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A Combinatorial Characterization of $H(4, q^2)$ †

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† Dedicated to Prof. Franco Eugeni on the occasion of their 80th birthday.

Abstract: In this paper, we remove the solid incidence assumption in a characterization of $H(4, q^2)$ by J. Schillewaert and J. A. Thasby proving that Hermitian plane incidence numbers imply Hermitian solid incidence numbers, except for a few possible small cases.

Keywords: Hermitian variety; three character sets; intersection number

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1. Introduction and Motivation

Let q denote a prime power p^h with exponent $h \geq 1$. In $PG(r, q)$, the projective space of dimension r and order q, let K denote a k-set, i.e., a set of k points. For each integer i such that $0 \leq i \leq \theta_d := \sum_{j=0}^{d} q^j$, let us denote by $t_i^d = t_i^d(K)$ the number of d-subspaces of $PG(r, q)$ meeting K in exactly i points. The nonnegative integers t_i^d are called the characters of K with respect to the dimension d, as can be seen in [1–3]. Let $m_1, m_2, \ldots, m_s$ be s integers such that $0 \leq m_1 < m_2 < \cdots < m_s \leq \theta_d$. A set K is said to be of class $[m_1, m_2, \ldots, m_s]_d$ if $t_i^d > 0$ only if $i \in \{m_1, m_2, \ldots, m_s\}$. Moreover, K is said to be of type $(m_1, m_2, \ldots, m_s)_d$ if $t_i^d > 0$ if and only if $i \in \{m_1, m_2, \ldots, m_s\}$. The nonnegative integers $m_1, m_2, \ldots, m_s$ are called intersection numbers with respect to the dimension d. Intersection numbers with respect to dimensions 2 and 3 will be called plane and solid intersection numbers, respectively. A full swing research topic is to recognize algebraic varieties by intersection numbers, as can be seen in [4–6]. The Hermitian variety $H(4, q^2)$ is the set of all absolute points of a non-degenerate unitary polarity in $PG(4, q^2)$; it is a non-singular algebraic hypersurface of degree $q + 1$ in $PG(4, q^2)$ with three plane intersection numbers and two solid intersection numbers (for more details, we refer the reader to Chapter 23 of [1]). The size and the solid intersection numbers are generally not sufficient to characterize Hermitian varieties due to the existence of quasi-Hermitian varieties, as can be seen in [7,8]. In [9], Theorem 4.2, J. Schillewaert and J. A. Thas proved the following

Result 1. *In $PG(4, q^2)$, any set of class $[q^2 + 1, q^3 + 1, q^3 + q^2 + 1]_2$ and of class $[q^5 + q^2 + 1, q^5 + q^3 + q^2 + 1]_3$ is the Hermitian variety $H(4, q^2)$.*

In this paper, we remove the solid incidence assumption of Result 1 by proving the following

Theorem 1. *In $PG(4, q^2)$, apart from possible cases with $q \in \{2, 3, 5\}$, any set of class $[q^2 + 1, q^3 + 1, q^3 + q^2 + 1]_2$ is the Hermitian variety $H(4, q^2)$.*

In order to remove the solid incidence assumption, we have to calculate the solid intersection numbers of a set of class $[q^2 + 1, q^3 + 1, q^3 + q^2 + 1]_2$ in $PG(4, q^2)$. To do this, in Section 2, we analyze the possible sizes of a set that have the same plane intersection numbers in $PG(3, q^2)$.

2. Sets of Class $[q^2+1, q^3+1, q^3+q^2+1]_2$ in $PG(3,q^2)$

We start by recalling the following

Result 2 (see [10] Lemma 2.2). *In $PG(r,q)$ with $r \geq 2$, let K be a k-set of class $[m_1, m_2, \ldots, m_s]_d$ and of class $[n_1, n_2, \ldots, n_u]_{d+1}$ with $1 \leq d < d+1 \leq r$. If there is an integer x such that for any $m_i \in \{m_1, m_2, \ldots, m_s\}$, we have $m_i \equiv x \bmod q$; then, for any $n_j \in \{n_1, n_2, \ldots, n_u\}$, we have $n_j \equiv x \bmod q$. Thus, $k \equiv x \bmod q$ as well, since K is of type $(k)_r$.*

In this section, we will prove the following:

Theorem 2. *In $PG(3,q^2)$, with $q = p^h$ a prime power, let K be a k-set of class $[q^2+1, q^3+1, q^3+q^2+1]_2$. Then, there is an integer a such that $k = aq^2+1$ with either $a \equiv 0 \pmod q$ or $a \equiv 1 \pmod q$. Furthermore:*

