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Banach Limit and Ulam Stability of Nonhomogeneous Cauchy Equation

El-sayed El-hady [1,2,*] and Janusz Brzdęk [3]

[1] Mathematics Department, College of Science, Jouf University, Sakaka P.O. Box 2014, Saudi Arabia
[2] Basic Science Department, Faculty of Computers and Informatics, Suez Canal University, Ismailia 41522, Egypt
[3] Faculty of Applied Mathematics, AGH University of Science and Technology, Mickiewicza 30, 30-059 Kraków, Poland; brzdek@agh.edu.pl
* Correspondence: eaelhady@ju.edu.sa

Abstract: We prove new results on Ulam stability of the nonhomogeneous Cauchy functional equation $f(x+y) = f(x) + f(y) + d(x,y)$ in the class of mappings f from a square symmetric groupoid $(H,+)$ into the set of reals $\mathbb{R}$. The mapping $d : H^2 \to \mathbb{R}$ is assumed to be given and satisfy some weak natural assumption. The equation arises naturally, e.g., in the theory of information in a description of generating functions of branching measures of information. Moreover, we provide a suitable example of application of our results in this area at the very end of this paper. The main tool used in the proofs is the Banach limit.

Keywords: Banach limit; Ulam stability; nonhomogeneous Cauchy functional equation

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1. Introduction

The problem of Ulam stability for equations (also known as Hyers–Ulam or Ulam–Hyers stability) can be roughly expressed as follows: how much a mapping satisfying an equation approximately (in a given sense) differs from a solution to the equation. This issue has become a very popular subject of research, and we refer to [1–5] for information on the historical background and the methods applied. The next theorem includes one of the most classical results concerning the Ulam stability of the additive Cauchy functional equation

$$\phi(s+t) = \phi(s) + \phi(t), \qquad s,t \in E, s+t \in E.$$

Theorem 1. *Let X_1 and X_2 be real normed spaces, $X_0 := X_1 \setminus \{0\}$, $c \geq 0$, $p \in \mathbb{R}$, $p \neq 1$, and $h\colon X_1 \to X_2$ be such that*

$$\|h(s+t) - h(s) - h(t)\| \leq c(\|s\|^p + \|t\|^p), \qquad s,t \in X_0. \tag{1}$$

Then the following two statements are valid.

(i) If X_2 is complete, then there is a unique mapping $g\colon X_1 \to X_2$ such that

$$g(s+t) = g(s) + g(t), \qquad s,t \in X_1, \tag{2}$$

and

$$\|h(s) - g(s)\| \leq \frac{c}{|1-2^{p-1}|}\|s\|^p, \qquad s \in X_0. \tag{3}$$

(ii) If $p < 0$, then h is additive, i.e., it is a solution to Equation (78).

This result for $p = 0$ was first proved by D.H. Hyers [6] as an answer to a question asked by S.M. Ulam in 1940. Next, an extension of it, for $p \in [0, 1)$, was obtained by T. Aoki [7]. A somewhat similar result (as that of Aoki), but for linear mappings was obtained (independently) nearly thirty years later by Th.M. Rassias [8], who also noticed that a similar reasoning works for $p < 0$. Z. Gajda [9] proved an analogous result for $p > 1$ and provided an example that for $p = 1$ a similar outcome is not possible. The statement (ii) has been proved first in [10] and next on restricted domain in [11].

In 1994, P. Găvruta [12] replaced (1) by a more general inequality

$$\|h(s+t) - h(s) - h(t)\| \le \psi(s,t), \qquad s,t \in X_1, \tag{4}$$

and obtained the following theorem.

Theorem 2. *Assume that $(X_1, +)$ is an abelian group, X_2 is a Banach space and $\psi : X_1^2 \to [0, \infty)$ fulfills*

$$\widetilde{\psi}(s,t) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\psi(2^n s, 2^n t) < \infty, \qquad s,t \in X_1.$$

If $h : X_1 \to X_2$ satisfies (4), then there is a unique additive $g : X_1 \to X_2$ with $\|h(s) - g(s)\| \le \widetilde{\psi}(s,s)$ for every $s \in X_1$.

In ([13], Theorem 1.2) it has been proved that the above mentioned results can also be extended to the Cauchy nonhomogeneous functional equation.

$$g(s+t) = g(s) + g(t) + d(s,t), \qquad s,t \in X_1, \tag{5}$$

with a given function $d : X_1^2 \to X_2$. Namely, we have for instance the following generalization of Theorem 1.

Theorem 3. *Let X_1 and X_2 be real normed spaces, $X_0 := X_1 \setminus \{0\}$, $d : X_1^2 \to X_2$ be such that Equation (5) has at least one solution $g \colon X_1 \to X_2$, $c \ge 0$, $p \in \mathbb{R}$, $p \ne 1$, and $h \colon X_1 \to X_2$ be a mapping with*

$$\|h(s+t) - h(s) - h(t) - d(x,y)\| \le c(\|s\|^p + \|t\|^p), \qquad s,t \in X_0. \tag{6}$$

Then the following two statements are valid.
(i) If X_2 is complete, then there is a unique solution $g \colon X_1 \to X_2$ of (5) such that

$$\|g(s) - h(s)\| \le \frac{c}{|1 - 2^{p-1}|}\|s\|^p, \qquad s \in X_0. \tag{7}$$

(ii) If $p < 0$, then h is a solution to Equation (5).

For $p = 1$ an analogous result is not possible (in the sense depicted in ([13], Theorem 1.2 (c))). Moreover, estimation (7) is optimum when $X_2 = \mathbb{R}$ (see ([13], Theorem 1.2 (b))).

Let us mention here that Equation (5) has also been called the Cauchy inhomogeneous functional equation in [13,14]. It is connected with the notion of cocycles (see, e.g., [15,16]) and arises in a natural way, e.g., in the theory of information (see [17]). For further information on its solutions we refer to [18–21].

In [22] (Theorem 8 and Remark 7) it has been proved that in the case $X_2 = \mathbb{R}$, the following finer results are possible for Equation (78).

Theorem 4. *Let E_1 be a normed space, $E_0 := E_1 \setminus \{0\}$, $\chi, \rho, p \in \mathbb{R}$, $p \ne 1$, $\chi \le \rho$ and $h \colon E_1 \to \mathbb{R}$ be a mapping with*

$$\chi(\|x\|^p + \|y\|^p) \le h(x+y) - h(x) - h(y) \le \rho(\|x\|^p + \|y\|^p), \qquad x,y \in E_0. \tag{8}$$

Then the following two statements are valid.

