambda_i^{(1)}h_i'^{(1)}\lambda_i^{(4)}h_j^{i(4)} - \lambda_i^{(1)}h_i'^{(1)}\lambda_i^{(4)}h_j^{i(3)} + \lambda_i^{(1)}h_i'^{(1)}\lambda_i^{(4)}h_j^{i(4)} + \lambda_i^{(1)}h_i'^{(2)}\lambda_i^{(3)}h_j^{i(4)} - \\
\lambda_i^{(1)}h_i'^{(2)}\lambda_i^{(4)}h_j^{i(4)} + \lambda_i^{(1)}h_i'^{(2)}\lambda_i^{(4)}h_j^{i(3)} - \lambda_i^{(1)}h_i'^{(2)}\lambda_i^{(4)}h_j^{i(4)} - \lambda_i^{(1)}h_j'^{(1)}\lambda_i^{(3)}h_j^{i(4)} + \lambda_i^{(1)}h_j'^{(1)}\lambda_i^{(4)}h_j^{i(4)} - \\
\lambda_i^{(1)}h_j'^{(1)}\lambda_i^{(4)}h_j^{i(3)} + \lambda_i^{(1)}h_j'^{(1)}\lambda_i^{(4)}\, h_j^{i(4)}\Big) + \lambda_i^{(1)}h_i'^{(1)}\lambda_i^{(4)}\ h_j^{i(4)}\Big\}\Big]
\end{gathered}$$

Now, $\{A_\alpha\} - \{(B(*)C)_\alpha\} = \left\{a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}\right\} = \left\{\begin{array}{l} -2A_1\alpha^2 - \left\{2B_1 - \left(f'^{(2)} - f'^{(1)}\right)\alpha\right\} + f'^{(1)} - C_1, \\ -2A_2\alpha^2 - \left\{2B_2 - \left(f'^{(3)} - f'^{(4)}\right)\alpha\right\} + f'^{(4)} - C_2 \end{array}\right\}$

Solving for α and then putting $\alpha = 0$ and $\alpha = 1$ which follows the domain of x and hence the MF of $\{A_\alpha(-)B_\alpha(*)C_\alpha\}$ is as follows

$$\mu_{A(-)B(*)C}(x) = \begin{cases} \frac{-\left(2B_1 - \left(f'^{(2)} - f'^{(1)}\right)\right) \pm \left(\left\{2B_1 - \left(f'^{(2)} - f'^{(1)}\right)\right\}^2 - 8A_1\left(f'^{(1)} - C_1\right)\right)^{\frac{1}{2}}}{4A_1}, \\ \qquad \text{for } f'^{(1)} - \lambda_i^{(1)} h_i'^{(1)} \le x \le f'^{(2)} - \lambda_i^{(2)} h_i'^{(2)} \\ 1, \qquad \text{for } f'^{(2)} - \lambda_i^{(2)} h_i'^{(2)} \le x \le f'^{(3)} - \lambda_i^{(3)} h_i'^{(3)} \\ \frac{-\left(2B_2 - \left(f'^{(3)} - f'^{(4)}\right)\right) \pm \left(\left\{2B_2 - \left(f'^{(3)} - f'^{(4)}\right)\right\}^2 - 8A_2\left(f'^{(4)} - C_2\right)\right)^{\frac{1}{2}}}{4A_2}, \\ \qquad \text{for } f'^{(3)} - \lambda_i^{(3)} h_i'^{(3)} \le x \le f'^{(4)} - \lambda_i^{(4)} h_i'^{(4)} \\ 0, \quad \text{otherwise} \end{cases}$$

where,

$$A_1 = \lambda_i^{(2)} h_i'^{(2)} \lambda_i^{(3)} h_j^{i(3)} - \lambda_i^{(2)} h_i'^{(2)} \lambda_i^{(3)} h_j^{i(4)} - \lambda_i^{(2)} h_i'^{(2)} \lambda_i^{(4)} h_j^{i(3)} + \lambda_i^{(2)} h_i'^{(2)} \lambda_i^{(4)} h_j^{i(4)} - \lambda_i^{(2)} h_i'^{(1)} \lambda_i^{(3)} h_j^{i(3)} + \lambda_i^{(2)} h_i'^{(1)} \lambda_i^{(3)} h_j^{i(4)} + \lambda_i^{(2)} h_i'^{(1)} \lambda_i^{(4)} h_j^{i(3)} - \lambda_i^{(2)} h_i'^{(1)} \lambda_i^{(4)} h_j^{i(4)} - \lambda_i^{(1)} h_i'^{(2)} \lambda_i^{(3)} h_j^{i(3)} + \lambda_i^{(1)} h_i'^{(2)} \lambda_i^{(3)} h_j^{i(4)} + \lambda_i^{(1)} h_i'^{(2)} \lambda_i^{(4)} h_j^{i(3)} - \lambda_i^{(1)} h_i'^{(2)} \lambda_i^{(4)} h_j^{i(4)} + \lambda_i^{(1)} h_i'^{(1)} \lambda_i^{(3)} h_j^{i(3)} - \lambda_i^{(1)} h_i'^{(1)} \lambda_i^{(3)} h_j^{i(4)} - \lambda_i^{(1)} h_i'^{(1)} \lambda_i^{(4)} h_j^{i(3)} + \lambda_i^{(1)} h_i'^{(1)} \lambda_i^{(4)} h_j^{i(4)}$$