1. *$t^2_{q^2+1} = 0$ if and only if $k = q^5+q^3+q^2+1$;*
   furthermore, K is of type $(q^3+1, q^3+q^2+1)_2$;
2. *If $t^2_{q^2+1} \geq 1$, then*
   - *$q = 2$ and $k \in \{25, 33, 49\}$; furthermore:*
     - *If $k = 25$, then K is a set of type $(5,9)_2$ of $PG(3,4)$;*
     - *If $k \in \{33, 49\}$, then K is a set of type $(5,9,13)_2$ of $PG(3,4)$;*
     - *If $k = 49$, then a line meets K in at most 4 points and therefore K contains no line;*
   - *$q = 3$ and $k = 244$; furthermore:*
     - *K is a set of type $(10,28,37)_2$ of $PG(3,9)$;*
     - *A line meets K in at most 8 points and therefore K contains no line;*
   - *$q = 5$ and $k = 3126$; furthermore:*
     - *K is a set of type $(26,126,151)_2$ of $PG(3,25)$;*
     - *A line meets K in at most 12 points and therefore K contains no line;*
   - *$k = q^5+q^2+1$ for any $q \geq 2$;*
     furthermore, K is a set of type $(q^2+1, q^3+1, q^3+q^2+1)_2$;
   - *$k = q^5+q^3+1$ for any $q \geq 2$; furthermore:*
     - *K is a set of type $(q^2+1, q^3+1, q^3+q^2+1)_2$;*
     - *If α is a (q^2+1)-plane, then $\alpha \cap K$ is not a line;*
   - *$k = q^5+q^4-q^3+q^2+1$ for any $q \geq 3$; furthermore:*
     - *K is a set of type $(q^2+1, q^3+1, q^3+q^2+1)_2$;*
     - *A line meets K in at most $2q+1$ points and therefore K contains no line.*

Theorem 2 will be a consequence of Lemmas 2–5.

Now let K be a k-set of $PG(3,q^2)$ of class $[l,m,n]_2$. Thus, by definition, $l < m < n$. By double counting the number of planes, the number of pairs (P,α) where $P \in K$ and α is a plane through P, and the number of pairs $((P,Q),\alpha)$ where P and Q are two distinct points of K and α is a plane through P and Q, we obtain the following equations on the integers $t_i = t_i^2(K)$

$$t_l + t_m + t_n = (q^2+1)(q^4+1) \tag{1}$$

$$lt_l + mt_m + nt_n = k(q^4+q^2+1) \tag{2}$$

$$l(l-1)t_l + m(m-1)t_m + n(n-1)t_n = k(k-1)(q^2+1) \tag{3}$$

Lemma 1. *Let K be a k-set of class $[q^2+1, q^3+1, q^3+q^2+1]_2$ in $PG(3,q^2)$ and let r_h be a line meeting K in exactly h points. Then:*

1. *$k \equiv 1 \pmod{q^2}$;*
2. *$h \leq q^3+q^2+q+2-\frac{k-1}{q^2} = [q^3+q+1-\frac{k-1}{q^2}]+q^2+1$;*

3. $h \leq [q^3 + q + 1 - \frac{k-1}{q^2}] + t_{q^3+q^2+1}$;
4. *If* $t_{q^2+1} \geq 2$, *then* $k \leq q^5 + q^4 - q^3 + 2q^2 + 1$.

Proof. By Result 2, we immediately have that $k \equiv 1 \ (mod\ q^2)$.

Now let r_h be a line meeting K in exactly h points and let us denote by u_i^h the number of i-planes passing through r_h with $i \in \{q^2+1, q^3+1, q^3+q^2+1\}$. Counting the number of points of $K \setminus r_h$ via the planes through r_h, we obtain

$$k - h = (q^2 + 1 - h)u^h_{q^2+1} + (q^3 + 1 - h)u^h_{q^3+1} + (q^3 + q^2 + 1 - h)u^h_{q^3+q^2+1} \tag{4}$$

Since $u^h_{q^2+1} + u^h_{q^3+1} + u^h_{q^3+q^2+1} = q^2 + 1$, we have that

$$h + u^h_{q^3+1} + qu^h_{q^2+1} = q^3 + q^2 + q + 2 - \frac{k-1}{q^2} \tag{5}$$

$$h + (q-1)u^h_{q^2+1} = q^3 + q + 1 - \frac{k-1}{q^2} + u^h_{q^3+q^2+1} \tag{6}$$

By (5), we immediately have that $h \leq q^3 + q^2 + q + 2 - \frac{k-1}{q^2}$. Since $u^h_{q^3+q^2+1} \leq t_{q^3+q^2+1}$, by (6), we have that $h \leq q^3 + q + 1 - \frac{k-1}{q^2} + t_{q^3+q^2+1}$.