(i) *If $p \geq 0$, then there is a unique additive mapping $T\colon E_1 \to \mathbb{R}$ such that, in the case $p < 1$,*

$$\frac{\chi}{1-2^{p-1}}\|x\|^p \leq T(x) - h(x) \leq \frac{\rho}{1-2^{p-1}}\|x\|^p, \qquad x \in E_0, \tag{9}$$

and, in the case $p > 1$,

$$\frac{\chi}{2^{p-1}-1}\|x\|^p \leq h(x) - T(x) \leq \frac{\rho}{2^{p-1}-1}\|x\|^p, \qquad x \in E_0. \tag{10}$$

(ii) *If $p < 0$, then h is additive (in view of (8) it is possible only when $\chi \leq 0 \leq \rho$).*

In this paper we show that an analog of Theorem 4 is also possible for Equation (5). First, we prove extensions (to Equation (5)) of two general results from ([22], Theorems 6 and 7), somewhat corresponding to Theorem 2.

For the convenience of readers, we recall below ([22], Theorem 6) (([22], Theorem 7) is analogous and complementary to it). To this end we need to remind the notion of a square symmetric groupoid.

So, let X be a nonempty set and $\star : X^2 \to X$ be a binary operation. We say that the operation is square symmetric if

$$s^2 \star t^2 = (s \star t)^2, \qquad s,t \in X, \tag{11}$$

where $s^2 := s \star s$. If $\star$ is square symmetric, then we say that the groupoid $(X, \star)$ is square symmetric.

In what follows, for the simplicity of notation, it is convenient to denote a square symmetric operation in a groupoid by the symbol $+$ (without assuming its commutativity) and then (11) can be written as

$$2s + 2t = 2(s+t), \qquad s,t \in X, \tag{12}$$

where $2s := s + s$. Next, we write $2^0 s := s$ and $2^{n+1}s := 2(2^n s + 2^n s)$ for $s \in X$ and $n \in \mathbb{N}$ ($\mathbb{N}$ stands for the set of positive integers). Further information on square symmetric operations is given in the next section.

Let us mention that the notion of Banach limit LIM (used in the next theorem) is defined in Section 3. Now, we are in a position to present in ([22], Theorem 6).

Theorem 5. *Let $(H,+)$ be a square symmetric groupoid, $E \subset H$ be nonempty, $2E := \{2s : s \in E\} \subset E$, $\Gamma, \Delta : E^2 \to \mathbb{R}$ be such that*

$$\liminf_{n\to\infty} \frac{\Delta(2^n s, 2^n t)}{2^n} = 0, \qquad \limsup_{n\to\infty} \frac{\Gamma(2^n s, 2^n t)}{2^n} = 0, \qquad s,t \in E, \tag{13}$$

and the sequences $(\Gamma_n(s))_{n\in\mathbb{N}}$ and $(\Delta_n(s))_{n\in\mathbb{N}}$ be bounded for every $s \in E$, where

$$\Gamma_n(s) = \sum_{j=0}^{n-1} \frac{\Gamma(2^j s, 2^j s)}{2^{j+1}}, \qquad \Delta_n(s) = \sum_{j=0}^{n-1} \frac{\Delta(2^j s, 2^j s)}{2^{j+1}}, \qquad n \in \mathbb{N}, s \in E. \tag{14}$$

Let $\psi : E \to \mathbb{R}$ satisfy

$$\Delta(s,t) \leq \psi(s+t) - \psi(s) - \psi(t) \leq \Gamma(s,t), \qquad s,t \in E, s+t \in E, \tag{15}$$

and

$$a_n(s) := \frac{\psi(2^n s)}{2^n}, \qquad n \in \mathbb{N}, s \in E. \tag{16}$$

Then the sequence $\big(a_n(s)\big)_{n\in\mathbb{N}}$ *is bounded for every* $s \in E$ *and the function* $\Psi : E \to \mathbb{R}$*, given by*

$$\Psi(s) := \mathrm{LIM}\big((a_n(s))_{n\in\mathbb{N}}\big), \qquad s \in E, \tag{17}$$

is a solution of the conditional Cauchy functional equation

$$\Psi(s+t) = \Psi(s) + \Psi(t), \qquad s,t \in E, s+t \in E, \tag{18}$$

and

$$\beta(s) := \liminf_{k\to\infty} \Delta_k(s) \leq \Psi(s) - \psi(s) \leq \limsup_{k\to\infty} \Gamma_k(s) =: \alpha(s), \qquad s \in E. \tag{19}$$

Moreover, if

$$\inf_{n\in\mathbb{N}} \frac{\alpha(2^n s) - \beta(2^n s)}{2^n} = 0, \qquad s \in E, \tag{20}$$

then $\Psi : E \to \mathbb{R}$ *is the unique solution to* (18) *that satisfies* (19).

Finally, let us add that a result, more general than Theorem 2, was obtained much earlier in [23]. Various further related outcomes can be found in [2,4,5,24–26]. For some useful information on solutions to functional equations we refer to monographs [27,28].

2. Square Symmetric Operations

Let $(X, \star)$ be a square symmetric groupoid. By induction it is very easy to show that

$$2^n s + 2^n t = 2^n(s+t), \qquad s,t \in X, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{21}$$

Remark 1. *Obviously, every commutative semigroup is a square symmetric groupoid. Next, let* W *be a linear space over a field* $\mathbb{K}$ *and fix* $c, d \in \mathbb{K}$*,* $e \in W$*. Write* $x \oplus y = cx + dy + e$ *for* $x, y \in W$*. Then it is easy to verify that* $(W, \oplus)$ *is a simple example of square symmetric groupoid, which in general (depending on* c *and* d*) is neither commutative nor associative.*

Finally, let us mention that a groupoid $(G,+)$ is uniquely divisible by 2 if for each $y \in G$ there is a unique $x \in G$ such that $x + x = y$; we denote such x by $2^{-1}y$ and recurrently we define $2^{-n-1}y := 2^{-1}(2^{-n}y)$ for every $n \in \mathbb{N}$. Clearly, the square symmetric groupoid depicted in Remark 1 is uniquely divisible by 2 if and only if $c + d \neq 0$.

If a square symmetric groupoid $(X,+)$ is uniquely divisible by 2, then it is easy to show by induction that

$$2^{-n}s + 2^{-n}t = 2^{-n}(s+t), \qquad s,t \in X, n \in \mathbb{N}. \tag{22}$$

For some further information on square symmetric operations we refer to [29].