$$B_1 = \lambda_i^{(2)} h_i'^{(1)} \lambda_i^{(3)} h_j^{i(4)} - \lambda_i^{(2)} h_i'^{(1)} \lambda_i^{(4)} h_j^{i(4)} + \lambda_i^{(2)} h_i'^{(1)} \lambda_i^{(4)} h_j^{i(3)} - \lambda_i^{(2)} h_i'^{(1)} \lambda_i^{(4)} h_j^{i(4)} - \lambda_i^{(1)} h_i'^{(1)} \lambda_i^{(3)} h_j^{i(4)} + \lambda_i^{(1)} h_i'^{(1)} \lambda_i^{(4)} h_j^{i(4)} - \lambda_i^{(1)} h_i'^{(1)} \lambda_i^{(4)} h_j^{i(3)} + \lambda_i^{(1)} h_i'^{(1)} \lambda_i^{(4)} h_j^{i(4)} + \lambda_i^{(1)} h_i'^{(2)} \lambda_i^{(3)} h_j^{i(4)} - \lambda_i^{(1)} h_i'^{(2)} \lambda_i^{(4)} h_j^{i(4)} + \lambda_i^{(1)} h_i'^{(2)} \lambda_i^{(4)} h_j^{i(3)} - \lambda_i^{(1)} h_i'^{(2)} \lambda_i^{(4)} h_j^{i(4)} - \lambda_i^{(1)} h_j'^{(1)} \lambda_i^{(3)} h_j^{i(4)} + \lambda_i^{(1)} h_j'^{(1)} \lambda_i^{(4)} h_j^{i(4)} - \lambda_i^{(1)} h_j'^{(1)} \lambda_i^{(4)} h_j^{i(3)} + \lambda_i^{(1)} h_j'^{(1)} \lambda_i^{(4)} h_j^{i(4)}$$

$C_1 = \lambda_i^{(1)} h_i'^{(1)} \lambda_i^{(4)} h_j^{i(4)}$ and

$$A_2 = \lambda_i^{(2)} h_j^{i(2)} \lambda_i^{(3)} h_i'^{(3)} - \lambda_i^{(2)} h_j^{i(2)} \lambda_i^{(3)} h_i'^{(4)} - \lambda_i^{(2)} h_j^{i(2)} \lambda_i^{(4)} h_i'^{(3)} + \lambda_i^{(2)} h_j^{i(2)} \lambda_i^{(4)} h_i'^{(4)} - \lambda_i^{(2)} h_j^{i(1)} \lambda_i^{(3)} h_i'^{(3)} + \lambda_i^{(2)} h_j^{i(1)} \lambda_i^{(3)} h_i'^{(4)} + \lambda_i^{(2)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(3)} - \lambda_i^{(2)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(4)} - \lambda_i^{(1)} h_j^{i(2)} \lambda_i^{(3)} h_i'^{(3)} + \lambda_i^{(1)} h_j^{i(2)} \lambda_i^{(3)} h_i'^{(4)} + \lambda_i^{(1)} h_j^{i(2)} \lambda_i^{(4)} h_i'^{(3)} - \lambda_i^{(1)} h_j^{i(2)} \lambda_i^{(4)} h_i'^{(4)} + \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(3)} h_i'^{(3)} - \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(3)} h_i'^{(4)} - \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(3)} + \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(4)}$$

$$B_2 = \lambda_i^{(2)} h_j^{i(1)} \lambda_i^{(3)} h_i'^{(4)} - \lambda_i^{(2)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(4)} + \lambda_i^{(2)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(3)} + \lambda_i^{(2)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(4)} - \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(3)} h_i'^{(4)} + \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(4)} - \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(3)} + \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(4)} + \lambda_i^{(1)} h_j^{i(2)} \lambda_i^{(3)} h_i'^{(4)} - \lambda_i^{(1)} h_j^{i(2)} \lambda_i^{(4)} h_i'^{(4)} + \lambda_i^{(1)} h_j^{i(2)} \lambda_i^{(4)} h_i'^{(3)} - \lambda_i^{(1)} h_j^{i(2)} \lambda_i^{(4)} h_i'^{(4)} - \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(3)} h_i'^{(4)} + \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(4)} - \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(3)} + \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(4)}$$

$C_2 = \lambda_i^{(1)} h_j^{i(1)} \lambda_i^{(4)} h_i'^{(4)}$.