Now let us suppose that $t_{q^2+1} \geq 2$. Let α and β be two (q^2+1)-planes and let r_h be the line $\alpha \cap \beta$. Equation (5) can be rewritten in the following way

$$q^3 + q^2 - q + 2 - \frac{k-1}{q^2} = h + q(u^h_{q^2+1} - 2) + u^h_{q^3+1} \tag{7}$$

Since $u^h_{q^2+1} - 2 \geq 0$, by (7), we have that $q^3 + q^2 - q + 2 - \frac{k-1}{q^2} \geq 0$ from which it immediately follows that $k \leq q^5 + q^4 - q^3 + 2q^2 + 1$. □

Lemma 2. *If K is a k-set of $PG(3, q^2)$ of class $[q^3+1, q^3+q^2+1]_2$, then $k = q^5 + q^3 + q^2 + 1$.*

Proof. A set of class $[m, n]_2$ is a set of class $[l, m, n]_2$ having $t_l = 0$. Putting $t_l = 0$, $m = q^3 + 1$ and $n = q^3 + q^2 + 1$ in Equations (1)–(3), we obtain

$$[k - (q^5 + q^3 + q^2 + 1)][k - (q^5 + q^4 - q^3 + 2q + 1 - \frac{2q}{q^2+1})] = 0 \tag{8}$$

Therefore, $k = q^5 + q^3 + q^2 + 1$ necessarily, since $q \geq 2$ and k is an integer. □

From now on, K will ever be a k-set of class $[q^2+1, q^3+1, q^3+q^2+1]_2$ having $t_{q^2+1} \geq 1$. By Lemma 1, there is an integer a such that $k = aq^2 + 1$.

Lemma 3. *We have that either $a \equiv 0 \ (mod\ q)$ or $a \equiv 1 \ (mod\ q)$.*

Proof. Putting $l = q^2 + 1$, $m = q^3 + 1$, $n = q^3 + q^2 + 1$ and $k = aq^2 + 1$ into Equations (1)–(3), we obtain

$$t_{q^2+1} = H + 3q^4 + (6 - 2a)q^3 - 3aq^2 - 7a - \alpha + \beta \tag{9}$$

$$t_{q^3+1} = -H + (a-4)q^4 + (3a-7)q^3 + 4aq^2 - (a-1)^2 q + 8a - \beta \tag{10}$$

$$t_{q^3+q^2+1} = q^6 + (2-a)q^4 + (1-a)q^3 + (1-a)q^2 + (a-1)^2 q - a + 1 + \alpha \tag{11}$$

with $H = q^6 + 2q^5 + 8q^2 + (9-5a)q + a^2 + 11$, $\alpha = \frac{a(a-1)}{q}$ and $\beta = \frac{2(a-2)(a-3)}{q-1}$. Since $\alpha = \frac{a(a-1)}{q}$ an integer, we have that $a(a-1) \equiv 0 \ (mod\ q)$ and hence either $a \equiv 0 \ (mod\ q)$ or $a \equiv 1 \ (mod\ q)$, since a and $a-1$ are coprime and q is a prime power. □

Lemma 4. *If $a \equiv 0 \ (mod\ q)$, then:*

1. $q = 2$ *and* $k \in \{25, 33, 49\}$*; furthermore:*
   - *If* $k = 25$*, then* K *is a set of type* $(5,9)_2$ *of* $PG(3,4)$*;*
   - *If* $k \in \{33, 49\}$*, then* K *is a set of type* $(5,9,13)_2$ *of* $PG(3,4)$*;*
   - *If* $k = 49$*, then a line meets* K *in at most 4 points and therefore* K *contains no line;*
2. $q = 3$ *and* $k = 244$*; furthermore:*
   - K *is a set of type* $(10, 28, 37)_2$ *of* $PG(3,9)$*;*
   - *A line meets* K *in at most 8 points and therefore* K *contains no line;*
3. $q = 5$ *and* $k = 3126$*; furthermore:*
   - K *is a set of type* $(26, 126, 151)_2$ *of* $PG(3,25)$*;*
   - *A line meets* K *in at most 12 points and therefore* K *contains no line;*
4. $k = q^5 + q^3 + 1$ *for any* $q \geq 2$*; furthermore:*
   - K *is a set of type* $(q^2+1, q^3+1, q^3+q^2+1)_2$*;*
   - *If* α *is a* (q^2+1)*-plane, then* $\alpha \cap K$ *is not a line.*

Proof. Putting $a = bq$ into Equations (9)–(11) we obtain

$$(q-1)t_{q^2+1} = q(q^2+1)b^2 - (2\theta_5 - q^4 - 1)b + \theta_7 + 2q^4 + q^3 + q \tag{12}$$

$$(q-1)t_{q^3+1} = -q^2(q^2+1)b^2 + (\theta_5 + q^4 + 2q^2 + 1)qb - (\theta_7 + q^5 + 2q^4 + q^2) \tag{13}$$

$$t_{q^3+q^2+1} = q(q^2+1)b^2 - (\theta_5 + q^2)b + \theta_6 - q^5 + q^4 \tag{14}$$

where $\theta_d := \sum_{i=0}^{d} q^i$.