3. Banach Limit

The Banach limit is a very important tool in the proofs of our main results. This notion was motivated by the efforts of mathematicians to extend the notion of the limit to a family larger than that of convergent sequences. Early information on it can be found in [30] (p. 103) with the proof published in Banach's monograph [31]. For more recent results concerning it we refer to [32,33] (see also [34–36]).

So, let ℓ^∞ denote the space of all bounded real sequences (with the supremum norm) and c mean the space of all convergent real sequences. There exists a real linear functional on ℓ^∞, called the Banach limit and usually denoted by LIM, which satisfies the following conditions:

$$\inf\{b_n : n \in \mathbb{N}\} \leq \mathrm{LIM}\big((b_n)_{n\in\mathbb{N}}\big) \leq \sup\{b_n : n \in \mathbb{N}\} \tag{23}$$

$$\mathrm{LIM}\big((b_{n+k})_{n\in\mathbb{N}}\big) = \mathrm{LIM}\big((b_n)_{n\in\mathbb{N}}\big), \tag{24}$$

for all $(b_n)_{n\in\mathbb{N}} \in \ell^\infty$ and $k \in \mathbb{N}$. Clearly, from (23) and (24) we get

$$\liminf_{n\to\infty} \, b_n \le \mathrm{LIM}\big((b_n)_{n\in\mathbb{N}}\big) \le \limsup_{n\to\infty} \, b_n, \qquad (b_n)_{n\in\mathbb{N}} \in \ell^\infty, \tag{25}$$

whence

$$\mathrm{LIM}((b_n)_{n\in\mathbb{N}}) = \lim_{n\to\infty} b_n, \qquad (b_n)_{n\in\mathbb{N}} \in c. \tag{26}$$

This functional is not unique (because in the proof of its existence the Hahn–Banach theorem is applied), which means that the Banach limit of a sequence is not defined unequivocally for all bounded real sequences; however, (26) holds and there exist other (non-convergent) sequences for which the Banach limit is uniquely determined. Such sequences are called almost convergent and an example is $b_n = (-1)^n$ for $n \in \mathbb{N}$.

4. Auxiliary Results

In this section X stands for a nonempty set. Next, given $f : X \to X$, by f^n (for $n \in \mathbb{N}_0$) we denote the nth iterate of f, i.e., $f^0 = id$ (the identity mapping) and $f^{n+1} = f \circ f^n$ (mapping composition).

The following result from ([22], Theorem 3) will be very useful in the sequel.

Theorem 6. *Let $\tau : X \to X$, $\beta, \chi, \kappa, \mu : X \to \mathbb{R}$ be such that $\beta(X) \subset (0,\infty)$ and the sequences $\big(\widehat{\kappa}_n(s)\big)_{n\in\mathbb{N}}$ and $\big(\widehat{\mu}_n(s)\big)_{n\in\mathbb{N}}$ are bounded for every $s \in X$, where*

$$\widehat{\kappa}_n(s) := \sum_{j=0}^{n-1} \frac{\kappa(\tau^j(s))}{\prod_{i=0}^{j} \beta(\tau^i(s))}, \qquad \widehat{\mu}_n(s) := \sum_{j=0}^{n-1} \frac{\mu(\tau^j(s))}{\prod_{i=0}^{j} \beta(\tau^i(s))}, \qquad s \in X, n \in \mathbb{N}. \tag{27}$$

Assume that $\phi : X \to \mathbb{R}$ satisfies the inequalities

$$\kappa(s) \le \phi(\tau(s)) - \beta(s)\phi(s) - \chi(s) \le \mu(s), \qquad s \in X, \tag{28}$$

and

$$b_n(s) = \frac{\phi(\tau^n(s))}{\prod_{i=0}^{n-1} \beta(\tau^i(s))} - \sum_{k=0}^{n-1} \frac{\chi(\tau^k(s))}{\prod_{j=0}^{k} \beta(\tau^j(s))}, \qquad s \in X, n \in \mathbb{N}.$$

Then the sequence $(b_n(s))_{n\in\mathbb{N}}$ is bounded for each $s \in X$, the mapping $\Phi : X \to \mathbb{R}$, given by the formula

$$\Phi(s) = \mathrm{LIM}\big((b_n)_{n\in\mathbb{N}}\big), \qquad s \in X, \tag{29}$$

satisfies the equation

$$\Phi(\tau(s)) = \beta(s)\Phi(s) + \chi(s), \qquad s \in X, \tag{30}$$

and

$$\widehat{\kappa}(s) := \liminf_{k\to\infty} \, \widehat{\kappa}_k(s) \le \Phi(s) - \phi(s) \le \limsup_{k\to\infty} \, \widehat{\mu}_k(s) =: \widehat{\mu}(s), \qquad s \in X. \tag{31}$$

Further, if

$$\inf_{n\in\mathbb{N}} \frac{\widehat{\mu}(\tau^n(s)) - \widehat{\kappa}(\tau^n(s))}{\prod_{i=0}^{n-1} \beta(\tau^i(s))} = 0, \qquad s \in X, \tag{32}$$

then such mapping Φ is unique.

For information on various similar results on Ulam stability of related functional equations in single variable we refer to [26,37–39].

5. The Main Results

In this section $(H,+)$ denotes a square symmetric groupoid. Moreover, we always assume that $d : H^2 \to \mathbb{R}$ is a solution of the functional equation:

$$\begin{aligned} d(s+t,s+t) - d(s,s) - d(t,t) \\ = d(2s,2t) - 2d(s,t), \qquad s,t \in H. \end{aligned} \tag{33}$$

The beginning of the next remark shows that this is not a very demanding assumption on d.

Remark 2. *It seems that it only makes sense to study Ulam stability of equations that have solutions. So, assume that the equation*

$$\rho(s+t) = \rho(s) + \rho(t) + d(s,t), \qquad s,t \in H, \tag{34}$$

has at least one solution $\rho : H \to \mathbb{R}$. Then

$$d(s,t) = \rho(s+t) - \rho(s) - \rho(t), \qquad s,t \in H. \tag{35}$$

Next, it is easy to verify that every mapping $d : H^2 \to \mathbb{R}$ that has form (35), with some $\rho : H \to \mathbb{R}$, is a solution to Equation (33).

In particular, note that if d is symmetric and biadditive (i.e., $d(s,t) = d(t,s)$ and $d(s,t+u) = d(s,t) + d(s,u)$ for $s,t,u \in H$), then (35) holds with $\rho(s) = \frac{1}{2}d(s,s)$ for $s \in H$, which means that (33) is fulfilled for every symmetric and biadditive mapping $d : H^2 \to \mathbb{R}$.