5. Numerical Illustration

This section outlines two illustrative examples that can be used to optimize the models for addressing the problem of fuzzy NLP using TFMF and its mathematical calculations [5–7,32]. In Case (i), the fuzzy model explains the procedure using the MF approach, and in Case (ii), the same problem was investigated using the robust ranking approach.

The NLP in the manner of fuzziness is as follows, and the fuzzified form of the considered NLPP can be stated as below:

Minimize

$$[-1,0,2,3]\left(x_1^{(k)}\right)^2+[-1,0,2,3]\left(x_2^{(k)}\right)^2+[-1,0,2,3]\left(x_3^{(k)}\right)^2, \text{for all } k=1,2,3,4.$$

Subject to the constraints,

$$[2,3,5,6]\left(x_1^{(k)}\right)+[-1,0,2,3]\left(x_2^{(k)}\right)^2+[0,1,3,4]\left(x_3^{(k)}\right)=[12,13,15,16], \text{for all } k=1,2,3,4.$$

$$\left(x_1^{(k)}\right),\left(x_2^{(k)}\right),\left(x_3^{(k)}\right)\geq 0, \text{for all } k=1,2,3,4.$$

5.1. Case (i): NLP with Fuzzy Membership Functions

The above NLP has been optimized with fuzziness by the necessary condition and sufficiency conditions of Lagrangian, as discussed earlier.

In fuzzified manner, the necessary conditions for the LMM for minimizing the above NLPP are as follows

$$[-2,0,4,6]\left(x_1^{(k)}\right)-[2,3,5,6]\left(\lambda^{(k)}\right)=0, \text{for all } k=1,2,3,4. \tag{4}$$

$$[-2,0,4,6]\left(x_2^{(k)}\right)-[-2,0,4,6]\left(x_2^{(k)}\right)\left(\lambda^{(k)}\right)=0, \text{for all } k=1,2,3,4. \tag{5}$$

$$[-2,0,4,6]\left(x_3^{(k)}\right)-[0,1,3,4]\left(\lambda^{(k)}\right)=0, \text{for all } k=1,2,3,4. \tag{6}$$

$$[2,3,5,6]\left(x_1^{(k)}\right)+[-1,0,2,3]\left(x_2^{(k)}\right)^2+[0,1,3,4]\left(x_3^{(k)}\right)-[12,13,14,15]=0, \text{for all } k=1,2,3,4. \tag{7}$$

The relevant computations for determining the FMFs for the first equation of the above Lagrangian necessary conditions are as follows:

$$\mu_{A(*)B}(x)=\begin{cases}\frac{-x_1+3\pm(x_1^2-2x_1+1)^{\frac{1}{2}}}{2} & \text{for } -2x_1\leq x\leq 0\\ 1 & \text{for } 0\leq x\leq 4x_1\\ \frac{x_1-5\pm(x_1^2-2x_1+1)^{\frac{1}{2}}}{2} & \text{for } 4x_1\leq x\leq 6x_1\\ 0 & \text{otherwise}\end{cases}$$

$$\mu_{C(*)D}(x)=\begin{cases}\frac{-\lambda\pm(\lambda^2-8\lambda+16)^{\frac{1}{2}}}{2} & \text{for } 2\lambda\leq x\leq 3\lambda\\ 1 & \text{for } 3\lambda\leq x\leq 5\lambda\\ \frac{\lambda+8\pm(\lambda^2-8\lambda+16)^{\frac{1}{2}}}{2} & \text{for } 5\lambda\leq x\leq 6\lambda\\ 0 & \text{otherwise}\end{cases}$$

Therefore,

$$\mu_{A(*)B-C(*)D}(x) = \begin{cases} \frac{-x_1+\lambda+3\pm(x_1^2+\lambda^2-2x_1-8\lambda+17)^{\frac{1}{2}}}{2} & \text{for } -2x_1-2\lambda \le x \le -3\lambda \\ 1 & \text{for } -3\lambda \le x \le 4x_1-5\lambda \\ \frac{x_1-\lambda-13\pm(x_1^2+\lambda^2-2x_1-8\lambda+17)^{\frac{1}{2}}}{2} & \text{for } 4x_1-5\lambda \le x \le 6x_1-6\lambda \\ 0 & \text{otherwise} \end{cases}$$

