

Article

The Due Window Assignment Problems with Deteriorating Job and Delivery Time

Jin Qian and Yu Zhan *

College of Science, Northeastern University, Shenyang 110819, China; qianjin@mail.neu.edu.cn

* Correspondence: zhanyu@mail.neu.edu.cn

Abstract: This paper considers the single machine scheduling problem with due window, delivery time and deteriorating job, whose goal is to minimize the window location, window size, earliness and tardiness. Common due window and slack due window are considered. The delivery time depends on the actual processing time of past sequences. The actual processing time of the job is an increasing function of the start time. Based on the small perturbations technique and adjacent exchange technique, we obtain the propositions of the problems. For common and slack due window assignment, we prove that the two objective functions are polynomial time solvable in $O(n \log n)$ time. We propose the corresponding algorithms to obtain the optimal sequence, window location and window size.

Keywords: scheduling; due window; deteriorating job; delivery time; earliness; tardiness

MSC: 90B35



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1. Introduction

In order to gain the competitive advantage, modern operations management advocates that companies improve customer service. Operationally, good customer service means fulfilling orders within a specified time period. The research on due window usually assumes three situations. First, the jobs that are completed before the start time of the due window have earliness costs. Second, the jobs that are completed after the end of the due window have tardiness costs. Third, the jobs that are completed within the due window do not incur any costs. For cost considerations, we need to take into account the comprehensive factors, which contain earliness costs, tardiness costs, starting time of the window and window size.

Based on the research gap found in the literature review, this paper studies scheduling problems involving both simple linear degradation and past sequence dependent delivery time under the common and slack due window assignment. To the best of our knowledge, such articles are rare for the common or slack due window assignment. In this paper, the single machine scheduling problem with delivery time and deteriorating job is considered. The actual processing time of job is an increasing function of the start time. The delivery time depends on the actual processing time of past sequence. The goal is to minimize the window location, window size, earliness and tardiness. Small perturbation technique and adjacent switching technique are effective methods to deal with scheduling problems. Based on the small perturbations technique and adjacent exchange technique, we obtain the propositions of the method. By the propositions, we propose the polynomial time solvable algorithms to obtain the optimal sequence, window location and window size. The complexity of the algorithms is $O(n \log n)$.

2. Literature Review

In the field of production sequencing and scheduling, the scheduling problem with due window has received a lot of attention. In 1955, Jackson first proposed the concept of due date [1]. The scheduling problem of the optimal due date was considered by

Seidmann [2] and Panwalkar [3]. In 2015, Li et al. studied the scheduling problem with the slack due window, resource allocation and learning effect [4]. In 2015, Yang et al. studied the scheduling problem with deteriorating jobs, learning effect and due window [5]. In 2016, Yin et al. studied the scheduling problem with resource allocation and slack due window [6]. In 2017, Wang et al. studied the scheduling problem with the slack due window and controllable processing times [7]. In 2020, Wang et al. considered the scheduling problem with due window and position-dependent weights [8]. In 2020, Sun et al. considered the proportional flow shop scheduling problem with due window and position-dependent weights [9]. In 2021, Yue et al. studied a scheduling model with due window and the processing time as a special function [10].

The processing time of deteriorating job is variable. As the jobs are processed, and the processing times of jobs increase. The concept of deteriorating jobs was proposed by Gupta [11]. Under the common due window assignment, Yue et al. considered the scheduling problem with deteriorating jobs [12]. In 2018, Wang et al. studied the unrelated parallel processors scheduling problem with a maintenance activity and deterioration effect [13]. In 2019, Sun et al. considered the single machine scheduling problem with machine maintenance and deteriorating jobs [14]. In 2020, Cheng et al. studied the single machine scheduling problem whose processing time was a stage function of the start time [15]. In 2020, Liang et al. studied the scheduling problem with resource allocation and deteriorating jobs [16].

After a job is processed, the job should be delivered to the customer, which is called the past sequence dependent (psd) delivery time. Koulamas and Kyparisis first proposed the past sequence dependent delivery time [17]. In 2012, Liu et al. studied some new results on single machine scheduling with past sequence dependent delivery times [18]. In 2014, Zhao et al. studied single machine scheduling problems with general position-dependent processing times and past sequence dependent delivery times [19]. In 2020, Mir et al. studied parallel machine problem with delivery time and deterioration effect [20]. In 2021, Toksari et al. studied some scheduling problems with learning effect and past sequence dependent delivery times [21]. In 2021, Wang et al. studied single machine scheduling with position-dependent weights and delivery times [22].

The problem is described in the third Section 3. The research method is given in the forth Section 4. The summary is given in the last Section 5.

3. Notation and Problem Statement

There are n independent jobs $S = \{J_1, \dots, J_n\}$ processed at time t_0 ($t_0 > 0$) on a machine. The actual processing time p_i of J_i is

$$p_i = b_i t_i, \quad (1)$$

where b_i is the deterioration rate, t_i is the start time. The delivery time $q_{[i]}$ of $J_{[i]}$ is

$$q_{[i]} = r w_{[i]} = r \sum_{j=0}^{i-1} p_{[j]}, \quad (2)$$

where r is the delivery rate, $p_{[i]}$ is the actual processing time of $J_{[i]}$ and $w_{[i]} = \sum_{j=0}^{i-1} p_{[j]} = t_0 \prod_{j=1}^{i-1} (1 + b_{[j]})$, $p_{[0]} = t_0$. The subscript $[i]$ indicates that the job is arranged at the i th position. The completion time $C_{[i]}$ of $J_{[i]}$ is

$$C_{[i]} = w_{[i]} + p_{[i]} + q_{[i]} = t_0(1 + b_{[i]} + r) \prod_{j=1}^{i-1} (1 + b_{[j]}). \quad (3)$$

The common due window (CONW) and slack due window (SLKW) are considered. Let $[d'_i, d''_i]$ be the due window of J_i , where d'_i and d''_i are the start time and end time of due window, respectively. For the common due window, all jobs have the same start time d' and end time d'' of due window. For the slack due window, $d'_i = p_i + q_1$ and $d''_i = p_i + q_2$. $D = d''_i - d'_i$ is the size of due window, $D = q_2 - q_1$, q_1 and q_2 are the decision variables. C_i represents the completion time of J_i . E_i is the earliness of J_i , $E_i = \max\{0, d'_i - C_i\}$; T_i is the tardiness of J_i , $T_i = \max\{0, C_i - d''_i\}$.