There also exist other solutions of (35). For, if $g_1, g_2 : H \to \mathbb{R}$ are additive, then it is easy to check that the function $d : H^2 \to \mathbb{R}$, given by

$$d(s,t) = g_1(s) + g_2(t), \qquad s,t \in H, \tag{36}$$

satisfies (33). We show that if $g_1(s) \neq 0$ or $g_2(s) \neq 0$ for some $s \in H$, then the function d given by (36) is not the of form (35).

So, suppose that (35) and (36) hold with some $\rho : H \to Y$ and some additive $g_1, g_2 : H \to \mathbb{R}$. Then

$$\rho(s+t) - \rho(s) - \rho(t) = g_1(s) + g_2(t), \qquad s,t \in H,$$

and consequently (with $s = 0$) we obtain $-\rho(0) = g_2(t)$ for every $t \in H$ and (with $t = 0$) $-\rho(0) = g_1(s)$ for every $s \in H$. As $-\rho(0) = g_1(2s) = 2g_1(s) = -2\rho(0)$ for every $s \in H$, we have $\rho(0) = 0$ and consequently $g_1(s) = g_2(s) = 0$ for every $s \in H$ (which means that $d(s,t) = 0$ for every $s,t \in H$).

At the end of this paper (Corollary 1) we also show that in the case where d is not of form (35) we can obtain some interesting results on the existence of approximate solutions to Equation (5).

The next theorem shows that Theorem 5 (i.e., ([22], Theorem 6)) can be extended to the case of Equation (5).

Theorem 7. *Let $E \subset H$ be nonempty, $2E := \{2s : s \in E\} \subset E$, $\Gamma, \Delta : E^2 \to \mathbb{R}$ be such that*

$$\liminf_{n\to\infty} \frac{\Delta(2^n s, 2^n t)}{2^n} = 0, \qquad \limsup_{n\to\infty} \frac{\Gamma(2^n s, 2^n t)}{2^n} = 0, \qquad s,t \in E, \tag{37}$$

and the sequences $(\Gamma_n(s))_{n\in\mathbb{N}}$ and $(\Delta_n(s))_{n\in\mathbb{N}}$ be bounded for every $s \in E$, where

$$\Gamma_n(s) = \sum_{j=0}^{n-1} \frac{\Gamma(2^j s, 2^j s)}{2^{j+1}}, \qquad \Delta_n(s) = \sum_{j=0}^{n-1} \frac{\Delta(2^j s, 2^j s)}{2^{j+1}}, \qquad n \in \mathbb{N}, s \in E. \tag{38}$$

Let $\phi : E \to \mathbb{R}$ satisfy

$$\Delta(s,t) \le \phi(s+t) - \phi(s) - \phi(t) - d(s,t) \le \Gamma(s,t), \qquad s,t \in E, s+t \in E, \tag{39}$$

and

$$b_n(s) := \frac{\phi(2^n s)}{2^n} - \sum_{k=0}^{n-1} \frac{d(2^k s, 2^k s)}{2^{k+1}}, \qquad n \in \mathbb{N}, s \in E. \tag{40}$$

Then the sequence $\big(b_n(s)\big)_{n\in\mathbb{N}}$ is bounded for every $s \in E$, the mapping $\Phi : E \to \mathbb{R}$, given by

$$\Phi(s) := \mathrm{LIM}\big((b_n(s))_{n\in\mathbb{N}}\big), \qquad s \in E, \tag{41}$$

is a solution of the conditional nonhomogeneous Cauchy functional equation

$$\Phi(s+t) = \Phi(s) + \Phi(t) + d(s,t), \qquad s,t \in E, s+t \in E, \tag{42}$$

and

$$\delta(s) := \liminf_{k\to\infty} \Delta_k(s) \le \Phi(s) - \phi(s) \le \limsup_{k\to\infty} \Gamma_k(s) =: \gamma(s), \qquad s \in E. \tag{43}$$

Moreover, if

$$\inf_{n\in\mathbb{N}} \frac{\gamma(2^n s) - \delta(2^n s)}{2^n} = 0, \qquad s \in E, \tag{44}$$

then $\Phi : E \to \mathbb{R}$ is the unique solution to (42) such that (43) is valid.

Proof. From (33) we obtain

$$\frac{d(s+t,s+t) - d(s,s) - d(t,t)}{2} = \frac{1}{2} d(2s,2t) - d(s,t), \qquad s,t \in H, \tag{45}$$

whence replacing s and t by $2^k s$ and $2^k t$ (with $k \in \mathbb{N}$), by (21) we obtain

$$\begin{aligned} &\frac{d(2^k(s+t), 2^k(s+t)) - d(2^k s, 2^k s) - d(2^k t, 2^k t)}{2^{k+1}} \\ &\qquad = \frac{1}{2^{k+1}} d(2^{k+1}s, 2^{k+1}t) - \frac{1}{2^k} d(2^k s, 2^k t), \qquad s,t \in H, k \in \mathbb{N}_0. \end{aligned} \tag{46}$$

Consequently

$$\begin{aligned} &\sum_{k=0}^{n-1} \frac{d(2^k(s+t), 2^k(s+t)) - d(2^k s, 2^k s) - d(2^k t, 2^k t)}{2^{k+1}} \\ &\qquad = \sum_{k=0}^{n-1} \Big(\frac{1}{2^{k+1}} d(2^{k+1}s, 2^{k+1}t) - \frac{1}{2^k} d(2^k s, 2^k t) \Big) \\ &\qquad = \frac{1}{2^n} d(2^n s, 2^n t) - d(s,t), \qquad s,t \in H, n \in \mathbb{N}. \end{aligned} \tag{47}$$

Next, replacing in (39) s and t by $2^k s$ and $2^k t$ (with $k \in \mathbb{N}$) we obtain the inequality

$$\begin{aligned} \frac{\Delta(2^k s, 2^k t)}{2^k} \le \frac{\phi(2^k(s+t))}{2^k} - \frac{\phi(2^k s) + \phi(2^k t)}{2^k} - \frac{d(2^k s, 2^k t)}{2^k} \le \frac{\Gamma(2^k s, 2^k t)}{2^k}, \\ s,t \in E, s+t \in E, k \in \mathbb{N}_0, \end{aligned} \tag{48}$$

which can be rewritten as

$$\frac{\Delta(2^k s, 2^k t)}{2^k} \le a_k(s+t) - a_k(s) - a_k(t) - \frac{d(2^k s, 2^k t)}{2^k} \le \frac{\Gamma(2^k s, 2^k t)}{2^k}, \qquad s,t \in E, s+t \in E, k \in \mathbb{N}_0, \tag{49}$$

where a_k is defined by

$$a_k(s) := \frac{\phi(2^k s)}{2^k}, \qquad k \in \mathbb{N}_0, s \in E. \tag{50}$$

Note that (39) (with $s = t$) yields

$$\Delta(s,s) \le \phi(2s) - 2\phi(s) - d(s,s) \le \Gamma(s,s), \qquad s \in E. \tag{51}$$

So, from Theorem 6 with $X = E$, $\kappa(s) = \Delta(s,s)$, $\mu(s) = \Gamma(s,s)$, $\tau(s) \equiv 2s$, $\beta(s) \equiv 2$ and $\chi(s) \equiv d(s,s)$, we obtain that the sequence $\big(b_n(s)\big)_{n\in\mathbb{N}}$ defined by (40) is bounded for every $s \in E$ and the mapping $\Phi : E \to \mathbb{R}$, given by (41), fulfills inequalities (43).