Similarly for the second, third, and fourth equations, we see

$$\mu_{A(*)B-C(*)D(*)E}(x) = \begin{cases} \frac{-x_2+3\pm(x_2^2-2x_2+4x_2\lambda+17)^{\frac{1}{2}}}{2} & \text{for } -2x_2+2x_2\lambda \le x \le 0 \\ 1 & \text{for } 0 \le x \le 4x_2-4x_2\lambda \\ \frac{x_2-5\pm(x_2^2-2x_2+4x_2\lambda+17)^{\frac{1}{2}}}{2} & \text{for } 4x_2-4x_2\lambda \le x \le 6x_2-6x_2\lambda \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{A(*)B-C(*)D}(x) = \begin{cases} \frac{-x_3+\lambda+1\pm(x_3^2+\lambda^2-2x_3-4\lambda+5)^{\frac{1}{2}}}{2} & \text{for } -2x_3 \le x \le -\lambda \\ 1 & \text{for } -\lambda \le x \le 4x_3-3\lambda \\ \frac{x_3-\lambda-11\pm(x_3^2+\lambda^2-2x_3-4\lambda+5)^{\frac{1}{2}}}{2} & \text{for } 4x_3-3\lambda \le x \le 6x_3-4\lambda \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{A(*)B+C(*)D(*)E+F(*)G-H}(x) = \begin{cases} \frac{-x_1-x_3-2x+26\pm(x_1^2+4x_2^2+x_3^2-8x_1-4x_3+36)^{\frac{1}{2}}}{2} & \\ & \text{for } 2x_1-x_2^2-12 \le x \le 3x_1+x_3-13 \\ 1 & \text{for } 3x_1+x_3-13 \le x \le 5x_1+2x_2^2+3x_3-15 \\ \frac{x_1+x_3+2x-18\pm(x_1^2+4x_2^2+x_3^2-8x_1-4x_3+36)^{\frac{1}{2}}}{2} & \\ & \text{for } 5x_1+2x_2^2+3x_3-15 \le x \le 6x_1+3x_2^2+4x_3-16 \\ 0 & \text{otherwise} \end{cases}$$

The fuzzified result of the four equations are as follows:

$$\begin{gathered} [-2,0,4,6]\left(x_1^{(k)}\right) - [2,3,5,6]\left(\lambda^{(k)}\right) = 0, \textit{for all } k = 1,2,3,4. \\ [-2x_1-2\lambda, -3\lambda, 4x_1-5\lambda, 6x_1-6\lambda] = 0 \end{gathered} \tag{8}$$

Similarly, the Equations (5) to (7) are as follows:

$$[-2x_2+2x_2\lambda, 0, 4x_2-4x_2\lambda, 6x_2-6x_2\lambda] = 0 \tag{9}$$

$$[-2x_3, -\lambda, 4x_3-3\lambda, 6x_3-4\lambda] = 0 \tag{10}$$

$$\left[2x_1-{x_2}^2-12, 3x_1+x_3-13, 5x_1+2{x_2}^2+3x_3-15, 6x_1+3{x_2}^2+4x_3-16\right] = 0 \tag{11}$$

Solving the above the equations results in the extreme points, they are;

$$\text{Extreme point } 1: (x_\circ, \lambda_\circ) = [(1, 1.5, 2.5, 3), (-2, 0, 4, 6), (0, 0.5, 1.5, 2), (-1, 0, 2, 3)] \tag{12a}$$

$$\text{Extreme point } 2: (x_\circ, \lambda_\circ) = [(1, 1.5, 2.5, 3), (-2, 0, 4, 6), (0, 0.5, 1.5, 2), (-1, 0, 2, 3)] \tag{12b}$$

$$\text{Extreme point } 3: (x_\circ, \lambda_\circ) = [(1.4, 2.1, 3.5, 4.2), (-2, -1, 1, 2), (0, 0.7, 2.1, 2.8), (0.8, 1.1, 1.7, 2)] \quad (12c)$$

By employing the sufficiency conditions to evaluate whether the extreme points are maximum or minimum. Hence, the sufficient conditions for the LMM for minimizing the above NLPP as

$$H^B = \begin{vmatrix} [-2,-1,1,2] & [2,3,5,6] & [0,1,3,4]x_2 & [0,1,3,4] \\ [2,3,5,6] & [0,1,3,4] & [-2,-1,1,2] & [-2,-1,1,2] \\ [0,1,3,4]x_2 & [-2,-1,1,2] & [0,1,3,4]-[0,1,3,4]\lambda & [-2,-1,1,2] \\ [0,1,3,4] & [-2,-1,1,2] & [-2,-1,1,2] & [0,1,3,4] \end{vmatrix}_{4\times 4}$$

Extreme point 1at H^B is

$$H^B{}_{at\,(i)} = \begin{vmatrix} [-2,-1,1,2] & [2,3,5,6] & [0,2,6,8] & [0,1,3,4] \\ [2,3,5,6] & [0,1,3,4] & [-2,-1,1,2] & [-2,-1,1,2] \\ [0,2,6,8] & [-2,-1,1,2] & [-4,-2,2,4] & [-2,-1,1,2] \\ [0,1,3,4] & [-2,-1,1,2] & [-2,-1,1,2] & [0,1,3,4] \end{vmatrix}_{4\times 4}$$

Therefore, the sufficient condition to follow the evaluation of the above matrices of sizes 3×3 *and* 4×4 *of* $|H^B|$. Here, the condition following the sign of the determinants is $(-1)^m = -1$.