If $q = 2$, then we obtain $t_5 = 10b^2 - 109b + 297$, $t_9 = -20b^2 + 176b - 323$ and $t_{13} = 10b^2 - 67b + 111$. Since $t_5 \geq 1$, $t_9 \geq 0$ and $t_{13} \geq 0$, it is easy to prove that $b \in \{3, 4, 5, 6\}$ necessarily.

If $b = 3$, then $a = bq = 6$ and $k = aq^2 + 1 = 25$. Furthermore, we obtain that $(t_5, t_9, t_{13}) = (60, 25, 0)$. K is a 25-set of type $(5,9)_2$ in $PG(3,4)$.

If $b = 4$, then $a = bq = 8$ and $k = aq^2 + 1 = 33$. Furthermore, we obtain $(t_5, t_9, t_{13}) = (21, 61, 3)$. K is therefore a 33-set of type $(5,9,13)_2$ in $PG(3,4)$.

If $b = 5$, then $a = bq = 10$ and $k = aq^2 + 1 = 41$. Let us note that in such a case, $k = 41 = 2^5 + 2^3 + 1 = q^5 + q^3 + 1$. This case is therefore included in item 4 in the statement of the lemma.

If $b = 6$, then $a = bq = 12$ and $k = aq^2 + 1 = 49$. Furthermore, we obtain that $(t_5, t_9, t_{13}) = (3, 13, 69)$. K is therefore a 49-set of type $(5,9,13)_2$ in $PG(3,4)$. Finally, by point (2) of Lemma 1, we obtain that $h \leq 4$.

Now let us study the case $q \geq 3$.

Since $t_{q^2+1} \geq 1$, by Equation (12), we obtain that:

$$f(b) := q(q^2+1)b^2 - (2\theta_5 - q^4 - 1)b + \theta_7 + 2q^4 + q^3 + 1 \geq 0 \tag{15}$$

It is easy to see that:

- $f(q^2+1) = q^3 - 2q^2 + 1 > 0$ for any q;
- $f(q^2+2) = -q(q-1)(q^2 - 3q + 1) < 0$ for any $q \geq 3$;
- $f(q^2+q-1) = -2q^2 + 3q + 3 < 0$ for any $q \geq 3$;
- $f(q^2+q) = (q-1)(q^3 - 2q - 2) > 0$ for any q.

For any $q \geq 3$, there are therefore two real numbers b_1 and b_2 such that $q^2 + 1 < b_1 < q^2 + 2$, $q^2 + q - 1 < b_2 < q^2 + q$ and $g(b_1) = g(b_2) = 0$. Thus, for any $q \geq 3$, we have $b \leq q^2 + 1$ or $b \geq q^2 + q$.

Since $t_{q^3+1} \geq 0$, by Equation (13), we obtain

$$g(b) := q^2(q^2+1)b^2 - (\theta_5 + q^4 + 2q^2 + 1)qb + \theta_7 + q^5 + 2q^4 + q^2 \leq 0 \quad (16)$$

It is easy to see that

- $g(q) = (q+1)(q^4+1) > 0$ for any q;
- $g(q+1) = -(q-1)(q^5 - q^3 + 1) < 0$ for any q;
- $g(q^2+q) = -2q^3 + q + 1 < 0$ for any q;
- $g(q^2+q+1) = (q-1)(q^5 + q^4 + 3q^3 - 1) > 0$ for any q.

Thus, for any q, there are two real numbers b_1 and b_2 such that $q < b_1 < q+1$, $q^2+q < b_2 < q^2+q+1$ and $g(b_1) = g(b_2) = 0$. For any q, we therefore have $q+1 \leq b \leq q^2+q$.

Since $t_{q^3+q^2+1} \geq 0$, by Equation (14), we obtain

$$h(b) := q(q^2+1)b^2 - (\theta_5 + q^2)b + \theta_6 - q^5 + q^4 \geq 0 \quad (17)$$

It is easy to see that:

- $h(q) = q^4 + 1 > 0$ for any q;
- $h(q+1) = -(q-2)q^4 < 0$ for any $q \geq 3$;
- $h(q^2-1) = -(q-2)(q^4 + q^3 + 2q^2 + 2q + 1) < 0$ for any $q \geq 3$;
- $h(q^2) = q + 1 > 0$ for any q.