The goal is to minimize the earliness, tardiness, start time of window and window size. The objective functions are

$$M = \sum_{i=1}^n [aE_{[i]} + cT_{[i]} + ed' + fD], \quad (4)$$

$$M = \sum_{i=1}^n [aE_{[i]} + cT_{[i]} + eq_1 + fD], \quad (5)$$

where a, c, e, f represent the unit cost of earliness, the unit cost of tardiness, and the unit cost of the start time and window size. If $C_{[h]} < d' < C_{[h+1]}$ and $C_{[l]} < d'' < C_{[l+1]}$ (CONW), the objective function is

$$\begin{aligned} M &= \sum_{i=1}^h a(d' - C_{[i]}) + \sum_{i=l+1}^n c(C_{[i]} - d'') + ned' + nf(d'' - d') \\ &= -a \sum_{i=1}^h C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [ha + n(e - f)]d' + [nf - (n - l)c]d''. \end{aligned} \quad (6)$$

If $d' = C_{[h+1]}$ and $d'' = C_{[l]}$ (CONW), the objective function is

$$\begin{aligned} M &= \sum_{i=1}^h a(C_{[h+1]} - C_{[i]}) + \sum_{i=l+1}^n c(C_{[i]} - C_{[l]}) + neC_{[h+1]} + nf(C_{[l]} - C_{[h+1]}) \\ &= -a \sum_{i=1}^h C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [ha + n(e - f)]C_{[h+1]} + [nf - (n - l)c]C_{[l]}. \end{aligned} \quad (7)$$

The objective function of SLKW model is similar to that of CONW model. By the three-region notation, the models can be defined as

$$\begin{aligned} 1|\text{CONW}, p_i = b_i t_i, q_{psd}| \sum_{i=1}^n [aE_{[i]} + cT_{[i]} + ed' + fD], \\ 1|\text{SLKW}, p_i = b_i t_i, q_{psd}| \sum_{i=1}^n [aE_{[i]} + cT_{[i]} + eq_1 + fD], \end{aligned}$$

where q_{psd} represents the past sequence dependent delivery time.

4. Research Method

In this section, we present several properties for an optimal schedule. First, we show the optimal common and slack due window positions by using the technique of small perturbations. Then, the sequence of jobs within different sets is determined by the adjacent exchange technique. Finally, we propose the polynomial time algorithms to obtain the optimal sequence.

4.1. The Problem 1|CONW, $p_i = b_i t_i, q_{psd} | \sum_{i=1}^n [aE_i + cT_i + ed' + fD]$

4.1.1. Optimal Properties of the Problem

Lemma 1. For any job sequence, d' of the optimal schedule is the start time of some job.

Proof. (a) When $C_{[h]} < d' < C_{[h+1]}$ and $d'' = C_{[l]}$, $0 \leq h < l \leq n$, $C_{[0]} = t_0$. The objective function is

$$M = -a \sum_{i=1}^h C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [ha + n(e - f)]d' + [nf - (n - l)c]C_{[l]}. \quad (8)$$

when $d' = C_{[h]}$, the objective function is

$$M_1 = -a \sum_{i=1}^{h-1} C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [(h - 1)a + n(e - f)]C_{[h]} + [nf - (n - l)c]C_{[l]}. \quad (9)$$

$$M - M_1 = [ha + n(e - f)][d' - C_{[h]}]. \quad (10)$$

when $d' = C_{[h+1]}$, the objective function is

$$M_2 = -a \sum_{i=1}^h C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [ha + n(e - f)]C_{[h+1]} + [nf - (n - l)c]C_{[l]}. \quad (11)$$

$$M - M_2 = [ha + n(e - f)][d' - C_{[h+1]}]. \quad (12)$$

when $ha + n(e - f) < 0$, $M > M_2$; otherwise, $M \geq M_1$.

(b) When $C_{[h]} < d' < C_{[h+1]}$ and $C_{[l]} < d'' < C_{[l+1]}$, $0 \leq h \leq l < n$. The objective function is

$$M = -a \sum_{i=1}^h C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [ha + n(e - f)]d' + [nf - (n - l)c]d''. \quad (13)$$

when $d' = C_{[h]}$, the objective function is

$$M_3 = -a \sum_{i=1}^{h-1} C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [(h - 1)a + n(e - f)]C_{[h]} + [nf - (n - l)c]d''. \quad (14)$$

$$M - M_3 = [ha + n(e - f)][d' - C_{[h]}]. \quad (15)$$

when $d' = C_{[h+1]}$, the objective function is

$$M_4 = -a \sum_{i=1}^h C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [ha + n(e - f)]C_{[h+1]} + [nf - (n - l)c]d''. \quad (16)$$

$$M - M_4 = [ha + n(e - f)][d' - C_{[h+1]}]. \quad (17)$$

when $ha + n(e - f) < 0$, $M > M_4$; otherwise, $M \geq M_3$. $\square$

Lemma 2. For any job sequence, d'' of the optimal schedule is the completion time of some job.