$$\begin{vmatrix} [-2,-1,1,2] & [2,3,5,6] & [0,2,6,8] \\ [2,3,5,6] & [0,1,3,4] & [-2,-1,1,2] \\ [0,2,6,8] & [-2,-1,1,2] & [-4,-2,2,4] \\ [0,1,3,4] & [-2,-1,1,2] & [-2,-1,1,2] \end{vmatrix}_{3\times 3} = -32$$

$$\begin{vmatrix} [-2,-1,1,2] & [2,3,5,6] & [0,2,6,8] & [0,1,3,4] \\ [2,3,5,6] & [0,1,3,4] & [-2,-1,1,2] & [-2,-1,1,2] \\ [0,2,6,8] & [-2,-1,1,2] & [-4,-2,2,4] & [-2,-1,1,2] \\ [0,1,3,4] & [-2,-1,1,2] & [-2,-1,1,2] & [0,1,3,4] \end{vmatrix}_{4\times 4} = -64$$

Therefore, extreme point 1 is a minimum point.
Extreme point 2 at H^B is

$$H^B{}_{at\,(ii)} = \begin{vmatrix} [-2,-1,1,2] & [2,3,5,6] & [0,2,6,8] & [0,1,3,4] \\ [2,3,5,6] & [0,1,3,4] & [-2,-1,1,2] & [-2,-1,1,2] \\ -[0,2,6,8] & [-2,-1,1,2] & [-4,-2,2,4] & [-2,-1,1,2] \\ [0,1,3,4] & [-2,-1,1,2] & [-2,-1,1,2] & [0,1,3,4] \end{vmatrix}_{4\times 4}$$

Similarly, extreme point 2 is also a minimum point.
Extreme point 3 at H^B is

$$H^B{}_{at\,(iii)} = \begin{vmatrix} [-2,-1,1,2] & [2,3,5,6] & [-2,-1,1,2] & [0,1,3,4] \\ [2,3,5,6] & [0,1,3,4] & [-2,-1,1,2] & [-2,-1,1,2] \\ [-2,-1,1,2] & [-2,-1,1,2] & [-4,-2.4,0.8,2.4] & [-2,-1,1,2] \\ [0,1,3,4] & [-2,-1,1,2] & [-2,-1,1,2] & [0,1,3,4] \end{vmatrix}_{4\times 4}$$

Since the extreme point may not fulfill the sufficiency conditions of both maximum and minimum.

Therefore, extreme point 3 is neither minimum nor maximum.

The Pictorial representation of the fuzzy vectors and the Lagrangian multiplier of the fuzzy NLPP is demonstrated in the below Figure 2.

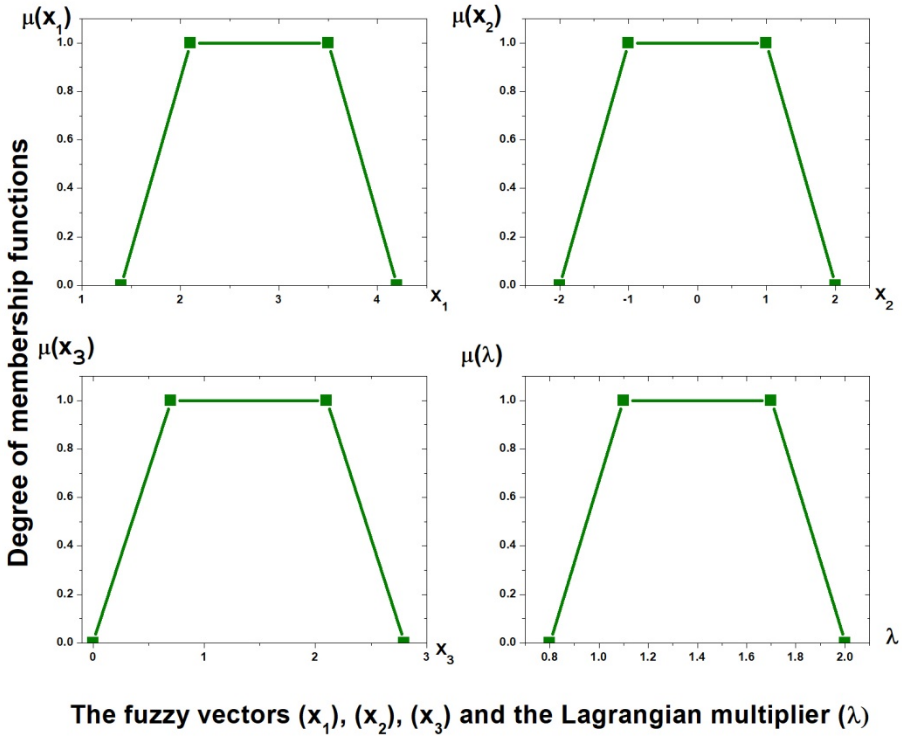

Figure 2. Pictorial representation of the fuzzy vectors and Lagrangian multiplier.