For any $q \geq 3$, there are therefore two real numbers b_1 and b_2 such that $q < b_1 < q+1$, $q^2 - 1 < b_2 < q^2$ and $g(b_1) = g(b_2) = 0$. Thus, for any $q \geq 3$, we have $b \leq q$ or $b \geq q^2$.

Finally, if $q \geq 3$, then $b \in \{q^2, q^2+1, q^2+q\}$ necessarily.

If $b = q^2$, then $k = q^5 + 1$, $t_{q^3+q^2+1} = q + 1$, $t_{q^3+1} = q^6 + q^4 - q^3 - 2q - 3 - \frac{4}{q-1}$, and $t_{q^2+1} = q^3 + q^2 + q + 3 + \frac{4}{q-1}$. Thus, $q-1$ must divide 4 with $q \geq 3$. Hence, $q \in \{3, 5\}$ and K is a 244-set of type $(10, 28, 37)_2$ in $PG(3,9)$ or $q = 5$ and K is a 3126-set of type $(26, 126, 151)_2$ in $PG(3,25)$. Furthermore, since $t_{q^3+q^2+1} = q+1$, by point (3) of Lemma 1, we have that $h \leq 2(q+1)$.

If $b = q^2+1$, then $k = q^5 + q^3 + 1$, $t_{q^2+1} = q^2 - q$, $t_{q^3+1} = q^6 - q^5 + 2q^4 - 2q^3 + 2q^2 + 1$, $t_{q^3+q^2+1} = q^5 - q^4 + 2q^3 - 2q^2 + q$; therefore, K is of type $(q^2+1, q^3+1, q^3+q^2+1)_2$. Now, let us suppose that there is a (q^2+1)-plane α such that $\alpha \cap K$ is a line r. Substituting $h = q^2+1$ into Equation (5), we obtain $u_{q^3+1} = 1 - qu_{q^2+1}$. Hence, $u_{q^2+1} = 0$ necessarily and no (q^2+1)-plane passes through line r, which is a contradiction.

If $b = q^2+q$, then $k = q^5 + q^4 + 1$. Since $t_{q^2+1} \geq 2$, by point (4) of Lemma 1, we have that $q^5 + q^4 + 1 = k \leq q^5 + q^4 - q^3 + 2q^2 + 1$. Thus, $q \leq 2$, which is a contradiction. □

Lemma 5. *If $a \equiv 1 \ (mod\ q)$, then:*

1. *$k = q^5 + q^2 + 1$ for any $q \geq 2$; furthermore:*
   K is a set of type $(q^2+1, q^3+1, q^3+q^2+1)_2$.
2. *$k = q^5 + q^4 - q^3 + q^2 + 1$ for any $q \geq 3$; furthermore:*
   - *K is a set of type $(q^2+1, q^3+1, q^3+q^2+1)_2$;*
   - *A line meets K in at most $2q+1$ points and therefore K contains no line.*

Proof. Putting $a = cq + 1$ into Equations (9)–(11), we obtain

$$(q-1)t_{q^2+1} = q(q^2+1)c^2 - (2\theta_5 - q^4 - 2q^2 - 3)c + \theta_7 - 2q^2 - 2 \quad (18)$$

$$(q-1)t_{q^3+1} = -q^2(q^2+1)c^2 + (\theta_6 + q^5 - q - 1)c - (\theta_7 - \theta_3) \quad (19)$$

$$t_{q^3+q^2+1} = q(q^2+1)c^2 - (\theta_5 - q^2 - 2)c + q^6 + q^4 \quad (20)$$

where $\theta_d := \sum_{i=0}^{d} q^i$.

Since $t_{q^2+1} \geq 1$, by Equation (18) we obtain:

$$f(c) := q(q^2+1)c^2 - (2\theta_5 - q^4 - 2q^2 - 3)c + \theta_7 - 2q^2 - q - 1 \geq 0 \quad (21)$$

It is easy to see that:

- $f(q^2) = (q-1)q^3 > 0$ for any q;
- $f(q^2+1) = -(q-1) < 0$ for any q;
- $f(q^2+q-2) = -q^4 + 4q^3 - 6q^2 + 9q - 2 < 0$ for any $q \geq 3$;
- $f(q^2+q-1) = (q-1)(q^2-3q+1) > 0$ for any $q \geq 3$.