Proof. (a) When $d' = C_{[h]}$ and $C_{[l]} < d'' < C_{[l+1]}$, $0 \leq h \leq l < n$. The objective function is

$$M = -a \sum_{i=1}^{h-1} C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [(h - 1)a + n(e - f)]C_{[h]} + [nf - (n - l)c]d''. \quad (18)$$

when $d'' = C_{[l]}$, the objective function is

$$M_1 = -a \sum_{i=1}^{h-1} C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [(h-1)a + n(e-f)]C_{[h]} + [nf - (n-l)c]C_{[l]}. \quad (19)$$

$$M - M_1 = [nf - (n-l)c][d'' - C_{[l]}]. \quad (20)$$

when $d'' = C_{[l+1]}$, the objective function is

$$M_2 = -a \sum_{i=1}^{h-1} C_{[i]} + c \sum_{i=l+2}^n C_{[i]} + [(h-1)a + n(e-f)]C_{[h]} + [nf - (n-l-1)c]C_{[l+1]}. \quad (21)$$

$$M - M_2 = [nf - (n-l)c][d'' - C_{[l+1]}]. \quad (22)$$

when $nf - (n-l)c < 0$, $M > M_2$; otherwise, $M \geq M_1$.

(b) When $C_{[h]} < d' < C_{[h+1]}$ and $C_{[l]} < d'' < C_{[l+1]}$, $0 \leq h \leq l < n$. The objective function is

$$M = -a \sum_{i=1}^h C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [ha + n(e-f)]d' + [nf - (n-l)c]d''. \quad (23)$$

when $d'' = C_{[l]}$, the objective function is

$$M_3 = -a \sum_{i=1}^h C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [ha + n(e-f)]d' + [nf - (n-l)c]C_{[l]}. \quad (24)$$

$$M - M_3 = [nf - (n-l)c][d'' - C_{[l]}]. \quad (25)$$

When $d'' = C_{[l+1]}$, the objective function is

$$M_4 = -a \sum_{i=1}^h C_{[i]} + c \sum_{i=l+2}^n C_{[i]} + [ha + n(e-f)]d' + [nf - (n-l-1)c]C_{[l+1]}. \quad (26)$$

$$M - M_4 = [nf - (n-l)c][d'' - C_{[l+1]}]. \quad (27)$$

when $nf - (n-l)c < 0$, $M > M_4$; otherwise, $M \geq M_3$. $\square$

Lemma 3. For the optimal schedule, d' is the completion time $C_{[h]}$, d'' is the completion time $C_{[l]}$, $h = \lceil \frac{n(f-e)}{a} \rceil$, $l = \lceil \frac{n(c-f)}{c} \rceil$.

Proof. when $d' = C_{[h]}$ and $d'' = C_{[l]}$ for the optimal schedule, the objective function is

$$M = -a \sum_{i=1}^{h-1} C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [(h-1)a + n(e-f)]C_{[h]} + [nf - (n-l)c]C_{[l]}. \quad (28)$$

(a) When $d' = C_{[h-1]}$ and $d'' = C_{[l]}$, the objective function is

$$M_1 = -a \sum_{i=1}^{h-2} C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [(h-2)a + n(e-f)]C_{[h-1]} + [nf - (n-l)c]C_{[l]}. \quad (29)$$

$$M - M_1 = [(h-1)a + n(e-f)][C_{[h]} - C_{[h-1]}] \leq 0. \quad (30)$$

So, $h \leq 1 + \frac{n(f-e)}{a}$.

When $d' = C_{[h+1]}$ and $d'' = C_{[l]}$, the objective function is

$$M_2 = -a \sum_{i=1}^h C_{[i]} + c \sum_{i=l+1}^n C_{[i]} + [ha + n(e-f)]C_{[h+1]} + [nf - (n-l)c]C_{[l]}. \quad (31)$$

$$M - M_2 = [ha + n(e-f)][C_{[h]} - C_{[h+1]}] \leq 0. \quad (32)$$

So $\frac{n(f-e)}{a} \leq h \leq 1 + \frac{n(f-e)}{a}$, $h = \lceil \frac{n(f-e)}{a} \rceil$.

(b) When $d'' = C_{[l-1]}$ and $d' = C_{[h]}$, the objective function is

$$M_3 = -a \sum_{i=1}^{h-1} C_{[i]} + c \sum_{i=l}^n C_{[i]} + [(h-1)a + n(e-f)]C_{[h]} + [nf - (n-l+1)c]C_{[l-1]}. \quad (33)$$

$$M - M_3 = [nf - (n-l+1)c][C_{[l]} - C_{[l-1]}] \leq 0. \quad (34)$$

So, $l \leq 1 + \frac{n(c-f)}{c}$.

when $d'' = C_{[l+1]}$ and $d' = C_{[h]}$, the objective function is

$$M_4 = -a \sum_{i=1}^{h-1} C_{[i]} + c \sum_{i=l+2}^n C_{[i]} + [(h-1)a + n(e-f)]C_{[h]} + [nf - (n-l-1)c]C_{[l+1]}. \quad (35)$$

$$M - M_4 = [nf - (n-l)c][C_{[l]} - C_{[l+1]}] \leq 0. \quad (36)$$

So $\frac{n(c-f)}{c} \leq l \leq 1 + \frac{n(c-f)}{c}$, $l = \lceil \frac{n(c-f)}{c} \rceil$. $\square$

Suppose $d' = C_{[h]}$ and $d'' = C_{[l]}$ for the optimal schedule. Three sets $\Omega_1 = \{J_i \in \Omega | i \leq h-1\}$, $\Omega_2 = \{J_i \in \Omega | i = h\}$, $\Omega_3 = \{J_i \in \Omega | h+1 \leq i \leq l-1\}$, $\Omega_4 = \{J_i \in \Omega | i = l\}$, $\Omega_5 = \{J_i \in \Omega | l+1 \leq i \leq n\}$, Ω is the job sequence.