Hence the optimum result of the NLPP is

$$\left(x_1^{(k)}\right) = [1.4, 2.1, 3.5, 4.2],\ \left(x_2^{(k)}\right) = [-2, -1, 1, 2],\ \left(x_3^{(k)}\right) = [0, 0.7, 2.1, 2.8]\ ,$$
$$\left(\lambda^{(k)}\right) = [0.8, 1.1, 1.7, 2]\ \&\ \left(Z^{(k)}\right) = [-9.8, 0, 19.6, 29.4]\ ,\ for\ all\ k = 1, 2, 3, 4.$$

5.2. Case (ii): The Robust Ranking Approach for NLP with Fuzzy MFs

The NLP in the manner of fuzziness is as follows, and the fuzzified form of the considered NLPP can be stated as below:

Minimize

$$[-1, 0, 2, 3]\left(x_1^{(k)}\right)^2 + [-1, 0, 2, 3]\left(x_2^{(k)}\right)^2 + [-1, 0, 2, 3]\left(x_3^{(k)}\right)^2, for\ all\ k = 1, 2, 3, 4.$$

Subject to the constraints,

$$[2, 3, 5, 6]\left(x_1^{(k)}\right) + [-1, 0, 2, 3]\left(x_2^{(k)}\right)^2 + [0, 1, 3, 4]\left(x_3^{(k)}\right) = [12, 13, 15, 16], for\ all\ k = 1, 2, 3, 4.$$
$$\left(x_1^{(k)}\right),\ \left(x_2^{(k)}\right), \left(x_3^{(k)}\right) \geq 0, for\ all\ k = 1, 2, 3, 4.$$

Let's use a robust ranking approach to solve the above NLPP [6]. Further, the ranking index of $R[-1,0,2,3]$ and its fuzzy MF are as follows

$$\mu_{R[-1,0,2,3]}(x) = \begin{cases} x+1\,, -1 \le x \le 0 \\ 1\,,\ 0 \le x \le 2 \\ -x+3\,,\ 2 \le x \le 3 \\ 0\,,\ otherwise \end{cases}$$

The confidence interval for each degree α & the trapezoidal structures will be characterized by the functions of α.

$$\text{here } \alpha = x_1^{(\alpha)} + 1 \ \&\ \alpha = -x_2^{(\alpha)} + 3$$

Therefore,

$$[x^{(1)},\ x^{(2)}] = \left[T^L_{\alpha,}T^U_{\alpha,}\right] = [(t_2 - t_1)\alpha + t_1, t_4 + (t_3 - t_4)\alpha] = [\alpha - 1\,, -\alpha + 3]$$
$$R(T) = R[-1,0,2,3] = \int_0^1 (0.5) * \left[T^L_{\alpha,}T^U_{\alpha,}\right] d\alpha = \int_0^1 (0.5)(2)\, d\alpha = 1$$

Similarly, the ranking index R[2,3,5,6] is calculated as below:

$$\mu_{R[2,3,5,6]}(x)(=)\begin{cases} x-2\,,2 \le x \le 3 \\ 1\,,\ 3 \le x \le 5 \\ -x+6\,,\ 5 \le x \le 6 \\ 0\,,\ otherwise \end{cases}$$

The confidence interval for each degree α & the trapezoidal structures will be characterized by the functions of α.
Therefore, $[x^{(1)},\ x^{(2)}] = \left[T^L_{\alpha,}T^U_{\alpha,}\right] = [\alpha + 2, -\alpha + 6]$

$$R(T) = R[2,3,5,6] = \int_0^1 (0.5) * \left[T^L_{\alpha,}T^U_{\alpha,}\right] d\alpha = \int_0^1 (0.5)(8)\, d\alpha = 4$$

Likewise, the other ranking index has been calculated below

$$R(T) = R[0,1,3,4] = 2 \ \&\ R[12,13,15,16] = 14.$$

Using the proposed approach in the previous section, the fuzzy NLPP can be modified to the conventional crisp problem; the crisp problem is

$$Minimize\ Z = {x_1}^2 + {x_2}^2 + {x_3}^2$$

Subject to the constraints,

$$4x_1 + {x_2}^2 + 2x_1 = 14; x_1, x_2, x_3 \ge 0.$$

Now apply the existing conventional approach to the NLPP by using necessary and sufficient conditions of the LMM and obtained the optimum solution for the above is $x_1 = 2.8, x_2 = 0, x_3 = 1.4\ \lambda = 1.4$ & Minimum $Z = 9.8$.

5.3. Models Performance Evaluation with Different Sets of Inputs

This section is encapsulated to determine the efficiency of the fuzzy model and its solutions. For this efficiency test, we have considered four different sets of inputs in fuzzy format and then, using the ranking function provided in the earlier section, we have defuzzified all these inputs to obtain the equivalent crisp number. The fuzzy inputs are available in Table 1. With the defuzzified value, we have solved the model for each set using LINGO software and we have obtained the optimal solution for the NLPP. The results are given in Table 1 and here it can be easily observed that for any arbitrary set of trapezoidal fuzzy inputs, the model is solvable and gives the optimal solution. this demonstrates that the efficiency level of the proposed model is of high impact. The code to solve the model is implemented in LINGO, and it comes with a collection of built-in solvers for various problems. The modeling environment is strictly aligned to the LINGO solver and because of this interconnectivity, it transmits problems directly to memory which results in the minimization of compatibility issues between the solver and modeling components. It uses multiple CPU cores for model simulation, thus giving faster results.