Thus, for any $q \geq 3$, there are two real numbers c_1 and c_2 such that $q^2 < c_1 < q^2+1$, $q^2+q-2 < c_2 < q^2+q-1$ and $g(c_1) = g(c_2) = 0$. For any $q \geq 3$, we therefore have $c \leq q^2$ or $c \geq q^2+q-1$.

If $q = 2$ and $c = q^2+q-2 = 4$, then $a = cq+1 = 9$ and $k = aq^2+1 = 37 = (2^3+1)2^2+1 = (q^3+1)q^2+1$. Thus, this case is included in item 1 in the statement of the lemma.

If $q = 2$ and $c = q^2+q-1 = 5$, then $a = cq+1 = 11$ and $k = aq^2+1 = 45 = 2^5+2^3+2^2+1 = q^5+q^3+q^2+1$. Therefore, $t_{q^2+1} = 0$, which is a contradiction.

Since $t_{q^3+1} \geq 0$, by Equation (19), we obtain

$$g(c) := q^2(q^2+1)c^2 - (\theta_6 + q^5 - q - 1)c + (\theta_7 - \theta_3) \leq 0 \quad (22)$$

It is easy to see that:

- $g(q) = (q-1)q^3 > 0$ for any q;
- $g(q+1) = -(q-1)(q+1)q^4 < 0$ for any q;
- $g(q^2+q-1) = -(q-1)(q^3+2)q^2 < 0$ for any q;
- $g(q^2+q) = (q-1)(q+1)q^3 > 0$ for any q.

Therefore, for any q, there are two real numbers c_1 and c_2 such that $q < c_1 < q+1$, $q^2+q-1 < c_2 < q^2+q$ and $g(c_1) = g(c_2) = 0$. For any q, we thus have $q+1 \leq c \leq q^2+q-1$.

Since $t_{q^3+q^2+1} \geq 0$, by Equation (20), we obtain

$$h(b) := q(q^2+1)c^2 - (\theta_5 - q^2 - 2)c + q^6 + q^4 \geq 0 \quad (23)$$

It is easy to see that:

- $h(q) = q(q^2-q+1) > 0$ for any q;
- $h(q+1) = -q^5 + q^4 + q^3 + q^2 + q + 1 < 0$ for any q;
- $h(q^2-1) = -q^5 + 2q^4 - q^3 + q^2 + 2q - 1 < 0$ for any q;
- $h(q^2) = q^2(q^2-q+1) > 0$ for any q.

Therefore, for any q there are two real numbers c_1 and c_2 such that $q < c_1 < q+1$, $q^2-1 < c_2 < q^2$ and $g(c_1) = g(c_2) = 0$. For any q, we thus have $c \leq q$ or $c \geq q^2$.

Finally, if $q \geq 3$, then $c \in \{q^2, q^2+q-1\}$ necessarily.

If $c = q^2$, then $k = q^5+q^2+1$, $t_{q^2+1} = q^3+1$, $t_{q^3+1} = q^6$, $t_{q^3+q^2+1} = q^4-q^3+q^2$; K is therefore a set of type $(q^2+1, q^3+1, q^3+q^2+1)_2$.

If $c = q^2+q-1$, then $k = q^5+q^4-q^3+q^2+1$, $t_{q^2+1} = (q-1)(q-2)$, $t_{q^3+1} = (q^3+2)q^2$, $t_{q^3+q^2+1} = q^6-q^5+q^4-2q^2+3q-1$; K is therefore a set of type $(q^2+1, q^3+1, q^3+q^2+1)_2$.

Furthermore, by point (2) of Lemma 1, we have that $h \leq 2q+1$. $\square$

3. The Proof of the Main Result

In this section, K is a k-set of class $[q^2+1, q^3+1, q^3+q^2+1]_2$ in $PG(4,q^2)$. By Result 2, we immediately have that $k \equiv 1\ (mod\ q^2)$.

We will prove that, apart from possible initial cases with $q \in \{2,3,5\}$ as in Corollary 1, K is the Hermitian variety $H(4,q^2)$.