Lemma 4. In the optimal schedule, the jobs in Ω_1 can be processed in descending order of b_j .

Proof. J_g is at the u th position and J_k is at the $(u+1)$ th position in Ω_1 . $\pi_1 = \{J_1, \dots, J_g, J_k, \dots, J_n\}$, $1 \leq u < u+1 \leq h-1$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$M_1 - M_2 = at_0(b_k - b_g)(1+r) \prod_{i=1}^{u-1} (1+b_{[i]}). \quad (37)$$

If $b_g > b_k$, $M_1 < M_2$. $\square$

Lemma 5. The deterioration rate of the job in Ω_2 is less than the deterioration rate of any job in Ω_1 .

Proof. J_g is at the $(h-1)$ th position, and J_k is at the h th position. $\pi_1 = \{J_1, \dots, J_g, J_k, \dots, J_n\}$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$M_1 - M_2 = \{[(h-1)a + n(e-f)]r - a\}t_0(b_g - b_k) \prod_{i=1}^{h-2} (1+b_{[i]}). \quad (38)$$

If $b_k < b_g$, $M_1 < M_2$. $\square$

Lemma 6. In the optimal schedule, the jobs in Ω_3 can be processed in any order of b_j .

Proof. J_g is at the u th position and J_k which is at the $(u + 1)$ th position in Ω_3 , $\pi_1 = \{J_1, \dots, J_g, J_k, \dots, J_n\}$, $h + 1 \leq u < u + 1 \leq l - 1$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$M_1 = M_2. \quad (39)$$

□

Lemma 7. The deterioration rate of any job in Ω_3 is less than the deterioration rate of the job in Ω_4 .

Proof. J_g is at the u th position, $h + 1 \leq u \leq l - 1$. J_k is at the l th position. $\pi_1 = \{J_1, \dots, J_g, \dots, J_k, \dots, J_n\}$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, \dots, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$M_1 - M_2 = [nf - (n - l)c]t_0r(b_g - b_k) \prod_{i=1, i \neq u}^{l-1} (1 + b_{[i]}). \quad (40)$$

If $b_g < b_k$, $M_1 < M_2$. □

Lemma 8. In the optimal schedule, the jobs in Ω_5 can be processed in ascending order of b_j .

Proof. J_g is at the u th position and J_k is at the $(u + 1)$ th position in Ω_5 , $\pi_1 = \{J_1, \dots, J_g, J_k, \dots, J_n\}$, $l + 1 \leq u < u + 1 \leq n$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$M_1 - M_2 = ct_0(1 + r)(b_g - b_k) \prod_{i=1}^{u-1} (1 + b_{[i]}). \quad (41)$$

If $b_g < b_k$, $M_1 < M_2$. □

Lemma 9. The deterioration rate of the job in Ω_4 is less than the deterioration rate of any job in Ω_5 .

Proof. J_g is at the l th position, and J_k is at the $(l + 1)$ th position. $\pi_1 = \{J_1, \dots, J_g, J_k, \dots, J_n\}$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$M_1 - M_2 = \{cr + [nf - (n - l)c]\}t_0(b_g - b_k) \prod_{i=1}^{l-1} (1 + b_{[i]}). \quad (42)$$

If $b_g < b_k$, $M_1 < M_2$. □

Suppose J_g is at the u th position and J_k is at the v th position in the sequence $\pi_1 = \{J_1, \dots, J_g, \dots, J_k, \dots, J_n\}$, $1 \leq u \leq h$, $l \leq v \leq n$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, \dots, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$\begin{aligned} M_1 - M_2 = & t_0(b_g - b_k) \prod_{i=1}^{u-1} (1 + b_{[i]}) \{-a - a \sum_{i=u+1}^h (1 + r + b_{[i]}) \prod_{m=u+1}^{i-1} (1 + b_{[m]}) \\ & + c \sum_{i=l}^{v-1} (1 + r + b_{[i]}) \prod_{m=u+1}^{i-1} (1 + b_{[m]}) + cr \prod_{i=u+1}^{v-1} (1 + b_{[i]}) \\ & + [ha + n(e - f)](1 + r + b_{[h]}) \prod_{i=u+1}^{h-1} (1 + b_{[i]}) \\ & + [nf - (n - l + 1)c](1 + r + b_{[l]}) \prod_{i=u+1}^{l-1} (1 + b_{[i]})\}. \end{aligned} \quad (43)$$

Define

$$\begin{aligned}\pi_{uv} = & -a - a \sum_{i=u+1}^h (1+r+b_{[i]}) \prod_{m=u+1}^{i-1} (1+b_{[m]}) + c \sum_{i=l}^{v-1} (1+r+b_{[i]}) \prod_{m=u+1}^{i-1} (1+b_{[m]}) \\ & + cr \prod_{i=u+1}^{v-1} (1+b_{[i]}) + [ha + n(e-f)](1+r+b_{[h]}) \prod_{i=u+1}^{h-1} (1+b_{[i]}) \\ & + [nf - (n-l+1)c](1+r+b_{[l]}) \prod_{i=u+1}^{l-1} (1+b_{[i]}).\end{aligned}\quad (44)$$

If $\pi_{uv} > 0$, J_g should be at the u th position; otherwise, J_g should be at the v th position.