Table 1. Optimum results comparison for a different set of fuzzy inputs with the proposed model.

	Fuzzy Input for Objective Functions' Coefficient	Fuzy Inputs for Coefficients in Constiants' and Right Side's Value	Optimal Objective Value	Solutions
Set-1	[2,4,7,11]; [6.5,12.3,16,19.98]; [5,9,11.5,15.07]	[4,7,10,13]; [2.5,4.9,7.9,11]; [1.2,3.4,6.7,10.5] and [11.8,14.9,19.2,24.4]	20.3700	$x_1 = 1.4754$, $x_2 = 0.5494$, $x_3 = 0.5596$
Set-2	[0.4,1.13,2.31,5.56]; [16.15,22.39,26.78,29.59]; [5.98,7.99,10.54,13.67]	[1.56,2.67,6.64,9.88]; [2.35,3.89,6.99,8.92]; [5.22,7.41,9.27,10.53] and	138.5768	$x_1 = 6.0683$, $x_2 = 0$, $x_3 = 2.3350$
Set-3	[-3.35,-0.93,-4.11,8.61]; [-5.11,-1.09,-3.11,10.19], [25.98,27.99,30.54,33.67]	[21.05,22.07,26.06,29.08], [12.03,13.09,16.09,18.02], [25.02,27.01,29.07,30.03] and [12,13,14,15]	0.0178	$x_1 = 0.5686$, $x_2 = 0$, $x_3 = 0.0012$
Set-4	[63.89,70.31,79.91,85.13]; [45.11,51.98,63.44,0.97]; [75.21,87.23,90.44,99.92]	[29.68,34.55,39.13,41.45], [12.03,14.09,17.09,19.02], [11.12,12.17,14.19,18.71] and [111.2,122.1,134.9,148.7]	412.3734	$x_1 = 0.9001$, $x_2 = 2.4381$, $x_3 = 0.2962$

5.4. Comparison Analysis

The table below (Table 2) provides a comparison of the optimum results attained by following the existing approach, the fuzzy MF approach, and the robust ranking approach for the fuzzy NLPP preferred in the numerical illustration above. From the results shown in the table, it is evident that the same results are given regardless of what existing or fuzzy membership and ranking approaches do. It shows the newness of the proposed model and also demonstrates that the decision-maker may use these kind of model to clear the vagueness of any suitable problem in order to achieve the best optimum values. Based on the above result, it is recommended to use either of the models given instead of the existing model, namely the fuzzy MF model or robust ranking approach, which is ideal [6,7].

Table 2. Optimum results comparison of existing and proposed models.

Optimum Values	The Existing Model Is Based on the Conventional Approach	The Proposed Model Is Based on the Conventional Approach in Terms of Fuzziness	The Proposed Model Is Based on the Robust Ranking Approach
x_1	2.8	$\left[x_1^{(k)}\right];\ k = 1,2,3,4 = [1.4, 2.1, 3.5, 4.2]$	$\left[x_1^{(k)}\right];\ k = 1,2,3,4 = [1.4, 2.1, 3.5, 4.2] = R[1.4, 2.1, 3.5, 4.2] = 2.8$
x_2	0	$\left[x_2^{(k)}\right];\ k = 1,2,3,4 = [-2, -1, 1, 2]$	$\left[x_2^{(k)}\right];\ k = 1,2,3,4 = [-2, -1, 1, 2] = R[-2, -1, 1, 2] = 0$
x_3	1.4	$\left[x_3^{(k)}\right];\ k = 1,2,3,4 = [0, 0.7, 2.1, 2.8]$	$\left[x_3^{(k)}\right];\ k = 1,2,3,4 = [0, 0.7, 2.1, 2.8] = R[0, 0.7, 2.1, 2.8] = 1.4$
λ	1.4	$\left[\lambda^{(k)}\right];\ k = 1,2,3,4 = [0.8, 1.1, 1.7, 2]$	$\left[\lambda^{(k)}\right];\ k = 1,2,3,4 = [0.8, 1.1, 1.7, 2] = R[0.8, 1.1, 1.7, 2] = 1.4$
Min Z	9.8	$\left[Z^{(k)}\right];\ k = 1,2,3,4 = [-9.8, 0, 19.6, 29.4]$	$\left[Z^{(k)}\right];\ k = 1,2,3,4 = [-9.8, 0, 19.6, 29.4] = R[-9.8, 0, 19.6, 29.4] = 9.8$

6. Results and Discussion

Employing the suggested model numerical illustrations demonstrate that the optimum value of the FNLPP is $[-9.8,\ 0,\ 19.6,\ 29.4]$, which might be a fresh attempt to clear the vagueness. The optimum solution for the fuzzified NLPPs will be continuously greater than -9.8 and less than 29.4, and the most likely outcome will be somewhere in the range of 0 and 19.6. The varieties in cost with significance probability can be seen in Figure 3. Additionally, obtained fuzzy optimum solutions x_{ij} might be empirically comprehended.