As an immediate consequence of Theorem 2, we have the following

Corollary 1. *Apart from the following initial possible cases:*

1. $q = 2$, *K is a set of class* $[25, 33, 37, 41, 45, 49]_3$, *and there is at least one* 25*-solid or one* 33*-solid or one* 49*-solid (otherwise K is of class* $[37, 41, 45]_3$ *as in the next general case); furthermore:*
   - *If S is a* 25*-solid, then* $K \cap S$ *is a set of type* $(5, 9)_2$ *of* $PG(3, 4)$*;*
   - *If S is a* 45*-solid, then* $K \cap S$ *is a set of type* $(9, 13)_2$ *of* $PG(3, 4)$*;*
   - *If S is a n-solid with* $n \in \{33, 37, 41, 49\}$*, then* $K \cap S$ *is a set of type* $(5, 9, 13)_2$ *of* $PG(3, 4)$.
2. $q = 3$, *K is a set of class* $[244, 253, 271, 280, 307]_3$ *and there is at least one* 244*-solid (otherwise K is of class* $[253, 271, 280, 307]_3$ *as in the next general case); furthermore:*
   - *If S is a* 280*-solid, then* $K \cap S$ *is a set of type* $(28, 37)_2$ *of* $PG(3, 9)$*;*
   - *If S is a n-solid with* $n \in \{244, 253, 271, 307\}$*, then* $K \cap S$ *is a set of type* $(10, 28, 37)_2$ *of* $PG(3, 9)$.
3. $q = 5$, *K a set is of class* $[3126, 3151, 3251, 3276, 3651]_3$ *and there is at least one* 3126*-solid (otherwise K is of class* $[3151, 3251, 3276, 3651]_3$ *as in the next general case); furthermore:*
   - *If S is a* 3276*-solid, then* $K \cap S$ *is a set of type* $(126, 151)_2$ *of* $PG(3, 25)$*;*
   - *If S is a n-solid with* $n \in \{3126, 3151, 3251, 3651\}$*, then* $K \cap S$ *is a set of type* $(26, 126, 151)_2$ *of* $PG(3, 25)$.

K is of class $[q^5 + q^2 + 1, q^5 + q^3 + 1, q^5 + q^3 + q^2 + 1, q^5 + q^4 - q^3 + q^2 + 1]_3$*; furthermore, if S is an n-solid, then*

- *If* $n = q^5 + q^3 + q^2 + 1$*, then* $K \cap S$ *is a set of type* $(q^3 + 1, q^3 + q^2 + 1)_2$ *of* $PG(3, q^2)$*; otherwise,* $K \cap S$ *is of type* $(q^2 + 1, q^3 + 1, q^3 + q^2 + 1)_2$*;*
- *If* $n \in \{q^5 + q^3 + 1, q^5 + q^4 - q^3 + q^2 + 1\}$*, then for any* $(q^2 + 1)$*-plane* α *of S the set* $\alpha \cap K$ *is not a line.*

Remark 1. *If H is a set of type* $(m)_d$ *of* $PG(r, q)$ *with* $1 \le d \le r$*, then* $m = 0$ *or* $m = \theta_d$*. Furthermore, in the first case, H is the empty set, while in the second one, H is the whole space.*

Lemma 6. *K is of type* $(q^2 + 1, q^3 + 1, q^3 + q^2 + 1)_2$.

Proof. If there is no $(q^2 + 1)$-plane, then K is of type $(q^5 + q^3 + q^2 + 1)_3$, which is a contradiction since $q^5 + q^3 + q^2 + 1 \neq 0$ and $q^5 + q^3 + q^2 + 1 \neq \theta_3$. There is therefore at least one $(q^2 + 1)$-plane.

If there is no $(q^3 + 1)$-plane, then K has no type with respect to solids, which is a contradiction. There is therefore at least one $(q^3 + 1)$-plane.

If there is no $(q^3 + q^2 + 1)$-plane, then $q = 2$, and K is a set of type $(25)_3$, which is a contradiction since $25 \neq 0$ and $25 \neq 15 = \theta_3$. There is therefore at least one $(q^3 + q^2 + 1)$-plane. □

Lemma 7. *Apart from the possible initial cases as in Corollary 1, at least one* $(q^5 + q^3 + q^2 + 1)$*-solid passes through each* $(q^3 + q^2 + 1)$*-plane.*

Proof. Let α be a h-plane with $h \in \{q^2 + 1, q^3 + 1, q^3 + q^2 + 1\}$ such that no $(q^5 + q^3 + q^2 + 1)$-solid passes through α and let:

- w be the number of $(q^5 + q^2 + 1)$-solids passing through α;
- x be the number of $(q^5 + q^3 + 1)$-solids passing through α;
- y be the number of $(q^5 + q^4 - q^3 + q^2 + 1)$-solids passing through α.