Further, for every $s,t \in E$ with $s+t \in E$,

$$\begin{aligned}\Phi(s+t) - \Phi(t) - \Phi(s) - d(s,t) &= \mathrm{LIM}\big((b_n(s+t) - b_n(t) - b_n(s))_{n\in\mathbb{N}}\big) - d(s,t)\\ &= \mathrm{LIM}\big((b_n(s+t) - b_n(t) - b_n(s) - d(s,t))_{n\in\mathbb{N}}\big)\end{aligned} \tag{52}$$

and according to (21), (40) and (47) we have

$$\begin{aligned}&b_n(s+t) - b_n(t) - b_n(s) - d(s,t)\\ &\qquad = \frac{\phi(2^n(s+t))}{2^n} - \frac{\phi(2^n t)}{2^n} - \frac{\phi(2^n s)}{2^n} - d(s,t)\\ &\qquad\quad - \sum_{k=0}^{n-1} \frac{d(2^k(s+t), 2^k(s+t)) - d(2^k s, 2^k s) - d(2^k t, 2^k t)}{2^{k+1}}\\ &\qquad = a_n(s+t) - a_n(s) - a_n(t) - \frac{d(2^n s, 2^n t)}{2^n}, \qquad n \in \mathbb{N},\end{aligned} \tag{53}$$

whence, by (49),

$$\frac{\Delta(2^n s, 2^n t)}{2^n} \le b_n(s+t) - b_n(t) - b_n(s) - d(s,t) \le \frac{\Gamma(2^n s, 2^n t)}{2^n}, \qquad n \in \mathbb{N}. \tag{54}$$

So, in view of (25),

$$\liminf_{n\to\infty} \frac{\Delta(2^n s, 2^n t)}{2^n} \le \Phi(s+t) - \Phi(t) - \Phi(s) - d(s,t) \le \limsup_{n\to\infty} \frac{\Gamma(2^n s, 2^n t)}{2^n}, \qquad s,t \in E, s+t \in E, \tag{55}$$

and consequently from (37) we derive that

$$\Phi(s+t) - \Phi(t) - \Phi(s) - d(s,t) = 0, \qquad s,t \in E, s+t \in E. \tag{56}$$

Finally we show the uniqueness of Φ. So, suppose that $\Phi_1, \Phi_2 : E \to \mathbb{R}$ are such that

$$\Phi_i(s+t) = \Phi_i(s) + \Phi_i(t) - d(s,t), \qquad s,t \in E, s+t \in E, i = 1,2, \tag{57}$$

and

$$\delta(s) \le \Phi_i(s) - \phi(s) \le \gamma(s), \qquad s \in E, i = 1,2. \tag{58}$$

Clearly

$$\delta(s) - \gamma(s) \le \Phi_1(s) - \Phi_2(s) \le \gamma(s) - \delta(s), \qquad s \in E, \tag{59}$$

whence

$$|\Phi_1(s) - \Phi_2(s)| \leq \gamma(s) - \delta(s), \qquad s \in E. \tag{60}$$

Further, (57) yields

$$\Phi_1(2^n s) - \Phi_2(2^n s) = 2^n(\Phi_1(s) - \Phi_2(s)), \qquad s \in E, n \in \mathbb{N}. \tag{61}$$

Hence, replacing s by $2^n s$ in (60), we obtain

$$|\Phi_1(s) - \Phi_2(s)| \leq \frac{\gamma(2^n s) - \delta(2^n s)}{2^n}, \qquad s \in E, n \in \mathbb{N}, \tag{62}$$

which (on account of (44)) implies that $\Phi_1 = \Phi_2$. This ends the proof. □

Arguing analogously as above we obtain the following complementary version of Theorem 7, i.e., an extension of ([22], Theorem 7) to the case of Equation (5).

Theorem 8. *Let $(H,+)$ be uniquely divisible by 2, $E \subset H$ be nonempty, $2^{-1}E := \{2^{-1}s : s \in E\} \subset E$, $\Gamma, \Delta : E^2 \to \mathbb{R}$ be such that*

$$\liminf_{n\to\infty} 2^n \Delta(2^{-n}s, 2^{-n}t) = 0, \qquad \limsup_{n\to\infty} 2^n \Gamma(2^{-n}s, 2^{-n}t) = 0, \qquad s,t \in E, \tag{63}$$

and the sequences $(\Gamma_n(s))_{n\in\mathbb{N}}$ and $(\Delta_n(s))_{n\in\mathbb{N}}$ be bounded for every $s \in E$, where

$$\Gamma_n(s) = \sum_{j=0}^{n-1} 2^j \Gamma(2^{-j-1}s, 2^{-j-1}s), \qquad \Delta_n(s) = \sum_{j=0}^{n-1} 2^j \Delta(2^{-j-1}s, 2^{-j-1}s), \qquad n \in \mathbb{N}, s \in E. \tag{64}$$

Let $\phi : E \to \mathbb{R}$ satisfy (39) and

$$b_n(s) := 2^n \phi(2^{-n}s) + \sum_{k=0}^{n-1} 2^k d(2^{-k-1}s, 2^{-k-1}s), \qquad n \in \mathbb{N}, s \in E. \tag{65}$$

Then the sequence $\big(b_n(s)\big)_{n\in\mathbb{N}}$ is bounded for every $s \in E$ and the function $\Phi : E \to \mathbb{R}$, given by (41), is a solution of Equation (42) and satisfies the inequalities

$$\delta(s) := \liminf_{k\to\infty} \Delta_k(s) \leq \phi(s) - \Phi(s) \leq \limsup_{k\to\infty} \Gamma_k(s) =: \gamma(s), \qquad s \in E. \tag{66}$$

Moreover, if (44) holds or

$$\inf_{n\in\mathbb{N}} 2^n \big(\gamma(2^{-n}s) - \delta(2^{-n}s)\big) = 0, \qquad s \in E, \tag{67}$$

then $\Phi : E \to \mathbb{R}$ is the unique solution to (42) such that (66) is valid.