4.1.2. Optimal Algorithm

The Algorithm 1 is summarized as follows:

Algorithm 1 1|CONW, $p_j = b_j t_j, q_{psd} | \sum_{i=1}^n [aE_{[i]} + cT_{[i]} + ed' + fD]$

Input: t_0, a, c, e, f, b_j, r

Output: The optimal sequence, d', d''

- 1: **First step :** Sorted by $b_{[1]} \leq \dots \leq b_{[n]}$.
 - 2: **Second step:** Determine $h = \lceil \frac{n(f-e)}{a} \rceil, l = \lceil \frac{n(c-f)}{c} \rceil, d' = C_{[h]}, d'' = C_{[l]}$.
 - 3: **Third step:** Determine set Ω_3 that contains $l-h-1$ jobs, i.e., $b_{[1]}, \dots, b_{[l-h-1]}$.
 - 4: **Last step:** Determine the jobs of sets $\Omega_1 \cup \Omega_2$ and $\Omega_4 \cup \Omega_5$ by π_{uv} .
-

Theorem 1. For the problem 1|CONW, $p_i = b_i t_i, q_{psd} | \sum_{i=1}^n [aE_{[i]} + cT_{[i]} + ed' + fD]$, the complexity of the algorithm is $O(n \log n)$.

Proof. The first step requires $O(n \log n)$ time. The second and third steps are completed in constant time. The last step requires $O(n)$ time. So the complexity of the algorithm is $O(n \log n)$. $\square$

Example 1. There are 4 jobs processed in sequence on a single machine. $t_0 = 1, r = 0.1, b_1 = 2, b_2 = 0.3, b_3 = 1, b_4 = 0.7, a = 4, c = 5, e = 1, f = 2$.

Step 1. Because $b_2 \leq b_4 \leq b_3 \leq b_1$, $J_2 \rightarrow J_4 \rightarrow J_3 \rightarrow J_1$.

Step 2. Calculate the values $h = \lceil \frac{n(f-e)}{a} \rceil = 1, l = \lceil \frac{n(c-f)}{c} \rceil = 3, d' = C_{[1]}, d'' = C_{[3]}$.

Step 3. J_2 is contained in set Ω_3 which is at the second position.

Step 4. (a) When $u = 1$ and $v = 3$, $\pi_{[1][3]} = -3.61 < 0$, J_4 is determined at the third position;

(b) When $u = 1$ and $v = 4$, $\pi_{[1][4]} = 4.125 > 0$, J_3 is determined at the first position.

Therefore, the optimal sequence is $J_3 \rightarrow J_2 \rightarrow J_4 \rightarrow J_1$.

4.2. The Problem 1|SLKW, $p_i = b_i t_i, q_{psd} | \sum_{i=1}^n [aE_{[i]} + cT_{[i]} + eq_1 + fD]$

4.2.1. Optimal Properties of the Problem

Lemma 10. For any job sequence, q_1 of the optimal schedule is the $(1+r)$ times the sum of actual processing time for some jobs or t_0 .

Proof. (a) When $(1+r) \sum_{i=0}^{h-1} p_{[i]} < q_1 < (1+r) \sum_{i=0}^h p_{[i]}$ and $q_2 = (1+r) \sum_{i=0}^{l-1} p_{[i]}, 1 \leq h < l \leq n$. The objective function is

$$M = -a(1+r) \sum_{i=1}^h w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [ha + n(e-f)]q_1 + [nf - (n-l)c](1+r) \sum_{i=0}^{l-1} p_{[i]}. \quad (45)$$

when $q_1 = (1+r) \sum_{i=0}^{h-1} p_{[i]}$, the objective function is

$$M_1 = -a(1+r) \sum_{i=1}^{h-1} w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [(h-1)a + n(e-f)](1+r) \sum_{i=0}^{h-1} p_{[i]} + [nf - (n-l)c](1+r) \sum_{i=0}^{l-1} p_{[i]}. \quad (46)$$

$$M - M_1 = [ha + n(e-f)][q_1 - (1+r) \sum_{i=0}^{h-1} p_{[i]}]. \quad (47)$$

when $q_1 = (1+r) \sum_{i=0}^h p_{[i]}$, the objective function is

$$M_2 = -a(1+r) \sum_{i=1}^{h+1} w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [(h+1)a + n(e-f)](1+r) \sum_{i=0}^h p_{[i]} + [nf - (n-l)c](1+r) \sum_{i=0}^{l-1} p_{[i]}. \quad (48)$$

$$M - M_2 = [ha + n(e-f)][q_1 - (1+r) \sum_{i=0}^h p_{[i]}]. \quad (49)$$

when $ha + n(e-f) < 0$, $M > M_2$; otherwise, $M \geq M_1$.

(b) When $(1+r) \sum_{i=0}^{h-1} p_{[i]} < q_1 < (1+r) \sum_{i=0}^h p_{[i]}$ and $(1+r) \sum_{i=0}^{l-1} p_{[i]} < q_2 < (1+r) \sum_{i=0}^l p_{[i]}$, $1 \leq h \leq l \leq n$. The objective function is

$$M = -a(1+r) \sum_{i=1}^h w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [ha + n(e-f)]q_1 + [nf - (n-l)c]q_2. \quad (50)$$

when $q_1 = (1+r) \sum_{i=0}^{h-1} p_{[i]}$, the objective function is

$$M_3 = -a(1+r) \sum_{i=1}^{h-1} w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [(h-1)a + n(e-f)](1+r) \sum_{i=0}^{h-1} p_{[i]} + [nf - (n-l)c]q_2. \quad (51)$$

$$M - M_3 = [ha + n(e-f)][q_1 - (1+r) \sum_{i=0}^{h-1} p_{[i]}]. \quad (52)$$

when $q_1 = (1+r) \sum_{i=0}^h p_{[i]}$, the objective function is

$$M_4 = -a(1+r) \sum_{i=1}^h w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [ha + n(e-f)](1+r) \sum_{i=0}^h p_{[i]} + [nf - (n-l)c]q_2. \quad (53)$$

$$M - M_4 = [ha + n(e-f)][q_1 - (1+r) \sum_{i=0}^h p_{[i]}]. \quad (54)$$

when $ha + n(e-f) < 0$, $M > M_4$; otherwise, $M \geq M_3$. $\square$

Lemma 11. For any job sequence, q_2 of the optimal schedule is $(1+r)$ times sum of actual processing time for some jobs or t_0 .