Counting the point of K via the $q^2 + 1$ solids passing through α, we have

$$k = h + w(q^5 + q^2 + 1 - h) + x(q^5 + q^3 + 1 - h) + y(q^5 + q^4 - q^3 + q^2 + 1 - h) \quad (24)$$

Substituting $x = q^2 + 1 - w - y$ and $k = aq^2 + 1$ into (24), we obtain

$$w = q^4 + q^3 + 3q^2 + (y+3)q - y + 4 - \frac{a+h-5}{q-1} \tag{25}$$

By Lemma 6, there is at least one (q^2+1)-plane α. By Corollary 1, no $(q^5+q^3+q^2+1)$-solid passes through α. Substituting $h = q^2 + 1$ into Equation (25), we obtain

$$w = q^4 + q^3 + 3q^2 + (y+2)q - y + 3 - \frac{a-3}{q-1} \tag{26}$$

Thus, $q-1$ divides $a-3$. Now, let β a (q^3+q^2+1)-plane and let us suppose that no $(q^5+q^3+q^2+1)$-solid passes through β. Substituting $h = q^3 + q^2 + 1$ into Equation (25), we obtain

$$w = q^4 + q^3 + 2q^2 + (y+1)q - y + 2 - \frac{a-2}{q-1} \tag{27}$$

Thus, $q-1$ divides $a-2$, which is a contradiction. Thus, the statement is true. □

Lemma 8. *Apart from the possible initial cases as in Corollary 1, a (q^3+q^2+1)-plane contains no external line.*

Proof. Let β be a (q^3+q^2+1)-plane and r_h be a line of β meeting K in exactly h points. In view of the previous Lemma, at least one $(q^5+q^3+q^2+1)$-solid S passes through β. By Corollary 1, S contains no (q^2+1)-plane. Substituting $u^h_{q^2+1} = 0$ and $k = q^5+q^3+q^2+1$ into Equation (6), we obtain $h = u^h_{q^3+q^2+1}$. Since $u^h_{q^3+q^2+1} \geq 1$, we have the statement. □

Lemma 9. *Apart from the possible initial cases as in Corollary 1, only $(q^5+q^3+q^2+1)$-solids can pass through an external line.*

Proof. Let r_0 be an external line and let S an n-solid passing through r_0. By the previous Lemma, we have that no (q^3+q^2+1)-plane passes through r_0. Substituting $h = 0$, $u^0_{q^3+q^2+1} = 0$ and $k = n$ into Equation (6), we obtain:

$$(q-1)u^0_{q^2+1} = q^3 + q + 1 - \frac{n-1}{q^2} \tag{28}$$

- If $n = q^5 + q^2 + 1$, then we have that $u^0_{q^2+1} = 1 + \frac{1}{q-1}$;
- If $n = q^5 + q^3 + 1$, then we have that $u^0_{q^2+1} = \frac{1}{q-1}$;
- If $n = q^5 + q^3 + q^2 + 1$, then we have that $u^0_{q^2+1} = 0$;
- If $n = q^5 + q^4 - q^3 + q^2 + 1$, then we have that $u^0_{q^2+1} = 1 - q + \frac{1}{q-1}$.

Since $q > 2$ and $u^0_{q^2+1}$ are integers, we necessarily obtain $n = q^5 + q^3 + q^2 + 1$. □

Lemma 10. *Apart from the possible initial cases as in Corollary 1, if α is a (q^2+1)-plane, then $K \cap \alpha$ is a line.*

Proof. By Lemma 6, there is at least one (q^2+1)-plane α. Let S be a solid passing through α. If $K \cap \alpha$ is not a line, then $K \cap \alpha$ is not a blocking-set with respect to the lines of α. Hence, in α (and hence in S), there is at least one line r_0 external to K. By the previous Lemma, S is a $(q^5+q^3+q^2+1)$-solid. Finally, by Corollary 1, S contains no (q^2+1)-plane, which is a contradiction. □

Lemma 11. *Apart from the possible initial cases as in Corollary 1, K is a set of class $[q^5+q^2+1, q^5+q^3+q^2+1]_3$. Furthermore, K has exactly $q^7+q^5+q^2+1$ points.*

Proof. By Corollary 1 and Lemma 10, we immediately have that K is of class $[q^5+q^2+1, q^5+q^3+q^2+1]_3$.

Now, let α be a (q^2+1)-plane. Again, by Corollary 1, we have that only (q^5+q^2+1)-solids pass through the plane α. Counting the points of K via these solids, we obtain $k = (q^2+1)q^5 + q^2 + 1 = q^7 + q^5 + q^2 + 1$. $\square$

Finally, Theorem 1 follows either by Result 1 or, as can be seen in [11], by the following:

Result 3. *In $PG(4, q^2)$ with $q > 2$, let K be a $(q^7 + q^5 + q^2 + 1)$-set having two solid intersection numbers and three plane intersection numbers. If the minimum plane intersection number is $q^2 + 1$, then K is $H(4, q^2)$.*

4. Conclusions

The principal aim of this paper was to prove that the lower dimensional incidence assumption is stronger that the higher one. Therefore, applications and future developments are improvements on other combined characterizations that are obtained through different dimensional assumptions that remove the higher dimensional one.

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