Proof. The reasoning is analogous as in the proof of Theorem 7, but for the convenience of readers we provide it.

Replacing s and t by $2^{-k-1}s$ and $2^{-k-1}t$ (with $k \in \mathbb{N}_0$) in (33), on account of (22) we easily obtain

$$\begin{aligned} &2^k\big(d(2^{-k-1}(s+t), 2^{-k-1}(s+t)) - d(2^{-k-1}s, 2^{-k-1}s) - d(2^{-k-1}t, 2^{-k-1}t)\big) \\ &\quad = 2^k d(2^{-k}s, 2^{-k}t) - 2^{k+1} d(2^{-k-1}s, 2^{-k-1}t), \qquad s,t \in H, s+t \in H, k \in \mathbb{N}_0, \end{aligned} \tag{68}$$

whence

$$\begin{aligned}
&\sum_{k=0}^{n-1} 2^k \big(d(2^{-k-1}(s+t), 2^{-k-1}(s+t)) - d(2^{-k-1}s, 2^{-k-1}s) - d(2^{-k-1}t, 2^{-k-1}t)\big)\\
&\quad = \sum_{k=0}^{n-1} \big(2^k d(2^{-k}s, 2^{-k}t) - 2^{k+1} d(2^{-k-1}s, 2^{-k-1}t)\big)\\
&\quad = d(s,t) - 2^n d(2^{-n}s, 2^{-n}t)\big), \qquad s,t \in H, s+t \in H, n \in \mathbb{N}_0.
\end{aligned}$$

Next, replacing in (39) s and t by $2^{-k}s$ and $2^{-k}t$ (with $k \in \mathbb{N}$) we obtain the inequality

$$\begin{aligned}
2^k\Delta(2^{-k}s, 2^{-k}t) &\le 2^k\phi(2^{-k}(s+t)) - 2^k\phi(2^{-k}s) - 2^k\phi(2^{-k}t) - 2^k d(2^{-k}s, 2^{-k}t)\\
&\le 2^k\Gamma(2^{-k}s, 2^{-k}t), \qquad s,t \in E, s+t \in E, k \in \mathbb{N}_0,
\end{aligned} \tag{69}$$

which can be rewritten as

$$\begin{aligned}
2^k\Delta(2^{-k}s, 2^{-k}t) &\le a_k(s+t) - a_k(s) - a_k(t) - 2^k d(2^{-k}s, 2^{-k}t)\\
&\le 2^k\Gamma(2^{-k}s, 2^{-k}t), \qquad s,t \in E, s+t \in E, k \in \mathbb{N}_0,
\end{aligned} \tag{70}$$

where a_k is defined by

$$a_k(s) := 2^k\phi(2^{-k}s), \qquad k \in \mathbb{N}_0, s \in E. \tag{71}$$

Note yet that from (39), with s and t replaced by $2^{-1}s$, for every $s \in E$ we obtain

$$2^{-1}\Delta\big(2^{-1}s, 2^{-1}s\big) \le 2^{-1}\phi(s) - \phi\big(2^{-1}s\big) - 2^{-1}d(2^{-1}s, 2^{-1}s) \le 2^{-1}\Gamma\big(2^{-1}s, 2^{-1}s\big),$$

which can be rewritten as

$$\begin{aligned}
-2^{-1}\Gamma\big(2^{-1}s, 2^{-1}s\big) &\le \phi\big(2^{-1}s\big) - 2^{-1}\phi(s) + 2^{-1}d(2^{-1}s, 2^{-1}s)\\
&\le -2^{-1}\Delta\big(2^{-1}s, 2^{-1}s\big).
\end{aligned}$$

Hence, according to Theorem 6 with $X = E$,

$$\kappa(s) = -\frac{1}{2}\Gamma(2^{-1}s, 2^{-1}s), \qquad \mu(s) = -\frac{1}{2}\Delta(2^{-1}s, 2^{-1}s), \qquad s \in E,$$

$\tau(s) = 2^{-1}s$, $\beta(s) \equiv 2^{-1}$ and $\chi(s) \equiv -2^{-1}d(2^{-1}s, 2^{-1}s)$, the sequence $\big(b_n(s)\big)_{n\in\mathbb{N}}$ defined by (65) is bounded for every $s \in D$ and the function $\Phi : D \to \mathbb{R}$, given by (41), fulfills the inequalities

$$\liminf_{k\to\infty}\,(-\Gamma_k(s)) \le \Phi(s) - \phi(s) \le \limsup_{k\to\infty}\,(-\Delta_k(s)), \qquad s \in E, \tag{72}$$

which implies (66). Moreover, for every $s,t \in E$ with $s+t \in E$,

$$\begin{aligned}
&\Phi(s+t) - \Phi(t) - \Phi(s) - d(s,t)\\
&\qquad = \mathrm{LIM}\big((b_n(s+t) - b_n(t) - b_n(s) - d(s,t))_{n\in\mathbb{N}}\big).
\end{aligned} \tag{73}$$

Next, according to (65), for every $n \in \mathbb{N}$ and $s, t \in E$ with $s + t \in E$

$$\begin{aligned} b_n(s+t) - b_n(s) - b_n(t) - d(s,t) \\ &= 2^n \phi(2^{-n}(s+t)) - 2^n \phi(2^{-n}s) - 2^n \phi(2^{-n}t) - d(s,t) \\ &\quad + \sum_{k=0}^{n-1} 2^k \big(d(2^{-k-1}(s+t), 2^{-k-1}(s+t)) \\ &\quad - d(2^{-k-1}s, 2^{-k-1}s) - d(2^{-k-1}t, 2^{-k-1}t) \big) \\ &= a_n(s+t) - a_n(s) - a_n(t) - 2^n d(2^{-n}s, 2^{-n}t) \end{aligned}$$

whence and by (70)

$$2^n \Delta(2^{-n}s, 2^{-n}t) \leq b_n(s+t) - b_n(t) - b_n(s) - d(s,t) \leq 2^n \Gamma(2^{-n}s, 2^{-n}t).$$

Hence, according to (25) and (73), we have

$$\begin{aligned} \liminf_{n\to\infty} 2^n \Delta(2^{-n}s, 2^{-n}t) &\leq \Phi(s+t) - \Phi(t) - \Phi(s) \\ &\leq \limsup_{n\to\infty} 2^n \Gamma(2^{-n}s, 2^{-n}t), \qquad s, t \in D, s+t \in D, \end{aligned}$$

and (63) now shows that (42) is valid.