Proof. (a) When $q_1 = (1+r) \sum_{i=0}^{h-1} p_{[i]}$ and $(1+r) \sum_{i=0}^{l-1} p_{[i]} < q_2 < (1+r) \sum_{i=0}^l p_{[i]}$, $1 \leq h \leq l \leq n$. The objective function is

$$M = -a(1+r) \sum_{i=1}^{h-1} w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [(h-1)a + n(e-f)](1+r) \sum_{i=0}^{h-1} p_{[i]} + [nf - (n-l)c]q_2. \quad (55)$$

when $q_2 = (1+r) \sum_{i=0}^{l-1} p_{[i]}$, the objective function is

$$M_1 = -a(1+r) \sum_{i=1}^{h-1} w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [(h-1)a + n(e-f)](1+r) \sum_{i=0}^{h-1} p_{[i]} + [nf - (n-l)c](1+r) \sum_{i=0}^{l-1} p_{[i]}. \quad (56)$$

$$M - M_1 = [nf - (n-l)c][q_2 - (1+r) \sum_{i=0}^{l-1} p_{[i]}]. \quad (57)$$

when $q_2 = (1+r) \sum_{i=0}^l p_{[i]}$, the objective function is

$$M_2 = -a(1+r) \sum_{i=1}^{h-1} w_{[i]} + c(1+r) \sum_{i=l+2}^n w_{[i]} + [(h-1)a + n(e-f)](1+r) \sum_{i=0}^{h-1} p_{[i]} + [nf - (n-l-1)c](1+r) \sum_{i=0}^l p_{[i]}. \quad (58)$$

$$M - M_2 = [nf - (n-l)c][q_2 - (1+r) \sum_{i=0}^l p_{[i]}]. \quad (59)$$

when $nf - (n-l)c < 0$, $M > M_2$; otherwise, $M \geq M_1$.

(b) When $(1+r) \sum_{i=0}^{h-1} p_{[i]} < q_1 < (1+r) \sum_{i=0}^h p_{[i]}$ and $(1+r) \sum_{i=0}^{l-1} p_{[i]} < q_2 < (1+r) \sum_{i=0}^l p_{[i]}$, $1 \leq h \leq l \leq n$. The objective function is

$$M = -a(1+r) \sum_{i=1}^h w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [ha + n(e-f)]q_1 + [nf - (n-l)c]q_2. \quad (60)$$

when $q_2 = (1+r) \sum_{i=0}^{l-1} p_{[i]}$, the objective function is

$$M_3 = -a(1+r) \sum_{i=1}^h w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [ha + n(e-f)]q_1 + [nf - (n-l)c](1+r) \sum_{i=0}^{l-1} p_{[i]}. \quad (61)$$

$$M - M_3 = [nf - (n-l)c][q_2 - (1+r) \sum_{i=0}^{l-1} p_{[i]}]. \quad (62)$$

when $q_2 = (1+r) \sum_{i=0}^l p_{[i]}$, the objective function is

$$M_4 = -a(1+r) \sum_{i=1}^h w_{[i]} + c(1+r) \sum_{i=l+2}^n w_{[i]} + [ha + n(e-f)]q_1 + [nf - (n-l-1)c](1+r) \sum_{i=0}^l p_{[i]}. \quad (63)$$

$$M - M_4 = [nf - (n-l)c][q_2 - (1+r) \sum_{i=0}^l p_{[i]}]. \quad (64)$$

when $nf - (n-l)c < 0$, $M > M_4$; otherwise, $M \geq M_3$. $\square$

Lemma 12. For the optimal schedule, q_1 is $(1+r) \sum_{i=0}^{h-1} p_{[i]}$, q_2 is $(1+r) \sum_{i=0}^{l-1} p_{[i]}$, $h = \lceil \frac{n(f-e)}{a} \rceil$, $l = \lceil \frac{n(c-f)}{c} \rceil$.

Proof. When $q_1 = (1+r) \sum_{i=0}^{h-1} p_{[i]}$ and $q_2 = (1+r) \sum_{i=0}^{l-1} p_{[i]}$ for the optimal schedule, the objective function is

$$M = -a(1+r) \sum_{i=1}^{h-1} w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [(h-1)a + n(e-f)](1+r) \sum_{i=0}^{h-1} p_{[i]} + [nf - (n-l)c](1+r) \sum_{i=0}^{l-1} p_{[i]}. \quad (65)$$

(a) When $q_1 = (1+r) \sum_{i=0}^{h-2} p_{[i]}$ and $q_2 = (1+r) \sum_{i=0}^{l-1} p_{[i]}$, the objective function is

$$M_1 = -a(1+r) \sum_{i=1}^{h-2} w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [(h-2)a + n(e-f)](1+r) \sum_{i=0}^{h-2} p_{[i]} + [nf - (n-l)c](1+r) \sum_{i=0}^{l-1} p_{[i]}. \quad (66)$$

$$M - M_1 = [(h-1)a + n(e-f)](1+r)p_{[h-1]} \leq 0. \quad (67)$$

So $h \leq 1 + \frac{n(f-e)}{a}$.

when $q_1 = (1+r) \sum_{i=0}^h p_{[i]}$ and $q_2 = (1+r) \sum_{i=0}^{l-1} p_{[i]}$, the objective function is

$$M_2 = -a(1+r) \sum_{i=1}^h w_{[i]} + c(1+r) \sum_{i=l+1}^n w_{[i]} + [ha + n(e-f)](1+r) \sum_{i=0}^h p_{[i]} + [nf - (n-l)c](1+r) \sum_{i=0}^{l-1} p_{[i]}. \quad (68)$$

$$M - M_2 = -[ha + n(e-f)](1+r)p_{[h]} \leq 0. \quad (69)$$

So, $\frac{n(f-e)}{a} \leq h \leq 1 + \frac{n(f-e)}{a}$, $h = \lceil \frac{n(f-e)}{a} \rceil$.