- The decision-maker perception, the entire value of the fuzzy NLPP, will be higher than -9.8 and less than 29.4.
- The decision-maker for the entire fuzzy NLPP estimations are going to be bigger than or sufficient to 0 and less than or equivalent to 19.6.
- The extent of the favors of the decision-maker for the rest of the estimations of the entire fuzzy NLPP value has frequently been attained as below:
- Here x describes the significance of the entire NLPP, and also the perception of decision-makers for $\mu_{min}(X)$, where

$$\mu_{\min}(\mathrm{X}) = \begin{cases} \frac{x+9.8}{9.8} & \text{for} \quad -9.8 \leq x \leq 0 \\ 1 & \text{for} \quad 0 \leq x \leq 19.6 \\ \frac{x-29.4}{9.8} & \text{for} \quad 19.6 \leq x \leq 29.4 \\ 0, & \text{otherwise} \end{cases}$$

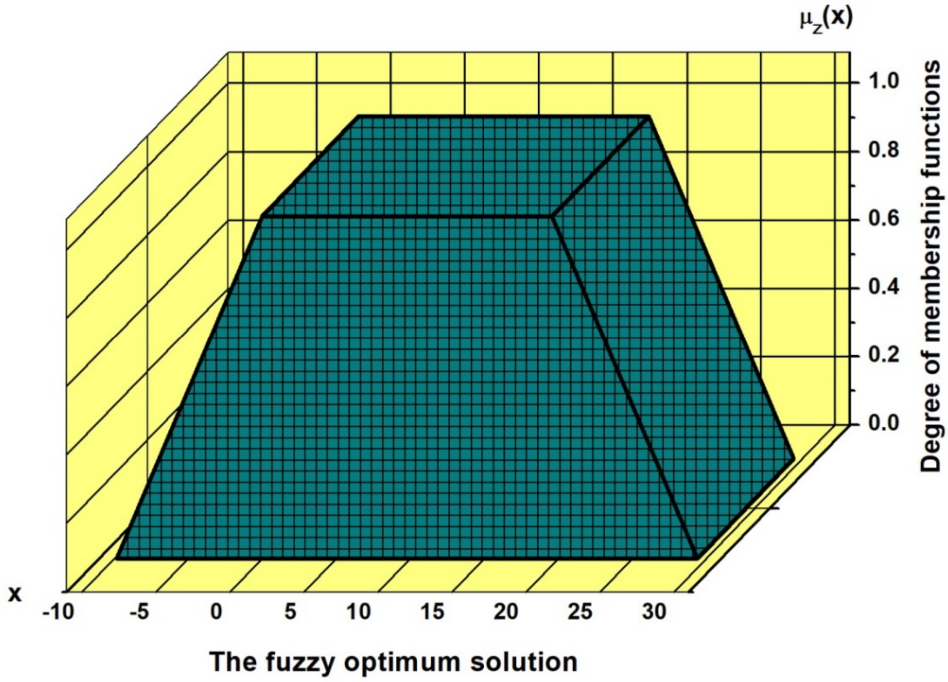

Figure 3. Pictorial representation of fuzzy optimum solution $\mu_Z(x)$.

7. Conclusions

Finally, an effort has been made to create a model that solves the problem of NLP in a fuzzy environment. The fuzzy version of the problem has been addressed using the necessary and sufficient conditions of Lagrangian multipliers in terms of fuzziness with the aid of a numerical illustration. This approach clarifies by solving two numerical illustrations; one is using MFs, and the other, the approach of robust rankings. MFs provide a significant role in the creation of a model in a fuzzy context. Most of the research techniques have been discussed in establishing only the MFs for the fuzzy objectives or constraints. However, this approach solved the mutually contradictory complexity of the objectives as well as constraints using MFs. This model offers an efficient approach to dealing with the problems of NLP. Therefore, the optimal solution has been signified through fuzziness within the result and discussion. Additionally, the solution is explained by the manner of TFMs which have models of performance evaluation with different sets of inputs. This shows that the efficiency level of the model is of high impact. The code to solve the model is implemented in LINGO, and comes with a collection of built-in solvers for various problems. Furthermore, the comparison analysis could be a newly-designed effort to solve NLPs under fuzziness. The model focuses on addressing the decision-makers uncertainties and subjective experiences, and can help to solve decision-making issues. The model's future scope suggests that the model be used in other types of NLPPs or suitable nonlinear optimization models in upcoming models, preferably optimization models, under numerous fuzzy situations.

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