We need yet to prove the uniqueness of Φ. So, suppose that $\Phi_1, \Phi_2 : D \to \mathbb{R}$ satisfy

$$\Phi_i(s+t) = \Phi_i(s) + \Phi_i(t) - d(s,t), \qquad s, t \in E, s+t \in E, i = 1, 2, \tag{74}$$

and

$$\delta(s) \leq \phi(s) - \Phi_i(s) \leq \gamma(s), \qquad s \in E, i = 1, 2.$$

Then (59) holds, whence we have

$$|\Phi_1(s) - \Phi_2(s)| \leq \gamma(s) - \delta(s), \qquad s \in E. \tag{75}$$

Note also that (74) yields

$$\Phi_1(s+t) - \Phi_2(s+t) = \Phi_1(s) - \Phi_2(s) + \Phi_1(t) - \Phi_2(t), \qquad s \in E,$$

which implies

$$\Phi_1(2^n s) - \Phi_2(2^n s) = 2^n (\Phi_1(s) - \Phi_2(s)), \qquad s \in E, n \in \mathbb{N},$$

$$\Phi_1(2^{-n} s) - \Phi_2(2^{-n} s) = 2^{-n} (\Phi_1(s) - \Phi_2(s)), \qquad s \in E, n \in \mathbb{N}.$$

Hence, replacing s by $2^n s$ and next by $2^{-n}s$ in (75), we obtain

$$|\Phi_1(s) - \Phi_2(s)| \leq \frac{\gamma(2^n s) - \delta(2^n s)}{2^n}, \qquad s \in E, n \in \mathbb{N},$$

and

$$|\Phi_1(s) - \Phi_2(s)| \leq 2^n \big(\gamma(2^{-n}s) - \delta(2^{-n}s) \big), \qquad s \in E, n \in \mathbb{N}.$$

Consequently, it is easily seen that, if (44) or (67) is valid, we must have $\Phi_1 = \Phi_2$. This ends the proof. □

Theorems 7 and 8 yield the following generalization of Theorem 1.

Theorem 9. *Let E_1 be a real normed space, $E \subset E_1 \setminus \{0\}$ be nonempty, $E = 2E$, $\chi, \nu, p \in \mathbb{R}$, $p \neq 1$, $\chi \leq \nu$ and $\phi\colon E \to \mathbb{R}$ be a mapping with*

$$\begin{aligned}\chi(\|s\|^p + \|t\|^p) &\leq \phi(s+t) - \phi(s) - \phi(t) - d(s,t)\\ &\leq \nu(\|s\|^p + \|t\|^p), \qquad s,t \in E, s+t \in E.\end{aligned}$$

Then there is a unique solution $\Phi\colon E \to \mathbb{R}$ of Equation (42) such that, in the case $p < 1$,

$$\frac{\chi}{1-2^{p-1}}\|s\|^p \leq \Phi(s) - \phi(s) \leq \frac{\nu}{1-2^{p-1}}\|s\|^p, \qquad s \in E, \tag{76}$$

and, in the case $p > 1$,

$$\frac{\chi}{2^{p-1}-1}\|s\|^p \leq \phi(s) - \Phi(s) \leq \frac{\nu}{2^{p-1}-1}\|s\|^p, \qquad s \in E. \tag{77}$$

Proof. If $p < 1$, then by Theorem 7 (with $H = E_1$, $\Delta(s,t) = \chi(\|s\|^p + \|t\|^p)$ and $\Gamma(s,t) = \nu(\|s\|^p + \|t\|^p)$ for $s,t \in E_0$), there exists a unique solution $\Psi\colon E \to \mathbb{R}$ of Equation (42) satisfying inequalities (43). It is very easy to check that in this case (43) is exactly (77).

If $p > 1$, then we use Theorem 8 in a similar way. □

Remark 3. *Let E_1 be a real normed space, $E \subset E_1 \setminus \{0\}$ be nonempty, $\chi, \nu, p \in \mathbb{R}$, $p \neq 1$, $\chi \leq \nu$, and $d : E^2 \to \mathbb{R}$ be such that $d(s,t) \in [-\nu(\|s\|^p + \|t\|^p), -\chi(\|s\|^p + \|t\|^p)]$ for $s,t \in E$. Let $\phi : E \to \mathbb{R}$ be such that*

$$\phi(s+t) = \phi(s) + \phi(t), \qquad s,t \in E, s+t \in E. \tag{78}$$

Then

$$\begin{aligned}\chi(\|s\|^p + \|t\|^p) &\leq \phi(s+t) - \phi(t) - \phi(s) - d(s+t)\\ &= -d(s+t) \leq \nu(\|s\|^p + \|t\|^p), \qquad s,t \in D, s+t \in E.\end{aligned}$$

This example shows that the families of mappings ϕ considered in Theorems 7–9 are very large.

Corollary 1. *Let E_1 be a real normed space, $E \subset E_1 \setminus \{0\}$ and $E = 2E$. Assume that there exist $s_0, t_0, u_0 \in E$ such that $s_0 + t_0, t_0 + u_0, s_0 + t_0 + u_0 \in E$ and*

$$d(s_0 + t_0, u_0) + d(s_0, t_0) \neq d(s_0, t_0 + u_0) + d(t_0, u_0). \tag{79}$$

Then for every $\chi, \nu, p \in \mathbb{R}$, $p \neq 1$, there does not exist any mapping $\phi\colon E \to \mathbb{R}$ with

$$\begin{aligned}\chi(\|s\|^p + \|t\|^p) &\leq \phi(s+t) - \phi(s) - \phi(t) - d(s,t)\\ &\leq \nu(\|s\|^p + \|t\|^p), \qquad s,t \in E, s+t \in E.\end{aligned} \tag{80}$$