(b) When $q_2 = (1+r) \sum_{i=0}^{l-2} p_{[i]}$ and $q_1 = (1+r) \sum_{i=0}^{h-1} p_{[i]}$, the objective function is

$$M_3 = -a(1+r) \sum_{i=1}^{h-1} w_{[i]} + c(1+r) \sum_{i=l}^n w_{[i]} + [(h-1)a + n(e-f)](1+r) \sum_{i=0}^{h-1} p_{[i]} \\ + [nf - (n-l+1)c](1+r) \sum_{i=0}^{l-2} p_{[i]}. \quad (70)$$

$$M - M_3 = [nf - (n-l+1)c](1+r)p_{[l-1]} \leq 0. \quad (71)$$

So, $l \leq 1 + \frac{n(c-f)}{c}$.

When $q_2 = (1+r) \sum_{i=0}^l p_{[i]}$ and $q_1 = (1+r) \sum_{i=0}^{h-1} p_{[i]}$, the objective function is

$$M_4 = -a(1+r) \sum_{i=1}^{h-1} w_{[i]} + c(1+r) \sum_{i=l+2}^n w_{[i]} + [(h-1)a + n(e-f)](1+r) \sum_{i=0}^{h-1} p_{[i]} \\ + [nf - (n-l-1)c](1+r) \sum_{i=0}^l p_{[i]}. \quad (72)$$

$$M - M_4 = -[nf - (n-l)c](1+r)p_{[l]} \leq 0. \quad (73)$$

So $\frac{n(c-f)}{c} \leq l \leq 1 + \frac{n(c-f)}{c}$, $l = \lceil \frac{n(c-f)}{c} \rceil$. $\square$

Suppose $q_1 = (1+r) \sum_{i=0}^{h-1} p_{[i]}$ and $q_2 = (1+r) \sum_{i=0}^{l-1} p_{[i]}$ for the optimal schedule. Three sets $\Omega_1 = \{J_i \in \Omega | i \leq h-1\}$, $\Omega_2 = \{J_i \in \Omega | h \leq i \leq l-1\}$, $\Omega_3 = \{J_i \in \Omega | l \leq i \leq n\}$, Ω is the job sequence.

Lemma 13. In the optimal schedule, the jobs in Ω_1 can be processed in descending order of b_j .

Proof. J_g is at the u th position and J_k is at the $(u+1)$ th position in Ω_1 , $\pi_1 = \{J_1, \dots, J_g, J_k, \dots, J_n\}$, $1 \leq u < u+1 \leq h-1$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$M_1 - M_2 = -a(1+r)t_0(b_g - b_k) \prod_{i=1}^{u-1} (1+b_{[i]}). \quad (74)$$

If $b_g > b_k$, $M_1 < M_2$. $\square$

Lemma 14. In the optimal schedule, the jobs in Ω_2 can be processed in any order of b_j .

Proof. J_g is at the u th position and J_k is at the $(u+1)$ th position in Ω_2 , $\pi_1 = \{J_1, \dots, J_g, J_k, \dots, J_n\}$, $h \leq u < u+1 \leq l-1$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$M_1 = M_2. \quad (75)$$

$\square$

Lemma 15. In the optimal schedule, the jobs in Ω_3 can be processed in ascending order of b_j .

Proof. J_g is at the u th position and J_k is at the $(u + 1)$ th position in Ω_3 , $\pi_1 = \{J_1, \dots, J_g, J_k, \dots, J_n\}$, $l \leq v < v + 1 \leq n$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$M_1 - M_2 = c(1 + r)t_0(b_g - b_k) \prod_{i=1}^{u-1} (1 + b_{[i]}). \quad (76)$$

If $b_g < b_k$, $M_1 < M_2$. $\square$

Lemma 16. The deterioration rate of any job in Ω_2 is less than the deterioration rate of any job in Ω_1 .

Proof. J_g is at the $(h - 1)$ th position, and J_k is at the h th position. $\pi_1 = \{J_1, \dots, J_g, J_k, \dots, J_n\}$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$M_1 - M_2 = [(h - 1)a + n(e - f)](1 + r)t_0(b_g - b_k) \prod_{i=1}^{h-2} (1 + b_{[i]}). \quad (77)$$

If $b_k < b_g$, $M_1 < M_2$. $\square$

Lemma 17. The deterioration rate of any job in Ω_2 is less than the deterioration rate of any job in Ω_3 .

Proof. J_g is at the $(l - 1)$ th position and J_k is at the l th position. $\pi_1 = \{J_1, \dots, J_g, J_k, \dots, J_n\}$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$M_1 - M_2 = [nf - (n - l)c](1 + r)t_0(b_g - b_k) \prod_{i=1}^{l-2} (1 + b_{[i]}). \quad (78)$$

If $b_g < b_k$, $M_1 < M_2$. $\square$

Suppose J_g is at the u th position and J_k is at the v th position in the sequence $\pi_1 = \{J_1, \dots, J_g, \dots, J_k, \dots, J_n\}$, $1 \leq u \leq h - 1$, $l \leq v \leq n$. The objective function of π_1 is M_1 . Swap J_g and J_k to get sequence $\pi_2 = \{J_1, \dots, J_k, \dots, J_g, \dots, J_n\}$. The objective function of π_2 is M_2 .