Proof. For the proof by contradiction suppose that there is $\phi\colon E_1 \to \mathbb{R}$ such that (80) holds. Then by Theorem 9 there exists a solution $\Phi\colon E \to \mathbb{R}$ of Equation (42), which means that

$$d(s,t) = \Phi(s+t) - \Phi(s) - \Phi(t), \qquad s,t \in E, s+t \in E,$$

and consequently

$$\begin{aligned}&d(s_0 + t_0, u_0) + d(s_0, t_0)\\ &\quad = \Phi(s_0 + t_0 + u_0) - \Phi(s_0 + t_0) - \Phi(u_0) + \Phi(s_0 + t_0) - \Phi(s_0) - \Phi(t_0)\\ &\quad = \Phi(s_0 + t_0 + u_0) - \Phi(s_0) - \Phi(t_0 + u_0) + \Phi(t_0 + u_0) - \Phi(t_0) - \Phi(u_0)\\ &\quad = d(s_0, t_0 + u_0) + d(t_0, u_0).\end{aligned}$$

Thus we obtain a contradiction to the assumption that (85) holds. □

Remark 4. *Equation (42) arises naturally in ([17], Theorem 2.2.4) in a description of generating functions of information measures (having certain branching property).*

Namely, let $k \in \mathbb{N}$ and

$$I = (0,1)^k := \{(t_1, \dots, t_k) \in \mathbb{R}^k : 0 < t_i < 1 \text{ for } i = 1, \dots, k\}.$$

According to ([17], Lemma 2.2.1 and Remark 2.2.3), every such generating function $G : I^2 \to \mathbb{R}$ is symmetric (i.e., $G(s,t) = G(t,s)$ for $s, t \in I$ with $s + t \in I$) and satisfies the cocycle equation

$$G(s+t, u) + G(s,t) = G(s, t+u) + G(t,u), \qquad s,t,u \in E, s+t+u \in E, \tag{81}$$

whence (see ([17], Theorem 2.2.4)) has the form

$$G(s,t) = g(s+t) - g(s) - g(t), \qquad s,t \in I, s+t \in I, \tag{82}$$

with some function $g : I \to \mathbb{R}$. Clearly, this function $g : I \to \mathbb{R}$ is a solution of Equation (42) (with $d = G$ and $E = I$), i.e.,

$$g(s+t) = g(s) + g(t) + G(s,t), \qquad s,t \in E, s+t \in E. \tag{83}$$

The next corollary shows that if two generating functions $G_1, G_2 : I^2 \to \mathbb{R}$ are 'close', then they can be represented in the form

$$G_i(s,t) = g_i(s+t) - g_i(s) - g_i(t), \qquad s,t \in I, s+t \in I, i = 1,2, \tag{84}$$

with functions $g_1, g_2 : I \to \mathbb{R}$ that are 'close'.

Corollary 2. *Let I be as in Remark 4 and $G_1, G_2 : I^2 \to \mathbb{R}$ be such that*

$$\begin{aligned} G_i(s+t,u) + G_i(s,t) = G_i(s,t+u) + G_i(t,u), \\ s,t,u \in E, s+t+u \in E, i = 1,2, \end{aligned} \tag{85}$$

and

$$G_i(s,t) = G_i(t,s), \qquad s,t \in I, s+t \in I, i = 1,2. \tag{86}$$

Assume that $g_1 : I \to \mathbb{R}$ fulfills the condition

$$G_1(s,t) = g_1(s+t) - g_1(s) - g_1(t), \qquad s,t \in I, s+t \in I, \tag{87}$$

and there are $\chi, \nu, p \in \mathbb{R}$, $\chi \leq \nu$, $p > 1$, with

$$\begin{aligned} \chi(\|s\|^p + \|t\|^p) &\leq G_1(s,t) - G_2(s,t) \\ &\leq \nu(\|s\|^p + \|t\|^p), \qquad s,t \in I, s+t \in I. \end{aligned} \tag{88}$$

Then there exists a unique $g_2 : I \to \mathbb{R}$ such that

$$G_2(s,t) = g_2(s+t) - g_2(s) - g_2(t), \qquad s,t \in I, s+t \in I, \tag{89}$$

$$\frac{\chi}{2^{p-1}-1}\|s\|^p \leq g_1(s) - g_2(s) \leq \frac{\nu}{2^{p-1}-1}\|s\|^p, \qquad s \in I. \tag{90}$$

Proof. Note that, in view of (87), inequality (88) can be written as

$$\begin{aligned} \chi(\|s\|^p + \|t\|^p) &\leq g_1(s+t) - g_1(s) - g_1(t) - G_2(s,t) \\ &\leq \nu(\|s\|^p + \|t\|^p), \qquad s,t \in I, s+t \in I. \end{aligned} \tag{91}$$

Hence, (arguing analogously as in the proof of Theorem 9) from Theorem 8 with $E = I$, we obtain that there exists a unique $g_2 : I \to \mathbb{R}$ such that (89) and (90) are valid. □

The result contained in Corollary 3 also can be expressed in the following somewhat different way.

Corollary 3. *Let I be as in Remark 4 and $G_1, G_2 : I^2 \to \mathbb{R}$ be such that (85) and (86) are valid. Assume that $g_1, g_2 : I \to \mathbb{R}$ fulfill (84) and there are $\chi, \nu, p \in \mathbb{R}$, $\chi \leq \nu$, $p > 1$, such that (88) holds.*

Then there exists $\alpha : I \to \mathbb{R}$ such that

$$\alpha(s+t) = \alpha(s) + \alpha(t), \qquad s,t \in I, s+t \in I, \tag{92}$$

$$\frac{\chi}{2^{p-1}-1}\|s\|^p \leq g_1(s) - g_2(s) - \alpha(s) \leq \frac{\nu}{2^{p-1}-1}\|s\|^p, \qquad s \in I. \tag{93}$$

Proof. As in the previous proof, in view of (87), inequality (88) implies (91). Hence, by Theorem 8 with $E = I$, (analogously as in the proof of Theorem 9) we obtain that there exists a unique $g_0 : I \to \mathbb{R}$ such that

$$G_2(s,t) = g_0(s+t) - g_0(s) - g_0(t), \qquad s,t \in I, s+t \in I, \tag{94}$$

$$\frac{\chi}{2^{p-1}-1}\|s\|^p \leq g_1(s) - g_0(s) \leq \frac{\nu}{2^{p-1}-1}\|s\|^p, \qquad s \in I. \tag{95}$$

Note that (84) and (94) imply that the function $\alpha = g_0 - g_2$ satisfies (92). Now, it is enough to notice that (95) yields (93). □

6. Conclusions

We presented new Ulam stability results for the nonhomogeneous Cauchy functional equation $f(x+y) = f(x) + f(y) + d(x,y)$ in the class of mappings f from a subset of a square symmetric groupoid $(H,+)$ into the set of reals $\mathbb{R}$. We employed the Banach limit as the main tool in our analysis. Moreover, we provided some interesting applications of our results. Potential future work could be to obtain analogous results for some other functional equations.

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