$$\begin{aligned} M_1 - M_2 = & t_0(1 + r)(b_g - b_k) \prod_{i=1}^{u-1} (1 + b_{[i]}) \left\{ -a - a \sum_{i=u+1}^{h-2} \prod_{m=u+1}^i (1 + b_{[m]}) \right. \\ & + c \sum_{i=l}^{v-1} \prod_{m=u+1}^i (1 + b_{[m]}) + [nf - (n - l)c] \prod_{i=u+1}^{l-1} (1 + b_{[i]}) \\ & \left. + [(h - 1)a + n(e - f)] \prod_{i=u+1}^{h-1} (1 + b_{[i]}) \right\}. \end{aligned} \quad (79)$$

Define

$$\begin{aligned} \pi_{uv} = & -a - a \sum_{i=u+1}^{h-2} \prod_{m=u+1}^i (1 + b_{[m]}) + [(h - 1)a + n(e - f)] \prod_{i=u+1}^{h-1} (1 + b_{[i]}) \\ & + c \sum_{i=l}^{v-1} \prod_{m=u+1}^i (1 + b_{[m]}) + [nf - (n - l)c] \prod_{i=u+1}^{l-1} (1 + b_{[i]}). \end{aligned} \quad (80)$$

If $\pi_{uv} > 0$, J_g should be at the u th position; otherwise, J_g should be at the v th position.

4.2.2. Optimal Algorithm

The Algorithm 2 is summarized as follows:

Algorithm 2 $1|SLKW, p_i = b_i t_i, q_{psd} | \sum_{i=1}^n [aE_{[i]} + cT_{[i]} + eq_1 + fD]$

Input: t_0, a, c, e, f, b_j, r

Output: The optimal sequence, q_1, q_2

- 1: **First step** : Sorted by $b_{[1]} \leq \dots \leq b_{[n]}$.
 - 2: **Second step**: When $c \leq f \leq e$, $q_1 = q_2 = t_0$, $\Omega_1 \cup \Omega_2 = \phi$, $\Omega_3 = \{J_{[1]}, \dots, J_{[n]}\}$.
 - 3: **Third step**: When $c > f$ and $f \leq e$, $q_1 = t_0$, $q_2 = (1+r) \sum_{i=0}^{l-1} p_{[i]}$, $l = \lceil \frac{n(c-f)}{c} \rceil$. $\Omega_1 = \phi$, $\Omega_2 = \{J_{[1]}, \dots, J_{[l]}\}$, $\Omega_3 = \{J_{[l+1]}, \dots, J_{[n]}\}$.
 - 4: **Forth step**: Determine $h = \lceil \frac{n(f-e)}{a} \rceil$, $l = \lceil \frac{n(c-f)}{c} \rceil$, $q_1 = (1+r) \sum_{i=0}^{h-1} p_{[i]}$, $q_2 = (1+r) \sum_{i=0}^{l-1} p_{[i]}$.
 - 5: **Last step**: $\Omega_2 = \{J_{[1]}, \dots, J_{[l-h]}\}$. Determine the jobs of sets Ω_1 and Ω_3 by π_{uv} .
-

Theorem 2. For the problem $1|SLKW, p_i = b_i t_i, q_{psd} | \sum_{i=1}^n [aE_{[i]} + cT_{[i]} + eq_1 + fD]$, the complexity of the algorithm is $O(n \log n)$.

Proof. The first step requires $O(n \log n)$ time. The second, third and forth steps are completed in constant time. The last step requires $O(n)$ time. So the complexity of the algorithm is $O(n \log n)$. $\square$

Example 2. There are 4 jobs processed in sequence on a single machine. $t_0 = 1, r = 0.1, b_1 = 2, b_2 = 0.3, b_3 = 1, b_4 = 0.7, a = 4, c = 5, e = 1, f = 2$.

Step 1. Because $b_2 \leq b_4 \leq b_3 \leq b_1$, $J_2 \rightarrow J_4 \rightarrow J_3 \rightarrow J_1$.

Step 2. Calculate the values $h = \lceil \frac{n(f-e)}{a} \rceil = 1$, $l = \lceil \frac{n(c-f)}{c} \rceil = 3$, $q_1 = (1+r)t_0$, $q_2 = (1+r)w_{[3]}$.

Step 3. J_2 and J_4 are contained in set Ω_2 which are at the first and second position.

Step 4. J_3 and J_1 are contained in set Ω_3 , which are at the third and forth positions, $\Omega_1 = \phi$.

Therefore, the optimal sequence is $J_2 \rightarrow J_4 \rightarrow J_3 \rightarrow J_1$.

4.3. Discussion

This paper studies scheduling problems involving both simple linear degradation and past sequence dependent delivery time under the common and slack due window assignment. To the best of our knowledge, such articles are rare for the common or slack due window assignment. Small perturbation technique and adjacent switching technique are effective methods to deal with scheduling problems. Based on the small perturbations technique and adjacent exchange technique, we obtain the propositions of the method. By the propositions, the polynomial time algorithms are proposed to obtain the optimal sequence. However, not all scheduling problems can be solved by these techniques. Specific problems need to be analyzed in detail. In the future research, it is worth investigating multi-machine scheduling problems.

5. Conclusions

The single machine scheduling problem with deteriorating jobs and delivery time is considered under due window assignment. The goal is to minimize the window location, window size, earliness and tardiness. Based on the small perturbations technique and adjacent exchange technique, we obtain the propositions of the problems. However, not all scheduling problems can be solved by these techniques. Specific problems need to be analyzed in detail. For common and slack due window assignment, we prove that the two objective functions are polynomial time solvable in $O(n \log n)$ time. We propose the corresponding algorithms to obtain the optimal sequence, window location and window size. In the future, the multi-machine environment can be considered to expand the research,

i.e., parallel machines and flow shop setting. The more general deterioration processing time is also considered for a single machine scheduling or the multi-machine scheduling.

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